Why Product of Probabilities (Masses) for Independent Events? A Remark

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Abstract

For independent events A and B, the probability $P(A\,\&\,B)$ is equal to the product of the corresponding probabilities: $P(A\&B) = P(A)\cdot P(B)$. It is well known that the product $f(a,b)=a\cdot b$ has the following property: once $\sum\limits_{i=1}^{n} P(A_i)=1$ and $\sum\limits_{j=1}^{m} P(B_j)=1$, the probabilities $P(A_i\,\&\,B_j) = f(P(A_i),P(B_j))$ also add to 1: $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} f(P(A_i),P(B_j)) = 1$. In 1986, D. Dubois, H. Prade, and R. Giles proved that the product is the only continuous function that satisfies this property, i.e., that if, vice versa, this property holds for some continuous function $f(a,b)$, then this function f is the product. This result provided an additional explanation of why for independent events, we multiply probabilities (or, in the Dempster-Shafer case, masses).

In this paper, we strengthen this result by showing that it holds for arbitrary (not necessarily continuous) functions $f(a,b)$.

Product is normally used as a combination rule for independent events. For independent events A and B, the probability $P(A\,\&\,B)$ is equal to the product of the corresponding probabilities: $P(A\&B) = f(P(A),P(B))$, where the combination function is the product $f(a,b)=a\cdot b$; see, e.g., [6].

Similarly, in Dempster-Shafer theory (see, e.g., [3, 7]) one of the ways to combine the masses from two independent knowledge bases is to multiply them.

A reasonable property of the combination rule. Due to the additivity property of probability, if the events $A_1,\ldots,A_n$ form a partition of the universal set, i.e., if one of these events always occurs and no two can occur at the same time, then $\sum\limits_{i=1}^{n} P(A_i) = 1$. If the events A_i form a partition and the events B_j form a partition, then their combinations $A_i \,\&\, B_j$ also form a partition; indeed:

- since A_i and B_j form a partition, any situation belongs to one of A_i and to one of B_j, thus, for this situation, the corresponding event $A_i \,\&\, B_j$ holds;
- similarly, since the events A_i are mutually exclusive and the events B_j are mutually exclusive, the combinations $A_i \,\&\, B_j$ are also mutually exclusive.

It is therefore reasonable to expect that if the events A_i form a partition, i.e., $\sum\limits_{i=1}^{n} P(A_i) = 1$, and if events B_j form a partition, i.e., $\sum\limits_{j=1}^{m} P(B_j) = 1$, then the events $A_i \,\&\, B_j$ should also form a partition, i.e., $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} f(P(A_i), P(B_j)) = 1$.

In formal terms, the function $f : [0,1] \times [0,1] \to [0,1]$ that describes the combination rule should satisfy the following property:

$$\text{For every two finite sequences}$$
$$\text{of non-negative real numbers } (a_1, \ldots, a_n) \text{ and } (b_1, \ldots, b_m), \tag{1}$$
$$\text{if } \sum_{i=1}^{n} a_i = 1 \text{ and } \sum_{j=1}^{m} b_j = 1, \text{ then } \sum_{i=1}^{n} \sum_{j=1}^{m} f(a_i, b_j) = 1.$$

What is known. It is well known that the product function $f(a,b) = a \cdot b$ satisfies the property (1). It is also known that many other possible combination functions, e.g., many t-norms that are different from the product (see, e.g., [4, 5]), do not satisfy this property.

D. Dubois, H. Prade, and R. Giles proved [2] that among *continuous* functions f, the product function is the only function that satisfies the above property.

This result provides an additional explanation of why for independent events, we multiply probabilities (or, in the Dempster-Shafer case, masses).

What we will prove. In this paper, we strengthen the result from [2] by showing that it holds for arbitrary (not necessarily continuous) functions $f(a,b)$.

We also extend this result to the case when we combine more than two events.

Theorem 1. *If a function $f : [0,1] \times [0,1] \to [0,1]$ satisfies the property* (1)*, then this function is the product: $f(a,b) = a \cdot b$ for all a and b.*

Case of several events. Let $k \geq 2$ be an integer, and let $f : [0,1]^k \to [0,1]$ be a function of k variables. For such functions, we will consider the following property:

$$\text{For every } k \text{ finite sequences}$$
$$\text{of non-negative real numbers } (a_1^{(1)}, \ldots, a_{n_1}^{(1)}), \ldots, (a_1^{(k)}, \ldots, a_{n_k}^{(k)}),$$
$$\text{if } \sum_{i_1=1}^{n_1} a_{i_1}^{(1)} = 1 \text{ and } \ldots \text{ and } \sum_{i_k=1}^{n_k} a_{i_k}^{(k)} = 1, \tag{2}$$
$$\text{then } \sum_{i_1=1}^{n_1} \ldots \sum_{i_k=1}^{n_k} f(a_{i_1}^{(1)}, \ldots, a_{i_k}^{(k)}) = 1.$$

Theorem 2. *If a function $f : [0,1]^k \to [0,1]$ satisfies the property* (2)*, then this function is the product: $f(a_1, \ldots, a_k) = a_1 \cdot \ldots \cdot a_k$ for all $a_1, \ldots, a_k$.*

Proof of the Theorems. The proof of Theorems 1 and 2 is based on the following Lemma:

Lemma. *Let a function $g : [0,1] \to R_0^+ \stackrel{\text{def}}{=} [0, \infty)$ satisfy the following property:*

$$\textit{For every finite sequence of non-negative real numbers } (a_1, \ldots, a_n),$$
$$\textit{if } \sum_{i=1}^{n} a_i = 1, \textit{ then } \sum_{i=1}^{n} g(a_i) = 1. \tag{3}$$

Then, $g(a) = a$ for every real number a.

Proof of the Lemma. Let us first consider the case when $n = 2$. In this case, the condition of the Lemma means that $a_1 + a_2 = 1$ implies $g(a_1) + g(a_2) = 1$, i.e., that $g(a_2) = 1 - g(a_1)$. The equality $a_1 + a_2 = 1$ means that $a_2 = 1 - a_1$, so the condition of the Lemma means that

$$g(1 - a_1) = 1 - g(a_1) \tag{4}$$

for all $a_1 \in [0,1]$.

For $n = 3$, we similarly conclude that $g(a_1) + g(a_2) + g(1 - (a_1 + a_2)) = 1$ for all $a_1 \geq 0$ and $a_2 \geq 0$ for which $a_1 + a_2 \leq 1$. Therefore, $g(a_1) + g(a_2) = 1 - g(1 - (a_1 + a_2))$. Due to (4), we have $1 - g(1 - (a_1 + a_2)) = g(a_1 + a_2)$, so the above property reads $g(a_1 + a_2) = g(a_1) + g(a_2)$. It is known (see, e.g., [1]) that every function g whose values are non-negative and which satisfies the above *additivity* property is linear, i.e., $g(a) = k \cdot a$ for some real number k. Substituting this expression for $g(a)$ into both sides of the formula (4), we conclude that $k = 1$, i.e., that $g(a) = a$. The Lemma is proven.

Completing the proof. Let us first prove Theorem 1. Let b_j be a sequence for which $\sum_{j=1}^{m} b_j = 1$. For this sequence, let us introduce an auxiliary function $g(a) \stackrel{\text{def}}{=} \sum_{j=1}^{m} f(a, b_j)$. In terms of this function, the double sum in (1) takes the form $\sum_{i=1}^{n} g(a_i)$, so the property (1) takes the form (3).

Since the values of the function f are non-negative, the new auxiliary function $g(a)$ has non-negative values as well. Due to Lemma, we now conclude that $g(a) = a$, i.e., that for every a, we have

$$\sum_{j=1}^{m} f(a, b_j) = a. \tag{5}$$

When $a = 0$, then, from the fact that $f(a, b) \geq 0$ for all b, we conclude that $f(a, b_j) = 0$ for all j – since the only way for a sum of non-negative numbers to be 0 is when each of these numbers is equal to 0. Thus, we conclude that $f(0, b) = 0$ for all b, i.e., that $f(a, b) = a \cdot b$ for $a = 0$.

When $a > 0$, then we can divide both sides of the formula (5) by a and get the following formula:

$$\sum_{j=1}^{m} \frac{f(a, b_j)}{a} = 1.$$

So, for every $a > 0$, the new auxiliary function $g(b) \stackrel{\text{def}}{=} \dfrac{f(a, b)}{a}$ satisfies the following property:

For every finite sequence of non-negative real numbers $(b_1, \ldots, b_m)$,

$$\text{if } \sum_{j=1}^{m} b_j = 1, \text{ then } \sum_{j=1}^{m} g(b_j) = 1.$$

This is exactly the property (3), so, due to Lemma, $g(b) = b$ for every real number b. Since $g(a) = f(a, b)/a$, we conclude that $f(a, b) = a \cdot b$ for all a and b.

Theorem 2 can be now proved by induction over k. We have already proven this theorem for $k = 2$ – this case corresponds exactly to Theorem 1. Let us now assume that we have proved this result for $k - 1$, let us show how to prove it for k. For that, we first fix $k - 1$ sequences $(a_1^{(2)}, \ldots, a_{n_2}^{(2)}), \ldots, (a_1^{(k)}, \ldots, a_{n_k}^{(k)})$, and consider an auxiliary function $g(a) \stackrel{\text{def}}{=} \sum_{i_2=1}^{n_2} \ldots \sum_{i_k=1}^{n_k} f(a, a_{i_2}^{(2)}, \ldots, a_{i_k}^{(k)})$. For this function, the condition (2) turns into (3), so, due to Lemma, we conclude that $g(a) = \sum_{i_2=1}^{n_2} \ldots \sum_{i_k=1}^{n_k} f(a, a_{i_2}^{(2)}, \ldots, a_{i_k}^{(k)}) = a$ for all a. Thus, for every a, the new function $f'(a_2,\ldots,a_k) \stackrel{\rm def}{=} f(a,a_2,\ldots,a_k)a$ of $k-1$ variables satisfies the following property:

$$\text{For every } k-1 \text{ finite sequences}$$

$$\text{of non-negative real numbers } (a_1^{(2)},\ldots,a_{n_2}^{(2)}),\ldots,(a_1^{(k)},\ldots,a_{n_k}^{(k)}),$$

$$\text{if } \sum_{i_2=1}^{n_2} a_{i_1}^{(2)} = 1 \text{ and } \ldots \text{ and} \sum_{i_k=1}^{n_k} a_{i_k}^{(k)} = 1,$$

$$\text{then } \sum_{i_2=1}^{n_2} \ldots \sum_{i_k=1}^{n_k} f'(a_{i_2}^{(2)},\ldots,a_{i_k}^{(k)}) = 1.$$

This is exactly the property (2) for $k-1$, so, due to induction assumption, we conclude that $f'(a_2,\ldots,a_k) = a_2 \cdot \ldots \cdot a_k$. Since $f'(a_2,\ldots,a_k) = f(a,a_2,\ldots,a_k)/a$, we thus conclude that $f(a,a_2,\ldots,a_n) = a \cdot f'(a_2,\ldots,a_k) = a \cdot a_2 \cdot \ldots \cdot a_k$. The induction step is proven, and so is the theorem.

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