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Orders on Intervals Over Partially Ordered Sets: Extending Allen's Algebra and Interval Graph Results

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Abstract To make a decision, we need to compare the values of quantities. In many practical situations, we know the values with interval uncertainty. In such situations, we need to compare intervals. Allen's algebra describes all possible relations between intervals on the real line which are generated by the ordering of endpoints; ordering relations between such intervals have also been well studied. In this paper, we extend this description to intervals in an arbitrary partially ordered set (poset). In particular, we explicitly describe ordering relations between intervals that generalize relation between points. As auxiliary results, we provide a logical interpretation of the relation between intervals, and extend the results about interval graphs to intervals over posets.

Keywords intervals in posets · Allen's algebra · interval orders · weak order · strong order · interval graph

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1 Introduction

Need to compare values. In order to compare different objects, we need to compare the values of their corresponding quantities. For example, one object is heavier than the other if its weight is larger than the weight of the other object, it is faster than the other if its velocity is larger, etc.

The result of comparing two values is often called a *relation* between the two values v and v':

- if $v < v'$, we say that v and v' are in relation $<$;
- if $v = v'$, we say that v and v' are in relation $=$; etc.

Important terminological comment. It should be mentioned that this usual use of the word "relation" can lead to confusion, since in mathematics, a *relation* is defined as a *set* of pairs: e.g., the relation $<$ between real numbers is defined as the set of all the pairs (a, b) for which $a < b$. To avoid confusion, in this paper, we will call the relation symbol between the two values an *individual relation*, or *i-relation*, for short.

Need to take into account interval uncertainty. In the ideal situation, when we represent the value in question as a real number $x \in \mathbb{R}$, and we know the exact values of the quantities for both objects $x, y \in \mathbb{R}$, we can compare these values and conclude either that the first value is smaller $x < y$, or that the first value is larger $y > x$, or that these values are equal $x = y$. In practice, we rarely know the exact values of the corresponding quantity: the values usually come from measurements, and measurement are never absolutely accurate – the measurement result $\widetilde{x}$ is, in general, different from the actual (unknown) value x of this quantity. In many practical situations, we only know the upper bound Δ on the absolute value $|\Delta x|$ of the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$.

In this case, once we know $\widetilde{x}$ and Δ, the only information that we have about the value x is that x belongs to the real interval $[\underline{x}, \overline{x}] \subseteq \mathbb{R}$, where $\underline{x} = \widetilde{x} - \Delta$ and $\overline{x} = \widetilde{x} + \Delta$.

i-Relations between values under interval uncertainty. If we know the two values with interval uncertainty, we may not be able to tell whether the first value is smaller or larger than the second value. For example, if the first value x is in the interval $[0.9, 1.1]$ and the second value y is in the interval $[1.0, 1.2]$, then it may be that $x = 0.9 < y = 1.2$, or it may be that $x = 1.1 > y = 1.0$.

Interval i-relations: what is known. Methods using intervals on the real line are prominent in quantitative analysis; see, e.g., Moore et al. (2009). Let $\mathbf{x} = [\underline{x}, \overline{x}] \subseteq \mathbb{R}$ be a generic real interval and let $\hat{\mathbb{R}}$ be the set of all real intervals. The possible i-relations between real intervals $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{R}}$ generated by i-relations between endpoints were explicated in the early 1980s in Allen (1983) (see also Nebel et al. (1995)); the class of such i-relations is known as *Allen's algebra.*

Specifically, if we are given two numbers x and y, then we have three possible i-relations between them: $<$, $=$, and $>$. In the interval case, instead of each number x (or y), we have *two* numbers: $\underline{x}$ and $\overline{x}$ (or $\underline{y}$ and $\overline{y}$). So, to fully describe the i-relation between the intervals, we need to describe all possible combinations of i-relations between these numbers. For non-degenerate intervals, we know the i-relation between x-bounds and the i-relation between the y-bounds: $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$. So, to describe possible i-relations between (non-degenerate) intervals, it is sufficient to describe the i-relations between the x-bounds and the y-bounds, i.e., the 4-tuple $(r_{--}, r_{-+}, r_{+-}, r_{++})$, where:

- r_{--} is the i-relation between $\underline{x}$ and $\underline{y}$;
- r_{-+} is the i-relation between $\underline{x}$ and $\overline{y}$;
- r_{+-} is the i-relation between $\overline{x}$ and $\underline{y}$;
- r_{++} is the i-relation between $\overline{x}$ and $\overline{y}$.

Each i-relation between numbers can have three possible values $<$, $=$, and $>$, so in principle, we can have 3^4 possible 4-tuples of i-relations. In our case, however, due to the properties of order, not all such 4-tuples are possible: e.g., if $\overline{x} < \underline{y}$, then, due to transitivity, we also have $\underline{x} < \overline{y}$, $\underline{x} < \overline{y}$, and $\underline{x} < \overline{y}$. What Allen did was described all possible 4-tuples of i-relations between the endpoints $\underline{x}$, $\overline{x}$, $\underline{y}$, $\overline{y}$ of two non-degenerate intervals $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$.

Possible orders between intervals: what is known. Allen's algebra defines many different relations between intervals. An important class of relations are orders. It is therefore natural to ask: which relations of Allen's algebra define orders?

At first glance, this question may seem easy to answer: we already have a full description of all possible i-relations between intervals, so we can simply check which of these i-relations define an order. However, the situation is not as simple: e.g., for real numbers, neither of the three i-relations $<$, $=$, and $>$ define an order. To get an order, we need to consider a *propositional combination* of these relations: e.g., the usual order $x \leq y$ means that $(x < y) \vee (x = y)$, and the order $x \geq y$ means that $(x > y) \vee (x = y)$.

Similarly, to describe orders between intervals, Allen considers propositional combinations. Specifically, since we know that the orders between numbers are $\leq$ and $\geq$, Allen considers propositional combinations of the corresponding relations $\leq$ and $\geq$. We want to describe orders that, in the degenerate case when $[\underline{x}, \overline{x}] = [x, x]$ and $[\underline{y}, \overline{y}] = [y, y]$, reduce to the usual numerical order $x \leq y$. Thus, it is reasonable to consider propositional combinations of the truth value of the following four relations: $\underline{x} \leq \underline{y}$, $\underline{x} \leq \overline{y}$, $\overline{x} \leq \underline{y}$, and $\overline{x} \leq \overline{y}$. It turns out that only two such combinations lead to orders that extend $x \leq y$:

- In the *strong order*, relation $\mathbf{x} \leq \mathbf{y}$ means that $\overline{x} \leq \underline{y}$, so that every value from the interval $[\underline{x}, \overline{x}]$ is smaller than or equal to every value from the interval $[\underline{y}, \overline{y}]$. This is the common and by far most prominent sense of "interval order", as advocated e.g., in Fishburn (1985).
- In the *weak order*, relation $\mathbf{x} \leq \mathbf{y}$ means that $\underline{x} \leq \underline{y}$ and $\overline{x} \leq \overline{y}$, so that the respective endpoints satisfy $\leq$ on the reals. This is a very natural sense of an interval order, for example saying that one event extended in time can be prior to another even if it is still underway when the subsequent event initiates.

If we do not require that the combination reduces to $x \leq y$ in the degenerate case, then we can additional orders, e.g., the *containment order* $\mathbf{x} \subseteq \mathbf{y}$ (see, e.g., Tanenbaum (1996)), in which $\mathbf{x} \leq \mathbf{y}$ means that $\underline{x} \geq \underline{y}$ and $\overline{x} \leq \overline{y}$.

Relation between different interval orders. It is worth mentioning that the strong order implies the weak order.

Also, the weak order and the containment order are generally conjugate, in that pairs of real intervals $\mathbf{x}, \mathbf{y} \in \hat{\mathbb{R}}$ are comparable in exactly one or the other[1]. In fact, the weak order is actually just the Cartesian

[1] Note that this is *almost* always true, in that endpoint equality also has to be taken into account, yielding intervals which are equal at *one* endpoint comparable in both orders.

product $\leq \times \leq$ of the natural order $\leq$ on $\mathbb{R}$, whereas the containment order is defined as $\geq \times \leq$ (Papadakis et al. (2010)).

Need to consider partially ordered sets. The set of all the real numbers is totally (linearly) ordered: for every two numbers x and y, either $x < y$ or $y < x$, or $x = y$. In many practical situations, however, we are interested in the quantities which are only partially ordered.

For example, in space-time geometry, we do not have the exact location of an event in space-time, we usually only know the events $\underline{x}$ that can causally affect the given event x ($\underline{x} \leq x$) and the events $\overline{x}$ that can causally be affected by x ($x \leq \overline{x}$). In this case, the only information that we have about the event x is that it belongs to the interval $[\underline{x}, \overline{x}] = \{x : \underline{x} \leq x \leq \overline{x}\}$. This description looks similar to the above interval case, but the important difference is that the causality relation is space-time is only a partial order: there exist events x and y for which $x \not\leq y$ and $y \not\leq x$; such events are called *incompatible* and denoted by $x \parallel y$; see, e.g., Feynman et al. (2005); Kronheimer et al. (1967); Misner et al. (1973); Zapata et al. (2012) and references therein.

There are other cases when we have intervals in partially ordered spaces: e.g., preferences often form only a partial order; see, e.g., Kosheleva et al. (1998); Tanenbaum et al. (2004); Xu et al. (2012). So, if we know the lower and upper bounds, we end up with an interval in a partially ordered space.

In order theory, i.e., in the mathematics of lattices and partially ordered sets (see, e.g., Davey at al. (1990)), intervals are readily available. Recall that for two elements x and y in a partially ordered set, we have the following possible relations: $x < y$, $x = y$, $x > y$, and $x \parallel y$ (meaning that x and y are incompatible, i.e., that $x \neq y$, $x \not< y$, and $x \not> y$). For two elements x, y where $x \leq y$, then we simply define the interval $\mathbf{x} = [x, y]$ as the set $\mathbf{x} = \{z : x \leq z \leq y\}$.

Need to extend interval orders to partially ordered sets. Since in practice, we encounter intervals in partially ordered spaces, it is desirable to describe possible relations between such intervals – i.e., to extend interval orders and Allen's algebra to partially ordered sets. In particular, we would like to list all possible ordering relations between two intervals in a partially ordered set.

Remaining open problem. In this paper, we consider the case when we know a preceding event $\underline{x}$ and a following event $\overline{x}$. In this case, the only information that we have about the event of interest x is that it belongs to the *interval* $[\underline{x}, \overline{x}]$. In principle, we may have *several* lower bounds and several upper bounds. In this more general case, the set of possible values of x is an intersection of several intervals. In other cases, we may have an even more general set. It is desirable to further generalize the results of this paper by extending these results from intervals to intersections of intervals – and to more general sets[2].

2 Possible i-Relations Between Intervals

Comparison between points x and y: reminder.

Definition 1 Let X be a partially ordered set. By an *i-relation* between elements $x, x' \in X$, we mean:

- a symbol $<$ if $x < x'$;
- a symbol $=$ if $x = x'$;
- a symbol $>$ if $x > x'$;
- a symbol $\parallel$ if $x \parallel x'$.

i-relations between points can be illustrated on the example of a 2-D analog of the causality relation of special relativity. In special relativity, it is assumed that all the speeds are limited by the speed of light c. Thus, an event (x_0, x_1) occurring at moment x_0 at a spatial point x_1 can influence an event (y_0, y_1) if and only if $y_0 > x_0$ and during the time $y_0 - x_0$, the signal traveling with speed of light c can cover the distance $|x_1 - y_1|$ between the corresponding spatial points, i.e., if

$$x = (x_0, x_1) \leq y = (y_0, y_1) \Leftrightarrow c \cdot (y_0 - x_0) \geq |x_1 - y_1|.$$

This relation is illustrated by Fig. 1, in which:

- the symbol $>$ marks all the points y for which $x > y$,
- the symbol $<$ marks all the points y for which $x < y$,
- etc.

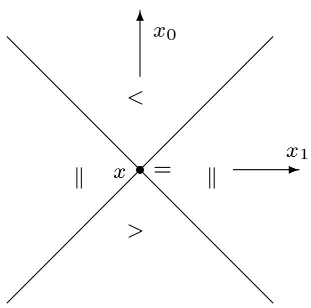

Fig. 1 Partial order corresponding to special relativity

[2] The authors are thankful to an anonymous referee for this interesting suggestion.

Comparison between a point x and an interval $[\underline{y}, \overline{y}]$. We have already described possible i-relations between points. Each point x can be viewed as a "degenerate" interval $[\underline{x}, \overline{x}]$. Thus, we have covered the case when both intervals are degenerate.

Before we consider the general case of comparing intervals, let us first consider the case when the first interval is still degenerate (i.e., is a point), but the second interval $[\underline{y}, \overline{y}]$ is non-degenerate (i.e., $\underline{y} < \overline{y}$). In this case, instead of a *single* i-relation r ($>$, $<$, $=$, or $\|$) between x and y, we have *two* i-relations:

- the i-relation r_- between x and $\underline{y}$ (for which $x r_- \underline{y}$), and
- the i-relation r_+ between x and $\overline{y}$ (for which $x r_+ \overline{y}$).

Our objective is to describe possible pairs $p = (r_-, r_+)$ of such i-relations. To come up with such a description, let us introduce the following order $\preceq$ between four possible relations:

$$> \prec =, \quad > \prec \|, \quad > \prec <, \quad = \prec <, \quad \| \prec <$$

This order is illustrated in Fig. 2. The order $\prec$ means that $>$ precedes all other i-relations, and $=$ and $\|$ precede $<$. Alternatively, we can say that $<$ follows all other relations, and $=$ and $\|$ follow $>$.

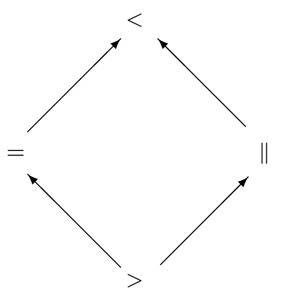

Fig. 2 Order $\prec$ between i-relations

Proposition 1 *Let X be a partially ordered set, and let x, $\underline{y}$, and $\overline{y}$, be elements of X for which $\underline{y} \le \overline{y}$. Then $r_- \preceq r_+$, where:*

- *r_- is the i-relation between x and $\underline{y}$, and*
- *r_+ is the i-relation between x and $\overline{y}$.*

Remark 1 For reader's convenience, all the proofs are placed in the special (last) Proofs section.

Proposition 2 *For a pair $p = (r_-, r_+)$ of i-relations, the following two conditions are equivalent to each other:*

- *there exists a partially ordered set and values $\underline{x} < \overline{x}$ and y from this set for which:*
  - *the i-relation r_- is the i-relation between x and $\underline{y}$, and*
  - *the i-relation r_+ is the i-relation between x and $\overline{y}$.*
- *the pair $p = (r_-, r_+)$ is equal to one of the following pairs:*

$$(<,<), \quad (=,<), \quad (\|,<), \quad (>,<),$$

$$(\|,\|), \quad (>,=), \quad (>,\|), \quad (>,>).$$

The possibility of all eight pairs can be illustrated on the example of the following points from the above-described 2-D analog of special relativity relation; see Fig. 3. Here, $\underline{y} = (-1, 0)$, $\overline{y} = (1, 0)$, and, in addition to points $x = \underline{y}$ and $x = \overline{y}$ that correspond to pairs $(=, <)$ and $(>, =)$, we have six more points x corresponding to six other possible pairs. Dashed lines describe ordering between these six points x.

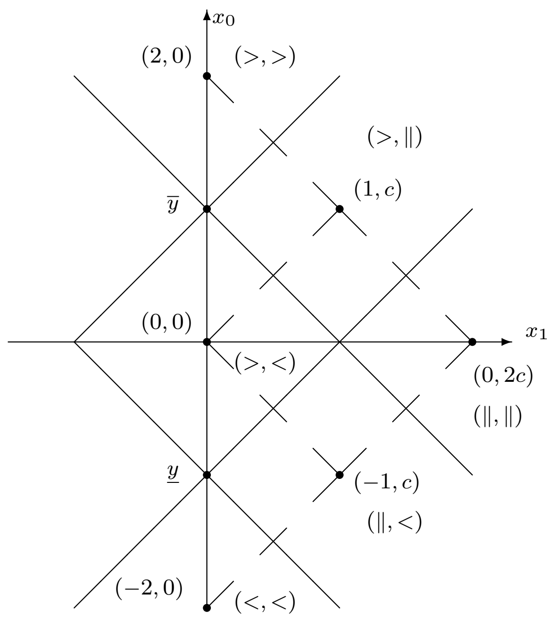

Fig. 3 All eight pairs $p = (p_-, p_+)$ are possible

Comment. To avoid confusion, it is worth mentioning that Fig. 2 and Fig. 3 describe *different* orders: Fig. 2 describes orders between *i-relations*, while Fig. 3 describes orders between *elements* of the original partially ordered set. As a result:

- In Fig. 2, the relation $<$ is on top, because once $x < \underline{y}$ (i.e., once the i-relation between x and $\underline{y}$ is $<$), then, due to transitivity, we also have $x < \overline{y}$, meaning that the i-relation between x and $\overline{y}$ is also $<$.

- In Fig. 3, the pair $(>,>)$ is on top, because for the largest elements x, we have $x > \underline{y}$ and $x > \overline{y}$ and thus, the corresponding pair of i-relations is $p = (r_-, r_+) = (>,>)$.

Comparison between two non-degenerate intervals. We would like to describe all possible i-relations between intervals generated by i-relation between endpoints. In the previous text, we have described all such i-relations for the situations in which at least one of the intervals is degenerate. So, to complete our description, it is sufficient to describe all possible i-relations between two non-degenerate intervals.

Specifically, we will use the above result about i-relations between a number and an interval to describe possible i-relations between two non-degenerate intervals $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$. In this case, instead of two i-relations r_- and r_+, we have four i-relations:

- the i-relation r_{--} between $\underline{x}$ and $\underline{y}$,
- the i-relation r_{-+} between $\underline{x}$ and $\overline{y}$,
- the i-relation r_{+-} between $\overline{x}$ and $\underline{y}$, and
- the i-relation r_{++} between $\overline{x}$ and $\overline{y}$.

Our objective is to describe possible combinations $(r_{--}, r_{-+}, r_{+-}, r_{++})$ of such relations.

Each such combination can be represented as a pair (p_-, p_+) of pairs $p_- \stackrel{\text{def}}{=} (r_{--}, r_{-+})$ and $p_+ \stackrel{\text{def}}{=} (r_{+-}, r_{++})$:

- the pair p_- describes the i-relations between the point $\underline{x}$ and the (endpoints of the) interval $[\underline{y}, \overline{y}]$, and
- the pair p_+ describes the i-relations between the point $\overline{x}$ and the (endpoints of the) interval $[\underline{y}, \overline{y}]$.

To come up with the desired description, let us introduce the order $\preceq$ between possible pairs as in Fig. 4. This means that the pair $(<,<)$ precedes all other pairs, etc.

Proposition 3 *Let X be a partially ordered set, and let $\underline{x} < \overline{x}$, and $\underline{y} < \overline{y}$ be elements of X. Then $p_- \preceq p_+$, where:*

- *p_- is a pair of i-relations between $\underline{x}$ and $\underline{y}$ and between $\underline{x}$ and $\overline{y}$, and*
- *p_+ is a pair of i-relations between $\overline{x}$ and $\underline{y}$ and between $\overline{x}$ and $\overline{y}$.*

Proposition 4 *For a combination of i-relations $(r_{--}, r_{-+}, r_{+-}, r_{++})$, the following two conditions are equivalent to each other:*

- *there exists a partially ordered set and values $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$ from this set for which:*
    - *r_{--} is the i-relation between $\underline{x}$ and $\overline{y}$,*
    - *r_{-+} is the i-relation between $\underline{x}$ and $\overline{y}$,*
    - *r_{+-} is the i-relation between $\overline{x}$ and $\underline{y}$, and*
    - *r_{++} is the i-relation between $\overline{x}$ and $\overline{y}$.*
- *the combination $(r_{--}, r_{-+}, r_{+-}, r_{++})$ is equal to one of the following combinations:*

$$(<,<,<,<),\quad (<,<,=,<),\quad (<,<,\|,<),$$
$$(<,<,>,<),\quad (<,<,\|,\|),\quad ,(<,<,>,=),$$
$$(<,<,>,\|),\quad (<,<,>,>),\quad (=,<,>,<),$$
$$(=,<,>,=),\quad (=,<,>,\|),\quad (=,<,>,>),$$
$$(\|,<,\|,<),\quad (\|,<,>,<),\quad (\|,<,\|,\|),$$
$$(\|,<,>,=),\quad (\|,<,>,\|),\quad (\|,<,>,>),$$
$$(>,<,>,<),\quad (>,<,>,=),\quad (>,<,>,\|),$$
$$(>,<,>,>),\quad (\|,\|,\|,\|),\quad (\|,\|,>,\|),$$
$$(\|,\|,>,>),\quad (>,=,>,>),\quad (>,\|,>,\|),$$
$$(>,\|,>,>),\quad (>,>,>,>).$$

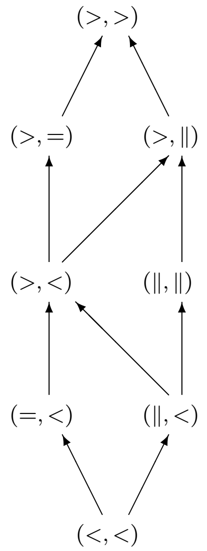

Fig. 4 Order $\preceq$ between pairs p

3 Possible Orders Between Intervals Generated By Orders Between Endpoints

It is desirable to describe all possible orders between intervals generated by orders between endpoints. In addition to describing all possible i-relations between intervals, we may also want to describe possible *orders* between intervals generated by orders between endpoints – in a general partially ordered case. Specifically, we would like to describe all i-relations that, in the degenerate case, when each interval consists of a single element, reduce to the order $x \leq y$ between the elements.

Why this problem is non-trivial. At first glance, this problem is simple, since we already have a full description of all possible i-relations, so we can simply check which of these i-relations describe order.

However, the situation is not as simple, because in addition to the original "basic" i-relations, we can have *propositional combinations* of these relations.

For example, the usual order $x \le y$ means

$$(x < y) \vee (x = y).$$

Similarly, the strong order $\overline{x} \le \underline{y}$ means that we have one of the following tuples $(r_{--}, r_{-+}, r_{+-}, r_{++})$:

$$(<,<,<,<), \quad (<,<,=,<), \quad (=,<,=,<),$$

$$(<,<,=,=), \text{ or } (=,=,=,=).$$

While the number of possible combinations is finite, it is huge, and simply checking all these combinations is not simple. Thus, instead of using the above classification, we start "from scratch", and use a different approach.

Towards describing all possible orders between intervals generated by orders between endpoints. In the interval case:

- instead of a single element x, we have two endpoints $\underline{x}$ and $\overline{x}$, and
- instead of a single element y, we have two endpoints $\underline{y}$ and $\overline{y}$.

Thus, instead of a single i-relation $x \le y$, we have $2 \times 2 = 4$ possible i-relations: $\overline{x} \le \underline{y}$, $\underline{x} \le \underline{y}$, $\overline{x} \le \overline{y}$, and $\underline{x} \le \overline{y}$.

In addition to these relations, we can also have propositional combinations of these i-relations, i.e., i-relations of the type

$$[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}] \Leftrightarrow P\left(\underline{x} \le \underline{y}, \underline{x} \le \overline{y}, \overline{x} \le \underline{y}, \overline{x} \le \overline{y}\right) \qquad (1)$$

for some propositional function $P : \{T, F\}^4 \to \{T, F\}$ that transforms four truth values of the four i-relations into a single truth value describing whether the intervals $[\underline{x}, \overline{x}]$ and $[\underline{y}, \overline{y}]$ are related.

Let us denote the truth value of the i-relation $\overline{x} \le \underline{y}$ between:

- the upper endpoint $\overline{x}$ of the first interval and
- the lower endpoint $\underline{y}$ of the second interval

by t_{+-}. Here:

- the first subscript $+$ means that we take the upper endpoint of the first interval, and
- the second subscript $-$ means that we take the lower endpoint of the second interval.

Similarly:

- the i-relation $\underline{x} \le \underline{y}$ between the lower endpoints will be denoted by t_{--};
- the i-relation $\underline{x} \le \overline{y}$ between the lower endpoint $\underline{x}$ of the first interval and the upper endpoint $\overline{y}$ of the second interval will be denoted by t_{-+}, and
- the i-relation $\overline{x} \le \overline{y}$ between the upper endpoints will be denoted by t_{++}.

In these terms, the strong order relation $\overline{x} \le \underline{y}$ means that $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = t_{+-}$, i.e., that $[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}]$ if and only if $\overline{x} \le \underline{y}$. Similarly, the weak order relation $\underline{x} \le \underline{y} \,\&\, \overline{x} \le \overline{y}$ corresponds to $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = t_{--} \,\&\, t_{++}$.

It is important to mention that not all combinations of truth values t_{--}, t_{-+}, t_{+-}, and t_{++} are possible: since the endpoints of each interval are related by the order, i.e., since $\underline{x} \le \overline{x}$ and $\underline{y} \le \overline{y}$, some of the four i-relations between endpoints imply each other. For example, by transitivity, $\underline{x} \le \overline{x}$ and $\overline{x} \le \underline{y}$ imply that $\underline{x} \le \underline{y}$. In general, we have the implications pictured in Fig. 5.

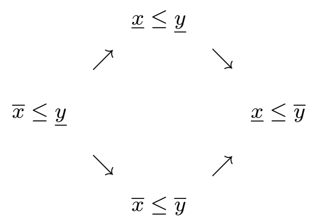

Fig. 5 Implications between truth values $t_{\pm\pm}$

Let us enumerate all possible combinations.

Proposition 5 *For a combination*

$$t = (t_{--}, t_{-+}, t_{+-}, t_{++})$$

of four truth values, the following two conditions are equivalent to each other:

- *there exists a partial ordered set and value $\underline{x} \le \overline{x}$ and $\underline{y} \le \overline{y}$ from this set for which:*
  - *t_{--} is the truth value of the relation $\underline{x} \le \underline{y}$,*
  - *t_{-+} is the truth value of the relation $\underline{x} \le \overline{y}$,*
  - *t_{+-} is the truth value of the relation $\overline{x} \le \underline{y}$, and*
  - *t_{++} is the truth value of the relation $\overline{x} \le \overline{y}$;*
- *the combination t is equal to one of the following combinations:*

$$(T,T,T,T), \quad (T,T,F,T), \quad (T,T,F,F),$$

$$(F,T,F,T), \quad (F,T,F,F), \quad (F,F,F,F).$$

In the following text, the set of all possible combination will be denoted by

$$\mathcal{S} = \{(T,T,T,T),(T,T,F,T),(T,T,F,F),$$
$$(F,T,F,T),(F,T,F,F),(F,F,F,F)\}.$$

Let us describe the general situation in precise terms.

Definition 2

- By a *propositional formula*, we mean a function $P : \mathcal{S} \to \{T,F\}$.
- Let X be a partially ordered set, and let P be a propositional formula. By a *relation corresponding to P* (or *P-relation*, for short), we mean the following relation between intervals $[\underline{x},\overline{x}]$ and $[\underline{y},\overline{y}]$:

$$[\underline{x},\overline{x}] \le_P [\underline{y},\overline{y}] \Leftrightarrow P\left(\underline{x} \le \underline{y}, \underline{x} \le \overline{y}, \overline{x} \le \underline{y}, \overline{x} \le \overline{y}\right).$$

We want the resulting i-relation between intervals to generalize the i-relation $x \le y$ between the elements: in the degenerate case when $\underline{x} = \overline{x} = x$ and $\underline{y} = \overline{y} = y$, the new i-relation should transform into the i-relation $x \le y$. In other words:

- If $x \le y$, then, in the degenerate case, all four i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$ coincide with $x \le y$ and are thus true. So, in this case, we should have $P(T,T,T,T) = T$.
- Similarly, if $x \not\le y$, then, in the degenerate case, all four i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$ coincide with $x \le y$ and are thus false. So, in this case, we should have $P(F,F,F,F) = F$.

Definition 3 We say that a P-relation $\le_P$ *extends the original order* if the corresponding propositional formula satisfies the condition

$$P(T,T,T,T) = T \text{ and } P(F,F,F,F) = F.$$

According to Definition 3, the ideal case is when all four i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$ are true. It may be possible, however, that the two intervals are related by a new interval i-relation $\le$ even when some of these relations are false. It is reasonable to require the following.

- Suppose that we have $[\underline{x},\overline{x}] \le [\underline{y},\overline{y}]$ for some case when some of the four i-relations are true and some are false.
- Then, if we keep true i-relations true and make some false i-relations true, we should have even fewer reasons not to conclude that that $[\underline{x},\overline{x}] \le [\underline{y},\overline{y}]$.
- Thus, we should be able to conclude that in the new situation, intervals are related.

In other words, if the formula $P(t_{--},t_{-+},t_{+-},t_{++})$ is true for some values t_{ij}, and we keep all the values $t_{ij} = T$ unchanged, but change some false values $t_{ij} = F$ to T, then, for the changed values t'_{ij}, the formula P should still be true.

Definition 4 We say that a P-relation $\le_P$ is *reasonable* if for every two sequences of truth values $t_{--},t_{-+},t_{+-},t_{++}$ and $t'_{--},t'_{-+},t'_{+-},t'_{++}$ for which $P(t_{--},t_{-+},t_{+-},t_{++}) = T$ and $t_{ij} = T$ implies $t'_{ij} = T$ for every i,j, we have

$$P(t'_{--},t'_{-+},t'_{+-},t'_{++}) = T.$$

This definition can be reformulated in more traditional mathematical terms

Definition 5 Let $F \le T$ be an ordering on the set of truth values. We say that a P-relation $\le_P$ is *monotonic* if $t_{ij} \le t'_{ij}$ for all i,j imply that

$$P(t_{--},t_{-+},t_{+-},t_{++}) \le P(t'_{--},t'_{-+},t'_{+-},t'_{++}).$$

Proposition 6 *A P-relation $\le_P$ is reasonable (in the sense of Definition 4) if and only if it is monotonic.*

Finally, since we want to define an order, we want to make sure that the relation (1) is transitive. The following result describes all possible monotonic transitive p-relations that extend the original order.

Proposition 7 *A P-relation $\le_P$ is monotonic, transitive, and extends the original order if and only if the corresponding propositional formula P has one of the following forms:*

1. *$P(T,T,T,T) = T$ and $P(t_{--},t_{-+},t_{+-},t_{++}) = F$ for all other tuples $(t_{+-},t_{--},t_{++},t_{-+})$;*
2. *$P(T,t_{-+},t_{+-},t_{++}) = T$ and $P(F,t_{-+},t_{+-},t_{++}) = F$ for all t_{-+}, t_{+-}, and t_{++};*
3. *$P(t_{--},t_{-+},t_{+-},T) = T$ and $P(t_{--},t_{-+},t_{-+},F) = F$ for all t_{--}, t_{-+}, and t_{+-};*
4. *$P(T,t_{-+},t_{+-},T) = T$ for all t_{+-} and t_{-+} and $P(t_{--},t_{-+},t_{+-},t_{++}) = F$ for all other tuples.*

As a result, we arrive at the following corollary:

Corollary 1 *There are four and only four monotonic transitive P-relations $\le_P$ that extends the original order:*

1. *$\overline{x} \le \underline{y}$ (strong order);*
2. *$\underline{x} \le \underline{y}$ (ordering of lower endpoints);*
3. *$\overline{x} \le \overline{y}$ (ordering of upper endpoints);*
4. *$\underline{x} \le \underline{y}$ and $\overline{x} \le \overline{y}$ (weak order).*

Remaining open problem. In this section, we only considered orders between intervals generated by orders between endpoints, i.e., generated by the truth values of the four ordering i-relations $\underline{x} \le \underline{y}$, $\underline{x} \le \overline{y}$, $\overline{x} \le \underline{y}$, and $\overline{x} \le \overline{y}$. In principle, we can add equalities to this list of i-relations, in which case we can have additional orders, such as $\underline{x} < \underline{y} \vee (\underline{x} = \underline{y} \,\&\, \overline{x} = \overline{y})$. It would be nice to describe all such possible orders[3].

4 First Auxiliary Topic: Interval Relations Reformulated In Logical Terms

It is worth mentioning that the four i-relations t_{ij} correspond to different selection of quantifiers:

Proposition 8

$$\overline{x} \le \underline{y} \Leftrightarrow \forall x_{\in[\underline{x},\overline{x}]} \, \forall y_{\in[\underline{y},\overline{y}]} \, (x \le y);$$

$$\underline{x} \le \underline{y} \Leftrightarrow \exists x_{\in[\underline{x},\overline{x}]} \, \forall y_{\in[\underline{y},\overline{y}]} \, (x \le y);$$

$$\overline{x} \le \overline{y} \Leftrightarrow \exists y_{\in[\underline{y},\overline{y}]} \, \forall x_{\in[\underline{x},\overline{x}]} \, (x \le y);$$

$$\underline{x} \le \overline{y} \Leftrightarrow \exists x_{\in[\underline{x},\overline{x}]} \, \exists y_{\in[\underline{y},\overline{y}]} \, (x \le y).$$

5 Second Auxiliary Topic: Extending Interval Graphs to Partially Ordered Sets

What is an interval graph. In many practical applications – e.g., in scheduling, in bioinformatics – it is useful to consider *interval graphs*, i.e., undirected graphs in which vertices are real-line intervals, and two vertices are connected by an edge if and only if the corresponding intervals intersect; see, e.g., Cormen at al. (2009); Fishburn (1985); Mandoiu et al. (2008).

In precise terms, an undirected graph is defined as a pair (V, E), where V is a set whose elements are called *vertices*, and E is a set of unordered pairs of vertices (v, v'); such pairs are called *edges*. A graph (V, E) is called an *interval graph* if it is possible to put into correspondence, to every vertex $v \in V$, an interval $I(v)$ so that the vertices v and v' are connected by an edge $(v, v') \in E$ if and only if the corresponding intervals have a non-empty intersection: $I(v) \cap I(v') = \emptyset$.

In view of the fact that the notion of an interval graph is practically important, efficient algorithms have been developed for checking whether a given graph can be represented as such an interval graph.

[3] The authors are thankful to an anonymous referee for this interesting suggestion.

Natural question. A natural question is: what if instead of real-valued intervals, we allow intervals in a general partially ordered set? It turns out that in this case, any undirected graph can be represented as an intersection graph of intervals:

Proposition 9 *For every undirected graph (V, E), there exists a poset $(X, \le)$ and a mapping I that maps $v \in V$ into intervals $I(v) \subseteq X$ so that vertices v and v' are connected by an edge if and only if corresponding intervals have a non-empty intersection: $I(v) \cap I(v') = \emptyset$.*

It is worth mentioning that this result holds for infinite graphs as well.

6 Proofs

Proof of Proposition 1. To prove this proposition, let us consider all possible values of the i-relation r_-: $<$, $=$, $\|$, and $>$.

1°. If r_- is $<$, i.e., if $x < \underline{y}$, then, since $\underline{y} \le \overline{y}$, by transitivity, we get $x < \overline{y}$, i.e., r_+ is $<$. Thus, we have $r_i \preceq r_+$.

2°. If the i-relation r_- is equality $=$, i.e., if $x = \underline{y}$, then, since $\underline{y} \le \overline{y}$, we have $x \le \overline{y}$, i.e., $x < \overline{y}$ or $x = \overline{y}$. In this case, the i-relation r_+ is either $<$ or $=$. In both cases, $r_- \preceq r_+$.

3°. If r_- is $\|$, i.e., $x \parallel \underline{y}$, then it is impossible to have $x \ge \overline{y}$. Indeed, in this case, we would have $x \ge \underline{y}$, while we have $x \parallel \underline{y}$. Thus, the i-relation r_+ between x and $\overline{y}$ can only be $\|$ or $<$. In both cases, we have $r_- \preceq r_+$.

4°. Finally, if r_- is $>$, then $r_- \preceq r_+$ for all possible i-relations r^+.

The proposition is proven. □

Proof of Proposition 2.

1°. An example presented in the main text shows for each of the eight pairs of i-relations (r_-, r_+) from the formulation of this proposition, there exists a partially ordered set and values x and $\underline{y} < \overline{y}$ for which

$$x r_- \underline{y} \text{ and } x r_+ \overline{y}.$$

2°. So, to complete the proof, it is sufficient to prove that for every partially ordered set and for all values x and $\underline{y} < \overline{y}$ from this set, the corresponding pair of i-relations (r_-, r_+) is equal to one of the pairs listed in the formulation of the Proposition.

To prove this, we will consider two possible cases: when x is equal to one of the points $\underline{y}$ and $\overline{y}$, and when x is different from both these points.

2.1°. When x is equal to one of the points $\underline{y}$ or $\overline{y}$, then, due to $\underline{y} < \overline{y}$, we get pairs $(=,<)$ and $(>,=)$.

2.2°. When x is different from both points $\underline{y}$ and $\overline{y}$, then for each of these points, we have three possible i-relations with x: $<$, $>$, and $\|$. In principle, there are $3 \times 3 = 9$ possible pairs, but the pairs

$$(\|,>), \quad (<,>), \text{ and } (<,\|)$$

are impossible due to Proposition 1. Thus, we get exactly one of the six remaining pairs – which are listed in the formulation of the Proposition. □

Proof of Proposition 3. We will prove that $p_- \preceq p_+$ by considering all possible pairs p_-.

1°. Let us first consider the case when $p_- = (>,>)$, i.e., when $\underline{x} > \underline{y}$ and $\underline{x} > \overline{y}$. Then, due to $\overline{x} > \underline{x}$, we have $\overline{x} > \underline{y}$ and $\overline{x} > \overline{y}$, i.e., $p_+ = (>,>)$. Thus, in this case, $p_- \preceq p_+$.

2°. For $p_- = (>,=)$, we have $\underline{x} > \underline{y}$ and $\underline{x} = \overline{y}$. In this case, from $\overline{x} > \underline{x}$, we conclude that $\overline{x} > \underline{y}$ and $\overline{x} > \overline{y}$, i.e., that $p_+ = (>,>)$. Thus, $p_- \preceq p_+$.

3°. For $p_- = (>,\|)$, we have $\underline{x} > \underline{y}$ and $\underline{x} \parallel \overline{y}$. In this case, from $\overline{x} > \underline{x}$, we conclude that $\overline{x} > \underline{y}$. We cannot have $\overline{x} \le \overline{y}$, because this would imply $\underline{x} < \overline{y}$ while we have $\underline{x} \parallel \overline{y}$. Thus, we can have either $\overline{x} > \overline{y}$ or $\overline{x} \parallel \overline{y}$, i.e., $p_+ = (>,>)$ or $p_+ = (>,\|)$. In both cases, $p_- \preceq p_+$.

4°. For $p_- = (>,<)$, we have $\underline{x} > \underline{y}$ and $\underline{x} < \overline{y}$. In this case, from $\overline{x} > \underline{x}$, we conclude that $\overline{x} > \underline{y}$. Thus, p_+ is equal to one of the pairs $(>,r_{++})$: $(>,>)$, $(>,=)$, $(>,<)$, and $(>,\|)$. In all four cases, $p_- \preceq p_+$.

5°. For $p_- = (\|,\|)$, we have $\underline{x} \parallel \underline{y}$ and $\underline{x} \parallel \overline{y}$. In this case, similarly to Part 3 of this proof, each of the i-relations r_{+-} and r_{++} is equal to either $>$ or to $\|$. If r_{++} is $>$, i.e., if $\overline{x} > \overline{y}$, then we have $\overline{x} > \underline{y}$, and $p_+ = (>,>)$. If $r_{++} = \|$, then we can have $p_+ = (\|,\|)$ and $p_+ = (>,\|)$. In all three cases, we have $p_- \preceq p_+$.

6°. For $p_- = (=,<)$, we have $\underline{x} = \underline{y}$ and $\underline{x} < \overline{y}$. In this case, $\underline{x} < \overline{x}$ implies that $\overline{x} > \underline{y}$. Thus, p_+ is equal to one of the pairs $(>,r_{++})$: $(>,>)$, $(>,=)$, $(>,<)$, and $(>,\|)$. In all four cases, $p_- \preceq p_+$.

7°. For $p_- = (\|,<)$, we have $\underline{x} \parallel \underline{y}$ and $\underline{x} < \overline{y}$. In this case, we cannot have $\overline{x} \le \underline{y}$, since then, due to $\underline{x} < \overline{x}$, we will have $\underline{x} < \underline{y}$, while we have $\underline{x} \parallel \underline{y}$. Thus, the first component r_{+-} of the pair $p_+ = (r_{+-}, r_{++})$ is either $>$ or $\|$. For all such pairs p_+, we have $p_- = (\|,<) \preceq p_+$.

8°. Finally, if $p_- = (<,<)$, then $p_- \preceq p_+$ for all pairs p_+.

The proposition is proven. □

Proof of Proposition 4.

1°. Let us first prove that if there exists a partially ordered set and values $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$, then the corresponding combination of i-relations $(r_{--}, r_{-+}, r_{+-}, r_{++})$ coincides with one of the combinations listed in the formulation of the Proposition.

Indeed, due to Proposition 3, we must have $p_- \preceq p_+$. In the formulation of the Proposition, we listed, for each pair p_-, all possible pairs $p_- \preceq p_+$, with two exceptions: combinations (p_-, p_+) corresponding to $p_- = p_+ = (=,<)$ and $p_- = p_+ = (>,=)$.

So, to prove the first implication, it is sufficient to prove that these two combinations are impossible. Let us do it case by case.

1.1°. If $p_- = (=,<)$, this means that $\underline{x} = \underline{y}$. Since we consider non-degenerate intervals, for which $\underline{x} < \overline{x}$, we cannot have $\overline{x} = \underline{y}$ and thus, we cannot have

$$p_+ = (=,<).$$

1.2°. Similarly, if $p_- = (>,=)$, this means that $\underline{x} = \overline{x}$. Since we consider non-degenerate intervals, for which $\underline{x} < \overline{x}$, we cannot have $\overline{x} = \overline{x}$ and thus, we cannot have

$$p_+ = (>,=).$$

The first implication is proven.

2°. To complete the proof of the Proposition, we must prove that for every combination (p_-, p_+) listed in the formulation, there exists a partially ordered set and values $\underline{x} < \overline{x}$ and $\underline{y} < \overline{y}$ that lead to this very combination.

For combinations for which $p_- \neq p_+$, we can have, as examples, points $\underline{y} = (-1,0) < \overline{y} = (1,0)$ described after the formulation of Proposition 2, and as the points $\underline{x} < \overline{x}$, points from this description corresponding to pairs p_- and p_+ (recall that in that example, we have one point y for each of the six pairs $p = (r_-, r_+)$).

For combinations for which $p_- = p_+$, we can take nearby points $\underline{x} < \overline{x}$ from the zone of all points x corresponding to this pair $p_- = p_+$; see Fig. 6.

The statement is proven. □

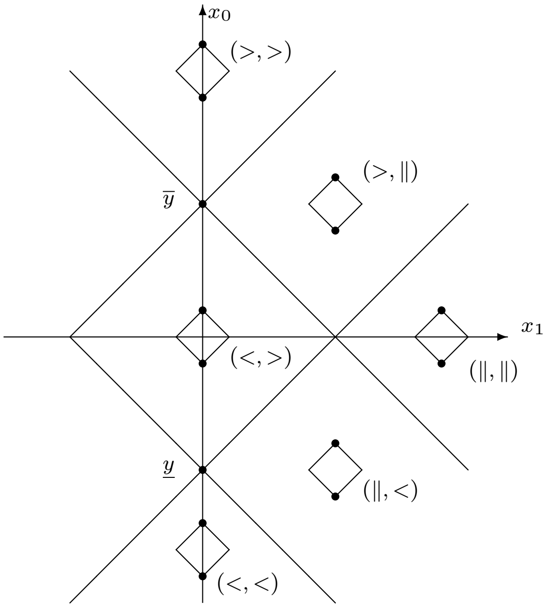

Fig. 6 Combinations corresponding to $p_- = p_+$.

Proof of Proposition 5.

1°. Let us first prove that for every ordered set, the corresponding combination of truth values coincides with one of the six combinations listed in the formulation of the proposition.

2°. Let us start our analysis with the truth value of the third variable t_{+-}. This value can take either the value T or the value F. Let us consider these two values one by one.

3°. Let us first consider the case when $t_{+-} = T$, i.e., the case of combinations $(t_{--}, t_{-+}, T, t_{++})$. In this case, $\overline{x} \le \underline{y}$, and so, due to the above implications, all three other i-relations t_{--}, t_{-+}, and t_{++} are also true. Thus, we get the combination (T, T, T, T).

4°. Let us now consider the case when $t_{+-} = F$, i.e., the case of combinations of the type $(t_{--}, t_{-+}, F, t_{++})$.

Let us consider possible truth values of the first variable t_{--}, first the value T and then the value F.

5°. Let us consider combinations of the type (T, t_{-+}, F, t_{++}), in which $t_{--} = T$. In such situations, $t_{--} = T$ implies that $t_{-+} = T$, so the value of the second variable t_{-+} is always true.

The fourth variable t_{++} can be either true or false. Thus, in this situation, we have two possible combinations: (T, T, F, T) and (T, T, F, F).

6°. Let us now consider situations of the type (F, t_{-+}, F, t_{++}) in which not only $t_{+-} = F$, but also $t_{--} = F$. In such situations, the fourth variable t_{++} can be either true or false. Let us consider these two cases one by one.

6.1°. If $t_{++} = T$, then, by the above implications, we get $t_{-+} = T$. Thus, we get a combination (F, T, F, T).

6.2°. If $t_{++} = F$, then we can have two possible values of t_{-+}: true and false. Thus, we get two possible combinations: (F, T, F, F) and (F, F, F, F).

7°. We have proven for all partially ordered sets, the combination of truth values coincides with one of the six given combinations. To complete the proof, it is sufficient to prove that all six combinations are indeed possible. Indeed:

7.1°. The combination (T, T, T, T) occurs, e.g., for $[\underline{x}, \overline{x}] = [0, 1]$ and $[\underline{y}, \overline{y}] = [2, 3]$.

7.2°. The combination (T, T, F, T) occurs, e.g., for $[\underline{x}, \overline{x}] = [0, 2]$ and $[\underline{y}, \overline{y}] = [1, 3]$.

7.3°. The combination (T, T, F, F) occurs, e.g., for $[\underline{x}, \overline{x}] = [0, 3]$ and $[\underline{y}, \overline{y}] = [1, 2]$.

7.4°. The combination (F, T, F, T) occurs, e.g., for $[\underline{x}, \overline{x}] = [1, 2]$ and $[\underline{y}, \overline{y}] = [0, 3]$.

7.5°. The combination (F, T, F, F) occurs, e.g., for $[\underline{x}, \overline{x}] = [1, 3]$ and $[\underline{y}, \overline{y}] = [0, 2]$.

7.6°. The combination (F, F, F, F) occurs, e.g., for $[\underline{x}, \overline{x}] = [2, 3]$ and $[\underline{y}, \overline{y}] = [0, 1]$. □

Proof of Proposition 6.

1°. Let us first prove that if the P-relation $\le_P$ is reasonable, then it is monotonic. Let us assume that $t_{ij} \le t'_{ij}$ for all i, j, and let us prove that

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}).$$

Our proof depends on the truth value of $P(t_{--}, t_{-+}, t_{+-}, t_{++})$.

1.1°. Let us first consider the case when

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) = F.$$

By definition of the order $\le$ on the set of truth values, the false value F is smaller than or equal to anything. Thus, in this case, the desired inequality

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++})$$

is indeed satisfied.

1.2°. Let us now consider the case when

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) = T.$$

In this case, if $t_{ij} = T$, then, by definition of the order $\le$ on the set of truth values, the inequality $t_{ij} \le t'_{ij}$ implies that $t'_{ij} = T$. Thus, due to the fact that the p-relation is reasonable, we get $P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}) = T$ and thus,

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}).$$

2°. Let us now prove that if the p-relation P is monotonic, then it is reasonable. Indeed, let us make the following two assumptions:

- let us assume that P is monotonic, i.e., that $t_{ij} \le t'_{ij}$ implies that

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}),$$

  and
- let us also assume that for every i, j, $t_{ij} = T$ implies that $t'_{ij} = T$.

Let us prove that in this case, we have $P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}) = T$.

To prove this, let us first prove that $t_{ij} \le t'_{ij}$ for all i, j. Indeed, if $t_{ij} = F$, then this inequality is satisfied because the false value F is smaller than or equal to anything. If $t_{ij} = T$, then, by our assumption, we have $t'_{ij} = T$ and thus, $t_{ij} \le t'_{ij}$. Since $t_{ij} \le t'_{ij}$ for all i, j, by monotonicity, we get

$$P(t_{--}, t_{-+}, t_{+-}, t_{++}) \le P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}).$$

Due to $P(t_{--}, t_{-+}, t_{+-}, t_{++}) = T$, this implies that $P(t'_{--}, t'_{-+}, t'_{+-}, t'_{++}) = T$. The statement is proven, and so is the proposition. □

Proof of Proposition 7.

1°. To describe a P-relation, we need to describe the propositional formula P, i.e., we need to describe the values of the function P on all six tuples from the set $\mathcal{S}$. We know, from the fact that the P-relation $\le_P$ extends the original order, that $P(T, T, T, T) = T$ and $P(F, F, F, F) = F$. So, to complete our description, it is sufficient to describe four remaining values: $P(F, T, T, T)$, $P(T, T, F, F)$, $P(F, T, F, T)$, and $P(F, T, F, F)$.

2°. Let us prove, by contradiction, that

$$P(F, T, F, F) = F.$$

Indeed, if we had $P(F, T, F, F) = T$, then we would have $[0, 2] \le [-3, 1]$. Indeed, in this case, out of four possible i-relations t_{ij}, only the i-relation t_{-+} $(0 \le 1)$ is true. Thus, the corresponding tuple is (F, T, F, F), and so,

$$[0, 2] \le [-3, 1] \Leftrightarrow P(F, T, F, F) = T.$$

Similarly, we conclude that $[-3, 1] \le [-2, -1]$. So, by transitivity, we would have $[0, 2] \le [-2, -1]$.

However, for the intervals $[0, 2]$ and $[-2, -1]$, all four i-relations are false, so we have $P(F, F, F, F) = F$ and

$$[0, 2] \le [-2, -1] \Leftrightarrow P(F, F, F, F) = F,$$

and thus, $[0, 2] \not\le [-2, -1]$. The contradiction shows that our assumption $P(F, T, F, F) = T$ is false, and thus, $P(F, T, F, F) = F$.

3°. Because of Part 2 of this proof, to describe a desired P-relation, it is sufficient to describe three remaining values: $P(T, T, F, T)$, $P(T, T, F, F)$, and $P(F, T, F, T)$.

Let us start with describing the last two values $P(T, T, F, F)$ and $P(F, T, F, T)$. Each of these values can be either true or false, so, in principle, we have four possible combinations of these values: (T, T), (T, F), (F, T), and (F, F). Let us consider these combinations one by one.

3.1°. Let us first consider the case when $P(T, T, F, F) = T$ and $P(F, T, F, T) = T$. We will prove that this case is impossible.

Indeed, in this case, the condition $P(T, T, F, F) = T$ implies that $[0, 3] \le [1, 1]$, and the condition $P(F, T, F, T) = T$ implies that $[1, 1] \le [-1, 2]$. Thus, by transitivity, we would conclude that $[0, 3] \le [-1, 2]$. However, for the intervals $[0, 3]$ and $[-1, 2]$, all four i-relations are false, so due to $P(F, F, F, F) = F$, we should get $[0, 3] \not\le [-1, 2]$. This contradiction shows that this case is indeed impossible.

3.2°. Let us now consider the case when $P(T, T, F, F) = T$ and $P(F, T, F, T) = F$. In this case, due to monotonicity, we get $P(T, T, F, T) = T$. The corresponding function P is thus fully defined. One can easily see that the corresponding P-relation

$$[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}] \Leftrightarrow P\left(\underline{x} \le \underline{y}, \underline{x} \le \overline{y}, \overline{x} \le \underline{y}, \overline{x} \le \overline{y}\right)$$

corresponds to ordering of lower endpoints.

3.3°. Similarly, when $P(T, T, F, F) = F$ and $P(F, T, F, T) = T$, due to monotonicity, we get $P(T, T, F, T) = T$. The corresponding function P is thus fully defined. One can easily see that the corresponding relation

$$[\underline{x}, \overline{x}] \le [\underline{y}, \overline{y}] \Leftrightarrow P\left(\underline{x} \le \underline{y}, \underline{x} \le \overline{y}, \overline{x} \le \underline{y}, \overline{x} \le \overline{y}\right)$$

corresponds to ordering of upper endpoints.

3.4°. The only remaining case is the case when $P(T,T,F,F) = P(F,T,F,T) = F$. In this case, the only value that we still need to define is the value $P(T,T,F,T)$. This value can be either true or false. One can see that:

- when $P(T,T,F,T) = T$, we get the weak order; and
- when $P(T,T,F,T) = F$, we get the strong order.

The proposition is proven. □

Proof of Proposition 8.

1°. Let us first prove that $\overline{x} \le \underline{y}$ if and only if $x \le y$ for all $x \in [\underline{x}, \overline{x}]$ and for all $y \in [\underline{y}, \overline{y}]$.

1.1°. If $\overline{x} \le \underline{y}$, then for every $x \in [\underline{x}, \overline{x}]$ and for every $y \in [\underline{y}, \overline{y}]$, we have $x \le \overline{x} \le \underline{y} \le y$. Thus, by transitivity, we get $x \le y$.

1.2°. Vice versa, if we have $x \le y$ for all $x \in [\underline{x}, \overline{x}]$ and for all $y \in [\underline{y}, \overline{y}]$, then, in particular, this inequality is true for $x = \overline{x} \in [\underline{x}, \overline{x}]$ and $\underline{y} \in [\underline{y}, \overline{y}]$. Thus, we get $\overline{x} \le \underline{y}$.

2°. Let us now prove that $\underline{x} \le \underline{y}$ if and only if there exists an $x \in [\underline{x}, \overline{x}]$ for which $x \le y$ for all $y \in [\underline{y}, \overline{y}]$.

2.1°. If $\underline{x} \le \underline{y}$, then for $x = \underline{x}$ and for all $y \in [\underline{y}, \overline{y}]$, we have $x \le \underline{y} \le y$ and thus, by transitivity, $x \le y$. Thus, there exists an $x \in [\underline{x}, \overline{x}]$ (namely, $x = \underline{x}$) for which $x \le y$ for all $y \in [\underline{y}, \overline{y}]$.

2.2°. Vice versa, let us assume that there exists an $x \in [\underline{x}, \overline{x}]$ for which $x \le y$ for all $y \in [\underline{y}, \overline{y}]$. In particular, this is true for $y = \underline{y} \in [\underline{y}, \overline{y}]$. Thus, we get $x \le \underline{y}$. Since $x \in [\underline{x}, \overline{x}]$, we conclude that $\underline{x} \le x$ and thus, by transitivity, we get $\underline{x} \le \underline{y}$.

3°. Let us prove that $\overline{x} \le \overline{y}$ if and only if there exists an $y \in [\underline{y}, \overline{y}]$ for which $x \le y$ for all $x \in [\underline{x}, \overline{x}]$.

3.1°. If $\overline{x} \le \overline{y}$, then for $y = \overline{y}$ and for all $x \in [\underline{x}, \overline{x}]$, we have $x \le \overline{x} \le y$ and thus, by transitivity, $x \le y$. Thus, there exists a $y \in [\underline{y}, \overline{y}]$ (namely, $y = \overline{x}$) for which $x \le y$ for all $x \in [\underline{x}, \overline{x}]$.

3.2°. Vice versa, let us assume that there exists a $y \in [\underline{y}, \overline{y}]$ for which $x \le y$ for all $x \in [\underline{x}, \overline{x}]$. In particular, this is true for $x = \overline{x} \in [\underline{x}, \overline{x}]$. Thus, we get $\overline{x} \le y$. Since $y \in [\underline{y}, \overline{y}]$, we conclude that $y \le \overline{y}$ and thus, by transitivity, we get $\overline{x} \le \overline{y}$.

4°. Finally, let us prove that $\underline{x} \le \overline{y}$ if and only if there exists an $x \in [\underline{x}, \overline{x}]$ and a $y \in [\underline{y}, \overline{y}]$ for which $x \le y$.

4.1°. If $\underline{x} \le \overline{y}$, then the inequality $x \le y$ holds for $x = \underline{x}$ and for $y = \overline{y}$. Thus, there exist an $x \in [\underline{x}, \overline{x}]$ (namely, $x = \underline{x}$) and a $y \in [\underline{y}, \overline{y}]$ (namely, $y = \overline{y}$) for which $x \le y$.

4.2°. Vice versa, let us assume that there exist $x \in [\underline{x}, \overline{x}]$ and $y \in [\underline{y}, \overline{y}]$ for which $x \le y$. Then, from $\underline{x} \le x$, $x \le y$, and $y \le \overline{y}$, by transitivity, we get $\underline{x} \le \overline{y}$. □

Proof of Proposition 9. In a (undirected) graph, an edge connecting a vertex v with a vertex v' can be identified with the 2-element set $\{v, v'\}$. As the poset X, let us take the union $X = E \cup (V \times \{-,+\})$ of the set E of all the edges and the set of all the pairs $(v,-)$ and $(v,+)$. On this set, we define the following partial order: $x \le x$ for all $x \in X$ plus the following relations:

- we require that $(v,-) < (v,+)$ for all v;
- for each edge $\{v, v'\} \in E$, we require that

$$(v,-) < (v,v') < (v,+),$$

$$(v',-) < (v,v') < (v',+),$$

$$(v,-) < (v',+), \text{ and } (v',-) < (v,+).$$

One can check that this relation is transitive and asymmetric, and is, thus, a partial order.

To each element $v \in V$, we put into correspondence an interval $I(v) = [(v,-),(v,+)]$. By definition of our order, the intervals $I(v)$ and $I(v')$, $v \ne v'$, have a non-empty intersection if and only if $\{v, v'\} \in E$, i.e., if and only if the vertices v and v' are connected by an edge in the original graph.

The statement is proven. □

Example 1 Let us illustrate this construction on the example of a simple fully connected graph with three vertices v_1, v_2, and v_3 described in Fig. 7.

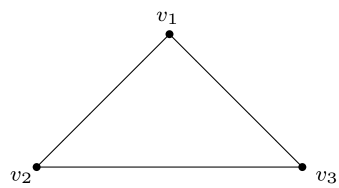

Fig. 7 An example of a graph G

In this case, the corresponding partially ordered set has the form described in Fig. 8.

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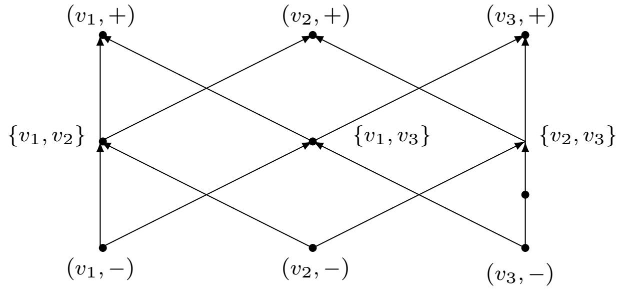

Fig. 8 A poset for which the interval graph is the original graph G

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