A REALISTIC (NON-ASSOCIATIVE) INTERVAL LOGIC AND HOW INTERVAL COMPUTATIONS HELP IN PROVING RESULTS ABOUT IT

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Abstract Experts' uncertainty about their statements S_i is described by probabilities p_i. The conclusion C of an expert system normally depends on several statements S_i, so to estimate the reliability $p(C)$, we must estimate the probability of Boolean combinations like $S_1 \& S_2$. We cannot ask experts about all 2^n ($> 10^{10}$) such combinations, so we must estimate $p(S_1 \& S_2)$ based on $p_1 = p(S_1)$ and $p_2 = p(S_2)$.

One can use the interval $\mathbf{p} = [\max(p_1 + p_2 - 1, 0), \min(p_1, p_2)]$ of possible values of $p(S_1 \& S_2)$, but this often leads to $\mathbf{p}(C) = [0, 1]$. A natural idea is to use a midpoint of $\mathbf{p}$ instead; this midpoint is a mathematical expectation of $p(S_1 \& S_2)$ over a uniform (second order) distribution on all possible probability distributions.

This midpoint operation & is not associative (which fits well with human reasoning). We show that some properties of this operation, like semi-associativity and the upper bound (1/9) on the difference $a \& (b \& c) - (a \& b) \& c$, can be derived by using interval computations.

1. FORMULATION OF THE PROBLEM

1.1. WE NEED ESTIMATES FOR THE DEGREES OF CERTAINTY OF $S_1 \,\&\, S_2$ AND $S_1 \vee S_2$

An expert system contains expert statements $S_1, \ldots, S_n$. The experts' uncertainty about each statement S_i is described by its *subjective probability* p_i. The conclusion C of an expert system normally depends on several statements S_i: e.g., if C follows from S_1 and from S_3, then $p(C) = p(S_1 \vee S_3)$.

To know the probability of every Boolean combination, we must know $2^n - 1$ values $p(S_1^{\varepsilon_1} \,\&\ldots\&\, S_n^{\varepsilon_n})$, where $\varepsilon_i \in \{+, -\}$, $S^+ \stackrel{\text{def}}{=} S$, and $S^- \stackrel{\text{def}}{=} \neg S$. For large n, this number is astronomical, so we cannot ask experts about all of these probabilities. We must thus estimate $p(S_1 \,\&\, S_2)$ or $p(S_1 \vee S_2)$ based only on the values $p_1 = p(S_1)$ and $p_2 = p(S_2)$.

1.2. INTERVAL ESTIMATES ARE POSSIBLE, BUT SOMETIMES NOT PERFECT

It is known that for given $p_1 = p(S_1)$ and $p_2 = p(S_2)$:

- possible values of $p(S_1 \,\&\, S_2)$ form an interval $\mathbf{p} = [p^-, p^+]$, where $p^- = \max(p_1 + p_2 - 1, 0)$ and $p^+ = \min(p_1, p_2)$; and
- possible values of $p(S_1 \vee S_2)$ form an interval $\mathbf{p} = [p^-, p^+]$, where $p^- = \max(p_1, p_2)$ and $p^+ = \min(p_1 + p_2, 1)$.

(see, e.g., a survey [20] and references therein).

In principle, we can use such interval estimates and get an interval $\mathbf{p}(C)$ of possible values of $p(C)$. Sometimes, this idea leads to meaningful estimates. Sometimes, however, it leads to a useless $\mathbf{p}(C) = [0, 1]$ [20, 21]. Then, we need *numerical* estimates for $p(S_1 \,\&\, S_2)$ and $p(S_1 \vee S_2)$.

1.3. NATURAL IDEA: SELECTING A MIDPOINT

It is natural to select a *midpoint* of each interval:

$$p_1 \,\&\, p_2 \stackrel{\text{def}}{=} \frac{1}{2} \cdot \max(p_1 + p_2 - 1, 0) + \frac{1}{2} \cdot \min(p_1, p_2); \tag{1}$$

$$p_1 \vee p_2 \stackrel{\text{def}}{=} \frac{1}{2} \cdot \max(p_1, p_2) + \frac{1}{2} \cdot \min(p_1 + p_2, 1). \tag{2}$$

This midpoint also has a deeper justification: For $n = 2$ statements S_1 and S_2, we need $2^2 = 4$ probabilities $x_1 = p(S_1 \,\&\, S_2)$, $x_2 = p(S_1 \,\&\, \neg S_2)$,

$x_3 = p(\neg S_1 \,\&\, S_2)$, and $x_4 = p(\neg S_1 \,\&\, \neg S_2)$. Each corresponding probability distribution is thus characterized by four real numbers $x_i \ge 0$ for which $x_1 + x_2 + x_3 + x_4 = 1$; the set of all such vectors $(x_1, \ldots, x_4)$ is, therefore, a *simplex* $\mathcal{S}$.

It is natural to assume that all points $(x_1, \ldots, x_4)$ from the simplex are "equally possible", i.e., that there is a uniform distribution ("second-order probability") on this set of probability distributions. Then, as a natural estimate for the probability $p(S_1 \,\&\, S_2)$ of $S_1 \,\&\, S_2$, we can take the conditional mathematical expectation of this probability under the condition that the values $p(S_1) = p_1$ and $p(S_2) = p_2$:

$$E(p(S_1 \,\&\, S_2) \,|\, p(S_1) = p_1 \,\&\, p(S_2) = p_2) =$$

$$P(x_1 \,|\, x_1 + x_2 = p_1 \,\&\, x_1 + x_3 = p_2).$$

(This idea was proposed and described in [1, 6, 7, 8, 9]; see also [2].)

From the geometric viewpoint, the two conditions $x_1 + x_2 = p_1$ and $x_1 + x_3 = p_2$ select a straight line segment within the simplex $\mathcal{S}$, a segment which can be parameterized by a parameter

$$x_1 \in [p^-, p^+] = [\max(p_1 + p_2 - 1, 0), \min(p_1, p_2)];$$

then, $x_2 = p_1 - x_1$, $x_3 = p_2 - x_1$, and $x_4 = 1 - (x_1 + x_2 + x_3)$. Since we start with a uniform distribution on $\mathcal{S}$, the conditional probability distribution on this segment is uniform, i.e., x_1 is uniformly distributed on the interval $[p^-, p^+]$. Thus, the conditional mathematical expectation of x_1 with respect to this distribution is equal to $(p^- + p^+)/2$, i.e., to the midpoint of this interval. Similarly, for an "or" operation, we can conclude that

$$E(p(S_1 \vee S_2) \,|\, p(S_1) = p_1 \,\&\, p(S_2) = p_2) =$$

$$\frac{1}{2} \cdot \max(p_1, p_2) + \frac{1}{2} \cdot \min(p_1 + p_2, 1).$$

1.4. MIDPOINT OPERATIONS: PROBLEM

The choice of a midpoint, however, comes with a problem. Intuitively, $(S_1 \,\&\, S_2) \,\&\, S_3$ means the same as $S_1 \,\&\, (S_2 \,\&\, S_3)$, so it is thus natural to require that the "and"-operation be associative: $(p_1 \,\&\, p_2) \,\&\, p_3 = p_1 \,\&\, (p_2 \,\&\, p_3)$. Alas, midpoint operations are *not* associative [2]: e.g., $(0.4 \,\&\, 0.6) \,\&\, 0.8 = 0.2 \,\&\, 0.8 = 0.1$, while $0.4 \,\&\, (0.6 \,\&\, 0.8) = 0.4 \,\&\, 0.5 = 0.2 \ne 0.1$.

By itself, a small non-associativity may not be so bad: associativity comes from the requirement that our reasoning be *rational*, while it is well known that our actual handling of uncertainty is *not* exactly following *rationality* requirements (see, e.g., [27]). So, it is desirable to find out how non-associative can these operations be.

2. MAIN RESULTS AND THEIR INTERPRETATION

Theorem 1. $\max\limits_{a,b,c} |(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)| = 1/9.$

Theorem 2. $\max\limits_{a,b,c} |(a \vee b) \vee c - a \vee (b \vee c)| = 1/9.$

Interpretation. Human experts do not use *all* the numbers from $[0, 1]$ to describe their possible degrees of belief. They use a few words like "very probable" etc. Each of words is a "granule" covering the entire subinterval of values. Since the largest possible non-associativity degree is $1/9$, non-associativity is negligible if the "granules" are of size $\geq 1/9$. No more than 9 such granules fit into $[0, 1]$. This may explain why humans are most comfortable with ≤ 9 items to choose from – the famous "7 ± 2" law; see, e.g., [17, 18].

This general psychological law has also been confirmed in our specific area of formalizing expert knowledge: namely, in [5], it was shown that this law explains why in intelligent control, experts normally use ≤ 9 different degrees (such as "small", "medium", etc.) to describe the value of each characteristic.

3. PROOFS: MAIN IDEA AND HOW INTERVAL COMPUTATIONS SIMPLIFY IT

One can easily see that the operation $\vee$ is *dual* to the operation & in the sense that $a \vee b = 1 - (1 - a) \,\&\, (1 - b)$. Because of this duality, we can easily deduce Theorem 2 from Theorem 1. Thus, it is sufficient to prove Theorem 1.

Every triple can be sorted: $a \leq b \leq c$. For these sorted real numbers, we want to know the relation between $t_a \stackrel{\text{def}}{=} a \,\&\, (b \,\&\, c)$, $t_b \stackrel{\text{def}}{=} b \,\&\, (a \,\&\, c)$, and $t_c \stackrel{\text{def}}{=} c \,\&\, (a \,\&\, b)$. The formulas for $a \,\&\, b$, $a \,\&\, c$, and $b \,\&\, c$ depend on the relation between $a + b$, $a + c$, $b + c$, and 1. Since $a \leq b \leq c$, we have $a + b \leq a + c \leq b + c$. Thus, there are exactly 4 possible locations of number 1 in relation to these three sums:

I. $a + b \le a + c \le b + c \le \mathbf{1}$;

II. $a + b \le a + c \le \mathbf{1} < b + c$;

III. $a + b \le \mathbf{1} < a + c \le b + c$;

IV. $\mathbf{1} < a + b \le a + c \le b + c$.

We prove by considering these cases one by one. In each case, we get expressions for $a \,\&\, b$, $a \,\&\, c$, and $b \,\&\, c$ which do not contain min and max.

We can subdivide each of these cases into subcases depending on which of the maximized and minimized terms in the expressions for $a \,\&\, (b \,\&\, c)$, $b \,\&\, (a \,\&\, c)$, and $c \,\&\, (a \,\&\, b)$ are larger.

For example, in case IV, all three sums $a + b$, $a + c$, and $b + c$ are greater than 1, so $a \,\&\, b = a + 0.5 \cdot b - 0.5$, $a \,\&\, c = a + 0.5 \cdot c - 0.5$, and $b \,\&\, c = b + 0.5 \cdot c - 0.5$.

- The value of $t_a = (b \,\&\, c) \,\&\, a$ depends on whether $(b \,\&\, c) + a \le 1$, i.e., whether $b + 0.5 \cdot c - 0.5 + a \le 1$. If we move terms which do not contain a, b, or c to the right hand-side, and rearrange terms which do contain a, b, or c, in alphabetic order, we get an equivalent inequality $a + b + 0.5 \cdot c \le 1.5$.
- Similarly, the value of $t_b = (a \,\&\, c) \,\&\, b$ depends on whether $(a \,\&\, c) + b \le 1$, i.e., whether $a + 0.5 \cdot c - (1 - \beta) + b \le 1$, which is also equivalent to the same inequality $a + b + 0.5 \cdot c \le 1.5$.
- Finally, the value of $t_c = (a \,\&\, b) \,\&\, c$ depends on whether $(a \,\&\, b) + c \le 1$, i.e., whether $a + 0.5 \cdot b - 0.5 + c \le 1$, which is equivalent to the inequality $a + 0.5 \cdot b + c \le 1.5$.

So, to find the expressions for t_a, t_b, and t_c, we must know where 1.5 stands in comparison with $a + b + 0.5 \cdot c$ and $a + 0.5 \cdot b + c$. Since $b \le c$, we have $0.5 \cdot b \le 0.5 \cdot c$, hence

$$a + b + 0.5 \cdot c = (a + b + c) - 0.5 \cdot c \le$$

$$(a + b + c) - 0.5 \cdot b = a + 0.5 \cdot b + c.$$

Due to this inequality, we have exactly three possibilities:

A. the number 1.5 can be larger than the largest of the above two expressions; in this case, both expressions are ≤ 1.5, i.e.,

$$a + b + 0.5 \cdot c \le a + 0.5 \cdot b + c \le 1.5;$$

B. the number 1.5 is in between the above two expressions; in this case,

$$a + b + 0.5 \cdot c \le 1.5 < a + 0.5 \cdot b + c;$$

C. the number 1.5 is smaller than the smallest of the above two expressions; in this case, both expressions are ≥ 1.5, i.e.,

$$1.5 < a + b + 0.5 \cdot c \le a + 0.5 \cdot b + c.$$

These subcases can be further subdivided, etc. For each of the resulting final subcases, all three combinations t_a, t_b, and t_c are described by linear expressions.

There are many such subcases, so the proof is possible but very lengthy. It turns out that interval computations can reduce this length.

Indeed, we want to find the maximum of the expression

$$|(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)|$$

when $a, b, c \in [0, 1]$. To help with the proof, we divided each interval $[0, 1]$ into 100 subintervals of length 0.01, thus generating $100^3 = 10^6$ sub-boxes. We use interval arithmetic with additional operations

$$\min([a^-, a^+], [b^-, b^+]) = [\min(a^-, b^-), \min(a^+, b^+)]$$

and

$$\max([a^-, a^+], [b^-, b^+]) = [\max(a^-, b^-), \max(a^+, b^+)].$$

For each subbox, we applied the "naive" interval computations technique to get the estimate $[m_i, M_i]$ for the range of the desired function on this subbox. Then, we eliminated all subboxes for which $M_i < 1/9$.

As a result, out of the original 10^6 boxes, we have only 80 possible locations of the global maximum. For these 80 boxes, $b \in [0.54, 0.58]$, and:

- either $a \in [0.43, 0.46]$ and $c \in [0.75, 0.79]$;
- or $a \in [0.75, 0.79]$ and $c \in [0.43, 0.46]$.

When we sort a, b, and c, we get $a \in [0.43, 0.46]$, $b \in [0.54, 0.58]$, and $c \in [0.75, 0.79]$. Hence, $a + c > 1$, and we only need proofs for *half* of the cases: Cases III and IV.

Some subcases of Case IV were also eliminated. Indeed, within the above interval bounds for a, b, and c, the upper bound for $a+b+0.5\cdot c$ is equal to $0.46+0.58+0.5\cdot 0.79 = 1.435 < 1.5$. Thus, to check that the value of the desired function cannot exceed $1/9$, we only need to consider cases when $a+b+0.5\cdot c < 1.5$. Hence, we can dismiss Subcase C when this inequality is not satisfied, and only consider Subcases A and B in our proof.

For each final subcase, the difference $(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)$ is a *linear* function, and the constraints describing this subcase are *linear* inequalities. Thus, for each subcase, we have a *linear programming* problem with rational coefficients. We can analytically solve each of these problems by computing the vertices of the corresponding polytope, and finding the vertex on which the objective function attains the largest value. As a result, we get the desired proof.

4. AUXILIARY RESULTS: ALTERNATIVE TO MIDPOINT

Instead of selecting a midpoint, we can make a more general selection of a value in the interval $\mathbf{p}$.

By a *choice function*, we mean a function s that maps every interval $\mathbf{u} = [u^-, u^+]$ into a point $s(\mathbf{u}) \in \mathbf{u}$ so that for every c and $\lambda > 0$:

- $s([u^- + c, u^+ + c]) = s([u^-, u^+]) + c$ (*shift-invariance*);
- $s([\lambda \cdot u^-, \lambda \cdot u^+]) = \lambda \cdot s([u^-, u^+])$ (*unit-invariance*).

Proposition. [19] *Every choice function has the form* $s([u^-, u^+]) = \alpha \cdot u^- + (1-\alpha)\cdot u^+$ *for some* $\alpha \in [0,1]$.

The combination $p = \alpha\cdot p^- + (1-\alpha)\cdot p^+$ (first proposed by Hurwicz [10]) has been successfully used in areas ranging from submarine detection [3, 4, 22, 23, 24] to petroleum engineering [26]; see also [11, 12, 13, 16, 25]. (In [28, 29], this approach is applied to second-order probabilities.)

With this approach, we get the following formulas which generalize (1) and (2):

$$p_1 \,\&\, p_2 \stackrel{\text{def}}{=} \alpha \cdot \max(p_1 + p_2 - 1, 0) + (1-\alpha)\cdot \min(p_1, p_2); \qquad (3)$$

$$p_1 \vee p_2 \stackrel{\text{def}}{=} \alpha \cdot \max(p_1, p_2) + (1-\alpha)\cdot \min(p_1 + p_2, 1). \qquad (4)$$

Theorem 3.

$$\max_{a,b,c} |(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)| = \frac{\alpha \cdot (1-\alpha)}{2 + \alpha \cdot (1-\alpha)}.$$

$$\max_{a,b,c} |(a \vee b) \vee c - a \vee (b \vee c)| = \frac{\alpha \cdot (1-\alpha)}{2 + \alpha \cdot (1-\alpha)}.$$

Comment. This non-associativity degree is the smallest ($=0$) when $\alpha = 0$ or $\alpha = 1$, and the largest ($=1/9$) for midpoint operations ($\alpha = 0.5$).

In our proof, it was useful to first show that the new operations have *some* properties of associativity:

Definition. *We say that a commutative operation $*$ is semi-associative if $a \le b \le c$ implies that*

$$a * (b * c) \ge b * (a * c) \ge c * (a * b).$$

Theorem 4. *For every $\alpha \in (0,1)$, both operations* (3) *and* (4) *are semi-associative.*

Proofs: main idea. To make it easier to follow the proofs, the reader is welcome to use the fact that the traditional fuzzy logic operation $\min(a,b)$ corresponds to $\alpha = 0$ and $1-\alpha = 1$; to make this following even easier, we introduce a new variable $\beta = 1-\alpha$; then, $\alpha = 1-\beta$.

Let us describe the proof of Theorems 3 and 4 for Case I of &. For Theorem 4, in this case, $a + b \le a + c \le b + c \le 1$, so $a + b \le 1$ and $b + c \le 1$. Hence, $b \,\&\, c = \beta \cdot b$, $a \,\&\, c = \beta \cdot a$, and $a \,\&\, b = \beta \cdot a$. Let us find the values of all three terms t_a, t_b, and t_c:

t_c: Since $a \,\&\, b \le a$ (by the properties of the new operation), and $a \le c$ (by our assumption), we conclude that $a \,\&\, b \le c$. Also, $(a \,\&\, b) + c = \beta \cdot a + c \le a + c \le 1$, so

$$(a \,\&\, b) \,\&\, c = \beta \cdot (a \,\&\, b) = \beta \cdot (\beta \cdot a) = \beta^2 \cdot a.$$

t_b: Similarly, $(a \,\&\, c) \le a \le b$, and $(a \,\&\, c) + b = \beta \cdot a + b \le a + b \le 1$, so

$$(a \,\&\, c) \,\&\, b = \beta \cdot (a \,\&\, c) = \beta \cdot (\beta \cdot a) = \beta^2 \cdot a.$$

t_a: Finally, $(b \,\&\, c) + a = \beta \cdot b + a \le a + b \le 1$, so

$$(b \,\&\, c) \,\&\, a = \beta \cdot \min(\beta \cdot b, a) = \min(\beta^2 \cdot b, \beta \cdot a).$$

Now we are ready to prove the desired inequalities:

$t_c \le t_b$: We have shown even that $t_b = (a \,\&\, c) \,\&\, b = (a \,\&\, b) \,\&\, c = t_c$.

$t_b \le t_a$: Since $b \ge a$, we have $\beta^2 \cdot b \ge \beta^2 \cdot a$; clearly, since $\beta < 1$, we have $\beta > \beta^2$, hence $\beta \cdot a \ge \beta^2 \cdot a$. Hence, $\min(\beta^2 \cdot b, \beta \cdot a) \ge \beta^2 \cdot a$.

Thus, for Case I, the desired inequalities are proven.

To prove Case I of Theorem 3, it is therefore sufficient to prove that the difference $t_a - t_c$ cannot exceed the desired bound M. We will prove this by reduction to a contradiction by assuming that $t_a - t_c > M$ and by getting a contradiction.

In Case I, as we have shown in the proof of Theorem 4, $t_a = \min(\beta^2 \cdot b, \beta \cdot a)$ and $t_c = \beta^2 \cdot a$. Thus, from the assumption that $t_a - t_c > M$, we can conclude that $\beta^2 \cdot \beta - \beta^2 \cdot a > M$ and that $\beta \cdot a - \beta^2 \cdot a > M$.

The second of these inequalities is equivalent to $\beta \cdot (1 - \beta) \cdot a > M$, i.e., to

$$a > \frac{M}{\beta \cdot (1-\beta)}.$$

By definition of M, we have

$$\frac{M}{\beta \cdot (1-\beta)} = \frac{1}{2 + \beta \cdot (1-\beta)} = a_0,$$

so this inequality leads to

$$a > a_0 = \frac{1}{2 + \beta \cdot (1-\beta).} \tag{5}$$

The first inequality $\beta^2 \cdot b - \beta^2 \cdot a = \beta^2 \cdot (b - a) > M$ is equivalent to

$$b - a > \frac{M}{\beta^2} = \frac{1-\beta}{\beta \cdot (2 + \beta \cdot (1-\beta))}. \tag{6}$$

From (5) and (6), we conclude that

$$a + b = (b - a) + 2a > \frac{1-\beta}{\beta \cdot (2 + \beta \cdot (1-\beta))} + \frac{2}{2 + \beta \cdot (1-\beta)} =$$

$$\frac{2\beta + (1-\beta)}{\beta \cdot (2 + \beta \cdot (1-\beta))} = \frac{1+\beta}{\beta \cdot (2 + \beta \cdot (1-\beta))}.$$

Since in Case I, $a + b \le 1$, we conclude that

$$\frac{1+\beta}{\beta \cdot (2 + \beta \cdot (1-\beta))} < 1,$$

i.e., that

$$1+\beta < \beta\cdot(2+\beta\cdot(1-\beta)) = 2\beta+\beta^2-\beta^3.$$

If we move β to the right-hand side and β^3 to the left-hand side, we get a simpler equivalent inequality $1+\beta^3 < \beta+\beta^2$. This inequality can be further simplified if we divide its both sides by $1+\beta > 0$, resulting in the following: $1-\beta+\beta^2 < \beta$. If we move β from the right-hand side to the left, we get $1-2\beta+\beta^2 = (1-\beta)^2 < 0$, which is impossible.

The contradiction shows that in Case I, we cannot have $t_a - t_c > M$; thus, for Case I, Theorem 3 is proven.

Other cases are treated similarly.

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