Egyptian Fractions Revisited

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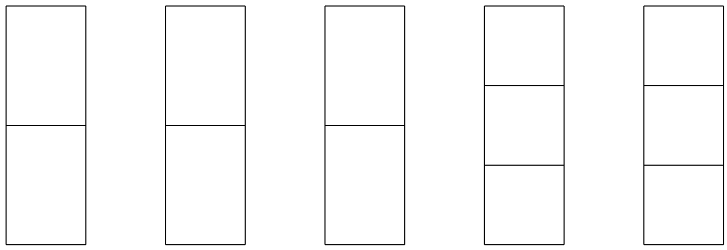
Abstract

It is well known that the ancient Egyptians represented each fraction as a sum of unit fractions – i.e., fractions with unit numerators; this is how they, e.g., divided loaves of bread. What is not clear is why they used this representation. In this paper, we propose a new explanation: crudely speaking, that the main idea behind the Egyptian fractions provides an optimal way of dividing the loaves. We also analyze the related properties of fractions.

1 What are Egyptian fractions

It is known that people of ancient Egypt represented fractions as sums of unit fractions – i.e., fractions of the type $1/n$. This representation is described, in detail, in the Rhind (Ahmes) Papyrus, the most extensive Egyptian mathematical papyrus; see, e.g., [1]. According to the papyrus, this was a method recommended, e.g., for dividing loaves of bread between several people.

For example, the number 5/6 can be represented as 1/2+1/3. In other words, $5 = 6 \cdot (1/2 + 1/3) = 6 \cdot (1/2) + 6 \cdot (1/3)$. So, according to the method described in the Rhind Papyrus, if we want to divide 5 loaves between 6 people, we must divide $6 \cdot (1/2) = 3$ loaves into two equal parts each, and $6 \cdot (1/3) = 2$ loaves into three equal parts each. As a result, we get six half-loaves and six third-loaves. Each of the six people receives one half and one third:



2 Why Egyptian fractions? A question

Most algorithms with Egyptian fractions are so complicated that it is puzzling why they were used in the first place. For example, according to [4], R. Graham (who wrote his Ph.D. dissertation on unit fractions) asked André Weil why, and A. Weil answered "They took a wrong turn".

3 Why Egyptian fractions: A possible answer

Let us assume that we can only divide loaves into equal pieces. One cut divides a loaf of bread into 2 equal pieces; in general, to divide a loaf into q equal pieces, we need $q-1$ cuts.

If we want to divide 5 loaves between 6 people, to give each of them $5/6$ of a loaf, then a natural way to do it is to divide each of 5 loaves into 6 equal pieces. To divide each loaf, we need $6-1=5$ cuts, so to divide all 5 loaves, we need $5\cdot 5=25$ cuts.

On the other hand, in the Egyptian fraction approach, we need to divide 3 loaves in half (1 cut each) and 2 loaves into three equal pieces (2 cuts each), to the total of $3+2\cdot 2=7\ll 25$ cuts.

4 General question: what is the smallest number of cuts?

Suppose that we want to divide a large number of loaves in such a way that every person gets p/q-th of a loaf. In other words, for some large number N, we have N people, and we want to distribute $N\cdot(p/q)$ loaves between these people.

The straightforward way would be to divide each loaf into q equal parts. A more general approach is to divide some loaves into q_1 equal parts, some loaves into q_2 equal parts, etc., and some loaves into q_k equal parts. Then, we give each person some pieces from each of these divisions: some number of parts $1/q_1$ (we will denote this number by p_1), some number of parts $1/q_2$ (we will denote this number by p_2), etc. In other words, we represent the desired ratio

p/q as a sum

$$p/q = p_1/q_1 + \ldots + p_k/q_k \quad (1)$$

for natural numbers p_i and q_i.

Each representation of this type corresponds to a possible way of cutting loaves of bread. To find out how we can minimize the number of cuts, let us find out how many cuts per loaf correspond to a representation (1). For N people, we need $N \cdot (p/q)$ loaves, out of which $N \cdot (p_1/q_1)$ are divided into q_1 equal pieces, $N \cdot (p_2/q_2)$ loaves are divided into q_2 equal pieces, etc. To divide a loaf into q_i pieces, we need $q_i - 1$ cuts, so the overall number of cuts is equal to

$$N \cdot (p_1/q_1) \cdot (q_1 - 1) + \ldots + N \cdot (p_k/q_k) \cdot (q_k - 1) =$$

$$N \cdot (p_1 + \ldots + p_k) - N \cdot (p_1/q_1 + \ldots + p_k/q_k).$$

Due to (1), we conclude that the for N persons, the overall number of cuts is equal to $N \cdot (p_1 + \ldots + p_k) - N \cdot (p/q)$. Thus, the average number of cuts per person is equal to $p_1 + \ldots + p_k - p/q$. So, the average number of cuts is the smallest if and only if the sum $p_1 + \ldots + p_k$ of the numerators in the representation (1) attains the smallest possible value.

Denotation. For every positive rational number $r = p/q$, let us denote, by $\|r\|$, the smallest possible sum $p_1 + \ldots + p_k$ among all representations of type (1).

In these terms, the smallest possible number of cuts per person is equal to $\|r\| - r$. What are the properties of this function $\|r\|$?

Proposition.

- *For every rational number,* $\|r\| \geq r$.
- *For every integer* n, $\|n\| = n$.
- *For every rational number* r *and for every integer* n, $\|r/n\| \leq \|r\|$.
- *For every two rational numbers* r *and* r', $\|r + r'\| \leq \|r\| + \|r'\|$ *and* $\|r \cdot r'\| \leq \|r\| \cdot \|r'\|$.

Proof. Since $\|r\| - r$ is the average number of cuts, i.e., a non-negative number, we have $\|r\| \geq r$. For integers n, we do not need any cuts, so $\|n\| - n = 0$ and $\|n\| = n$.

Let $r = p_1/q_1 + \ldots + p_k/q_k$ be a representation corresponding to $\|r\|$, i.e., representations for which $\|r\| = p_1 + \ldots + p_k$. Then,

$$r/n = p_1/(n \cdot q_1) + \ldots + p_k/(n \cdot q_k).$$

For this representation of r/n, the sum of the numerators is the same, i.e., it is equal to $\|r\|$. Thus, the smallest possible sum $\|r/n\|$ of the numerators in the representation of r/n cannot exceed $\|r\|$.

If $r = p_1/q_1 + \ldots + p_k/q_k$ and $r' = p'_1/q'_1 + \ldots + p'_{k'}/q'_{k'}$ are representations corresponding to $\|r\|$ and $\|r'\|$, i.e., representations for which $\|r\| = p_1 + \ldots + p_k$ and $\|r'\| = p'_1 + \ldots + p'_{k'}$, then for the sum of these representations, we get

$$r + r' = p_1/q_1 + \ldots + p_k/q_k + p'_1/q'_1 + \ldots + p'_{k'}/q'_{k'}$$

with $p_1 + \ldots + p_k + p'_1 + \ldots + p'_{k'} = \|r\| + \|r'\|$. Thus, the smallest possible sum $\|r + r'\|$ of the numerators in the representation of $r + r'$ cannot exceed $\|r\| + \|r'\|$.

Similarly, for the product

$$r \cdot r' = (p_1/q_1 + \ldots + p_k/q_k) \cdot (p'_1/q'_1 + \ldots + p'_{k'}/q'_{k'}) = \sum_{i,j} (p_i \cdot p'_j)/(q_i \cdot q'_j),$$

the sum of the numerators is equal to

$$\sum_{i,j} (p_i \cdot p'_j) = \sum_i p_i \cdot \sum_j p'_j = \|r\| \cdot \|r'\|,$$

so $\|r \cdot r'\| \le \|r\| \cdot \|r'\|$. The proposition is proven.

Comment. We can thus say that $\| \cdot \|$ is an integer-valued additive and multiplicative norm on the set of all positive rational numbers.

5 Computing the smallest number of cuts: algorithm

How can we actually compute the smallest number of cuts, i.e., the norm $\| \cdot \|$? This problem is not trivial, because, from the Egyptian papyri, it is known that even for small q, we often need large numbers q_k.

Nevertheless, an algorithm for computing $\|r\|$ is possible. What we will show is that for every integer n, there exists an algorithm A_n that checks whether $\|r\| \le n$, i.e., whether there exists a representation of r as a sum of $p_1 + \ldots + p_k \le n$ unit fractions $1/q_i$. Once we have such an algorithm, we can compute $\|r\|$ by checking whether $\|r\| \le 1$, whether $\|r\| \le 2$, whether $\|r\| \le 3$, etc., until we reach the smallest integer n for which $\|r\| \le n$ – this integer n is the desired norm $\|r\|$.

We will build the algorithm A_n by induction over n.

For $n = 1$, the algorithm A_1 follows from the fact that the only way to get $p_1 + \ldots + p_k = \|r\| = 1$ is to have $p_1 = 1$ and $k = 1$. So, only fractions of the type $1/n$ have $\|r\| = 1$. Thus, to check whether $\|r\| = 1$, it is sufficient to check whether r is a unit fraction.

Let us now suppose that we have already designed an algorithm A_n. Let us use this algorithm to design a new algorithm A_{n+1} for checking whether $\|r\| \le n + 1$. Indeed, the fact that $\|r\| \le n + 1$ means that the given fraction $r = p/q$

can be represented as the sum of $\leq n+1$ unit fractions $p/q = 1/q_1 + \ldots + q_M$ for some $M \leq n+1$.

Without losing generality, we can assume that $q_1 \leq q_2 \leq \ldots \leq q_M$. Thus, $1/q_i \leq 1/q_1$ for all i and hence, $1/q_1 \leq p/q \leq M/q_1$. Since $M \leq n+1$, we conclude that $1/q_1 \leq p/q \leq (n+1)/q_1$. Thus, $q/p \leq q_1 \leq (n+1) \cdot q/p$. There are only finitely many integers q_1 in the interval $[q/p, (n+1) \cdot q/p]$. So, to check whether $\|r\| \leq n+1$, it is sufficient to try all of them and for each of them, check whether the difference $p/q - 1/q_1$ can be represented as a sum of $\leq n$ unit fractions, i.e., whether $\|p/q - 1/q_1\| \leq n$. This auxiliary checking can be done by the algorithm A_n.

6 Computing the smallest number of cuts: example

Let us illustrate this algorithm on the example of $p/q = 4/5$. Here, q does not divide p, so $4/5$ is not a unit fraction and $\|4/5\| > 1$.

Let us check whether $\|4/5\| \leq 2$. According to the above algorithm, the value q_1 must satisfy the inequality $5/4 \leq q_1 \leq 2 \cdot 5/4$, i.e., $1.25 \leq q_1 \leq 2.5$. There is only one integer in the corresponding interval, $q_1 = 2$. We must now use the algorithm A_1 to check whether the difference $p/q - 1/q_1 = 4/5 - 1/2 = 3/10$ has $\|r\| \leq 1$. This difference is not a unit fraction, so $\|p/q - 1/q_1\| > 1$, hence $\|p/q\| > 2$.

Let us now check whether $\|4/5\| \leq 3$. According to the above algorithm, the value q_1 must satisfy the inequality $5/4 \leq q_1 \leq 3 \cdot 5/4$, i.e., $1.25 \leq q_1 \leq 3.75$. There are two integers in the corresponding interval, $q_1 = 2$ and $q_1 = 3$. We must now apply the algorithm A_2 to the differences $p/q - 1/q_1$. For $q_1 = 2$, the difference is equal to $4/5 - 1/2 = 3/10$. Let us apply the algorithm A_2 to this difference.

According to the algorithm A_2, we must select an integer q'_1 for which $10/3 \leq q'_1 \leq 2 \cdot 10/3$, i.e., we must consider $q'_1 = 4$, $q'_1 = 5$, and $q'_1 = 6$. Already for $q'_1 = 4$, the difference $3/10 - 1/4 = 1/20$ is a unit fraction, so $\|3/10\| = 2$, with $3/10 = 1/4 + 1/20$.

Since $3/10 = 4/5 - 1/2$, we thus conclude that $\|4/5\| = 3$, with $4/5 = 1/2 + 1/4 + 1/20$.

Comment. This is not the only possible representation of $4/5$ as a sum of three unit fractions: for $q'_1 = 5$, we also get $4/5 = 1/2 + 1/5 + 1/10$. The value $q'_1 = 6$ (and the values q'_1 corresponding to $q_1 = 3$) do not lead to a sum of three fractions.

7 Comment: Are Egyptian fractions optimal?

A natural question is: are actual Egyptian fraction representations given in the Rhind Papyrus optimal? Not always. For example, the Egyptians did not allow identical unit fractions in their representations and had olther unclear preferences. As a result, e.g., instead of $2/13 = 1/13 + 1/3$, they used a representation $2/13 = 1/8 + 1/52 + 1/104$. From the viewpoint of the smallest number of cuts, this representation does not make sense: it replaces a representation corresponding to $p_1 + \ldots + p_k = 2$ with a representation for which $p'_1 + \ldots + p'_{k'} = 1 + 1 + 1 = 3 > 2$ – i.e., with a representation with more cuts.

So, we do not claim that the ancient Egyptian always had it right, what we claim is that their general idea of reducing the sum of the numerators as much as possible seems to be right. From this viewpoint, it will be interesting to further analyze the properties of the norm $\|\cdot\|$.

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