How to Make a Solution to a Territorial Dispute More Realistic: Taking into Account Uncertainty, Emotions, and Step-by-Step Approach

Mahdokht Afravi and Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
mafravi@miners.utep.edu, vladik@utep.edu

Abstract—**In many real-life situations, it is necessary to divide a disputed territory between several interested parties. The usual way to perform this division is by using Nash's bargaining solution, i.e., by finding a partition that maximizes the product of the participants' utilities. However, this solution is based on several idealized assumptions: that we know the exact values of all the utilities, that division is performed on a purely rational basis, with no emotions involved, and that the entire decision is made once. In practice, we only know the utilities with some uncertainty, emotions are often involved, and the solution is often step-by-step. In this paper, we show how to make a solution to a territorial dispute more realistic by taking all this into account.**

I. Formulation of the Problem

Dividing a disputed territory: a real-life problem. In many real-life situations, from conflicts between neighboring farms to conflict between states, there is a dispute over a territory, as a result of which none of the sides can use this territory efficiently. In such situations, it is desirable to come up with a mutually beneficial agreement.

Current solution. The current solution is based on the work by the Nobelist J. Nash, who showed that under reasonable assumptions, the best mutually beneficial solution is the one that maximizes the product of utilities of all the sides [8], [9].

Nash's solution is a good agreement with common sense – as formalized by fuzzy logic. The above solution – known as Nash's *bargaining solution* – is in perfect agreement with informal common sense description as formalized by fuzzy logic; see, e.g., [5], [12], [15]. Indeed, we want all participants to be happy, i.e., we want the first participant to be happy *and* the second participant to be happy, etc. The degree of happiness of each participant can be described by his or her utility. To represent "and", it is reasonable to use one of the most frequently used fuzzy "and"-operations – the product.

Thus, the degree to which our overall objective – to make everyone happy – is satisfies can be estimated as the product of all the utilities. Out of all possible partitions, we want to select a partition for which this degree of overall happiness. This is exactly what Nash's bargaining solution is about.

Nash's solution: from a theoretical formulation to practical recommendations. Now that we are reasonably convinced that Nash's bargaining solution is a reasonable idea, let us recall what exactly partitions it leads to.

Let $u_i(x)$ be the utility (per area) of the i-th participant at location x. Then, we should select a partition for which the product $\prod\limits_{i=1}^{n} U_i$ is the largest, where $U_i \stackrel{\text{def}}{=} \int_{S_i} u_i(x)\,dx$ and S_i is the set allocated to the i-th participant. The solution to this optimization problem is, for some thresholds t_i, to assign each location x to a participant with the largest ratio $u_i(x)/t_i$; see, e.g., [6], [11].

The parameters t_i must be determined from the requirement that the product of all the utilities is the largest possible.

In particular, for two participants, $x \in S_1$ if

$$\frac{u_1(x)}{u_2(x)} \geq t \stackrel{\text{def}}{=} \frac{t_1}{t_2}.$$

How to make this solution more realistic.

- The above solution assumes that we know the exact values of all the utilities, i.e., that for every location x and for every participant i, we know the value $u_i(x)$. In reality, we know the values $u_i(x)$ only approximately; e.g., we only know the interval $[\underline{u}_i(x), \overline{u}_i(x)]$ containing $u_i(x)$. How can we take this uncertainty into account?
- The above solution assumes that all the sides are making their decisions on a purely rational basis, that no emotions are involved. In reality, there are often emotions. How can we take these emotions into account?
- Finally, while the above formula proposes an immediate solution, participants often follow step-by-step approach, where they first divide a small part, then another part, etc. This also needs to be taken into account.

What we do in this paper. In this paper, we describe how to take all this into account.

Comment. Preliminary results appeared in [1], [2].

II. How to Take Uncertainty into Account

Formulation of the problem. Let us take into account that instead of the exact values $u_i(x)$, we only know the bounds $\underline{u}_i(x)$ and $\overline{u}_i(x)$ on the actual (unknown) value $u_i(x)$ of the corresponding utility: $\underline{u}_i(x) \le u_i(x) \le \overline{u}_i(x)$.

In this case, for each allocation S_i, the only thing that we know about the i-th participant's utility $U_i = \int_{S_i} u_i(x)\,dx$ is that this utility is bounded by the integrals corresponding to $\underline{u}_i(x)$ and $\overline{u}_i(x)$:

$$\underline{U}_i \stackrel{\text{def}}{=} \int_{S_i} \underline{u}_i(x)\,dx \le U_i = \int_{S_i} u_i(x)\,dx \le$$

$$\overline{U}_i \stackrel{\text{def}}{=} \int_{S_i} \overline{u}_i(x)\,dx.$$

In other words, instead the *exact* utility U_i, we only know the *interval* $[\underline{U}_i, \overline{U}_i]$ of possible values of utility.

How to make decisions under interval uncertainty: reminder. In situations when the utility is only know with interval uncertainty, decision making theory recommends using Hurwicz's optimism-pessimism criterion (see, e.g., [4], [7], [8]), according to which, in our decisions, we should use the utility

$$\widetilde{U}_i = \alpha_i \cdot \overline{U}_i + (1 - \alpha_i) \cdot \underline{U}_i,$$

where $\alpha_i \in [0,1]$ describe the i-th participant's degree of optimism.

Let us apply this general recommendation to our problem. Hurwicz's idea means, in particular, that we should use the utilities $\widetilde{U}_i$ in Nash's bargaining solution, i.e., that we should maximize the product $\prod\limits_i \widetilde{U}_i$ of these utilities.

We need to go from the theoretical description to a practical recommendation. We have formulated territory partition as an optimization problem. To be able to use this formulation in practical situations, we need to come up with efficient algorithms for solving this optimization problem.

To come up with such algorithms, we can take into account that the utility $\widetilde{u}_i(x)$ can be represented in a form which is similar to the original formula $U_i = \int_{S_i} u_i(x)\,dx$. Namely, from the definition

$$\widetilde{U}_i = \alpha_i \cdot \overline{U}_i + (1 - \alpha_i) \cdot \underline{U}_i,$$

taking into account that $\overline{U}_i = \int_{S_i} \overline{u}_i(x)\,dx$ and $\underline{U}_i = \int_{S_i} \underline{u}_i(x)\,dx$, we conclude that

$$\widetilde{U}_i = \alpha_i \cdot \int_{S_i} \overline{u}_i(x)\,dx + (1 - \alpha_i) \cdot \int_{S_i} \underline{u}_i(x)\,dx$$

and thus,

$$\widetilde{U}_i = \int_{S_i} \widetilde{u}_i\,dx,$$

where we denoted

$$\widetilde{u}_i(x) \stackrel{\text{def}}{=} \alpha_i \cdot \overline{u}_i(x) + (1 - \alpha_i) \cdot \underline{u}_i(x).$$

Thus, from the mathematical viewpoint, the problem of territorial division under interval uncertainty is similar to the original territorial division problem, with the only difference that:

- instead of the utility function $u_i(x)$ used in the original problem,
- we now use the combined utility function

$$\widetilde{u}_i(x) = \alpha_i \cdot \overline{u}_i(x) + (1 - \alpha_i) \cdot \underline{u}_i(x).$$

Thus, we can use the known solution to the original territorial division problem to come up with the following solution to the problem of territorial division under interval uncertainty.

Resulting solution. The solution to this optimization problem is, for some thresholds t_i, to assign each location x to a participant with the largest ratio $\widetilde{u}_i(x)/t_i$.

The parameters t_i must be determined from the requirement that the product of all the utilities is the largest possible.

In particular, for two participants, $x \in S_1$ if

$$\frac{\widetilde{u}_1(x)}{\widetilde{u}_2(x)} \ge t \stackrel{\text{def}}{=} \frac{t_1}{t_2}.$$

Example. Let us illustrate the above decision on the example when:

- different participants assign the same utility to all the locations, i.e., when $\underline{u}_i(x) = \underline{u}_j(x)$ and $\overline{u}_i(x) = \overline{u}_j(x)$ for all i and j, and
- the only difference between the participants is that they have different optimism degrees $\alpha_i \ne \alpha_j$.

Let us use the above solution to describe the borderline between the locations assigned to participants i and j. According to the above solution, out of all locations that are assigned to either participant i or participant j, this location is:

- assigned to the participant i if
$$\frac{\widetilde{u}_i(x)}{t_i} \ge \frac{\widetilde{u}_j(x)}{t_j},$$
and
- assigned to the participant j if
$$\frac{\widetilde{u}_i(x)}{t_i} \le \frac{\widetilde{u}_j(x)}{t_j}.$$

Here,

$$\widetilde{u}_i(x) = \alpha_i \cdot \overline{u}_i(x) + (1 - \alpha_i) \cdot \underline{u}_i(x)$$

and

$$\widetilde{u}_j(x) = \alpha_j \cdot \overline{u}_j(x) + (1 - \alpha_j) \cdot \underline{u}_j(x).$$

Let us denote the common utility bounds by $\underline{u}(x)$ and $\overline{u}(x)$. Then, the above formulas can be reformulated as

$$\widetilde{u}_i(x) = \alpha_i \cdot \overline{u}(x) + (1 - \alpha_i) \cdot \underline{u}(x)$$

and

$$\widetilde{u}_j(x) = \alpha_j \cdot \overline{u}(x) + (1 - \alpha_j) \cdot \underline{u}_(x).$$

Substituting these expressions into the above formula, we conclude that the location x is assigned to the i-th participant if

$$\frac{\alpha_i}{t_i}\cdot\overline{u}(x)+\frac{1-\alpha_i}{t_i}\cdot\underline{u}(x)\ge\frac{\alpha_j}{t_j}\cdot\overline{u}(x)+\frac{1-\alpha_j}{t_j}\cdot\underline{u}(x).$$

Without losing generality, we can assume that

$$\frac{\alpha_i}{t_i}>\frac{\alpha_j}{t_j}.$$

In this case, moving the terms containing $\overline{u}$ to the left-hand side and the terms containing $\underline{u}$ to the right-hand side, we conclude that $A\cdot\overline{u}\ge B\cdot\underline{u}$ for some $A>0$. Dividing both sides by $A\cdot\underline{u}>0$, we get an equivalent inequality

$$\frac{\overline{u}}{\underline{u}}\ge\text{const}.$$

Subtracting one from both sides, we get

$$\frac{\overline{u}(x)-\underline{u}(x)}{\underline{u}(x)}\ge\text{const}.$$

The left-hand side is a relative uncertainty with which we know the utility.

Thus, one of the participants i and j gets all the locations with higher relative uncertainty, while the other gets all the locations with the lower relative uncertainty. One can easily check that in the optimal solution,

- the more optimistic participant gets locations with higher uncertainty, while
- the more pessimistic one gets locations with lower uncertainty.

This makes perfect sense, since:

- the optimistic participant, with $\alpha_i\approx 1$, for whom $\widetilde{u}_i(x)=\alpha_i\cdot\overline{u}(x)+(1-\alpha_i)\cdot\underline{u}(x)$, benefits the most from situations with higher uncertainty (for which $\overline{u}\gg\underline{u}$),
- as compared to a pessimistic participant, for whom $\alpha_i\approx 0$ and thus, $\widetilde{u}_i(x)\approx\underline{u}(x)$.

III. How to Take Emotions into Account

How to describe emotional involvement. In decision theory, emotional involvement is usually described as follows: if U_i is the utility of i-th participant that does not take emotions into account, then with emotions, decisions are determined by the modified utility

$$U_i^{\text{emo}}=U_i+\sum_j\alpha_{ij}\cdot U_j,$$

where the coefficients α_{ij} describe the feelings of the i-th participant about the j-th one:

- the value $\alpha_{ij}>0$ indicate positive feelings,
- the value $\alpha_{ij}<0$ indicate negative feelings, and
- the value $\alpha_{ij}=0$ indicate indifference;

see, e.g., [3], [10], [13], [14].

How to take into account emotional involvement in territorial partition problems. Without emotional involvement, the utility of the i-th participant is equal to $U_i=\int_{S_i}u_i(x)\,dx$. Thus, if we take into account emotional involvement, we will get updated utility values

$$U_i^{\text{emo}}=U_i+\sum_{j\ne i}\alpha_{ij}\cdot U_j.$$

The goal is to maximize the product of the updated utilities, i.e., the product $\prod\limits_i U_i^{\text{emo}}$.

Solving the resulting optimization problem. To solve the resulting optimization problem, we will use the same ideas as used in [6], [11] to solve the original optimization problem.

Let us select two participants i_0 and j_0. Suppose that in the optimal solution, a location x_0 is assigned to i_0-th participant. Optimality means that if we re-assign a small neighborhood of this location to some other participant j_0, the desired product will decrease. After this reassignment, the utilities U_i change as follows:

- the utility $U_{i_0}=\int_{S_{i_0}}u_{i_0}(x)\,dx$ decreases by $u_{i_0}(x_0)\cdot\delta$, where δ is the area of the selected small neighborhood:

$$\Delta U_{i_0}=-u_{i_0}(x_0)\cdot\delta;$$

- the utility $U_{j_0}=\int_{S_{j_0}}u_{j_0}(x)\,dx$ increases by $u_{j_0}(x_0)\cdot\delta$:

$$\Delta U_{j_0}=u_{j_0}(x_0)\cdot\delta;$$

  and
- all other utilities U_i remain unchange: $\Delta U_i=0$ for $i\ne i_0,j_0$.

Thus, for the changes ΔU_i^{emo} in the actual emotions U_i^{emo}, we get the following formula:

- for $i=i_0$, we have $\Delta U_{i_0}^{\text{emo}}=\Delta U_{i_0}+\alpha_{i_0j_0}\cdot\Delta U_{j_0}$;
- for $i=j_0$, we have $\Delta U_{j_0}^{\text{emo}}=\Delta U_{i_0}+\alpha_{j_0i_0}\cdot\Delta U_{i_0}$;
- for all other i, we have $\Delta U_i^{\text{emo}}=\alpha_{ii_0}\cdot\Delta U_{i_0}+\alpha_{ij_0}\cdot\Delta U_{j_0}$.

Maximizing the product $\prod\limits_i U_i^{\text{emo}}$ is equivalent to maximizing its logarithm $L\stackrel{\text{def}}{=}\sum\limits_i\ln(U_i^{\text{emo}})$. For small changes, $\Delta(\ln(x))=\dfrac{\Delta x}{x}$; thus, for small δ, the change ΔL in L is equal to

$$\Delta L=\sum_i\frac{\Delta U_i^{\text{emo}}}{U_i^{\text{emo}}}.$$

Substituting the above formulas for ΔU_i^{emo} into this expressions and taking into account that $\Delta U_{i_0}=-u_{i_0}(x_0)\cdot\delta$, $\Delta U_{j_0}=u_{j_0}(x_0)\cdot\delta$, and $\delta>0$, we conclude that the desired inequality $\Delta L\le 0$ is equivalent to a linear equality $a\cdot u_{i_0}(x_0)+b\cdot u_{j_0}(x_0)\le 0$, i.e., equivalently, to

$$\frac{u_{i_0}(x_0)}{u_{j_0}(x_0)}\ge c$$

for an appropriate constant c.

On the other hand, if a point x originally was assigned to the j_0-th participant, then by re-assigning this location to the i_0-th participant, we will get exactly the same changes as before, but with $-\delta$ instead of δ. In this case, after dividing the inequality $\Delta L \leq 0$ by the negative number $-\Delta$, we get an inequality

$$\frac{u_{i_0}(x_0)}{u_{j_0}(x_0)} \leq c$$

for the same constant c.

Thus, for the locations x that are assigned either to i_0-th or to the j_0-th participant, the location is assigned to the i_0-th one if and only if the ratio $\dfrac{u_{i_0}(x)}{u_{j_0}(x)}$ does not exceed a certain threshold value c.

By combining these conclusions for all possible pairs, we conclude that the solution to this problem is similar to the solution to the original problem:

Resulting solution. The solution to this optimization problem is, for some thresholds t_i, to assign each location x to a participant with the largest ratio $u_i(x)/t_i$.

The parameters t_i must be determined from the requirement that the product of all the updated utilities $U_i^{\rm emo}$ is the largest possible.

In particular, for two participants, $x \in S_1$ if

$$\frac{\widetilde{u}_1(x)}{\widetilde{u}_2(x)} \geq t \stackrel{\rm def}{=} \frac{t_1}{t_2}.$$

Discussion. The solution looks the same as when we did not take emotions into account, but there is a difference: the *thresholds* t_i *are* now *different*, since we are selecting them by maximizing a *different objective function*.

For example, one can check that when

- one of the two participants is positive towards the second one, while
- the second one is neutral towards towards the first one,

then the optimal solution allocated more location to the second participant – since the first one gets positive just because he or she feels good about the second one.

Indeed, let us consider the simplest case when $u_i(x) = 1$ for all i and x. In this case, the utility U_i is simply equal to the area A_i of the corresponding set S_i.

Let us assume that we are dividing the area of area A. Let A_1 be the area allocated to the first participant. Then the area allocated to the second participant is $A - A_1$. In this case, $U_1 = A_1$, $U_2 = A - A_1$ and so, if there are no emotions, we need to maximize the product $A_1 \cdot (A - A_1)$. Differentiating this expression relative to A_1 and equating the derivative to 0, we conclude that $A_1 = A/2$. This makes perfect sense: we have an absolutely symmetric situation, so both participants should get the exact same utilities.

Let us consider the case when $\alpha_{12} > 0$ but $\alpha_{21} = 0$. In this case, we need to maximize a slightly different product

$$(U_1 + \alpha_{12} \cdot U_2) \cdot U_2 = (A_1 + \alpha_{21} \cdot (A - A_1)) \cdot (A - A_1).$$

Differentiating this expression relative to A_1 and equating the derivative to 0, we conclude that

$$(1 - \alpha_{12} \cdot (A - A_1) - (A + \alpha_{12} \cdot (A - A_1)) = 0,$$

i.e., that

$$A \cdot (1 - 2\alpha_{12}) = 2A_1 \cdot (1 - \alpha_{12}),$$

and

$$A_1 = \frac{A}{2} \cdot \frac{1 - 2\alpha_{12}}{1 - \alpha_{12}}.$$

When $\alpha_{12} > 0$, we have $1 - 2\alpha_{12} < 1 - \alpha_{12}$, and thus, indeed, $A_1 < A/2$.

Comment. When emotions are negative, i.e., when $\alpha_{ij} < 0$, then, somewhat surprisingly, we get a positive effect out of it: namely, it stimulates equality. Indeed, all the sides agree to a division only if their utilities $U_i^{\rm emo}$ are non-negative. For example, when $\alpha_{12} = \alpha_{21} = -1$, then the only way to guarantee that both values $U_1^{\rm emo} = U_1 - U_2$ and $U_2^{\rm emo} = U_2 - U_1$ are non-negative is when the values U_1 and U_2 are equal to each other.

For other values of α_{ij}, we do not get exact equality, but we still get bounds limiting how much the utilities U_i can differ from each other; this is described, in detail, in [2].

IV. Immediate Solution vs. Step-by-Step Approach

Step-by-step vs. immediate solution. While it is desirable to make an immediate solution, in international affairs, solutions are often step-by-step. Namely, instead of making a bold decision about *all* parts of the disputed border, a decision is often make location-by-location.

What we do in this section. In this section, we analyze the effect of such a strategy.

Analysis of the problem. In a small vicinity of each location x, utility functions $u_i(x)$ do not change much – and the smaller the vicinity, the less they change. Thus, we can safely assume that in each such vicinity, each utility function is a constant $u_i(x) = u_i$. In this case, the resulting utility $U_i = \int_{S_i} u_i(x)\,dx$ is simply proportional to the area A_i of the set S_i allocated to the i-th participant: $U_i = u_i \cdot A_i$.

Thus, the optimal division of a territory S of area A between n participants means that among the tuples $(A_1, \ldots, A_n)$ for which $A_1 + \ldots + A_n = A$, we need to select a tuple for which the product

$$\prod_{i=1}^n U_i = \prod_{i=1}^n (u_i \cdot A_i)$$

is the largest possible. This product, however, has the form

$$\left(\prod_{i=1}^n u_i\right) \cdot \left(\prod_{i=1}^n A_i\right),$$

so its maximization is equivalent to maximizing the product $\prod\limits_{i=1}^n A_i$ under the condition that $\sum\limits_{i=1}^n A_i = A$. This problem is symmetric with respect to all A_i, and its solution is symmetric: $A_i = \dfrac{A}{n}$.

So, in step-by-step approach, each small vicinity is divided equally. As a result, in each vicinity, the resulting utility of the i-th participant is exactly $1/n$ of what he or she would have got if this participant would get the whole vicinity. Thus, overall, for each participant, we get

$$U_i = \frac{1}{n} \cdot \int_S u_i(x)\,dx.$$

Conclusion: step-by-step solution can be very non-optimal. On each step, we optimized, but the resulting solution is not optimal at all. Indeed, let us consider the following simple example: an area S consists of two equal parts:

- the first part is useless for the 1st participant, but valuable to the second one, and
- the second part is valuable for the first participant but useless for the second one.

In this case, a clear optimal solution is to allocate the first part to the second participant and the second part to the first participant, and this is exactly what a one-step immediate optimization would lead to.

But this is not what we get from a step-by-step solution: in this solution, we divide each part equally. As a result, each participant gets only half od the area which is useful to this participant. This is clearly very non-optimal.

Resulting recommendation. So, our recommendation is to try to solve the problem as a whole, and avoid step-by-step solutions.

But do we need to divide? At first glance, it may seem that instead of dividing a disputed territory, it is desirable to show a brotherly spirit and control it jointly. Sometimes, this may work, but, as we have shown in [1], in general, this strategy will lead to a suboptimal solution: in almost all cases, the product of utilities is the largest when we divide, not when we share the control.

Acknowledgment

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

References

[1] M. Afravi and V. Kreinovich, "How to Divide a Territory: an Argument in Favor of Private Property", *Proceedings of the 3rd International Conference on Mathematical and Computer Modeling*, Omsk, Russia, November 12, 2015, pp. 20–22.

[2] M. Afravi and V. Kreinovich, "Positive Consequences of Negative Attitude: Game-Theoretic Analysis", *International Journal of Contemporary Mathematical Sciences*, 2016, Vol. 11, No. 3, pp. 113–118.

[3] G. S. Becker, *A Treatise on the Family*, Harvard University Press, Cambridge, Massachusetts, 1991.

[4] L. Hurwicz, *Optimality Criteria for Decision Making under Ignorance*, Cowles Commission Discussion Paper, Statistics, No. 370, 1951.

[5] G. Klir and B. Yuan, "Fuzzy Sets and Fuzzy Logic", Prentice Hall, Upper Saddle River, New Jersey, 1995.

[6] V. Kreinovich, "Nash's solution for the territory division problem", *Proceedings of the 3rd USSR National Conference on Game Theory*, Odessa, 1974, pp. 117–118 (in Russian).

[7] V. Kreinovich, "Decision making under interval uncertainty (and beyond)", In: P. Guo and W. Pedrycz (eds.), *Human-Centric Decision-Making Models for Social Sciences*, Springer Verlag, 2014, pp. 163–193.

[8] R. D. Luce and H. Raiffa, *Games and Decisions: Introduction and Critical Survey*, New York, Dover, 1989.

[9] J. F. Nash, "The bargaining problem", *Econometrica*, 1951, Vol. 18, pp. 155-162.

[10] H. T. Nguyen, O. Kosheleva, and V. Kreinovich, "Decision making beyond Arrow's 'Impossibility Theorem', with the analysis of effects of collusion and mutual attraction", *International Journal of Intelligent Systems*, 2009, Vol. 24, No. 1, pp. 27–47.

[11] H. T. Nguyen and V. Kreinovich, "How to divide a territory? A new simple differential formalism for optimization of set functions", *International Journal of Intelligent Systems*, 1999, Vol. 14, No. 3, pp. 223–252.

[12] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2006.

[13] A. Rapoport, "Some game theoretic aspects of parasitism and symbiosis", *Bull. Math. Biophysics*, 1956, Vol. 18, No. 1, pp. 15–30.

[14] A. Rapoport, *Strategy and Conscience*, Harper & Row, New York, 1964.

[15] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.