INVERSE PROBLEMS IN THEORY AND PRACTICE OF MEASUREMENTS AND METROLOGY

SEMENOV K. K., SOLOPCHENKO G. N.
Department of measurement informational technologies, St. Petersburg State Polytechnical University, 29, Polytechnicheskaya str., St. Petersburg, 195251, Russia

KREINOVICH V. YA.
Department of Computer Science, University of Texas at El Paso, 500 W. University, El Paso, TX 79968, USA

In this paper, we consider the role of inverse problems in metrology. We describe general methods of solving inverse problems which are useful in measurements practice. We also discuss how to modify these methods in situations in which there is a need for real-time data processing.

1. Introduction

What mathematical physics calls inverse problems is, in effect, the class of problems which are fundamental in measurement theory and practice [1, 2]. The main objective of such problems is to develop procedures for acquiring information about objects and phenomena, by decreasing the distortion caused by the measuring instruments. Lord Rayleigh was the first to formulate such problem in 1871, on the example of spectroscopy. His purpose was to maximally decrease the influence of diffraction. Rayleigh showed that in mathematical terms, the problem of reconstructing the actual spectrum $x(\nu)$ from the measured signal $y(u)$ can be reformulated as the problem of solving an integral equation

$$y(u) = \int_{-\infty}^{\infty} K(u-\nu)x(\nu)\, d\nu \quad , \tag{1}$$

where $K(u-\nu)$ is the apparatus function of the spectrometer – which describes the distortion caused by diffraction.

The relation between inverse problems and measurements was emphasized by G. I. Vasilenko [3], who explicitly stated that the main objective of the inverse problem is "restoring the signals" or "reduction to the ideal instrument".

Eq. (1) is the integral Fredholm's equation of first type; it can be represented in the form $y(u)=\mathbf{A}\,x(v)$, where **A** is a compact linear operator of convolution – which describes a generic analog transformation of a signal inside a measuring instrument – and $K(u-\text{v})$ is the kernel of this operator. From the mathematical viewpoint, the solution of Eq. (1) can be expressed as $x(v)=\mathbf{A}^{-1}\,y(u)$, where $\mathbf{A}^{-1}$ is the inverse operator to the compact operator **A**. From the practical viewpoint, however, we have a problem: it is known that such inverse operators are not bounded (see [4, p. 509]); as a result, a small noise in the measured signal can lead to drastic changes in the reconstructed solution $x(\text{v})$. Such problems are known as ill-posed. A general approach of generating a physically reasonable solution to this problem – known as regularization – was formulated by A. N. Tikhonov in 1963 [5].

2. Inverse problems in metrology

If we take into account the inaccuracy *e*(*u*) with which we register the output signal registration and the inaccuracy $\varepsilon(u-v)$ with which we know the apparatus function of the measurement device, then Eq. (1) will have the form $y(u)=\int_{-\infty}^{\infty}K_{\varepsilon}(u-\text{v})x(\text{v})d\text{v}+e(u)$. This equation with infinite (symmetric) integration limits describes spatial distortion processes in spectroscopy, chromatography, and in acoustic and other antenna-based measurements. For dynamic measurements – i.e., for measuring dynamic signals – the measurement result can only depend on the past values of the signal, so integration starts at 0:

$$y(t)=\int_{0}^{\infty}K_{\varepsilon}(t-\tau)x(\tau)d\tau+e(t)=\mathbf{A}_{\varepsilon}\,x(t)+e(t), \tag{2}$$

where $\mathbf{A}_{\varepsilon}$ is the convolution operator with the kernel $K_{\varepsilon}(t-\tau)$ (known with inaccuracy $\varepsilon(t-\tau)$) and *e*(*t*) is the additive noise.

The main idea behind Tikhonov's regularization is that we look for an (approximate) solution $\tilde{x}(t)$ to Eq. (2) by minimizing an appropriate stabilizing functional $\Omega(x(t))$ in Sobolev's space of smooth functions [5]. Usually, a functional $\Omega(x(t))=\beta_0\int_{0}^{\infty}\tilde{x}^2(t)dt+\beta_1\int_{0}^{\infty}[\tilde{x}'(t)]^2dt$, $\beta_1>0$ and $\beta_2>0$, is used on the condition that the difference between *y*(*t*) and $\mathbf{A}\,\tilde{x}(t)$ is of the same order as the error Δ caused by *e*(*t*) and $\varepsilon(t)$: $\|\mathbf{A}\tilde{x}(t)-y(t)\|^2=\Delta^2$. The Lagrange multiplier techniques reduces this constrained optimization problem to the unconstrained optimization of the functional [5]:

$$\min_{x(t)}\left[\left\|\mathbf{A}x(t)-y(t)\right\|^2+\alpha\Omega(x(t))\right], \tag{3}$$

where α is called a regularization parameter.

2.1. *The minimal modulus principle*

When we have an *a priori* information about the norm of the solution and/or its derivative, we can find α. In particular, we can use fuzzy (imprecise) expert *a priori* information [6]. In the absence of such *a priori* information, we can use the principle of minimal modulus [7,8] to select α.

This method is based on the fact that in the frequency domain, the stabilizing functional takes the form $\Omega(x(j\omega))=\beta_0\int_0^\infty |x(j\omega)|^2 d\omega+\beta_1\int_0^\infty \omega^2|x(j\omega)|^2 d\omega$, where j is imaginary unit and ω is circular frequency. The minimum of this functional is attained when the modulus $|x(j\omega)|$ is minimal.

Fourier transform of Eq. (2) leads to $y(j\omega)=K_\varepsilon(j\omega)\cdot x(j\omega)+e(j\omega)$. Based on 95% confidence intervals $K(\tau)-\varepsilon_{0,95}(\tau)\le K_\varepsilon(\tau)\le K(\tau)+\varepsilon_{0,95}(\tau)$ and $-e_{0,95}(t)\le e(t)\le e_{0,95}(t)$ in time domain, we can find the ellipses describing uncertainty in the frequency domain [9].

As a result, for every frequency ω_i we obtain two error-related ellipses in the complex plane: the first one centered in $y(j\omega_i)$ (Fourier transform of output signal) and another one centered at the value $K_\varepsilon(j\omega_i)$ (Fourier transform of apparatus function), as shown on Fig. 1. As shown in [7], for all values ω_i the value $\tilde{x}(j\omega_i)$ corresponding to the regularized solution is equal to $\tilde{x}(j\omega_i)=y_*(j\omega_i)/K^*(j\omega_i)$, where $K^*(j\omega_i)$ is point on the ellipse which is the farthest from the coordinates origin, and $y_*(j\omega)$ is the point on the corresponding ellipse that is the closest to the coordinates origin.

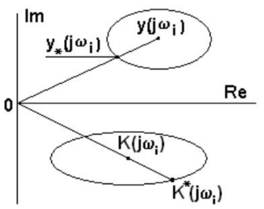

Fig. 1.

From Fig. 1, it is clear that this solution indeed minimizes the modulus $|x(j\omega)|$, and the condition $\left\|\mathbf{A}\tilde{x}(t)-y(t)\right\|^2=\Delta^2$ holds. After applying the inverse Fourier transform to the solution $\tilde{x}(j\omega_i)$, we get the desired regularized solution to the inverse problem – in other words, we achieve the desired reduction to the ideal measuring instrument. We have shown that this method works very well in many practical situations. This method also allows us to take into account the "objective" prior information about errors and also "subjective" information – as described by (possibly imprecise) expert estimates [6].

2.2. *The inverse filter*

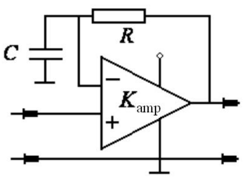

Fig. 2.

The principle of minimal modulus can only be used after the whole signal is measured. This is reasonable in spectroscopy and chromatography, but in processing dynamic signals, we often need to produce results in real time, before all the measurements are finished. This can be achieved by using an inverse filter, which can be physically implemented as one or several sequential dynamically stable circuits. An example is given on Fig. 2. If the amplifier gain is K_{amp} and *R* and *C* are the resistance and capacitance of inertial *RC*-circuit, then the complex frequency characteristic (CFC) of circuit on Fig. 2 is equal to

$$K(j\omega) = \frac{K_{yc}}{1+K_{yc}}(1+j\omega RC)\left[1+j\omega\frac{RC}{1+K_{\text{yc}}}\right]^{-1}.$$

This filter can be used if the modulus of CFC of the measuring instrument is monotonically decreasing. If the order of CFC for the measurement device is larger than one, then the positive result can be achieved with individual tuning of gain and parameters *R* and *C* for every circuit in the inverse filter.

In the report examples will be presented of using such inverse filters of different orders for ΣΔ – Analog-Digital Conversion.

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