From [0,1]-Based Logic To Interval Logic

(From known description of all possible [0,1]-based logical operations to a description of all possible interval-based logical operations)

Hung T. Nguyen[1] and Vladik Kreinovich[2]

[1]Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003, USA
hunguyen@nmsu.edu

[2]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
vladik@cs.utep.edu

Abstract

Since early 1960s, we have a complete description of all possible $[0,1]$-based logical operations, namely of "and"-operations (t-norms) and of "or"-operations (t-conorms). In some real-life situations, intervals provide a more adequate way of describing uncertainty, so we need to describe interval-based logical operations. Usually, researchers followed a pragmatic path and simply derived these operations from the $[0,1]$-based ones. From the foundational viewpoint, it is desirable not to *a priori* restrict ourselves to such derivative operations but, instead, to get a description of all interval-based operations which satisfy reasonable properties.

Such description is presented in this paper. It turns out that all such operations can be described as the result of applying interval computations to the corresponding $[0,1]$-based ones.

1 Introduction

1.1 [0,1]-Based Logical Operations: Reminder

In many areas of expertise, such as medicine, geophysics, etc., human experts are needed. Usually, there are very few top level experts, and it is not physically possible for these few experts to solve all numerous related problems. It is therefore desirable to develop computer-based system which incorporate the knowledge of the top experts and use this knowledge either to directly solve the related problems – or, at least, to provide high-level advise to people trying to solve these problems.

Experts can describe their knowledge in terms of statements and rules, but this formulation often comes with uncertainty and ambiguity: experts are often not 100% confident in the statements which form their knowledge, and even when they are, these statements are formulated in terms of words of natural language (such as "large") which do not have precise meaning. To adequately describe the expert knowledge, we must therefore store, in the knowledge base, not only the statements themselves, but also the indication of the degree to which the experts are confident in these statements.

This degree is usually described by a number from the interval [0,1]. An expert's degree of confidence $d(A)$ in a statement A can be determined, if, e.g., we ask an expert to estimate his degree of confidence on a scale from 0 to 10. If he selects 8, then we take $d(A) = 8/10$.

Suppose now that we know the degrees of confidence $d(A)$ and $d(B)$ in statements A and B, and we know nothing else about A and B. Suppose also that we are interested in the degree of confidence of the composite statement $A \,\&\, B$. Since the only information available consists of the values $d(A)$ and $d(B)$, we must compute $d(A \,\&\, B)$ based on these values. We must be able to do that for arbitrary values $d(A)$ and $d(B)$. Therefore, we need a *function* that transforms the values $d(A)$ and $d(B)$ into an estimate for $d(A \,\&\, B)$. Such a function is called an *"and"-operation* (*t-norm*). If an "and"-operation $f_\& : [0,1] \times [0,1] \to [0,1]$ is fixed, then we take $f_\&(d(A), d(B))$ as an estimate for $d(A \,\&\, B)$.

Similarly, to estimate the degree of confidence in $A \vee B$, we need an *"or"-operation* (*t-conorm*) $f_\vee : [0,1] \times [0,1] \to [0,1]$.

The first "and" and "or" operations were proposed by L. Zadeh in [17]: $f_\&(x,y) = \min(x,y)$, $f_\&(x,y) = x \cdot y$, $f_\vee(x,y) = \max(x,y)$, and $f_\vee(x,y) = x + y - x \cdot y$. Later, numerous other operations have been proposed: e.g., in [2], Giles proposed "bold and" $f_\&(a,b) = \max(a+b-1,0)$ and "bold or" $f_\vee(a,b) = \min(a+b,1)$.

It was also shown that we can get new t-norm if we consider different "scales" on the interval $[0,1]$ of all possible degrees of certainty. Namely, the assignment of different numerical degrees to words expressing uncertainty is rather arbitrary. Let us assume that we assign new values to these words, and let $\varphi(a)$ be a new value assigned to the word to which we originally assigned the value a. In this new scale, to each statement A, instead of the original degree of certainty $d(A)$, we assign a new degree of certainty $d'(A) = \varphi(d(A))$. In the new scale, the same "and"-operation will look differently. Namely, if we know the degrees $a' = d'(A)$ and $b' = d'(B)$ in the new scale, and we want to find $d'(A \,\&\, B)$, then we must do the following:

- first, we compute the degrees $a = d(A)$ and $b = d(B)$ in the old scale as $a = \varphi^{-1}(a')$ and $b = \varphi^{-1}(b')$ (where φ^{-1} denotes the inverse function);
- second, we use the known t-norm $f_\&(a, b)$ to compute the degree of certainty $c = f_\&(a, b) = f_\&(\varphi^{-1}(a'), \varphi^{-1}(b'))$ of the composite statement $A \,\&\, B$ in the old scale;
- finally, we transform the degree c back into the new scale, resulting in $c' = \varphi(c) = \varphi(f_\&(\varphi^{-1}(a'), \varphi^{-1}(b')))$.

This three-step procedure is equivalent to using an operation $f'_\&(a', b') = \varphi(f_\&(\varphi^{-1}(a'), \varphi^{-1}(b')))$. This new operation is called *isomorphic* to the original t-norm $f_\&(a, b)$. Isomorphic operations provide numerous new *examples* of t-norms and t-conorms.

Later on, natural *general* requirements for "and"- and "or"-operations were formulated:

Definition 1.

- *By an "and"-operation, we mean a commutative, associative, monotonic, continuous operation* $f_\& : [0, 1] \times [0, 1] \to [0, 1]$ *for which* $f_\&(1, a) = a$ *and* $f_\&(0, a) = 0$.
- *By an "or"-operation, we mean a commutative, associative, monotonic, continuous operation* $f_\vee : [0, 1] \times [0, 1] \to [0, 1]$ *for which* $f_\vee(1, a) = 1$ *and* $f_\vee(0, a) = a$.

These properties are easy to explain. For example, commutativity $f_\&(a, b) = f_\&(b, a)$ comes from the fact that, from a common sense viewpoint, composite statements $A \,\&\, B$ and $B \,\&\, A$ are equivalent; therefore, we expect our "and"-operation to lead to the same degree of certainty for both composite statements. In precise terms, this means that we expect $f_\&(d(A), d(B)) = f_\&(d(B), d(A))$ for every two statements A and B. If we denote $d(A)$ by a and $d(B)$ by b, we can therefore conclude that $f_\&(a, b) = f_\&(b, a)$ for every a and b.

Similarly, associativity $f_\&(a, f_\&(b, c)) = f_\&(f_\&(a, b), c)$ comes from the fact that from the common sense viewpoint, the composite statements $A \,\&\, (B \,\&\, C)$ and $(A \,\&\, B) \,\&\, C$ are equivalent.

Monotonicity, i.e., the fact that $a_1 \le a_2$ and $b_1 \le b_2$, then $f_\&(a_1, b_1) \le f_\&(a_2, b_2)$ (and $f_\vee(a_1, b_1) \le f_\vee(a_2, b_2)$), comes from the fact that if our degree of confidence in A_1 is smaller than the degree of confidence in A_2, and the degree of confidence in B_1 is smaller than the degree of confidence in B_2, then our confidence in $A_1 \,\&\, B_1$ must be smaller (or at least equal, but not larger) than our confidence in $A_2 \,\&\, B_2$.

A complete description of operations that satisfy these properties has been given, in effect, in [9] (see also [6, 8, 12, 14]). It turns out that for every t-norm, we can subdivide the interval $[0, 1]$ into subintervals on each of which t-norm is isomorphic either to the "algebraic" t-norm $a \cdot b$, or to $\max(a + b - 1, 0)$, or to $\min(a, b)$, and $f_\&(a, b) = \min(a, b)$ when a and b belong to two different sub-intervals.

A similar description is known for t-conorms.

1.2 Interval-Based Logical Operations: A Problem

Experts cannot describe their degrees of confidence precisely. At best, they can give an *interval* of possible values. For example, an expert can point to 8 on a scale from 0 to 10, but this same expert will hardly be able to pinpoint a value on a scale from 0 to 100. As a result, the only thing that we know about the expert's degree of confidence is that it is closer to 8 than to 7 or to 9, i.e., that it is in the interval $[0.75, 0.85]$.

So, to describe degrees of confidence more adequately, we must use *intervals* $\mathbf{a} = [a^-, a^+]$ instead of real numbers. In this representation, real numbers can be viewed as particular – degenerate – cases of intervals $[a, a]$. The idea of using intervals have been originally proposed by Zadeh himself and further developed by Bandler and Kohout [1], Türkşen [16], and others; for a recent survey, see, e.g., [11].

Since we went from numbers to intervals in our description of degrees of certainty, we must have "and" and "or" operations as functions from intervals to intervals. Traditionally, researchers followed a pragmatic path and simply derive these operations are derived from the $[0, 1]$-based ones. Namely, when an expert says that his degree of certainty in a statement A belongs to the interval $[a^-, a^+]$, we can interpret it as meaning that the (unknown) actual degree of confidence can be any number from this interval. With this interpretation in mind, it is natural to define, e.g., an interval "and"-operation as follows:

- First, we select a $[0, 1]$-based "and"-operation (t-norm) $f_\&(a, b)$. This operation corresponds to the case when an expert knows the exact values of his degrees of certainty, i.e., when the intervals $\mathbf{a} = [a^-, a^+]$ and $\mathbf{b} = [b^-, b^+]$ are degenerate ($a^- = a^+$ and $b^- = b^+$).

- Next, when we know the interval degrees $\mathbf{a}$ and $\mathbf{b}$, we interpret these intervals by saying that a can take any value from $\mathbf{a}$ and b can take any value from $\mathbf{b}$. Thus, as the degree corresponding to $A \,\&\, B$, it is natural to take the set of all possible values of $f_\&(a, b)$ when $a \in \mathbf{a}$ and $b \in \mathbf{b}$. In precise terms, we define $f_\&(\mathbf{a}, \mathbf{b})$ as follows:

$$f_\&(\mathbf{a}, \mathbf{b}) = \{f_\&(a, b) \mid a \in \mathbf{a}, b \in \mathbf{b}\}.$$

This formula is a particular case of the so-called *interval computations* [3, 4, 5, 13]. Since the function $f_\&(a,b)$ is monotonically increasing and continuous, the resulting set is easy to describe:

$$f_\&([a^-,a^+],[b^-,b^+]) = [f_\&(a^-,b^-), f_\&(a^+,b^+)].$$

We can use a similar "pragmatic" approach and define an interval-based "or" operation as

$$f_\vee([a^-,a^+],[b^-,b^+]) = [f_\vee(a^-,b^-), f_\vee(a^+,b^+)].$$

From the *pragmatic* viewpoint, we get a good class of reasonable "and"- and "or"-operations. However, our experience with $[0,1]$-based operations has shown that it is not sufficient to just list some of them: the more operations we can choose from, the better we can adjust to different specific problems, and the better quality results we can get; see, e.g., [15], the surveys [7, 10] and references therein. From this viewpoint, it is desirable not to *a priori* restrict ourselves to such "derivative" interval operative but, instead, to get a complete description of all possible interval-based operations.

The task of obtaining such a description was started in a pioneer paper by Zuo [18] who described all interval-based operations which are *strictly monotonic* (in some reasonable sense). In this paper, we extend Zuo's results and find a description of *all possible* interval-based logical operations (which satisfy reasonable properties like commutativity and monotonicity).

Specifically, we show that the above interval-computation operations are the only ones possible. Thus, we provide a fundamental justification for the traditional pragmatic approach.

2 Towards Formalization of the Problem: How to Define Monotonicity for Interval Operations?

2.1 Why Is It Important to Define Monotonicity?

An important part of the definition of t-norm and t-conorm is the requirement that these operations are *monotonic*, i.e., that if $a_1 \le a_2$ and $b_1 \le b_2$, then $f_\&(a_1,b_1) \le f_\&(a_2,b_2)$ and $f_\vee(a_1,b_1) \le f_\vee(a_2,b_2)$. For $[0,1]$-based operations, these properties are easy to formalize, because the order $\le$ is well defined on the interval $[0,1]$. For interval degrees, however, the situation is less clear.

If we know the interval degrees $\mathbf{a}_1 = [a_1^-, a_1^+]$ and $\mathbf{a}_2 = [a_2^-, a_2^+]$ for two statements A_1 and A_2, this means that the actual degree of confidence a_1 in A_1 can take any value from the interval $\mathbf{a}_1$, and the actual degree of confidence a_2 in A_2 can take any value from the interval $\mathbf{a}_2$. If the intervals $\mathbf{a}_1$ and $\mathbf{a}_2$

intersect, then, depending on the selection of the values $a_i \in \mathbf{a}_i$, we may have $a_< a_2$ and we may also have $a_2 < a_1$.

For example, if $\mathbf{a}_1 = [0.7, 0.9]$ and $\mathbf{a}_2 = [0.8, 1.0]$, then:

- on one hand, we may have $a_1 = 0.7 \in \mathbf{a}_1$ and $a_2 = 1.0 \in \mathbf{a}_2$, in which case $a_1 < a_2$;
- on the other hand, we may have $a_1 = 0.9 \in \mathbf{a}_1$ and $a_2 = 0.8 \in \mathbf{a}_2$, in which case $a_1 > a_2$.

2.2 Solution: Operations "Necessarily $\leq$" and "Possibly $\leq$"

We have already mentioned that for interval degrees $\mathbf{a}_1$ and $\mathbf{a}_2$, it is sometimes not clear whether $\mathbf{a}_1 \leq \mathbf{a}_2$ or not. However, the situation is not hopeless: we have the following two natural order-like relations:

Definition 2. *Let* $\mathbf{a}_1 = [a_1^-, a_1^+]$ *and* $\mathbf{a}_2 = [a_2^-, a_2^+]$ *be two intervals.*

- *We say that* $\mathbf{a}_1$ *is necessarily* $\leq \mathbf{a}_2$ *(and denote it by* $\mathbf{a}_1 \leq^\square \mathbf{a}_2$*) if* $a_1 \leq a_2$ *for every* $a_1 \in \mathbf{a}_1$ *and for every* $a_2 \in \mathbf{a}_2$.
- *We say that* $\mathbf{a}_1$ *is possibly* $\leq \mathbf{a}_2$ *(and denote it by* $\mathbf{a}_1 \leq^\diamond \mathbf{a}_2$*) if* $a_1 \leq a_2$ *for some* $a_1 \in \mathbf{a}_1$ *and* $a_2 \in \mathbf{a}_2$.

It is therefore natural to require that the desired interval-based logical operations be monotonic relative to *both* these operations: i.e., that:

- if $\mathbf{a}_1 \leq^\square \mathbf{a}_2$ and $\mathbf{b}_1 \leq^\square \mathbf{b}_2$, then $f_\&(\mathbf{a}_1, \mathbf{b}_1) \leq^\square f_\&(\mathbf{a}_2, \mathbf{b}_2)$;
- if $\mathbf{a}_1 \leq^\diamond \mathbf{a}_2$ and $\mathbf{b}_1 \leq^\diamond \mathbf{b}_2$, then $f_\&(\mathbf{a}_1, \mathbf{b}_1) \leq^\diamond f_\&(\mathbf{a}_2, \mathbf{b}_2)$.

2.3 Solution Simplified

At first glance, the above solution may seem somewhat complicated. Indeed, if we try to use the above definitions to check, e.g., whether $\mathbf{a}_1$ is necessarily $\leq$ than $\mathbf{a}_2$, then we will have to check *infinitely many* inequalities $a_1 \leq a_2$ for all possible pairs $a_1 \in \mathbf{a}_1$ and $a_2 \in \mathbf{a}_2$. Luckily, the above definition can be easily simplified; indeed, the following result can be easily proven:

Proposition 1.

- $\mathbf{a}_1 \leq^\square \mathbf{a}_2 \leftrightarrow a_1^+ \leq a_2^-$*;*
- $\mathbf{a}_1 \leq^\diamond \mathbf{a}_2 \leftrightarrow a_1^- \leq a_2^+$.

2.4 Simple to Check But Not Easy to Analyze

The above reformulation shows that both relations $\le^{\Box}$ and $\le^{\Diamond}$ are easy to check. However, this same result shows that these relations are not easy to analyze, because they are *not* orders (We are thankful to Carol and Elbert Walkers who attracted our attention to this fact.)

Indeed, an order $\le$ is *reflexive* (i.e., $a \le a$ for every a), but the relation $\le^{\Box}$ is *not* reflexive: if $a^- < a^+$, then $[a^-, a^+] \not\le^{\Box} [a^-, a^+]$.

One might suspect that $\le^{\Box}$ is a *strict order*, i.e., a anti-reflexive relation (for which $a \not< a$ for all a), but this is not true either: for degenerate intervals, the relation is reflexive: $\mathbf{a} \le^{\Box} \mathbf{a}$.

Similarly, the order $\le$ should be *transitive* (if $a \le b$ and $b \le c$, then $a \le c$), but the relation $\le^{\Diamond}$ is *not* transitive: e.g., $[0.9, 1.0] \le^{\Diamond} [0, 1]$, $[0, 1] \le^{\Diamond} [0, 0.1]$, but $[0.9.1] \not\le^{\Diamond} [0, 0.1]$.

Since these relations are *not* orders, we cannot use standard results about monotonicity, and we therefore have to prove everything "from scratch". This is what we will do in the next section.

An interesting auxiliary question – originally formulated by Walkers – is to give a complete algebraic characterization of these relations. This characterization is given in Section 4.

2.5 Additional Monotonicity Property: Inclusion Monotonicity

Let us show that, in addition to $\le^{\Box}$- and $\le^{\Diamond}$-monotonicity, it is natural to require one more monotonicity property for interval operations.

Indeed, suppose that initially, we had $\mathbf{a}_2$ and $\mathbf{b}_2$ as sets of possible values of degrees of confidence in A and B. Then, by applying the interval "and"-operation $f_\&$, we can conclude that the degree of confidence in $A \,\&\, B$ is in $f_\&(\mathbf{a}_2, \mathbf{b}_2)$.

Suppose now that we have narrowed down our degrees of confidence to $\mathbf{a}_1 \subseteq \mathbf{a}_2$ and $\mathbf{b}_1 \subseteq \mathbf{b}_2$. If we apply the same interval "and"-operation to the new degrees of confidence, we get a new interval $f_\&(\mathbf{a}_1, \mathbf{b}_1)$. Since we have narrowed down our intervals of possible degrees of confidence, it can happen that some previously possible degrees of confidence in $A \,\&\, B$ are not possible anymore. But it is reasonable to require that if a value is now possible, then it was possible earlier as well (when we had even fewer knowledge about degrees of confidence). In other words, we require that every number from $f_\&(\mathbf{a}_1, \mathbf{b}_1)$ should belong to $f_\&(\mathbf{a}_2, \mathbf{b}_2)$.

In other words, we require that if $\mathbf{a}_1 \subseteq \mathbf{a}_2$ and $\mathbf{b}_1 \subseteq \mathbf{b}_2$, then $f_\&(\mathbf{a}_1, \mathbf{b}_1) \subseteq f_\&(\mathbf{a}_2, \mathbf{b}_2)$. In mathematical terms, we require that the interval "and"-operation $f_\&(\mathbf{a}, \mathbf{b})$ be monotonic relative to set inclusion $\subseteq$, i.e., in short, *inclusion monotonic.*

Now, we are ready for the main result.

3 Main Result

Although our main interest is in binary operations over subintervals of the interval $[0, 1]$, we will formulate this result in the most general terms: as a result about operations of arbitrary arity over subintervals of an arbitrary ordered set.

Definition 3. *Let L be an arbitrary (partially) ordered set.*

- *Let a^- and a^+ be two points from L for which $a^- \le a^+$. The set*

$$\{b \mid a^- \le b \le a^+\}$$

will be called an interval and denoted by $[a^-, a^+]$.

- *The set of all intervals over L will be denoted by $\mathbb{I}(L)$.*

For intervals over an arbitrary ordered set L, we can use Definition 2 to define relations $\le^{\Box}$ and $\le^{\Diamond}$; Proposition 1 holds for this case as well.

Definition 4. *Let n be an arbitrary positive integer. We say that an n-ary interval operation $F : \mathbb{I}(L) \times \ldots \times \mathbb{I}(L) \to \mathbb{I}(L)$ is obtained by interval computations if there exists an n-ary $\le$-monotonic function $f : L \times \ldots \times L \to L$ for which*

$$F([a^-, a^+], \ldots, [b^-, b^+]) = [f(a^-, \ldots, b^-), f(a^+, \ldots, b^+)].$$

Theorem 1.

- *Every operation F obtained by interval computations is $\le^{\Box}$-, $\le^{\Diamond}$-, and inclusion-monotonic.*
- *Every $\le^{\Box}$-, $\le^{\Diamond}$-, and inclusion-monotonic interval operation F is obtained by interval computations.*

The second part of this theorem says that *every interval-based operation which satisfies the above natural monotonicity requirement is obtained by interval computations.* Thus, for binary operations over $\mathbb{I}([0, 1])$, we did provide a fundamental justification for the traditional pragmatic approach to interval-valued operations.

Editorial Comment. For the convenience of the readers who are interested in the results but not in the technical details of the proofs, all the proofs are placed in the special Proofs section located at the end of the paper.

Technical Comment. In the second part of Theorem 1, we required that the interval operation F be both $\le^{\Box}$- and $\le^{\Diamond}$-monotonic. As one can see from the proof, it is sufficient to require that F is $\le^{\Box}$-monotonic; then $\le^{\Diamond}$-monotonicity follows automatically.

4 Auxiliary Results

Normally, we require that the relation $\leq$ between degrees of certainty is an order, i.e., a relation which satisfies the following three properties:

- it is *reflexive* ($a \leq a$);
- it is *transitive* ($a \leq b$ and $b \leq c$ imply $a \leq c$); and
- it is *antisymmetric* ($a \leq b$ and $b \leq a$ imply $a = b$).

In the previous section, we mentioned that neither $\leq^{\square}$ not $\leq^{\diamond}$ are orders. What are they?

In this section, we give exact algebraic characterizations of these two relations.

To describe these results, let us recall the definition of a restriction of a relation to a subset. Let S be an arbitrary set, let R be an arbitrary relation on this set, and let $S' \subseteq S$ be a subset of S. Then, we define a *restriction* $R_{|S'}$ of R to S' as follows: if $a, b \in S'$ then $a\, R_{|S'}\, b$ if and only if $a\, R\, b$.

Theorem 2.

- *Let L be an arbitrary partially ordered set, and let S be an arbitrary subset of $\mathbb{I}(L)$. Then, the restriction of $\leq^{\square}$ on S is transitive and antisymmetric.*
- *Let S be an arbitrary set with a transitive antisymmetric relation R. Then, there exists a partially ordered set L and a subset S' of the interval set $\mathbb{I}(L)$ such that the relation R on S is isomorphic to the restriction of $\leq^{\square}$ to S'.*

Theorem 3.

- *Let L be an arbitrary partially ordered set, and let S be an arbitrary subset of $\mathbb{I}(L)$. Then, the restriction of $\leq^{\diamond}$ on S is reflexive.*
- *Let S be an arbitrary set with a reflexive relation R. Then, there exists a partially ordered set L and a subset S' of the interval set $\mathbb{I}(L)$ such that the relation R on S is isomorphic to the restriction of $\leq^{\diamond}$ to S'.*

So, both relations appear naturally if we divide the three properties describing order into two groups: reflexivity in one group, and transitivity and antisymmetry in another group.

- If we only keep properties from the first group, we get $\leq^{\diamond}$.
- If we only keep properties from the second group, we get $\leq^{\square}$.
- If we keep properties from both groups, we get a normal order relation.

5 Proofs

5.1 Proof of Theorem 1

1°. The first part is reasonable straightforward: if the interval operation F is obtained by interval computations from some monotonic operation

$$f : L \times \ldots \times L \to L,$$

then from $\le$-monotonicity of f, one can easily prove that F is $\le^\Box$-, $\le^\Diamond$-, and inclusion-monotonic.

2°. To complete the proof of the theorem, we must therefore prove its second part: that every $\le^\Box$-, $\le^\Diamond$-, and inclusion-monotonic interval operation F is obtained by interval computations.

We will actually prove this result without requiring that F is $\le^\Diamond$-monotonic. Then, from the first part, it will follow that $\le^\Diamond$-monotonicity is automatically satisfied.

So, let F be $\le^\Box$- and inclusion-monotonic. The result of applying F is an interval. Let us denote its lower endpoint by F^- and its upper endpoint by F^+.

2.1°. Let us first prove that when all inputs to F are degenerate intervals, then the output is also degenerate, i.e., for every $a, \ldots, b \in L$, we have

$$F^-([a,a], \ldots, [b,b]) = F^+([a,a], \ldots, [b,b]).$$

Indeed, by definition of $\le^\Box$, for every $a \in L$, we have

$$[a,a] \le^\Box [a,a].$$

So, $[a,a] \le^\Box [a,a]$, $\ldots$, $[b,b] \le^\Box [b,b]$, and due to $\le^\Box$-monotonicity of the operation F, we conclude that

$$F([a,a], \ldots, [b,b]) \le^\Box F([a,a], \ldots, [b,b]).$$

By definition of $\le^\Box$, from

$$F^-([a,a], \ldots, [b,b]) \in F([a,a], \ldots, [b,b])$$

and

$$F^+([a,a], \ldots, [b,b]) \in F([a,a], \ldots, [b,b]),$$

we can conclude that

$$F^+([a,a], \ldots, [b,b]) \le F^-([a,a], \ldots, [b,b]).$$

On the other hand, since F^- and F^+ are endpoints of the interval, we have

$$F^-([a,a],\ldots,[b,b]) \le F^+([a,a],\ldots,[b,b]).$$

Thus,

$$F^-([a,a],\ldots,[b,b]) = F^+([a,a],\ldots,[b,b]).$$

The statement is proven.

2.2°. Let us define a function $f : L \times \ldots \times L \to L$ as follows: for every $a,\ldots,b \in L$, we define

$$f(a,\ldots,b) \stackrel{\text{def}}{=} F^-([a,a],\ldots,[b,b]) = F^+([a,a],\ldots,[b,b]).$$

Then, for degenerate intervals, we have

$$F([a,a],\ldots,[b,b]) = [f(a,\ldots,b), f(a,\ldots,b)].$$

We will complete the proof of the theorem by showing two things:

- that thus defined function f is monotonic, and
- that

  $$F([a^-,a^+],\ldots,[b^-,b^+]) = [f(a^-,\ldots,b^-), f(a^+,\ldots,b^+)]$$

  for all possible intervals $[a^-,a^+],\ldots,[b^-,b^+] \in \mathbb{I}(L)$.

2.3°. Let us prove that the function f (defined in Part 2.2 of this proof) is monotonic. In other words, let us prove that if $a_1 \le a_2$, …, $b_1 \le b_2$, then $f(a_1,\ldots,b_1) \le f(a_2,\ldots,b_2)$.

Indeed, let $a_1 \le a_2$, …, $b_1 \le b_2$. By definition of $\le^\Box$, we can therefore conclude that $[a_1,a_1] \le^\Box [a_2,a_2]$, …, $[b_1,b_1] \le^\Box [b_2,b_2]$. Due to $\le^\Box$-monotonicity of the operation F, we conclude that

$$F([a_1,a_1],\ldots,[b_1,b_1]) \le^\Box F([a_2,a_2],\ldots,[b_2,b_2]).$$

We already know, from Part 2.3 of this proof, that

$$F([a_1,a_1],\ldots,[b_1,b_1]) = [f(a_1,\ldots,b_1), f(a_1,\ldots,b_1)]$$

and

$$F([a_2,a_2],\ldots,[b_2,b_2]) = [f(a_2,\ldots,b_2), f(a_2,\ldots,b_2)].$$

Thus, the above "necessarily $\le$" relation means that

$$f(a_1,\ldots,b_1) \le f(a_2,\ldots,b_2).$$

The statement is proven.

2.4°. Let us now prove that

$$F([a^-,a^+],\ldots,[b^-,b^+]) = [f(a^-,\ldots,b^-), f(a^+,\ldots,b^+)]$$

for all possible intervals $[a^-,a^+],\ldots,[b^-,b^+] \in \mathbb{I}(L)$, i.e., that for all possible intervals,

$$F^-([a^-,a^+],\ldots,[b^-,b^+]) = f(a^-,\ldots,b^-)$$

and

$$F^+([a^-,a^+],\ldots,[b^-,b^+]) = f(a^+,\ldots,b^+).$$

2.4.1°. Let us first prove that

$$f(a^-,\ldots,b^-) \le F^-([a^-,a^+],\ldots,[b^-,b^+]).$$

Indeed, from the definition of $\le^\Box$, we can easily conclude that for every interval $[a^-,a^+]$, we have $[a^-,a^-] \le^\Box [a^-,a^+]$.

From the fact that $[a^-,a^-] \le^\Box [a^-,a^+]$, $\ldots$, $[b^-,b^-] \le^\Box [b^-,b^+]$, and that F is $\le^\Box$-monotonic, we conclude that

$$F([a^-,a^-],\ldots,[b^-,b^-]) \le^\Box F([a^-,a^+],\ldots,[b^-,b^+]).$$

According to Proposition 1, this means that

$$F^+([a^-,a^-],\ldots,[b^-,b^-]) \le F^-([a^-,a^+],\ldots,[b^-,b^+]).$$

We already know, from Part 2.2 of this proof, that

$$F^+([a^-,a^-],\ldots,[b^-,b^-]) = f(a^-,\ldots,b^-).$$

Thus, the above inequality is exactly what we want to prove. The statement is proven.

2.4.2°. Let us now prove that

$$F^-([a^-,a^+],\ldots,[b^-,b^+]) \le f(a^-,\ldots,b^-).$$

Indeed, for each of the input intervals, we have $[a^-,a^-] = \{a^-\} \subseteq [a^-,a^+]$, $\ldots$, $[b^-,b^-] = \{b^-\} \subseteq [b^-,b^+]$. Since the operation F is inclusion-monotonic, we conclude that

$$F([a^-,a^-],\ldots,[b^-,b^-]) \subseteq F([a^-,a^+],\ldots,[b^-,b^+]) =$$

$$[F^-([a^-,a^+],\ldots,[b^-,b^+]), F^+([a^-,a^+],\ldots,[b^-,b^+])].$$

Due to Parts 2.1 and 2.2. of this proof, we have

$$F([a^-,a^-],\ldots,[b^-,b^-]) = \{f(a^-,\ldots,b^-)\}.$$

Thus, the above inclusion means that

$$f(a^-,\ldots,b^-) \in [F^-([a^-,a^+],\ldots,[b^-,b^+]), F^+([a^-,a^+],\ldots,[b^-,b^+])].$$

By definition of an interval, this means, in particular, that

$$F^-([a^-,a^+],\ldots,[b^-,b^+]) \le f(a^-,\ldots,b^-).$$

The statement is proven.

2.4.3°. From Parts 2.4.1 and 2.4.2 of this proof, we can now conclude that

$$F^-([a^-,a^+],\ldots,[b^-,b^+]) = f(a^-,\ldots,b^-).$$

2.4.4°. Let us now start proving the second inequality from Part 2.4 by first proving that

$$F^+([a^-,a^+],\ldots,[b^-,b^+]) \le f(a^+,\ldots,b^+).$$

Indeed, from the definition of $\le^\square$, we can easily conclude that for every interval $[a^-,a^+]$, we have $[a^-,a^+] \le^\square [a^+,a^+]$.

From the fact that $[a^-,a^+] \le^\square [a^+,a^+]$, …, $[b^-,b^+] \le^\square [b^+,b^+]$, and that F is $\le^\square$-monotonic, we conclude that

$$F([a^-,a^+],\ldots,[b^-,b^+]) \le^\square F([a^+,a^+],\ldots,[b^+,b^+]).$$

According to Proposition 1, this means that

$$F^+([a^-,a^+],\ldots,[b^-,b^+]) \le F^-([a^+,a^+],\ldots,[b^+,b^+]).$$

We already know, from Part 2.2 of this proof, that

$$F^-([a^+,a^+],\ldots,[b^+,b^+]) = f(a^+,\ldots,b^+).$$

Thus, the above inequality is exactly what we want to prove. The statement is proven.

2.4.5°. Let us now prove that

$$f(a^+,\ldots,b^+) \le F^+([a^-,a^+],\ldots,[b^-,b^+]).$$

Indeed, for each of the input intervals, we have $[a^+, a^+] = \{a^+\} \subseteq [a^-, a^+]$, ..., $[b^+, b^+] = \{b^+\} \subseteq [b^-, b^+]$. Since the operation F is inclusion-monotonic, we conclude that

$$F([a^+, a^+], \ldots, [b^+, b^+]) \subseteq F([a^-, a^+], \ldots, [b^-, b^+]) =$$

$$[F^-([a^-, a^+], \ldots, [b^-, b^+]), F^+([a^-, a^+], \ldots, [b^-, b^+])].$$

Due to Parts 2.1 and 2.2. of this proof, we have

$$F([a^+, a^+], \ldots, [b^+, b^+]) = \{f(a^+, \ldots, b^+)\}.$$

Thus, the above inclusion means that

$$f(a^+, \ldots, b^+) \in [F^-([a^-, a^+], \ldots, [b^-, b^+]), F^+([a^-, a^+], \ldots, [b^-, b^+])].$$

By definition of an interval, this means, in particular, that

$$f(a^+, \ldots, b^+) \leq F^+([a^-, a^+], \ldots, [b^-, b^+]).$$

The statement is proven.

2.4.6°. From Parts 2.4.4 and 2.4.5 of this proof, we can now conclude that

$$F^+([a^-, a^+], \ldots, [b^-, b^+]) = f(a^+, \ldots, b^+).$$

The theorem is proven.

5.2 Proof of Theorem 2

The first part of the theorem easily follows from Proposition 1, so it is sufficient to prove the second part.

Let S be a set with a transitive antisymmetric relation R. Let

$$S_r \stackrel{\text{def}}{=} \{a \in S \mid a\, R\, a\}$$

denote the set of all reflexive elements of S, and let

$$S_i \stackrel{\text{def}}{=} \{a \in S \mid \neg a\, R\, a\}$$

denote the set of all irreflexive elements of S. Let us define L as

$$L \stackrel{\text{def}}{=} S_r \cup (S_i \times \{-, +\}),$$

i.e., as a set consisting of:

- all reflexive element of S, and

- of pairs $\langle a, -\rangle$ and $\langle a, +\rangle$, where $a \in S_i$,

and let us define the relation $\leq$ on L as follows:

- for every $a, b \in S_r$, we have $a \leq b$ if and only if $a\,R\,b$;
- for $a \in S_r$ and $b \in S_i$, we have:
  - $a \leq \langle b, -\rangle$ if and only if $a\,R\,b$, and
  - $a \leq \langle b, +\rangle$ if and only if $a\,R\,b$;
- for $a \in S_i$ and $b \in S_r$, we have:
  - $\langle a, -\rangle \leq b$ if and only if $a\,R\,b$, and
  - $\langle a, +\rangle \leq b$ if and only if $a\,R\,b$.
- for every $a \in S_i$, $\langle a, -\rangle \leq \langle a, -\rangle$, $\langle a, -\rangle \leq \langle a, +\rangle$, and $\langle a, +\rangle \leq \langle a, +\rangle$;
- finally, for $a, b \in S_i$, $a \neq b$, we have:
  - $\langle a, -\rangle \leq \langle b, -\rangle$ if and only if $a\,R\,b$;
  - $\langle a, -\rangle \leq \langle b, +\rangle$ if and only if $a\,R\,b$;
  - $\langle a, +\rangle \leq \langle b, -\rangle$ if and only if $a\,R\,b$;
  - $\langle a, +\rangle \leq \langle b, +\rangle$ if and only if $a\,R\,b$.

One can easily check that this relation is an order.

Let us now assign, to every element $a \in S$, an interval from $\mathbb{I}(L)$. Specifically, we assign:

- to every element $a \in S_r$, a degenerate interval $[a, a] \in \mathbb{I}(L)$, and
- to every element $a \in S_i$, an interval $[\langle a, -\rangle, \langle a, +\rangle] \in \mathbb{I}(L)$.

Due to Proposition 1 and the definition of the order on L, we have the following equivalences:

- when $a, b \in S_r$, then $[a, a] \leq^{\Box} [b, b]$ if and only if $a\,R\,b$;
- when $a \in S_r$ and $b \in S_i$, then $[a, a] \leq^{\Box} [\langle b, -\rangle, \langle b, +\rangle]$ if and only if $a\,R\,b$;
- when $a \in S_i$ and $b \in S_r$, then $[\langle a, -\rangle, \langle a, +\rangle] \leq^{\Box} [b, b]$ if and only if $a\,R\,b$;
- finally, when $a, b \in S_i$, then $[\langle a, -\rangle, \langle a, +\rangle] \leq^{\Box} [\langle b, -\rangle, \langle b, +\rangle]$ if and only if $a\,R\,b$.

Thus, the original relation R on S is isomorphic to the restriction of $\leq^{\Box}$ to the set S' of all intervals assigned to elements of S. The theorem is proven.

5.3 Proof of Theorem 3

The first part of this theorem easily follows from the definition of $\leq^{\diamond}$, so it is sufficient to prove the second part.

Let S be a set with a reflexive relation R. Let us define L as $S \times \{-,+\}$, i.e., as the set of all pairs $\langle a,-\rangle$ and $\langle a,+\rangle$, where $a \in S$, and let us define the relation $\leq$ on L as follows:

- for every $a \in S$, $\langle a,-\rangle \leq \langle a,-\rangle$, $\langle a,-\rangle \leq \langle a,+\rangle$, and $\langle a,+\rangle \leq \langle a,+\rangle$;
- for $a \neq b$, we have $\langle a,-\rangle \leq \langle b,+\rangle$ if and only if $a\,R\,b$;
- for every a and b, $\langle a,+\rangle \not\leq \langle b,-\rangle$.

One can easily check that this relation is an order.

Let us now assign, to every element $a \in S$, an interval $[\langle a,-\rangle,\langle a,+\rangle] \in \mathbb{I}(L)$. Due to Proposition 1 and the definition of the order on L, we have $[\langle a,-\rangle,\langle a,+\rangle] \leq^{\diamond} [\langle b,-\rangle,\langle b,+\rangle]$ if and only if $a\,R\,b$. Thus, the original relation R on S is isomorphic to the restriction of $\leq^{\diamond}$ to the set S' of all intervals $[\langle a,-\rangle,\langle a,+\rangle]$. The theorem is proven.

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