What Decision to Make In a Conflict Situation under Interval Uncertainty: Efficient Algorithms for the Hurwzcz Approach

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Abstract. In this paper, we show how to take interval uncertainty into account when solving conflict situations. Algorithms for conflict situations under interval uncertainty are know under the assumption that each side of the conflict maximizes its worst-case expected gain. However, it is known that a more general Hurwicz approach provides a more adequate description of decision making under uncertainty. In this approach, each side maximizes the convex combination of the worst-case and the best-case expected gains. In this paper, we describe how to resolve conflict situations under the general Hurwicz approach to interval uncertainty.

1 Conflict Situations Under Interval Uncertainty: Formulation of the Problem and What Is Known So Far

How conflict situations are usually described. In many practical situations – e.g., in security – we have conflict situations in which the interests of the two sides are opposite. For example, a terrorist group wants to attack one of our assets, while we want to defend them. In game theory, such situations are described by *zero-sum games*, i.e., games in which the gain of one side is the loss of another side; see, e.g., [9].

To fully describe such a situation, we need to describe:

– for each possible strategy of one side and
– for each possible strategy of the other side,

what will be the resulting gain to the first side (and, correspondingly, the loss to the other side). Let us number all the strategies of the first side, and all the strategies of the second side, and let u_{ij} be the gain of the first side (negative if this is a loss). Then, the gain of the second side is $v_{ij} = -u_{ij}$.

While zero-sum games are a useful approximation, they are not always a perfect description of the situation. For example, the main objective of the terrorists may be publicity. In this sense, a small attack in the country's capital may not cause much damage but it will bring them a lot of media attention, while a more serious attack in a remote location may be more damaging to the country, but not as media-attractive. To take this difference into account, we need, for each pair of strategies (i, j), to describe both:

– the gain u_{ij} of the first side and
– the gain v_{ij} of the second side.

In this general case, we do not necessarily have $v_{ij} = -u_{ij}$ [9].

How to describe this problem in precise terms. It is a well-known fact that in conflict situations, instead of following one of the deterministic strategies, it is beneficial to select a strategy at random, with some probability. For example, if we only have one security person available and two objects to protect, then we have two deterministic strategies:

– post this person at the first objects and
– post him/her at the second object.

If we exactly follow one of these strategies, then the adversary will be able to easily attack the other – unprotected – object. It is thus more beneficial to every time flip a coin and assign the security person to one of the objects at random. This way, for each object of attack, there will be a 50% probability that this object will be defended.

In general, each corresponding strategy of the first side can be described by the probabilities $p_1, \ldots, p_n$ of selecting each of the possible strategies, so that

$$\sum_{i=1}^{n} p_i = 1. \tag{1.1}$$

Similarly, the generic strategy of the second side can be described by the probabilities $q_1, \ldots, q_m$ for which

$$\sum_{j=1}^{m} q_j = 1. \tag{1.2}$$

If the first side selects the strategy $p = (p_1, \ldots, p_n)$ and the second side selects the strategy $q = (q_1, \ldots, q_m)$, then the expected gain of the first side is equal to

$$g_1(p, q) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \cdot q_j \cdot u_{ij}, \tag{1.3}$$

while the expected gain of the second side is equal to

$$g_2(p, q) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \cdot q_j \cdot v_{ij}. \tag{1.4}$$

Based on this, how can we select a strategy? It is reasonable to assume that once a strategy is selected, the other side knows the corresponding probabilities – simply by observing the past history. So, if the first side selects the strategy p, the second side should select a strategy for which, under this strategy of the first side, their gain is the largest possible, i.e., the strategy $q(p)$ for which

$$g_2(p, q(p)) = \max_q g_2(p, q). \tag{1.5}$$

In other words,

$$q(p) = \arg\max_q g_2(p, q). \tag{1.6}$$

Under this strategy of the second side, the first side gains the value $g_1(p, q(p))$. A natural idea is to select the strategy p for which this gain is the largest possible, i.e., for which

$$g_1(p, q(p)) \to \max_p, \text{ where } q(p) \stackrel{\text{def}}{=} \arg\max_q g_2(p, q). \tag{1.7}$$

Similarly, the second side select a strategy q for which

$$g_2(p(q), q) \to \max_q, \text{ where } p(q) \stackrel{\text{def}}{=} \arg\max_p g_1(p, q). \tag{1.8}$$

Towards an algorithm for solving this problem. Once the strategy p of the first side is selected, the second side selects q or which its expected gain $g_2(p, q)$ is the largest possible.

The expression $g_2(p, q)$ is linear in terms of q_j. Thus, for every q, the resulting expected gain is the convex combination

$$g_2(p, q) = \sum_{j=1}^{m} q_j \cdot q_{2j}(p) \tag{1.9}$$

of the gains

$$g_{2j}(p) \stackrel{\text{def}}{=} \sum_{i=1}^{n} p_i \cdot v_{ij} \tag{1.10}$$

corresponding to different deterministic strategies of the second side. Thus, the largest possible gain is attained when q is a deterministic strategy.

The j-th deterministic strategy will be selected by the second side if its gain at this strategy are larger than (or equal to) gains corresponding to all other deterministic strategies, i.e., under the constraint that

$$\sum_{i=1}^{n} p_i \cdot v_{ij} \geq \sum_{i=1}^{n} p_i \cdot v_{ik} \tag{1.11}$$

for all $k \neq j$.

For strategies p for which the second side selects the j-th response, the gain of the first side is

$$\sum_{i=1}^{n} p_i \cdot u_{ij}. \tag{1.12}$$

Among all strategies p with this "j-property", we select the one for which the expected gain of the first side is the largest possible. This can be found by optimizing a linear function under constraints which are linear inequalities – i.e., by solving a *linear programming* problem. It is known that for linear programming problems, there are efficient algorithms; see, e.g., [6].

In general, we thus have m options corresponding to m different values $j = 1, \ldots, m$. Among all these m possibility, the first side should select a strategy for which the expected gain is the largest possible. Thus, we arrive at the following algorithm.

An algorithm for solving the problem. For each j from 1 to m, we solve the following linear programming problem:

$$\sum_{i=1}^{n} p_i^{(j)} \cdot u_{ij} \to \max_{p_i^{(j)}} \tag{1.13}$$

under the constraints

$$\sum_{i=1}^{n} p_i^{(j)} = 1, \quad p_i^{(j)} \ge 0, \quad \sum_{i=1}^{n} p_i^{(j)} \cdot v_{ij} \ge \sum_{i=1}^{n} p_i^{(j)} \cdot v_{ik} \text{ for all } k \ne j. \tag{1.14}$$

Out of the resulting m solutions $p^{(j)} = \left(p_1^{(j)}, \ldots, p_n^{(j)}\right)$, $1 \le j \le m$, we select the one for which the corresponding value $\sum\limits_{i=1}^{n} p_i^{(j)} \cdot u_{ij}$ is the largest.

Comment. Solution is simpler in zero-sum situations, since in this case, we only need to solve one linear programming problem; see, e.g., [9].

Need for parallelization. For simple conflict situations, when each side has a small number of strategies, the corresponding problem is easy to solve.

However, in many practical situations, especially in security-related situations, we have a large number of possible deterministic strategies of each side. This happens, e.g., if we assign air marshals to different international flights. In this case, the only way to solve the corresponding problem is to perform at least some computations in parallel.

Good news is that the above problem allows for a natural parallelization: namely, all m linear programming problems can be, in principle, solved on different processors. (Not so good news is that this exhausts the possibility of parallelization: once we get to the linear programming problems, they are P-hard, i.e., provably the hardest to parallelize; see, e.g., [8].)

Need to take uncertainty into account. The above description assumed that we know the exact consequence of each combination of strategies. This is rarely the case. In practice, we rarely know the exact gains u_{ij} and v_{ij}. At best, we know the *bounds* on these gains, i.e., we know:

- the interval $[\underline{u}_{ij}, \overline{u}_{ij}]$ that contains the actual (unknown) values u_{ij}, and
- the interval $[\underline{v}_{ij}, \overline{v}_{ij}]$ that contains the actual (unknown) values v_{ij}.

It is therefore necessary to decide who to do in such situations of interval uncertainty.

How interval uncertainty is taken into account now. In the above description of a conflict situation, we mentioned that when we select the strategy p, we maximize the worst-case situation, i.e., the smallest possible gain $g_1(p,q)$ under all possible actions of the second side. It seems reasonable to apply the same idea to the case of interval uncertainty, i.e., to maximize the smallest possible gain $g_1(p,q)$ over all possible strategies of the second side *and* over all possible values $u_{ij} \in [\underline{u}_{ij}, \overline{u}_{ij}]$.

For some practically important situations, efficient algorithms for such worst-case formulation have indeed been proposed; see, e.g., [3].

Need for a more adequate formulation of the problem. In the case of adversity, it makes sense to consider the worst-case scenario: after all the adversary wants to minimize the gain of the other side.

However, in case of interval uncertainty, using the worst-case scenario may not be the most adequate idea. The problem of decision making under uncertainty, when for each alternative a, instead of the exact value $u(a)$, we only know the interval $[\underline{u}(a), \overline{u}(a)]$ of possible values of the gain, has been thoroughly analyzed.

It is known that in such situations, the most adequate decision strategy is to select an alternative a for which the following expression attains the largest possible value:

$$u^H(a) \stackrel{\text{def}}{=} \alpha \cdot \overline{u}(a) + (1-\alpha) \cdot \underline{u}(a), \tag{1.5}$$

where $\alpha \in [0,1]$ describes the decision maker's attitude; see, e.g., [1, 4, 5]. This expression was first proposed by the Nobelist Leonid Hurwicz and is thus, known as the Hurwicz approach to decision making under interval uncertainty.

In the particular case of $\alpha = 0$, this approach leads to optimizing the worst-case value $\underline{u}(a)$, but for other values α, we have different optimization problems.

What we do in this paper. In this paper, we analyze how to solve conflict situations under this more adequate Hurwicz approach to decision making under uncertainty.

In this analysis, we will assume that each side know the other's parameter α, i.e., that both sides know the values α_u and α_v that characterize their decision making under uncertainty. This can be safely assumed since we can determine these values by analyzing past decisions of each side.

2 Conflict Situation under Hurwicz-Type Interval Uncertainty: Analysis of the Problem

Once the first side selects a strategy, what should the second side do? If the first side selects the strategy p, then, for each strategy q of the second side, the actual (unknown) gain of the second side is equal to $\sum_{i=1}^{n}\sum_{j=1}^{m} p_i \cdot q_j \cdot v_{ij}$. We do not know the exact values v_{ij}, we only know the bounds $\underline{v}_{ij} \le v_{ij} \le \overline{v}_{ij}$. Thus, once:

– the first side selects the strategy p and
– the second side selects the strategy q,

the gain of the second side can take any value from

$$\underline{g}_2(p,q) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_i \cdot q_j \cdot \underline{v}_{ij} \qquad (2.1)$$

to

$$\overline{g}_2(p,q) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_i \cdot q_j \cdot \overline{v}_{ij}. \qquad (2.2)$$

According to Hurwicz's approach, the second side should select a strategy q for which the Hurwicz combination

$$g_2^H(p,q) \stackrel{\text{def}}{=} \alpha_v \cdot \overline{g}_2(p,q) + (1-\alpha_v) \cdot \underline{g}_2(p,q) \qquad (2.3)$$

attains the largest possible value.

Substituting the expressions (2.1) and (2.2) into the formula (2.3), we conclude that

$$g_2^H(p,q) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_i \cdot q_j \cdot v_{ij}^H, \qquad (2.4)$$

where we denoted

$$v_{ij}^H \stackrel{\text{def}}{=} \alpha_v \cdot \overline{v}_{ij} + (1-\alpha_v) \cdot \underline{v}_{ij}. \qquad (2.5)$$

Thus, once the first side selects its strategy p, the second side should select a strategy $q(p)$ for which the corresponding Hurwicz combination $g_2^H(p,q)$ is the largest possible, i.e., the strategy $q(p)$ for which

$$g_2^H(p,q(p)) = \max_q g_2^H(p,q). \qquad (2.6)$$

In other words,

$$q(p) = \arg\max_q g_2^H(p,q). \qquad (2.7)$$

Based on this, what strategy should the first side select? Under the above strategy $q = q(p)$ of the second side, the first side gains the value

$$g_1(p, q(p)) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \cdot q_j \cdot u_{ij}. \tag{2.8}$$

Since we do not know the exact values u_{ij}, we only know the bounds $\underline{u}_{ij} \le u_{ij} \le \overline{u}_{ij}$, we therefore do not know the exact gain of the first side. All we know is that this gain will be between

$$\underline{g}_1(p, q(p)) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \cdot q_j \cdot \underline{u}_{ij} \tag{2.9}$$

and

$$\overline{g}_1(p, q(p)) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \cdot q_j \cdot \overline{u}_{ij}. \tag{2.10}$$

According to Hurwicz's approach, the first side should select a strategy p for which the Hurwicz combination

$$g_1^H(p, q) \stackrel{\text{def}}{=} \alpha_u \cdot \overline{g}_1(p, q(p)) + (1 - \alpha_u) \cdot \underline{g}_1(p, q(p)) \tag{2.11}$$

attains the largest possible value.

Substituting the expressions (2.9) and (2.10) into the formula (2.11), we conclude that

$$g_1^H(p, q) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i \cdot q_j \cdot u_{ij}^H, \tag{2.12}$$

where we denoted

$$u_{ij}^H \stackrel{\text{def}}{=} \alpha_u \cdot \overline{u}_{ij} + (1 - \alpha_u) \cdot \underline{u}_{ij}. \tag{2.13}$$

What strategy should the second side select? Thus, the first side will select the strategy p for which this Hurwicz combination is the largest possible, i.e., for which

$$g_1^H(p, q(p)) \to \max_p, \text{ where } q(p) \stackrel{\text{def}}{=} \arg\max_q g_2^H(p, q). \tag{2.14}$$

Similarly, the second side select a strategy q for which

$$g_2^H(p(q), q) \to \max_q, \text{ where } p(q) \stackrel{\text{def}}{=} \arg\max_p g_1^H(p, q). \tag{2.15}$$

We thus reduce the interval-uncertainty problem to the no-uncertainty case. One can easily see that the resulting optimization problem is exactly the same as in the no-uncertainty case described in Section 1, with the gains u_{ij}^H and v_{ij}^H described by the formulas (2.13) and (2.5).

Thus, we can apply the algorithm described in Section 1 to solve the interval-uncertainty problem.

3 Algorithm for Solving Conflict Situation under Hurwicz-Type Interval Uncertainty

What is given. For every deterministic strategy i of the first side and for every deterministic strategy j of the second side, we are given:

- the interval $[\underline{u}_{ij}, \overline{u}_{ij}]$ of the possible values of the gain of the first side, and
- the interval $[\underline{v}_{ij}, \overline{v}_{ij}]$ of the possible values of the gain of the second side.

We also know the parameters α_u and α_v characterizing decision making of each side under uncertainty.

Preliminary step: forming appropriate combinations of gain bounds. First, we compute the values

$$u^H_{ij} \stackrel{\text{def}}{=} \alpha_u \cdot \overline{u}_{ij} + (1 - \alpha_u) \cdot \underline{u}_{ij} \tag{3.1}$$

and

$$v^H_{ij} \stackrel{\text{def}}{=} \alpha_v \cdot \overline{v}_{ij} + (1 - \alpha_v) \cdot \underline{v}_{ij}. \tag{3.2}$$

Main step. For each j from 1 to m, we solve the following linear programming problem:

$$\sum_{i=1}^n p_i^{(j)} \cdot u^H_{ij} \to \max_{p_i^{(j)}} \tag{3.3}$$

under the constraints

$$\sum_{i=1}^n p_i^{(j)} = 1, \quad p_i^{(j)} \ge 0, \quad \sum_{i=1}^n p_i^{(j)} \cdot v^H_{ij} \ge \sum_{i=1}^n p_i^{(j)} \cdot v^H_{ik} \text{ for all } k \ne j. \tag{3.4}$$

Final step. Out of the resulting m solutions $p^{(j)} = \left(p_1^{(j)}, \ldots, p_n^{(j)}\right)$, $1 \le j \le m$, we select the one for which the corresponding value

$$\sum_{i=1}^n p_i^{(j)} \cdot u^H_{ij} \tag{3.5}$$

is the largest.

Comment. In view of the fact that in the no-uncertainty case, zero-sum games are easier process, let us consider zero-sum games under interval uncertainty. To be more precise, let us consider situations in which possible values v_{ij} are exactly values $-u_{ij}$ for possible u_{ij}:

$$[\underline{v}_{ij}, \overline{v}_{ij}] = \{-u_{ij} : \underline{u}_{ij} \in [\underline{u}_{ij}, \overline{u}_{ij}]\}. \tag{3.6}$$

One can easily see (see, e.g., [2, 7]) that this condition is equivalent to

$$\underline{v}_{ij} = -\overline{u}_{ij} \text{ and } \overline{v}_{ij} = -\underline{u}_{ij}. \tag{3.7}$$

In this case, we have

$$v_{ij}^H = \alpha_v \cdot \overline{v}_{ij} + (1 - \alpha_v) \cdot \underline{v}_{ij} = \alpha_v \cdot (-\underline{v}_{ij}) + (1 - \alpha_v) \cdot (-\overline{u}_{ij}), \tag{3.8}$$

and thus,

$$v_{ij}^H = -((1 - \alpha_v) \cdot \overline{u}_{ij} + \alpha_v \cdot \underline{u}_{ij}). \tag{3.9}$$

By comparing this expression with the formula (3.1) for u_{ij}^H, we can conclude that the resulting game is zero-sum (i.e., $v_{ij}^H = -u_{ij}^H$) only when $\alpha_u = 1 - \alpha_v$.

In all other cases, even if we start with a zero-sum interval-uncertainty game, the no-uncertainty game to which we reduce that game will *not* be zero-sum – and thus, the general algorithm will be needed, without a simplification that is available for zero-sum games.

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