

Towards Decision Making under Interval, Set-Valued, Fuzzy, and Z-Number Uncertainty: A Fair Price Approach

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Abstract—In this paper, we explore one of the possible ways to make decisions under uncertainty: namely, we explain how to define a fair price for a participation in such a decision, and then select an alternative for which the corresponding fair price is the largest. This idea is explained on the examples of interval uncertainty, set-valued, fuzzy, and Z-number uncertainty.

I. FORMULATION OF THE PROBLEM

Need for decision making under uncertainty. In many practical situations, we need to make a decision in situations when we have an incomplete information about consequences of different decisions.

Fair price approach: an idea. When we have a full information about an object, we can express our desirability of each possible situation, e.g., by declaring a price that we are willing to pay to get involved in this situation. Once these prices are set, selecting the most preferable alternative is easy: we just select the alternative for which the participation price is the highest – since this is clearly the most desirable alternative.

In decision making under uncertainty, the situation is not so clear, since it is not easy to come up with a fair price. A natural idea is to develop techniques for producing such fair prices – these prices can then be used in decision making, to select an appropriate alternative.

Comment. An alternative approach to decision making under uncertainty – based on the extension of the notion of utility to interval, fuzzy, and Z-number uncertainty – is described in [1].

II. CASE OF INTERVAL UNCERTAINTY

Interval uncertainty: description. Let us start with a simple case of uncertainty, in which, instead of knowing the exact gain u of selecting an alternative, we only know the lower bound \underline{u} and the upper bound \bar{u} on this gain – and we have no information on which values from the corresponding interval $[\underline{u}, \bar{u}]$ are more probable or less probable. This situation, in which the only information that we have about the gain u is that this gain belongs to the interval $[\underline{u}, \bar{u}]$, is called *interval uncertainty*.

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Assigning fair price under interval uncertainty: description of the problem. We want to assign, to each interval $[\underline{u}, \bar{u}]$, a number $P([\underline{u}, \bar{u}])$ describing the fair price of this interval. There are several reasonable requirements that this function must satisfy.

First, since in all cases, the gain is larger than or equal to \underline{u} and is smaller than or equal to \bar{u} , it is reasonable to require that the price should also be larger than or equal to \underline{u} and smaller than or equal to \bar{u} : $\underline{u} \leq P([\underline{u}, \bar{u}]) \leq \bar{u}$.

Second, if we keep the lower endpoint \underline{u} intact but increase the upper bound, this means that we are keeping all the previous possibilities, but we are allowing new possibilities with a higher gain. In this case, it is reasonable to require that the corresponding price increases (or at least that it does not decrease). In other words, if $\underline{u} = \underline{v}$ and $\bar{u} < \bar{v}$, then $P([\underline{u}, \bar{u}]) \leq P([\underline{v}, \bar{v}])$.

Similar, if we dismiss some low-gain alternatives, this should increase (or at least not decrease) the fair price. So, in general, if $\underline{u} \leq \underline{v}$ and $\bar{u} \leq \bar{v}$, then we should have $P([\underline{u}, \bar{u}]) \leq P([\underline{v}, \bar{v}])$.

To describe the third requirement, let us consider the situation when we have two different alternatives. If these alternatives are independent, in the sense that the overall gain $u + v$ of selecting both alternatives is equal to the sum of the gains u and v obtained by selecting each of them, then it is reasonable to require that the fair price of the joint selection should be equal to the sums of fair prices corresponding to both alternatives. Let us describe this requirement for the case when the consequences of each alternative are only known with interval uncertainty. About the gain u from the first alternative, we only know that this (unknown) gain value belongs to the interval $[\underline{u}, \bar{u}]$. About the gain v from the second alternative, we only know that this gain belongs to the interval $[\underline{v}, \bar{v}]$. The overall gain $u + v$ can thus take any value from the interval $[\underline{u}, \bar{u}] + [\underline{v}, \bar{v}] = [\underline{u} + \underline{v}, \bar{u} + \bar{v}]$. Thus, the above requirement takes the form

$$P([\underline{u} + \underline{v}, \bar{u} + \bar{v}]) = P([\underline{u}, \bar{u}]) + P([\underline{v}, \bar{v}]).$$

Thus, we arrive at the following definition.

Definition 1. By a fair price under interval uncertainty, we mean a function $P([\underline{u}, \bar{u}])$ that assigns, to every interval, a real number, and which satisfies the following properties:

- $\underline{u} \leq P([\underline{u}, \bar{u}]) \leq \bar{u}$ for all u (conservativeness);
- if $\underline{u} = \underline{v}$ and $\bar{u} < \bar{v}$, then $P([\underline{u}, \bar{u}]) \leq P([\underline{v}, \bar{v}])$ (monotonicity);
- for all $\underline{u}, \bar{u}, \underline{v}, \bar{v}$, we have

$$P([\underline{u} + \underline{v}, \bar{u} + \bar{v}]) = P([\underline{u}, \bar{u}]) + P([\underline{v}, \bar{v}])$$

(additivity).

Proposition 1 [6]. *Each fair price under interval uncertainty has the form*

$$P([\underline{u}, \bar{u}]) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}$$

for some real number $\alpha_H \in [0, 1]$.

Comment. This expression was first proposed by the Nobelist L. Hurwicz; it is known as the *Hurwicz optimism-pessimism criterion* [3], [5].

Proof.

1°. Let us first consider the value $\alpha_H \stackrel{\text{def}}{=} P([0, 1])$ corresponding to the simplest possible interval $[0, 1]$. Due to conservativeness, we have $0 \leq \alpha_H \leq 1$.

2°. Let us now compute the value $P([0, m])$ for positive integer values m .

The interval $[0, m]$ can be represented as the sum of m intervals equal to $[0, 1]$: $[0, m] = [0, 1] + \dots + [0, 1]$ (m times). Thus, due to additivity, we have $P([0, m]) = P([0, 1]) + \dots + P([0, 1])$ (m times) $= \alpha_H + \dots + \alpha_H$ (m times) $= \alpha_H \cdot m$.

3°. Now, let us compute the value $z \stackrel{\text{def}}{=} P\left(\left[0, \frac{1}{n}\right]\right)$ for a positive integer n .

In this case, the interval $[0, 1]$ can be represented as the sum of n intervals equal to $\left[0, \frac{1}{n}\right]$: $[0, 1] = \left[0, \frac{1}{n}\right] + \dots + \left[0, \frac{1}{n}\right]$ (n times). Thus, due to additivity, we have $\alpha_H = z + \dots + z$ (n times), i.e., $\alpha_H = z \cdot n$ and hence, $z = \alpha_H \cdot \frac{1}{n}$.

4°. For every two positive integers $m > 0$ and $n > 0$, the interval $\left[0, \frac{m}{n}\right]$ can be represented as the sum of m intervals equal to $\left[0, \frac{1}{n}\right]$. Thus, $P\left(\left[0, \frac{m}{n}\right]\right) = m \cdot P\left(\left[0, \frac{1}{n}\right]\right) = m \cdot \left(\alpha_H \cdot \frac{1}{n}\right) = \alpha_H \cdot \frac{m}{n}$.

5°. We have proved that for rational values $r = \frac{m}{n}$, we have $P([0, r]) = \alpha_H \cdot r$. Let us prove that the same property $P([0, x]) = \alpha_H \cdot x$ holds for every positive real value x .

To prove this property, we use monotonicity. Each real number x can be approximated, with arbitrary accuracy, by two rational numbers $r < x < r'$. Due to monotonicity, we have $P([0, r]) \leq P([0, x]) \leq P([0, r'])$. Due to Part 4 of this proof, we thus conclude that $\alpha_H \cdot r \leq P([0, x]) \leq \alpha_H \cdot r'$. When $r \rightarrow x$ and $r' \rightarrow x$, we get $\alpha_H \cdot r \rightarrow \alpha_H \cdot x$ and $\alpha_H \cdot r' \rightarrow \alpha_H \cdot x$ and thus, $P([0, x]) = \alpha_H \cdot x$.

6°. Now, we are ready to prove the proposition. For each \underline{u} and \bar{u} , we have $[\underline{u}, \bar{u}] = [\underline{u}, \underline{u}] + [0, \bar{u} - \underline{u}]$. Thus, due to additivity,

$$P([\underline{u}, \bar{u}]) = P([\underline{u}, \underline{u}]) + P([0, \bar{u} - \underline{u}]).$$

For the first term, due to conservativeness, we have $\underline{u} \leq P([\underline{u}, \underline{u}]) \leq \underline{u}$ and thus, $P([\underline{u}, \underline{u}]) = \underline{u}$. For the second term, due to Part 5 of this proof, we get $P([0, \bar{u} - \underline{u}]) = \alpha_H \cdot (\bar{u} - \underline{u})$. Thus, the above additivity formula leads to

$$P([\underline{u}, \bar{u}]) = \underline{u} + \alpha_H \cdot (\bar{u} - \underline{u}),$$

which is exactly $\alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}$. The proposition is proven.

III. CASE OF SET-VALUED UNCERTAINTY

Description of the case. In some cases, in addition to knowing that the actual gain belongs to the interval $[\underline{u}, \bar{u}]$, we also know that some values from this interval cannot be possible values of this gain. For example, if we gamble on a situation with an unknown probability, we either get the price or lose the money but we cannot gain any value in between.

In general, instead of a (bounded) *interval* of possible values, we can consider a more general bounded *set* of possible values. It makes sense to consider *closed* bounded sets, i.e., bounded sets S that contain all their limits points. Indeed, if $x_n \in S$ for all n and $x_n \rightarrow x$, then, for any given accuracy, x is undistinguishable from some possible value x_n – thus, in effect, the value x itself is possible.

Assigning fair price under set-valued uncertainty: description of the problem. We want to assign, to each bounded closed set S , a number $P(S)$ describing the fair price of this set. There are several reasonable requirements that this function must satisfy.

First, for the case when the set S is an interval, we must get the fair price as described by Proposition 1.

Second, if we have two independent alternatives described by sets S and S' , then we should have $P(S + S') = P(S) + P(S')$, where

$$S + S' \stackrel{\text{def}}{=} \{x + x' : x \in S \& x' \in S'\}$$

is the set of all possible sums $x + x'$.

Thus, we arrive at the following definition.

Definition 2. *By a fair price under set-valued uncertainty, we mean a function $P(S)$ that assigns, to every bounded closed set A , a real number, and which satisfies the following properties:*

- *a restriction of this function to intervals $S = [\underline{u}, \bar{u}]$ is a fair price under interval uncertainty – in the sense of Definition 1 (conservativeness);*
- *for every two sets S and S' , we have $P(S + S') = P(S) + P(S')$ (additivity).*

Proposition 2. *Each fair price under interval uncertainty has the form $P([\underline{u}, \bar{u}]) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S$ for some real number $\alpha_H \in [0, 1]$.*

Proof. Due to conservativeness, for intervals $S = [\underline{u}, \bar{u}]$, the function $P(S)$ is a fair price under interval uncertainty and thus, due to Proposition 1, has the form $P([\underline{u}, \bar{u}]) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}$.

For each bounded set S , its infimum $\underline{s} \stackrel{\text{def}}{=} \inf S$ and its supremum $\bar{s} \stackrel{\text{def}}{=} \sup S$ are finite. By definition, $\inf S$ is a lower bound (it is the greatest lower bound) and $\sup S$ is an upper bound (it is the least upper bound); thus, we have $S \subseteq [\underline{s}, \bar{s}]$. Both $\inf S$ and $\sup S$ are limits of points from the set S ; since the set S is closed, it contains these limits: $\{\underline{s}, \bar{s}\} \subseteq S \subseteq [\underline{s}, \bar{s}]$.

Let us prove that $[\underline{s}, \bar{s}] + S = [2\underline{s}, 2\bar{s}]$. Indeed, by definition of set addition, if $S' \subseteq S''$, then $S + S' \subseteq S + S''$. In particular, $\{\underline{s}, \bar{s}\} \subseteq S \subseteq [\underline{s}, \bar{s}]$ implies that

$$[\underline{s}, \bar{s}] + \{\underline{s}, \bar{s}\} \subseteq [\underline{s}, \bar{s}] + S \subseteq [\underline{s}, \bar{s}] + [\underline{s}, \bar{s}].$$

Here, $[\underline{s}, \bar{s}] + \{\underline{s}, \bar{s}\} = [2\underline{s}, 2\bar{s}]$ and similarly $[\underline{s}, \bar{s}] + [\underline{s}, \bar{s}] = [2\underline{s}, 2\bar{s}]$. Thus,

$$[2\underline{s}, 2\bar{s}] \subseteq [\underline{s}, \bar{s}] + S \subseteq [2\underline{s}, 2\bar{s}]$$

and so, indeed, $[\underline{s}, \bar{s}] + S = [2\underline{s}, 2\bar{s}]$.

Now, additivity implies that $P(S) = P([2\underline{s}, 2\bar{s}]) - P([\underline{s}, \bar{s}])$. Substituting the expression $P([\underline{u}, \bar{u}]) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}$ for the fair price of intervals into this formula, we get the desired expression for $P(S)$. The proposition is proven.

IV. (CRISP) Z-NUMBERS, Z-INTERVALS, AND Z-SETS: CASES WHEN THE PROBABILITIES ARE CRISP

Description of the case. In the previous sections, we assumed that we are 100% certain that the actual gain is contained in the given interval (or set). In reality, mistakes are possible, so usually, we are only certain that u belongs to the corresponding interval or set with some probability $0 < p < 1$. In such situations, to fully describe our knowledge, we need to describe both the interval (or set) and this probability p .

In the general context, after supplementing the information about a quantity with the information of how certain we are about this piece of information, we get what L. Zadeh calls a *Z-number* [11]. Because of this:

- we will call a pair consisting of a (crisp) number and a (crisp) probability a *crisp Z-number*;
- we will call a pair consisting of an interval and a probability a *Z-interval*; and
- we will call a pair consisting of a set and a probability a *Z-set*.

In this section, we will describe fair prices for crisp Z-numbers, Z-intervals, and Z-sets for situations when the probability p is known exactly.

Operations on the corresponding pairs. When we have two independent sequential decisions, and we are 100% sure that the first decision leads to gain u and the second decision leads to gain v , then, as we have mentioned earlier, the user's total gain is equal to the sum $u + v$. In this section, we consider the situation in which:

- for the first decision, our degree of confidence in the gain estimate u is described by some probability p ;
- for the second decision, our degree of confidence in the gain estimate v is described by some probability q .

The estimate $u + v$ is valid only if both gain estimates are correct. Since these estimates are independent, the probability that they are both correct is equal to the product $p \cdot q$ of the corresponding probabilities. Thus:

- for crisp Z-numbers (u, p) and (v, q) , the sum is equal to $(u + v, p \cdot q)$;
- for Z-intervals $([\underline{u}, \bar{u}], p)$ and $([\underline{v}, \bar{v}], q)$, the sum is equal to $([\underline{u} + \underline{v}, \bar{u} + \bar{v}], p \cdot q)$;
- finally, for Z-sets (S, p) and (S', q) , the sum is equal to $(S + S', p \cdot q)$.

Let us analyze these cases one by one.

Case of crisp Z-numbers. Since the probability p is usually known with some uncertainty, it makes sense to require that the fair price of a crisp Z-number (u, p) continuously depend on p , so that small changes in p lead to small change in the fair price – and the closer our estimate to the actual value of the probability, the closer the estimated fair price should be to the actual fair price.

Thus, we arrive at the following definitions.

Definition 3. By a crisp Z-number, we mean a pair (u, p) of two real numbers such that $0 < p \leq 1$.

Definition 4. By a fair price under crisp Z-number uncertainty, we mean a function $P(u, p)$ that assigns, to every crisp Z-number, a real number, and which satisfies the following properties:

- $P(u, 1) = u$ for all u (conservativeness);
- for all u, v, p , and q , we have $P(u+v, p \cdot q) = P(u, p) + P(v, q)$ (additivity);
- the function $P(u, p)$ is continuous in p (continuity).

Proposition 3. Each fair price under crisp Z-number uncertainty has the form $P(u, p) = u - k \cdot \ln(p)$ for some real number k .

Proof.

1°. By additivity, we have $P(u, p) = P(u, 1) + P(0, p)$. By conservativeness, we have $P(u, 1) = u$; thus, $P(u, p) = u + P(0, p)$. So, to find the general expression for the fair price function $P(u, p)$, it is sufficient to find the values $P(0, p)$ corresponding to $u = 0$.

2°. For the values $P(0, p)$, additivity implies that $P(0, p \cdot q) = P(0, p) + P(0, q)$.

3°. Let us first consider the value $p = e^{-1}$ which corresponds to $\ln(p) = -1$. The corresponding value of $P(0, p)$ will be denoted by $k \stackrel{\text{def}}{=} P(0, e^{-1})$. Then, for $p = e^{-1}$, we have the desired expression $P(0, p) = -k \cdot \ln(p)$.

4°. Let us now consider the values $P(0, e^{-m})$ for positive integer values m . The probability e^{-m} can be represented as a product of m values e^{-1} :

$$e^{-m} = e^{-1} \cdot \dots \cdot e^{-1} (m \text{ times}).$$

Thus, due to additivity, we have

$$P(0, e^{-m}) = P(0, e^{-1}) + \dots + P(0, e^{-1}) (m \text{ times}) = m \cdot k.$$

Since for $p = e^{-m}$, we have $\ln(p) = -m$, we thus have $P(0, p) = -k \cdot \ln(p)$ for these values p .

5°. Now, let us estimate the value $P(0, p)$ for $p = e^{-1/n}$, for a positive integer n .

In this case, the value e^{-1} can be represented as a product of n probabilities equal to $e^{-1/n}$: $e^{-1} = e^{-1/n} \cdot \dots \cdot e^{-1/n}$ (n times). Thus, due to additivity, we have $k = P(0, e^{-1}) = P(0, e^{-1/n}) + \dots + P(0, e^{-1/n})$ (n times), i.e.,

$$k = n \cdot P(0, e^{-1/n})$$

and hence, $P(0, e^{-1/n}) = \frac{k}{n}$. Therefore, for $p = e^{-1/n}$, we also have $P(0, p) = -k \cdot \ln(p)$.

6°. For every two positive numbers $m > 0$ and $n > 0$, the probability $e^{-m/n}$ can be represented as the product of m probabilities equal to $e^{-1/n}$. Thus, due to additivity, we have $P(0, e^{-m/n}) = m \cdot P(0, e^{-1/n}) = k \cdot \frac{m}{n}$. Hence, for the values $p = e^{-m/n}$ for which the logarithm $\ln(p)$ is a rational number, we have $P(0, p) = -k \cdot \ln(p)$.

7°. Every real number $\ell \stackrel{\text{def}}{=} \ln(p)$ can be approximated, with arbitrary accuracy, by rational numbers $\ell_n \rightarrow \ell$ for which $p_n \stackrel{\text{def}}{=} e^{-\ell_n} \rightarrow e^{-\ell} = p$. For these rational numbers, we have $P(0, p_n) = -k \cdot \ln(p_n)$. Thus, when $n \rightarrow \infty$ and $p_n \rightarrow p$, by continuity, we have $P(0, p) = -k \cdot \ln(p)$.

From Part 1, we know that $P(u, p) = u + P(0, p)$; thus, indeed, $P(u, p) = u - k \cdot \ln(p)$. The proposition is proven.

Cases of Z-intervals and Z-sets. Similar results hold for Z-intervals and Z-sets; in both results, we will use the fact that we already know how to set a fair price for the case when $p = 1$.

Definition 5. By a Z-interval, we mean a pair $([\underline{u}, \bar{u}], p)$ consisting of an interval $[\underline{u}, \bar{u}]$ and a real numbers p such that $0 < p \leq 1$.

Definition 6. By a fair price under Z-interval uncertainty, we mean a function $P([\underline{u}, \bar{u}], p)$ that assigns, to every Z-interval, a real number, and which satisfies the following properties:

- for some $\alpha_H \in [0, 1]$ and for all $\underline{u} \leq \bar{u}$, we have $P([\underline{u}, \bar{u}], 1) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}$ (conservativeness);
- for all $\underline{u}, \bar{u}, \underline{v}, \bar{v}, p$, and q , we have

$$P([\underline{u} + \underline{v}, \bar{u} + \bar{v}], p \cdot q) = P([\underline{u}, \bar{u}], p) + P([\underline{v}, \bar{v}], q)$$

(additivity).

Proposition 4. Each fair price under Z-interval uncertainty has the form $P([\underline{u}, \bar{u}], p) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u} - k \cdot \ln(p)$ for some real numbers $\alpha_H \in [0, 1]$ and k .

Proof. By additivity, we have $P([\underline{u}, \bar{u}], p) = P([\underline{u}, \bar{u}], 1) + P(0, p)$. By conservativeness, we have

$$P([\underline{u}, \bar{u}], 1) = \alpha_H \cdot \bar{u} + (1 - \alpha_H) \cdot \underline{u}.$$

For $P(0, p)$, similarly to the proof of Proposition 3, we conclude that $P(0, p) = -k \cdot \ln(p)$ for some real number k . The proposition is proven.

Definition 7. By a Z-set, we mean a pair (S, p) consisting of a closed bounded set S and a real numbers p such that $0 < p \leq 1$.

Definition 8. By a fair price under Z-set-valued uncertainty, we mean a function $P(S, p)$ that assigns, to every Z-interval, a real number, and which satisfies the following properties:

- for some $\alpha_H \in [0, 1]$ and for all sets S , we have

$$P(S, 1) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S$$

(conservativeness);

- for all S, S', p , and q , we have $P(S + S', p \cdot q) = P(S, p) + P(S', q)$ (additivity).

Proposition 5. Each fair price under Z-set-valued uncertainty has the form

$$P(S, p) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S - k \cdot \ln(p)$$

for some real numbers $\alpha_H \in [0, 1]$ and k .

Proof. By additivity, we have $P(S, p) = P(S, 1) + P(\{0\}, p)$. By conservativeness, we have $P(S, 1) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S$. For $P(\{0\}, p)$, similarly to the proof of Proposition 3, we conclude that $P(\{0\}, p) = -k \cdot \ln(p)$ for some real number k . The proposition is proven.

V. (CRISP) Z-NUMBERS, Z-INTERVALS, AND Z-SETS: CASES WHEN PROBABILITIES ARE KNOWN WITH INTERVAL OR SET-VALUED UNCERTAINTY

Motivations. When we know the exact probabilities p and q that the corresponding estimates are correct, then the probability that both estimates are correct is equal to the product $p \cdot q$.

Similarly to the fact that we often do not know the exact gain, we often do not know the exact probability p . Instead, we may only know the interval $[p, \bar{p}]$ of possible values of p , or, more generally, a set \mathcal{P} of possible values of p . If we know p and q with such uncertainty, what can we then conclude about the product $p \cdot q$?

For positive values p and q , the function $p \cdot q$ is increasing as a function of both variables: if we increase p and/or increase q , the product increases. Thus, if the only information that we have the probability p is that this probability belongs to the interval $[p, \bar{p}]$, and the only information that we have the probability q is that this probability belongs to the interval $[q, \bar{q}]$, then:

- the smallest possible of $p \cdot q$ is equal to the product $\underline{p} \cdot \underline{q}$ of the smallest values;
- the largest possible of $p \cdot q$ is equal to the product $\bar{p} \cdot \bar{q}$ of the largest values; and
- the set of all possible values $p \cdot q$ is the interval $[\underline{p} \cdot \underline{q}, \bar{p} \cdot \bar{q}]$.

For sets \mathcal{P} and \mathcal{Q} , the set of possible values $p \cdot q$ is the set

$$\mathcal{P} \cdot \mathcal{Q} \stackrel{\text{def}}{=} \{p \cdot q : p \in \mathcal{P} \text{ and } q \in \mathcal{Q}\}.$$

Let us find the fair price under such uncertainty.

Case of crisp Z-numbers. Let us start with the case of crisp Z-numbers under such uncertainty.

Definition 9. By a crisp Z-number under interval p -uncertainty, we mean a pair $(u, [\underline{p}, \bar{p}])$ consisting of a real number u and an interval $[\underline{p}, \bar{p}] \subseteq (0, 1]$.

Definition 10. By a fair price under crisp Z-number p -interval uncertainty, we mean a function $P(u, [\underline{p}, \bar{p}])$ that assigns, to every crisp Z-number under interval p -uncertainty, a real number, and which satisfies the following properties:

- for some real number k , we have

$$P(u, [p, p]) = u - k \cdot \ln(p)$$

for all u and p (conservativeness);

- for all $u, v, \underline{p}, \bar{p}, \underline{q},$ and \bar{q} , we have

$$P(u + v, [\underline{p} \cdot \underline{q}, \bar{p} \cdot \bar{q}]) = P(u, [\underline{p}, \bar{p}]) + P(v, [\underline{q}, \bar{q}])$$

(additivity);

- the function $P(u, [\underline{p}, \bar{p}])$ is continuous in \underline{p} and \bar{p} (continuity).

Proposition 6. Each fair price under crisp Z-number p -interval uncertainty has the form

$$P(u, [\underline{p}, \bar{p}]) = u - (k - \beta) \cdot \ln(\bar{p}) - \beta \cdot \ln(\underline{p})$$

for some real number $\beta \in [0, 1]$.

Proof.

1°. By additivity, we have $P(u, [\underline{p}, \bar{p}]) = P(u, \bar{p}) + P(0, [p, 1])$, where $p \stackrel{\text{def}}{=} \underline{p}/\bar{p}$. By conservativeness, we know that $P(u, \bar{p}) = u - k \cdot \ln(\bar{p})$. Thus, $P(u, p) = u - k \cdot \ln(\bar{p}) + P(0, [p, 1])$. So, to find the general expression for the fair price function $P(u, [\underline{p}, \bar{p}])$, it is sufficient to find the values $P(0, [p, 1])$ corresponding to $u = 0$ and $\bar{p} = 1$.

2°. For the values $P(0, [p, 1])$, additivity implies that

$$P(0, [p \cdot q, 1]) = P(0, [p, 1]) + P(0, [q, 1]).$$

3°. Let us first consider the value $p = e^{-1}$ which corresponds to $\ln(p) = -1$. The corresponding value of $P(0, [e^{-1}, 1])$ will be denoted by $\beta = P(0, [e^{-1}, 1])$. Then, for $p = e^{-1}$, we have the expression $P(0, [p, 1]) = -\beta \cdot \ln(p)$.

4°. Let us now consider the values $P(0, [e^{-m}, 1])$ for positive integer values m . The probability e^{-m} can be represented as a product of m values e^{-1} : $e^{-m} = e^{-1} \cdot \dots \cdot e^{-1}$ (m times). Thus, due to additivity, we have

$$P(0, [e^{-m}, 1]) =$$

$$P(0, [e^{-1}, 1]) + \dots + P(0, [e^{-1}, 1]) \quad (m \text{ times}) = m \cdot \beta.$$

Since for $p = e^{-m}$, we have $\ln(p) = -m$, we thus have $P(0, [p, 1]) = -\beta \cdot \ln(p)$ for these values p .

5°. Now, let us estimate the value $P(0, [p, 1])$ for $p = e^{-1/n}$, for a positive integer n .

In this case, the value e^{-1} can be represented as a product of n probabilities equal to $e^{-1/n}$:

$$e^{-1} = e^{-1/n} \cdot \dots \cdot e^{-1/n} \quad (n \text{ times})$$

Thus, due to additivity, we have $\beta = P(0, [e^{-1}, 1]) = P(0, [e^{-1/n}, 1]) + \dots + P(0, [e^{-1/n}, 1])$ (n times), i.e.,

$$\beta = n \cdot P(0, [e^{-1/n}, 1])$$

and hence, $P(0, [e^{-1/n}, 1]) = \frac{\beta}{n}$. Therefore, for $p = e^{-1/n}$, we also have $P(0, [p, 1]) = -\beta \cdot \ln(p)$.

6°. For every two positive numbers $m > 0$ and $n > 0$, the probability $e^{-m/n}$ can be represented as the product of m probabilities equal to $e^{-1/n}$. Thus, due to additivity, we have $P(0, [e^{-m/n}, 1]) = m \cdot P(0, [e^{-1/n}, 1]) = \beta \cdot \frac{m}{n}$. Hence, for the values $p = e^{-m/n}$ for which the logarithm $\ln(p)$ is a rational number, we have $P(0, [p, 1]) = -k \cdot \ln(p)$.

7°. Every real number $\ell \stackrel{\text{def}}{=} \ln(p)$ can be approximated, with arbitrary accuracy, by rational numbers $\ell_n \rightarrow \ell$ for which $p_n \stackrel{\text{def}}{=} e^{-\ell_n} \rightarrow e^{-\ell} = p$. For these rational numbers, we have $P(0, [p_n, 1]) = -\beta \cdot \ln(p_n)$. Thus, when $n \rightarrow \infty$ and $p_n \rightarrow p$, by continuity, we have $P(0, [p, 1]) = -\beta \cdot \ln(p)$.

From Part 1, we know that

$$P(u, [\underline{p}, \bar{p}]) = u - k \cdot \ln(\bar{p}) + P(0, [p, 1]);$$

thus,

$$P(u, [\underline{p}, \bar{p}]) = u - k \cdot \ln(\bar{p}) - \beta \cdot \ln(\underline{p}).$$

Substituting $p = \underline{p}/\bar{p}$ into this formula and taking into account that $\ln(p) = \ln(\underline{p}) - \ln(\bar{p})$, we get the desired formula.

Definition 11. By a crisp Z-number under set-valued p -uncertainty, we mean a pair (u, \mathcal{P}) consisting of a real number u and a bounded closed set $\mathcal{P} \subseteq (0, 1]$ for which $\inf \mathcal{P} > 0$.

Definition 12. By a fair price under crisp Z-number p -set-valued uncertainty, we mean a function $P(u, \mathcal{P})$ that assigns, to every crisp Z-number under set-valued p -uncertainty, a real number, and which satisfies the following properties:

- for some real numbers k and β , we have $P(u, [\underline{p}, \bar{p}]) = u - (k - \beta) \cdot \ln(\bar{p}) - \beta \cdot \ln(\underline{p})$ for all $u, \underline{p},$ and \bar{p} (conservativeness);
- for all $u, v, \mathcal{P},$ and \mathcal{Q} , we have

$$P(u + v, \mathcal{P} \cdot \mathcal{Q}) = P(u, \mathcal{P}) + P(v, \mathcal{Q}) \quad (\text{additivity}).$$

Proposition 7. Each fair price under crisp Z-number p -set-valued uncertainty has the form

$$P(u, \mathcal{P}) = u - (k - \beta) \cdot \ln(\sup \mathcal{P}) - \beta \cdot \ln(\inf \mathcal{P})$$

for some real number $\beta \in [0, 1]$.

Proof. By additivity, we have $P(u, \mathcal{P}) = P(u, 1) + P(0, \mathcal{P})$, i.e., due to conservativeness, $P(u, \mathcal{P}) = u + P(0, \mathcal{P})$. So, to find the expression for $P(u, \mathcal{P})$, it is sufficient to find the values $P(0, \mathcal{P})$. Similarly to prove of Proposition 2, we can prove that $\mathcal{P} \cdot [\inf \mathcal{P}, \sup \mathcal{P}] = [(\inf \mathcal{P})^2, (\sup \mathcal{P})^2]$. Due to additivity, this implies that

$$P(0, [(\inf \mathcal{P})^2, (\sup \mathcal{P})^2]) = P(0, \mathcal{P}) + P(0, [\inf \mathcal{P}, \sup \mathcal{P}]),$$

hence

$$P(0, \mathcal{P}) = P(0, [(\inf \mathcal{P})^2, (\sup \mathcal{P})^2]) - P(0, [\inf \mathcal{P}, \sup \mathcal{P}]).$$

Due to conservativeness, we know the values in the right-hand side of this equality. Substituting these values, we get the desired formula.

Case of Z-intervals and Z-sets. Let us extend the above results to Z-sets (and to their particular case: Z-intervals).

Definition 13. By a Z-set under set-valued p -uncertainty, we mean a pair (S, \mathcal{P}) consisting of a bounded closed set S and a bounded closed set $\mathcal{P} \subseteq (0, 1]$ for which $\inf \mathcal{P} > 0$.

Definition 14. By a fair price under Z-set p -set-valued uncertainty, we mean a function $P(S, \mathcal{P})$ that assigns, to every Z-set under set-valued p -uncertainty, a real number, and which satisfies the following properties:

- for some real number $\alpha_H \in [0, 1]$, we have

$$P(S, 1) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S$$

for all S (conservativeness);

- for some real numbers k and β , we have

$$P(u, \mathcal{P}) = u - (k - \beta) \cdot \ln(\sup \mathcal{P}) - \beta \cdot \ln(\inf \mathcal{P})$$

for all u and \mathcal{P} (conservativeness);

- for all S, S', \mathcal{P} , and \mathcal{Q} , we have

$$P(S + S', \mathcal{P} \cdot \mathcal{Q}) = P(S, \mathcal{P}) + P(S', \mathcal{Q}) \quad (\text{additivity}).$$

Proposition 8. Each fair price under Z-set p -set-valued uncertainty has the form

$$P(S, \mathcal{P}) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S - (k - \beta) \cdot \ln(\bar{p}) - \beta \cdot \ln(\underline{p}).$$

VI. CASE OF FUZZY AND Z-NUMBER UNCERTAINTY

Fuzzy numbers: reminder. In the above text, we first considered situations when about each value of gain u , the expert is either absolutely sure that this value is possible or absolutely sure that this value is not possible. Then, we took into account the possibility that the expert is not 100% certain about that – but we assumed that the expert either knows the exact probability p describing his/her degree of certainty, or that the expert is absolutely sure which probabilities can describe his/her uncertainty and which cannot.

In reality, an expert is often uncertain about the possible values, and uncertain about possible degrees of uncertainty. To take this uncertainty into account, L. Zadeh introduced the notion of a *fuzzy set* [4], [9], [10], where, to each possible value of u , we assign a degree $\mu(u) \in [0, 1]$ to which this value u is possible. Similarly, a fuzzy set $\mu_p : [0, 1] \rightarrow [0, 1]$ can describe the degrees to which different probability values are possible.

In this paper, we restrict ourselves to *fuzzy numbers* s , i.e., fuzzy sets for which the membership function is different from 0 only on a bounded set, where it first monotonically

increases until it reaches a point \bar{s} at which $\mu(\bar{s}) = 1$, and then monotonically decreases from 1 to 0.

Operations on fuzzy numbers. Operations on fuzzy numbers are usually described in terms of *Zadeh's extension principle*: if two quantities u and v are described by membership functions $\mu_1(u)$ and $\mu_2(v)$, then their sum $w = u + v$ is described by the membership function $\mu(w) = \max_{u,v: u+v=w} \min(\mu_1(u), \mu_2(v))$, and their product $w = u \cdot v$ is described by the membership function $\mu(w) = \max_{u,v: u \cdot v=w} \min(\mu_1(u), \mu_2(v))$.

It is known that these operations can be equivalently described in terms of the α -cuts. An α -cut of a fuzzy number $\mu(u)$ is defined as an interval $\mathbf{u}(\alpha) = [u^-(\alpha), u^+(\alpha)]$, where

$$u^-(\alpha) \stackrel{\text{def}}{=} \inf\{u : \mu(u) \geq \alpha\} \text{ and}$$

$$u^+(\alpha) \stackrel{\text{def}}{=} \sup\{u : \mu(u) \geq \alpha\}.$$

The α -cuts corresponding to $w = u + v$ can be described, for every α , as

$$[w^-(\alpha), w^+(\alpha)] = [u^-(\alpha), u^+(\alpha)] + [v^-(\alpha), v^+(\alpha)],$$

or, equivalently, as

$$[w^-(\alpha), w^+(\alpha)] = [u^-(\alpha) + v^-(\alpha), u^+(\alpha) + v^+(\alpha)].$$

Similarly, when both fuzzy numbers u and v are limited to the interval $[0, 1]$, the α -cuts corresponding to the product $w = u \cdot v$ can be described as

$$[w^-(\alpha), w^+(\alpha)] = [u^-(\alpha), u^+(\alpha)] \cdot [v^-(\alpha), v^+(\alpha)],$$

or, equivalently, as

$$[w^-(\alpha), w^+(\alpha)] = [u^-(\alpha) \cdot v^-(\alpha), u^+(\alpha) \cdot v^+(\alpha)].$$

Fair price of fuzzy numbers. Let us start with describing the fair price of fuzzy numbers. Similarly to the interval case, a natural requirement is monotonicity: if for all α , we have $s^-(\alpha) \leq t^-(\alpha)$ and $s^+(\alpha) \leq t^+(\alpha)$, then the fair price of t should be larger than or equal to the fair price of s . It is also reasonable to require continuity: that small changes in $\mu(u)$ should lead to small changes in the fair price.

Definition 15. By a fair price under fuzzy uncertainty, we mean a function $P(s)$ that assigns, to every fuzzy number s , a real number, and which satisfies the following properties:

- if a fuzzy number s is located between \underline{u} and \bar{u} , then $\underline{u} \leq P(s) \leq \bar{u}$ (conservativeness);
- if a fuzzy number w is the sum of fuzzy numbers u and v , then we have $P(w) = P(u) + P(v)$ (additivity);
- if for all α , we have

$$s^-(\alpha) \leq t^-(\alpha) \text{ and } s^+(\alpha) \leq t^+(\alpha),$$

then we have $P(s) \leq P(t)$ (monotonicity);

- if a sequence of membership functions μ_n uniformly converges to μ , then we should have $P(\mu_n) \rightarrow P(\mu)$ (continuity).

Riemann-Stieltjes integral: reminder. As we will see, the fair price of a fuzzy number is described in terms of a Riemann-Stieltjes integral. This integral is a natural generalization of the usual (Riemann) integral.

In general, an intuitive meaning of a Riemann integral $\int_a^b f(x) dx$ is that it is an area under the curve $y = f(x)$. To compute this integral, we select points $a = x_1 < x_2 < \dots < x_{n-1} < x_n = b$, and approximate the curve by a piecewise constant function $\tilde{f}(x) = f(x_i)$ for $x \in [x_i, x_{i+1})$. The subgraph of this piecewise constant function is a union of several rectangles, so its area is equal to the sum of the areas of these rectangles $\sum f(x_i) \cdot (x_{i+1} - x_i)$. This sum is known as the *integral sum* for the integral $\int_a^b f(x) dx$. Riemann's integral can be formally defined as a limit of such integral sums when $\max(x_{i+1} - x_i) \rightarrow 0$.

A Riemann-Stieltjes integral $\int_a^b f(x) dg(x)$ is similarly defined as the limit of the sums $\sum f(x_i) \cdot (g(x_{i+1}) - g(x_i))$ when $\max(x_{i+1} - x_i) \rightarrow 0$.

Proposition 9. For a fuzzy number s with a continuous membership function $\mu(x)$, α -cuts $[s^-(\alpha), s^+(\alpha)]$ and a point s_0 at which $\mu(s_0) = 1$, the fair price is equal to

$$P(s) = s_0 + \int_0^1 k^-(\alpha) ds^-(\alpha) - \int_0^1 k^+(\alpha) ds^+(\alpha),$$

for appropriate functions $k^-(\alpha)$ and $k^+(\alpha)$.

Discussion. When the function $g(x)$ is differentiable, the integral $\int_a^b f(x) dg(x)$ is equal to the usual integral $\int_a^b f(x) \cdot g'(x) dx$, where $g'(x)$ denotes the derivative. When the function $f(x)$ is also differentiable, we can use integration by part and get yet another equivalent form

$$f(b) \cdot g(b) - f(a) \cdot g(a) + \int_a^b F(x) \cdot g(x) dx,$$

with $F(x) = -f'(x)$. In general, a Stieltjes integral can be represented in a similar form for some *generalized function* $F(x)$ (see, e.g., [2]; generalized function are also known as *distributions*; we do not use this term to avoid confusion with probability distributions). Thus, the above general formula can be described as

$$P(s) = \int_0^1 K^-(\alpha) \cdot s^-(\alpha) d\alpha + \int_0^1 K^+(\alpha) \cdot s^+(\alpha) d\alpha$$

for appropriate generalized functions $K^-(\alpha)$ and $K^+(\alpha)$.

Conservativeness means that for a crisp number located at s_0 , we should have $P(s) = s_0$. For the above formula, this means that

$$\int_0^1 K^-(\alpha) d\alpha + \int_0^1 K^+(\alpha) d\alpha = 1.$$

For a fuzzy number which is equal to the interval $[\underline{u}, \bar{u}]$, the above formula leads to

$$P(s) = \left(\int_0^1 K^-(\alpha) d\alpha \right) \cdot \underline{u} + \left(\int_0^1 K^+(\alpha) d\alpha \right) \cdot \bar{u}.$$

Thus, Hurwicz optimism-pessimism coefficient α_H is equal to $\int_0^1 K^+(\alpha) d\alpha$. In this sense, the above formula is a generalization of Hurwicz's formula to the fuzzy case.

Proof.

1°. For every two real numbers $u \geq 0$ and $\gamma \in [0, 1]$, let us define a fuzzy number $s_{\gamma, u}(x)$ with the following membership function: $\mu_{\gamma, u}(0) = 1$, $\mu_{\gamma, u}(x) = \gamma$ for $x \in (0, u]$, and $\mu_{\gamma, u}(x) = 0$ for all other x . For this fuzzy numbers, α -cuts have the following form: $s_{\gamma, u}(\alpha) = [0, 0]$ for $\alpha > \gamma$, and $s_{\gamma, u}(\alpha) = [0, u]$ for $\alpha \leq \gamma$.

Based on the α -cuts, one can easily check that $s_{\gamma, u+v} = s_{\gamma, u} + s_{\gamma, v}$. Thus, due to additivity, $P(s_{\gamma, u+v}) = P(s_{\gamma, u}) + P(s_{\gamma, v})$. Due to monotonicity, the value $P(s_{\gamma, u})$ monotonically depends on u . Thus, similarly to the proof of Proposition 1, we can conclude that $P(s_{\gamma, u}) = k^+(\gamma) \cdot u$ for some value $k^+(\gamma)$.

By definition, the fuzzy number $s_{\gamma, u}$ is located between 0 and u , so, due to conservativeness, we have $0 \leq P(s_{\gamma, u}) \leq u$ for all u . This implies that $0 \leq k^+(\gamma) \leq 1$.

2°. Let us now consider a fuzzy number s whose membership function is equal to 0 for $x < 0$, jumps to 1 for $x = 0$, and then continuously decrease to 0. For this fuzzy number, all α -cuts have the form $[0, s^+(\alpha)]$ for some $s^+(\alpha)$. By definition of an α -cut, the value $s^+(\alpha)$ decreases with α .

For each sequence of values $\alpha_0 = 1 < \alpha_1 < \alpha_2 < \dots < \alpha_{n-1} < \alpha_n = 1$, we can define a fuzzy number s_n with the following α -cuts:

- $s_n^-(\alpha) = 0$ for all α ; and
- when $\alpha \in [\alpha_i, \alpha_{i+1})$, then $s_n^+(\alpha) = s^+(\alpha_i)$.

Since the membership function of s is continuous, when $\max(\alpha_{i+1} - \alpha_i) \rightarrow 0$, we have $s_n \rightarrow s$, and thus, $P(s_n) \rightarrow P(s)$.

One can check that the fuzzy number s_n can be represented as a sum of n fuzzy numbers

$$s_n = s_{\alpha_{n-1}, s^+(\alpha_{n-1})} + s_{\alpha_{n-2}, s^+(\alpha_{n-2}) - s^+(\alpha_{n-1})} + \dots + s_{\alpha_1, \alpha_1 - \alpha_2}.$$

Thus, due to additivity, we have

$$P(s_n) = P(s_{\alpha_{n-1}, s^+(\alpha_{n-1})}) +$$

$$P(s_{\alpha_{n-2}, s^+(\alpha_{n-2}) - s^+(\alpha_{n-1})}) + \dots + P(s_{\alpha_1, \alpha_1 - \alpha_2}).$$

Substituting the expression for $P(s_{\gamma, u})$ from Part 1 of this proof, we conclude that

$$P(s_n) = k^+(\alpha_{n-1}) \cdot s^+(\alpha_{n-1}) + k^+(\alpha_{n-2}) \cdot (s^+(\alpha_{n-2}) - s^+(\alpha_{n-1})) + \dots + k^+(\alpha_1) \cdot (\alpha_1 - \alpha_2).$$

The right-hand side is minus the integral sum for the Riemann-Stieltjes integral $\int_0^1 k^+(\gamma) ds^+(\gamma)$. Since we have $P(s_n) \rightarrow P(s)$, this means that the integral sums always converges, the Riemann-Stieltjes integral is defined, and the limit $P(s)$ is equal to this integral.

3°. Similarly, for fuzzy numbers s whose membership function $\mu(x)$ continuously increases from 0 to 1 as x increases to 0 and is equal to 0 for $x > 0$, the α -cuts are equal to

$[s^-(\alpha), 0]$, and $P(s) = \int_0^1 k^-(\gamma) ds^-(\gamma)$ for an appropriate function $k^-(\gamma)$.

4°. A general fuzzy number g , with α -cuts $[g^-(\alpha), g^+(\alpha)]$ and a point g_0 at which $\mu(g_0) = 1$, can be represented as a sum of three fuzzy numbers:

- a crisp number g_0 ;
- a fuzzy number whose α -cuts are equal to

$$[0, g^+(\alpha) - g_0]; \text{ and}$$

- a fuzzy number whose α -cuts are equal to

$$[g_0 - g^-(\alpha), 0].$$

By conservativeness, the fair price of the crisp number is equal to g_0 . The fair prices of the second and the their fuzzy numbers can be obtained by using the formulas from Parts 2 and 3 of this proof. By additivity, the fair price of the sum is equal to the sum of the prices. By taking into account that for every constant g_0 , $d(g(x) - g_0) = dg(x)$ and thus, $\int f(x) d(g(x) - g_0) = \int f(x) dg(x)$, we get the desired expression.

Case of Z-number uncertainty. In this case, we have two fuzzy numbers: the fuzzy number s which describes the values and the fuzzy number p which describes our degree of confidence in the piece of information described by s .

Definition 16. By a fair price under Z-number uncertainty, we mean a function $P(s, p)$ that assigns, to every pair of two fuzzy numbers s and p such that p is located on an interval $[p_0, 1]$ for some $p_0 > 0$, a real number, and which satisfies the following properties:

- if a fuzzy number s is located between \underline{u} and \bar{u} , then $\underline{u} \leq P(s, 1) \leq \bar{u}$ (conservativeness);
- if $w = u + v$ and $r = p \cdot q$, then

$$P(w, r) = P(u, p) + P(v, q)$$

(additivity);

- if for all α , we have

$$s^-(\alpha) \leq t^-(\alpha) \text{ and } s^+(\alpha) \leq t^+(\alpha),$$

then we have $P(s, 1) \leq P(t, 1)$ (monotonicity);

- if $s_n \rightarrow s$ and $p_n \rightarrow p$, then $P(s_n, p_n) \rightarrow P(p, s)$ (continuity).

Proposition 10. For a fuzzy number s with α -cuts $[s^-(\alpha), s^+(\alpha)]$ and a fuzzy number p with α -cuts $[p^-(\alpha), p^+(\alpha)]$, we have

$$P(s, p) = \int_0^1 K^-(\alpha) \cdot s^-(\alpha) d\alpha + \int_0^1 K^+(\alpha) \cdot s^+(\alpha) d\alpha + \int_0^1 L^-(\alpha) \cdot \ln(p^-(\alpha)) d\alpha + \int_0^1 L^+(\alpha) \cdot \ln(p^+(\alpha)) d\alpha$$

for appropriate generalized functions $K^\pm(\alpha)$ and $L^\pm(\alpha)$.

Proof. Due to additivity, we have

$$P(s, p) = P(s, 1) + P(0, p).$$

We already know the expression for $P(s, 1)$; we thus need to find the expression for $P(0, p)$. For logarithms, we have $\ln(p \cdot q) = \ln(p) + \ln(q)$, so in terms of logarithms, additivity takes the usual form

$$P(0, \ln(p) + \ln(q)) = P(0, \ln(p)) + P(0, \ln(q)).$$

Thus, similarly to the proof of Proposition 9, we conclude that

$$P(0, p) = \int_0^1 L^-(\alpha) \cdot \ln(p^-(\alpha)) d\alpha +$$

$$\int_0^1 L^+(\alpha) \cdot \ln(p^+(\alpha)) d\alpha.$$

By applying additivity to this expression and to the known expression for $P(s, 1)$, we get the desired formula.

VII. REMAINING PROBLEMS

In this paper, we described how to define fair price when we have one piece of information: a fuzzy set S of gains with a fuzzy set \mathcal{P} describing how confident we are in S . In practice, we may several such pieces of information. It is desirable to come up with formulas which describe fair price under such multiple pieces of information – formulas which are uniquely determined by additivity and similar reasonable conditions.

Another open question is how to extend the above formulas for fair price to the case of interval-valued and, more general, type-2 fuzzy sets; see, e.g., [7], [8].

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