Interval-Valued and Set-Valued Extensions of Discrete Fuzzy Logics, Belnap Logic, and Color Optical Computing*

Victor L. Timchenko[1], Yury P. Kondratenko[2], and Vladik Kreinovich[3][0000−0002−1244−1650]

[1] Admiral Makarov National University of Shipbuilding, Mykolaiv, Ukraine, vl.timchenko58@gmail.com

[2] Petro Mohyla Black Sea National University, Mykolaiv, Ukraine, y_kondrat2002@yahoo.com

[3] University of Texas at El Paso, El Paso, Texas, USA, olgak@utep.edu, vladik@utep.edu

Abstract. It has been recently shown that in some applications, e.g., in ship navigation near a harbor, it is convenient to use combinations of basic colors – red, green, and blue – to represent different fuzzy degrees. In this paper, we provide a natural explanation for the efficiency of this empirical fact: namely, we show that it is reasonable to consider discrete fuzzy logics, it is reasonable to consider their interval-valued and set-valued extensions, and that a set-valued extension of the 3-values logic is naturally equivalent to the use of color combinations.

Keywords: Fuzzy logic · Set-valued extension · Interval-valued extension · Color optical computing

1 Formulation of the Problem

Color optical computing representation of fuzzy degrees. It has been recently shown that in some practical applications of fuzzy logic – e.g., in ship navigation near a harbor – it is convenient to represent different fuzzy degrees by colors, namely, by combinations of the three basic colors: red, green, and blue; see, e.g., [10–13].

Question. This empirical success prompts a natural question: why is this representation efficient?

* This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), and by the AT&T Fellowship in Information Technology. It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

What we do in this paper. In this paper, we explain the empirical success of color optical computing representation by showing how the main ideas behind fuzzy logic naturally lead to this representation.

2 Why Interval-Valued and Set-Valued Extensions of Discrete Fuzzy Logics

Fuzzy degrees: a brief reminder. One of the main ideas behind fuzzy logic is to assign, to each imprecise natural-language statement such as "John is tall", a degree describing to what extent this statement is true – e.g., to what extent John is tall; see, e.g., [3–5, 8, 9, 14].

Need for discrete fuzzy logic. In the original fuzzy logic, these degrees were represented by numbers from the interval $[0, 1]$. From the mathematical viewpoint, this interval contains infinitely many numbers. When the numbers are significantly different, they represent different degrees of certainty. However, when the two numbers are very close, we cannot distinguish the corresponding degrees: e.g., hardly anyone can distinguish between degrees 0.80 and 0.81.

In general, according to psychological experiments, we can meaningfully distinguish at most 7 ± 2 different degrees: some of us can only distinguish $7 - 2 = 5$ different degrees, some can distinguish $7 + 2 = 9$ different degrees; see, e.g., [6, 7]. In other words, in pracctice, we use, in effect, a discrete set of fuzzy degrees.

Fuzzy degrees come with uncertainty. In the ideal case, we have a single perfect expert who selects a single degree – and experts are perfect in the sense that other experts would assign the exact same degree. In practice, the situation is more complicated.

- First, an expert can be not sure what exactly degree to assign. At best, the expert can provide a lower bound a and an upper bound b for this degree – just like when estimating the height of a person entering the room, the expert will not produce an exact value but rather a range of values. In this case, possible degrees form an *interval* $[a, b] \stackrel{\text{def}}{=} \{x : a \le x \le b\}$.
- Second, even if an expert produces an exact degree, other experts may produce different degrees. In this case, to describe uncertainty, it is reasonable to list all these degrees, i.e., to produce the *set* of experts' estimates. This extension of fuzzy logic is known as *hesitant* fuzzy logic.

In the following text, we will analyze such interval-valued and set-valued versions of the simplest discrete fuzzy logics, and we will show that this analysis indeed naturally leads to color optical computing.

Comment. Following this line of reasoning, it is also possible to have several experts producing intervals. This option may be worth exploring.

3 Interval-Valued and Set-Valued Extensions of 2-Valued Logic

Why 2-valued logic. In general, a discrete fuzzy logic is a finite subset of the interval $[0,1]$ that co0ntains both 0 ("false") and 1 ("true"). From this viewpoint, the simplest case is when this subset contains only 0 and 1, i.e., when we have a usual 2-valued logic.

Interval-values extension of 2-valued logic. In a logic consisting of two elements $0 < 1$, there are exactly three possible intervals:

- two degenerate intervals $[0,0] = \{0\}$ and $[1,1] = \{1\}$ consisting of a single original value, and
- a non-degenerate interval $[0,1] = \{0,1\}$ containing both values.

The general interpretation of interval-valued extensions – that was described in the previous section – provides the following explanation for the new truth value $[0,1]$: this truth value corresponds to the case when we do not know whether the statement is true or false – i.e., corresponds to uncertainty. Thus, we get a usual 3-valued logic with three possible truth values: true, false, and uncertain. These values can be naturally described as 1, 0, and an intermediate value 0.5.

Set-valued extension of 2-valued logic. In a 2-valued logic with the set of truth values $\{0,1\}$, there are four subsets:

- two 1-elements subsets $\{0\}$ and $\{1\}$;
- the original set $\{0,1\}$, and
- the empty set $\emptyset$.

The general interpretation of set-valued extensions – that was described in the previous section – provides the following interpretation of these four subsets:

- the set $\{0\}$ means that all experts agree that the statement is false;
- the set $\{1\}$ means that all experts agree that the statement is true;
- the set $\{0,1\}$ means that some experts believe that the statement is true, while some other experts believe that the statement is false;
- finally, the empty set means that no experts have any opinion about this statement.

Here, both the set $\{0,1\}$ and the empty set correspond to uncertainty, but there is a difference between the two cases:

- the empty set means, in effect, that we know nothing about the statement;
- in contrast, the set $\{0,1\}$ means, in effect, that we have some arguments in favor of the given statement, and some arguments against this statement.

How is this related to interval-valued fuzzy techniques. The need to distinguish between these two types of uncertainty is often emphasized as the need to go from the traditional fuzzy logic to its interval-values version. Indeed, in the traditional fuzzy logic, the same value 0.5 can mean two different things:

– it can mean that we know nothing about the given statement, and
– it can also mean that we have as many arguments in favor of this statement as against it.

In the interval-valued case:

– the first situation – when we know nothing, the statement can be false or true – is naturally described by the interval $[0, 1]$ containing all possible truth values, while
– for the second situation, a value 0.5 – corresponding to the degenerate (1-point) interval $[0.5, 0.5]$ seems to be a better match.

How is this related to Belnap logic. The above four truth values have been analyzed in a non-fuzzy context, under the name of Belnap logic [1, 2]. In this context, instead of expert opinions about the truth of a statement, we consider the actual validity of this statement. In this interpretation, the set $\{0, 1\}$ corresponds to inconsistency – when our knowledge base mistakenly contains both the information that this statement is true and the information that this same statement is false.

The need to consider this logic was caused by the fact that in the usual 2-valued logic, once we have a single contradiction, we can conclude that all statements are true – and that all statements are false. So, if we use the usual logic, one wrong statement added to the database – e.g., that the train leaves at 1 pm and that this same train leaves at 1.01 pm – would make the whole knowledge base useless.

4 Interval-Valued and Set-Valued Extensions of 3-Valued Logic and Their Relation to Color Optical Computing

3-valued logic. After the simplest 2-valued logic, the next simplest is 3-valued logic, when we add, to the usual 0 (“false") and 1 (”true”), and additional intermediate degree corresponding to uncertainty. For simplicity, let us denote this degree by 0.5.

Interval-valued extension of 3-valued logic. For this logic, with 3 truth values $0 < 0.5 < 1$, there are six possible intervals:

– the degenerate interval $[0, 0] = \{0\}$ meaning that the expert believes that the given statement is false;
– the degenerate interval $[1, 1] = \{1\}$ meaning that the expert believes that the given statement is true;
– the degenerate interval $[0.5, 0.5] = \{0.5\}$ meaning that the expert is uncertain;
– the interval $[0, 0.5] = \{0, 0.5\}$ meaning the expert is uncertain but is leaning towards “false”;
– the interval $[0.5, 1] = \{0.5, 1\}$ meaning the expert is uncertain but is leaning towards “true”; and

– the interval $[0,1] = \{0, 0.5, 1\}$ meaning that the expert is uncertain, but has some arguments in favor and against the given statement.

In this case, already the interval extension captures the difference between:

– not having any information about a statement and
– having arguments for and against,

the distinction that for extensions of 2-valued logic, required the use of set-valued extension.

Set-valued extension of 3-valued logic. In this case, in addition to the above six sets, we have two more sets:

– the empty set $\emptyset$ corresponding to situations in which no expert has any opinion, and
– the set $\{0, 1\}$ corresponding to the polarized case when some experts strongly believe that the given statement is true while others as strongly believe that this statement is false – case typical in politics.

Set-valued extension of 3-valued logic naturally leads to color optical computing. In color optical computing, we start with three basic colors read (R), green (G), and blue (B) whose position on the spectrum is described as $R < G < B$, and we consider combinations of some of these colors, i.e., all subsets of the set $\{R, G, B\}$:

– we can have three pure colors corresponding to three 1-element sets $\{R\}$, $\{G\}$, and $\{B\}$;
– we can have white – a combination of all three basic colors – corresponding to the set $\{R, G, B\}$;
– we can have black – where there are no colors at all – corresponding to the empty set; and
– we can also have combinations of two of three colors.

These $2^3 = 8$ combinations are in natural 1-to-1 correspondence with eight subsets that form the set-valued extension of the 3-valued logic. This provides a natural explanation of the color optical interpretation of fuzzy logic.

References

1. Belnap, N.: How computers should think. In: Ryle, G. (ed.), Contemporary Aspects of Philosophy. Oriel Press, London, UK, pp. 30–56 (1975)
2. Belnap, N.: A useful four-valued logic. In: Dunn, J. M., Epstein, G. (ed.). Modern Uses of Multiple-Valued Logic. Springer, New York (1977)
3. Belohlavek, R., Dauben, J. W., Klir, G. J.: Fuzzy Logic and Mathematics: A Historical Perspective. Oxford University Press, New York (2017)
4. Klir, G., Yuan, B.: Fuzzy Sets and Fuzzy Logic. Prentice Hall, Upper Saddle River, New Jersey (1995)

5. Mendel, J. M.: Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions. Springer, Cham, Switzerland (2017)
6. Miller, G. A.: The magical number seven plus or minus two: some limits on our capacity for processing information. Psychological Review **63**(2), 81–97 (1956)
7. Reed, S. K.: Cognition: Theories and Application. SAGE Publications, Thousand Oaks, California (2022)
8. Nguyen, H. T., Walker, C. L., Walker, E. A.: A First Course in Fuzzy Logic. Chapman and Hall/CRC, Boca Raton, Florida (2019)
9. Novák, V., Perfilieva, I., Močkoř, J.: Mathematical Principles of Fuzzy Logic. Kluwer, Boston, Dordrecht (1999)
10. Timchenko, V., Kondratenko, Y., Kreinovich, V.: Efficient optical approach to fuzzy data processing based on colors and light filter. International Journal of Problems of Control and Informatics **52**(4), 89–105 (2022)
11. Timchenko, V., Kondratenko, Y., Kreinovich, V.: Decision support system for the safety of ship navigation based on optical color logic gates. Proceedings of the IX International Conference "Information Technology and Implementation" IT&I-2022, Kyiv, Ukraine, November 30 – December 2, 2022 (2022)
12. Timchenko, V., Kondratenko, Y., Kreinovich, V.: Implementation of optical logic gates based on color filters. Proceedings of the he 6th International Conference on Computer Science, Engineering and Education Applications ICCSEEA2023, Warsaw, Poland, March 17–19, 2023 (2023)
13. Timchenko, V., Kondratenko, Y., Kreinovich, V.: Why color optical computing, In: Phuong, N. H., Kreinovich, V. (eds.), Deep Learning and Other Soft Computing Techniques: Biomedical and Related Applications. Springer, Cham, Switzerland (to appear)
14. Zadeh, L. A.: Fuzzy sets. Information and Control **8**, 338–353 (1965)