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# Trade-Off Between Sample Size and Accuracy: Case of Dynamic Measurements under Interval Uncertainty

Hung T. Nguyen<sup>1</sup>, Olga Kosheleva<sup>2</sup>, Vladik Kreinovich<sup>2</sup>, and Scott Ferson<sup>3</sup>

<sup>1</sup> New Mexico State University, Las Cruces, NM 88003, USA [hunguyen@nmsu.edu](mailto:hunguyen@nmsu.edu)

<sup>2</sup> University of Texas at El Paso, El Paso, TX 79968, USA [vladik@utep.edu](mailto:vladik@utep.edu)

<sup>3</sup> Applied Biomathematics, Setauket, NY 11733, USA, [scott@ramas.com](mailto:scott@ramas.com)

**Summary.** In many practical situations, we are not satisfied with the accuracy of the existing measurements. There are two possible ways to improve the measurement accuracy:

- first, instead of a *single* measurement, we can make *repeated* measurements; the additional information coming from these additional measurements can improve the accuracy of the result of this series of measurements;
- second, we can replace the *current* measuring instrument with a *more accurate* one; correspondingly, we can use a more accurate (and more expensive) measurement procedure provided by a measuring lab – e.g., a procedure that includes the use of a higher quality reagent.

In general, we can combine these two ways, and make *repeated* measurements with a *more accurate* measuring instrument. What is the appropriate trade-off between sample size and accuracy? In our previous paper, we solved this problem for the case of static measurements. In this paper, we extend the results to the case of dynamic measurements.

## 1 Formulation of the problem

In some practical situations, we want to know the value of the measured quantity with the accuracy which is higher than the guaranteed accuracy of a single measurement. There are two possible ways to improve the measurement accuracy:

- first, instead of a *single* measurement, we can make several ( $n$ ) measurements;
- second, we can replace the *current* measuring instrument with a *more accurate* one.

What is the appropriate trade-off between sample size and accuracy? In our previous paper [11], we analyzed this problem for the case when we measure a

*static* quantity, i.e., a quantity which does not change with time. In this paper, we extend the results from [11] to the general case of *dynamic* measurements, when the measured quantity changes over time.

For such dynamic quantities, we may have two different objectives:

- We may be interested in knowing the *average* value of the measured quantity, e.g., the average concentration of a pollutant in a lake or the average day temperature. In addition to knowing the average, we may also want to know the standard deviation and/or other statistical characteristics.
- We may also want to know not only the average, but also the actual dependence of the measured quantity on space location and/or time.

For example:

- If we interested in general weather patterns, e.g., as a part of the climatological analysis, then it is probably sufficient to measure the average temperature (or the average wind velocity) in a given area.
- On the other hand, if our intent is to provide the meteorological data to the planes flying in this area, then we would rather know how exactly the wind velocity depends on the location, so that the plane will be able to avoid locations where the winds are too strong.

In this paper, we analyze the trade-off between accuracy and sample size for both objectives.

## 2 First objective: measuring the average value of a varying quantity

**Case of ideal measuring instruments: analysis.** Let us start to analyze this situation with the case of an ideal measuring instrument, i.e., a measuring instrument for which the measurement errors are negligible.

By using this ideal instrument, we can measure the value of the quantity of interest at different points and at different moments of time. After we perform  $n$  measurements and get  $n$  measurement results  $x_1, \dots, x_n$ , a natural way to estimate the desired mean value  $x_0 = E[x]$  of  $x$  is to use the arithmetic average  $E \stackrel{\text{def}}{=} \frac{x_1 + \dots + x_n}{n}$  of these measured values. It is reasonable to assume that the differences  $x_i - x_0$  are independent random variables, with a known standard deviation  $\sigma_0$ .

In this case, due to the Central Limit Theorem, for large  $n$ , the difference  $\Delta x_0 \stackrel{\text{def}}{=} E - x_0$  between the estimate  $E$  and the desired value  $x_0$  is approximately normally distributed with 0 average and standard deviation  $\sigma_0/\sqrt{n}$ .

So, even for measurements with the ideal measuring instrument, the result  $E$  of measuring  $x_0$  is not exact; we can only guarantee (with the corresponding

level of confidence) that the measurement error  $\Delta x_0$  is bounded by the value  $k_0 \cdot \sigma_0 / \sqrt{n}$ .

*Comment.* If we do not know this standard deviation, we can estimate it based on the measurement results  $x_1, \dots, x_n$ , by using the standard statistical formulas, such as

$$\sigma_0 \approx \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^n (x_i - E)^2}.$$

**Case of ideal measuring instruments: recommendations.** In the case of ideal measuring instruments, if we want to achieve the desired overall accuracy  $\Delta_0$  with a given confidence, then the sample size  $n$  must be determined by the condition that  $k_0 \cdot \sigma_0 / \sqrt{n} \leq \Delta_0$ , where  $k_0$  corresponds to this confidence:

- 95% confidence corresponds to  $k_0 = 2$ ,
- 99.9% corresponds to  $k_0 = 3$ , and
- confidence  $1 - 10^{-6}\%$  corresponds to  $k_0 = 6$ .

The above condition is equivalent to  $\sqrt{n} \geq \frac{k_0 \cdot \sigma_0}{\Delta_0}$ , i.e., to  $n \geq \frac{k_0^2 \cdot \sigma_0^2}{\Delta_0^2}$ . To minimize the measurement costs, we must select the smallest sample size for which this inequality holds, i.e., select  $n \approx \frac{k_0^2 \cdot \sigma_0^2}{\Delta_0^2}$ .

**Case of realistic measuring instruments: description.** In practice, measuring instruments are not perfect, they have measurement errors. Usually, we assume that we know the standard deviation  $\sigma$  of the corresponding measurement error, and we know the upper bound  $\Delta$  on the possible values of the mean (systematic) error  $\Delta - s$ :  $|\Delta_s| \leq \Delta$ ; see, e.g., [14].

**Case of realistic measuring instruments: analysis.** For realistic measuring instruments, for each measurement, the difference  $\Delta x_i = \tilde{x}_i - x_i$  between the measured and actual values of the quantity of interest is no longer negligible.

In this case, based on  $n$  measurement results  $\tilde{x}_1, \dots, \tilde{x}_n$ , we do not get the arithmetic average  $E$  of the *actual* values, we only get the average

$$\tilde{E} = \frac{\tilde{x}_1 + \dots + \tilde{x}_n}{n}$$

of the *measured* values. We are using this average  $\tilde{E}$  as an estimate for the desired average  $x_0$ . There are two reasons why  $\tilde{E}$  is different from  $x_0$ :

- first, due to measurement errors,  $\tilde{x}_i \neq x_i$ , hence  $\tilde{E} \neq E$ ;
- second, due to the finite sample size,  $E \neq x_0$ .

As a result, the error  $\Delta x_0$  with which this procedure measures  $x_0$ , i.e., the difference  $\Delta x_0 \stackrel{\text{def}}{=} \tilde{E} - x_0$ , can be represented as the sum of two error components:

$$\tilde{E} - x_0 = (\tilde{E} - E) + (E - x_0). \quad (1)$$

If we use a measuring instrument whose mean (systematic) error is  $\Delta_s$  and standard deviation is  $\sigma$ , then for the difference of arithmetic averages, the mean is the same value  $\Delta_s$  (systematic error) and the standard deviation is  $\sqrt{n}$  times smaller: it is equal to  $\sigma/\sqrt{n}$ . We have just described the difference  $E - x_0$ : it is a random variable with 0 mean and standard deviation  $\sigma_0/\sqrt{n}$ .

Since the mean value of  $E - x_0$  is 0 (by definition of  $x_0$  as the mean of  $x_i$ ), the mean value of the sum (1) is equal to the mean value of the first error component, i.e., to  $\Delta_s$ .

It is reasonable to assume that the measurement errors  $\tilde{x}_i - x_i$  (caused by the imperfections of the measurement procedure) and the deviations  $x_i - x_0$  (caused by variability of the quantity of interest) are independent random variables. In this case, the variance of the sum (1) is equal to the sum of the corresponding variances, i.e., to

$$\frac{\sigma^2}{n} + \frac{\sigma_0^2}{n} = \frac{\sigma_t^2}{n},$$

where we denoted  $\sigma_t \stackrel{\text{def}}{=} \sqrt{\sigma^2 + \sigma_0^2}$ . Hence, the standard deviation of the total error is equal to  $\sigma_t/\sqrt{n}$ .

So, the measurement error  $\tilde{E} - x_0$  is approximately normally distributed, with the mean  $\Delta_s$  (about which we know that  $|\Delta_s| \leq \Delta$ ) and the standard deviation  $\sigma_t/\sqrt{n}$ . Thus, we can conclude that with a selected degree of confidence, the overall error cannot exceed  $\Delta + k_0 \cdot \frac{\sigma_t}{\sqrt{n}}$ .

**Case of realistic measuring instruments: recommendations.** From the purely mathematical viewpoint, when the standard deviation  $\sigma$  of a measuring instrument is fixed, then, to determine  $\Delta$  and  $n$ , we get exactly the same formulas as in the case of static measurements, with the only difference that:

- instead of the standard deviation  $\sigma$  of the random error component of the measuring instrument,
- we now have the combined standard deviation  $\sigma_t = \sqrt{\sigma^2 + \sigma_0^2}$  of the measuring instrument and of the measured quantity.

So, all the recommendations that we have developed in [11] for static measurements are also applicable here.

**Example.** If we want to achieve a given accuracy  $\Delta_0$  with the smallest possible cost, then, according to [11], we should use the measuring instrument with accuracy  $\Delta \approx (1/3) \cdot \Delta_0$ . The sample size  $n$  is then determined by the formula  $k_0 \cdot \frac{\sigma_t}{\sqrt{n}} = (2/3) \cdot \Delta_0$ .

For measuring average, the optimal accuracy  $\Delta$  is the same as for static measurements, but the optimal sample size is now determined by a new formula  $n_{\text{opt}} = \frac{9 \cdot k_0^2 \cdot \sigma_t^2}{4 \cdot \Delta_0^2}$ , with  $\sigma_t$  instead of  $\sigma$ . Since  $\sigma_t > \sigma$ , we will need a larger sample size  $n$ .

### 3 Second objective: measuring the actual dependence of the measured quantity on space location and/or on time

**Formulation of the problem.** In many real-life situations, we are interested not only in the average value of the measured quantity  $x$ , we are also interested in the actual dependence of this quantity on space and/or time.

Within this general scheme, there are several possible situations:

- We may have a quantity that does not depend on a spatial location but does depend on time – e.g., we may be interested in the temperature at a given location. In this case, we are interested to learn how this quantity  $x$  depends on the time  $t$ , i.e., we are interested to know the dependence  $x(t)$ .
- We may be interested in a quantity that does not change with time but does change from one spatial location to the other. For example:
  - in a geographic analysis, we may be interested in how the elevation  $x$  depends on the 2-D spatial location  $t = (t_1, t_2)$ ;
  - in a geophysical analysis, we may be interested how in the density depends on a 3-D location  $t = (t_1, t_2, t_3)$  inside the Earth.
- Finally, we may be interested in a quantity that changes both with time and from one spatial location to the other. For example:
  - we may be interested in learning how the surface temperature depends on time  $t_1$  and on the 2-D spatial location  $(t_2, t_3)$ ;
  - we may be also interested in learning how the general temperature in the atmosphere depends on time  $t_1$  and on the 3-D spatial location  $(t_2, t_3, t_4)$ .

In all these cases, we are interested to know the dependence  $x(t)$  of a measured quantity on the point  $t = (t_1, \dots, t_d)$  in  $d$ -dimensional space, where the dimension  $d$  ranges from 1 (for the case when we have a quantity depending on time) to 4 (for the case when we are interested in the dependence both on time and on the 3-D spatial location).

#### Measurement inaccuracy caused by the finiteness of the sample.

In practice, we can only measure the values of  $x$  at finitely many different locations, and we must use extrapolation to find the values at other locations. So, even if we use a perfect measuring instrument, for which the measurement error can be ignored, we still have an error cause by extrapolation.

For example, suppose that we have measured the values  $x(t^{(i)})$  of the quantity  $x$  at moments of time  $t^{(1)} < t^{(2)} < \dots < t^{(n)}$ , and we want to

describe the value  $x(t)$  of this quantity at a different moment of time  $t \neq t^{(i)}$ , a moment of time at which no measurement has been made.

In practice, for most systems, we know the limit  $g$  on how fast the value of the quantity  $x$  can change with time (or from one spatial location to the other). So, when, e.g.,  $t^{(1)} < t < t^{(2)}$ , we can conclude that  $|x(t) - x(t^{(1)})| \leq g \cdot |t - t^{(1)}|$ , i.e., that  $x(t) \in [x(t^{(1)}) - g \cdot |t - t^{(1)}|, x(t^{(1)}) + g \cdot |t - t^{(1)}|]$ . Thus, even when we have an ideal measuring instrument, the fact that we only have a finite sample  $t^{(1)}, \dots, t^{(n)}$  leads to uncertainty in our knowledge of the values  $x(t)$  for  $t \neq x^{(i)}$ .

**Estimate of the measurement uncertainty for a given measurement accuracy and given sample size.** Let us consider a general situation when we perform measurements with a guaranteed accuracy  $\Delta$ , and when we measure the quantity  $x$  at  $n$  different points  $t^{(1)}, \dots, t^{(n)}$  in the  $d$ -dimensional space. As a result of this measurement, we get  $n$  values  $\tilde{x}_i$  that are  $\Delta$ -close to the actual values of the quantity  $x$  at the corresponding point  $t^{(i)}$ :  $|\tilde{x}_i - x(t^{(i)})| \leq \Delta$ .

If we are interested in the value  $x(t)$  of the quantity  $x$  at a point  $t \neq t^{(i)}$ , then we have to use one of the measured values  $\tilde{x}_i$ .

We assume that we know the rate  $g$  with which  $x(t)$  changes with  $t$ . Thus, if we use the result  $\tilde{x}_i$  of measuring  $x(t^{(i)})$  to estimate  $x(t)$ , we can guarantee that  $|x(t^{(i)}) - x(t)| \leq g \cdot \rho(t, t^{(i)})$ , where  $\rho(a, b)$  denotes the distance between the two points in the  $d$ -dimensional space. Since  $|\tilde{x}_i - x(t^{(i)})| \leq \Delta$ , we can thus conclude that  $|\tilde{x}_i - x(t)| \leq |\tilde{x}_i - x(t^{(i)})| + |x(t^{(i)}) - x(t)| \leq \Delta + g \cdot \rho(t, t^{(i)})$ , i.e.,

$$|\tilde{x}_i - x(t)| \leq \Delta + g \cdot \rho(t, t^{(i)}). \quad (2)$$

Thus, the smaller the distance between  $t$  and  $t^{(i)}$ , the smaller the resulting error. So, to get the most accurate estimate for  $x(t)$ , we must select, for this estimate, the point  $t^{(i)}$  which is the closest to  $t$ .

In general, once we fix the accuracy  $\Delta$ , the sample size  $n$ , and the points  $t^{(1)}, \dots, t^{(n)}$  at which the measurement are performed, we can guarantee that for every  $t$ , the value  $x(t)$  can be reconstructed with the accuracy  $\Delta + g \cdot \rho_0$ , where  $\rho_0$  is the largest possible distance between a point  $t$  and the sample set  $\{t^{(1)}, \dots, t^{(n)}\}$ .

Thus, once we fixed  $\Delta$  and  $n$ , we should select the points  $t^{(i)}$  in such a way that this “largest distance”  $\rho_0$  attains the smallest possible value.

In the 1-D case, the corresponding allocation is easy to describe. Indeed, suppose that we want to allocate such points  $t^{(i)}$  on the interval  $[0, T]$ . We want to minimize the distance  $\rho_0$  corresponding to a given sample size  $n$  – or, equivalently, to minimize the sample size given a distance  $\rho_0$ . Every point  $t$  is  $\rho_0$ -close to one of the sample points  $t^{(i)}$ , so it belongs to the corresponding interval

$$[t^{(i)} - \rho_0, t^{(i)} + \rho_0].$$

Thus, the interval  $[0, T]$  of width  $T$  is covered by the union of  $n$  intervals  $[t^{(i)} - \rho_0, t^{(i)} + \rho_0]$  of widths  $2\rho_0$ . The width  $T$  of the covered interval cannot

exceed the sum of the widths of the covering intervals, so we have  $T \leq n \cdot (2\rho_0)$ , hence always  $\rho_0 \geq T/(2n)$ . Actually, we can have  $\rho_0 = T/2n$  if we select the points  $t^{(i)} = (i - 0.5) \cdot (T/n)$ . Then:

- for the values  $t \in [0, T/n]$ , we take, as the estimate for  $x(t)$ , the result  $\tilde{x}_1$  of measuring  $x(t^{(1)}) = x(T/(2n))$ ;
- for the values  $t \in [T/n, 2T/n]$ , we take, as the estimate for  $x(t)$ , the result  $\tilde{x}_2$  of measuring  $x(t^{(2)}) = x((3/2) \cdot (T/n))$ ;
- ...
- for the values  $t \in [(i - 1) \cdot T/n, i \cdot T/n]$ , we take, as the estimate for  $x(t)$ , the result  $\tilde{x}_i$  of measuring  $x(t^{(i)}) = x((i - 1/2) \cdot (T/n))$ ;
- ...

So, the optimal location of points is when they are on a grid  $t^{(1)} = 0.5 \cdot T/n$ ,  $t^{(2)} = 1.5 \cdot T/n$ ,  $t^{(3)} = 2.5 \cdot T/n$ , ..., and each point  $t^{(i)}$  “serves” the values  $t$  from the corresponding interval  $[(i - 1) \cdot T/n, i \cdot T/n]$  (the interval that contains this point  $t^{(i)}$  as its center), serves in the sense that for each point  $t$  from this interval, as the measured value of  $x(t)$ , we take the value  $x^{(i)}$ . These intervals corresponding to individual points  $t^{(i)}$  cover the entire interval  $[0, T]$  without intersection,

In this optimal location, when we perform  $n$  measurements, we get  $\rho_0 = T/(2n)$ .

Similarly, in the general  $d$ -dimensional case, we can place  $n$  points on a  $d$ -dimensional grid. In this case, each point  $t^{(i)}$  “serves” the corresponding cube; these cubes cover the whole domain without intersection. If we denote, by  $V$ , the  $d$ -dimensional volume of the spatial (or spatio-temporal) domain that we want to cover, then we can conclude that each point  $x^{(i)}$  serves the cube of volume  $V/n$ . Since the volume of a  $d$ -dimensional cube of linear size  $\Delta t$  is equal to  $(\Delta t)^d$ , we can thus conclude that the linear size of each of the cubes served by a measurement point is  $(V/n)^{1/d}$ .

Within this cube, each point  $t^{(i)}$  is located at the center of the corresponding cube. Thus, for each point  $t$  within this cube and for each coordinate  $j$ , the absolute value  $|t_j - t_j^{(i)}|$  between the  $j$ -th coordinate of this point  $t$  and the  $j$ -th coordinate of the cube’s center  $t^{(i)}$  does not exceed one half of the cube’s linear size:  $|t_j - t_j^{(i)}| \leq (1/2) \cdot (V/n)^{1/d}$ . Therefore, for

$$\rho(t, t^{(i)}) = \sqrt{(t_1 - t_1^{(i)})^2 + \dots + (t_d - t_d^{(i)})^2},$$

we get

$$\rho(t, t^{(i)}) \leq \rho \stackrel{\text{def}}{=} \sqrt{d \cdot \left( \frac{1}{2} \cdot \left( \frac{V}{n} \right)^{1/d} \right)^2} = \sqrt{d} \cdot \frac{1}{2} \cdot \frac{V^{1/d}}{n^{1/d}}.$$

We have already mentioned that for every point  $t$ , the accuracy with which we can reconstruct  $x(t)$  is bounded by the value  $\Delta + g \cdot \rho_0$ . Thus, this accuracy

$$\text{is bounded by } \Delta + g \cdot \sqrt{d} \cdot \frac{1}{2} \cdot \frac{V^{1/d}}{n^{1/d}}.$$

We are now ready to formally describe the corresponding trade-off problems.

**Trade-off problems for engineering and science: formulation.** In engineering applications, we know the overall accuracy  $\Delta_0$ , and we want to minimize the cost of the resulting measurement. In this case, the trade-off problem takes the following form:

$$\text{Minimize } n \cdot F(\Delta) \rightarrow \min_{\Delta, n} \text{ under the constraint } \Delta + \frac{g_0}{n^{1/d}} = \Delta_0, \quad (3)$$

where  $F(\Delta)$  is a cost of a single measurement made by a measuring instrument with accuracy  $\Delta$ , and we denoted

$$g_0 \stackrel{\text{def}}{=} g \cdot \sqrt{d} \cdot \frac{1}{2} \cdot V^{1/d}. \quad (4)$$

In scientific applications, when we are given the cost  $F_0$ , and the problem is to achieve the highest possible accuracy within this cost. In this case, we arrive at the following problem

$$\text{Minimize } \Delta + \frac{g_0}{n^{1/d}} \rightarrow \min_{\Delta, n} \text{ under the constraint } n \cdot F(\Delta) = F_0. \quad (5)$$

**Engineering situation: solution.** For the basic cost model  $F(\Delta) = c/\Delta$  [11], the engineering problem (3) has the following solution:

$$\Delta_{\text{opt}} = \frac{1}{d+1} \cdot \Delta_0; \quad n_{\text{opt}} = \left( \frac{g_0}{\Delta_0} \cdot \frac{d+1}{d} \right)^d. \quad (6)$$

Similarly to the static case [11], the optimal trade-off between accuracy and the sample size is attained when both error components are of approximately the same size.

**Science situation: solution.** For the basic cost model  $F(\Delta) = c/\Delta$ , the science problem (5) has the following solution:

$$n_{\text{opt}} = \left( \frac{F_0}{c} \cdot \frac{g_0}{d} \right)^{d/(d+1)}; \quad \Delta_{\text{opt}} = \frac{n_{\text{opt}} \cdot c}{F_0}. \quad (7)$$

In this case too, in the optimal trade-off, the error bound coming from the accuracy of individual measurements is approximately equal to the error bound coming from the finiteness of the sample.

**Case of non-smooth processes: how to describe them.** In the above text, we considered the case the dependence of the quantity  $x$  on time and/or space  $t$  is smooth. In this case, for small changes  $\Delta t$ , this dependence can be approximately described by a linear function  $x(t + \Delta t) = x(t) + g_1 \cdot \Delta t_1 +$



$\dots + g_d \cdot \Delta t_d$ . So, if we know the upper bound  $g$  on the length  $\|(g_1, \dots, g_d)\|$  of the gradient of  $x(t)$ , we can bound the difference  $x(t + \Delta t) - x(t)$  between the values of the quantity  $x$  at close points  $t + \Delta t$  and  $t$  by the product  $g \cdot \|\Delta t\| = g \cdot \rho(t, t + \Delta t)$ .

In practice, we often encounter non-smooth processes. For example, meteorological data exhibit random change (similar to the Brownian motion); as the result of this, the dependence of the corresponding quantities  $x$  on time and spatial coordinates is not smooth.

For the particular case of a Brownian motion, the difference between the values of the quantity  $x$  at nearby points grows as the square root of the distance between these points:  $|x(t + \Delta t) - x(t)| \leq C \cdot \|\Delta t\|^{1/2}$  for some real number  $C$ . In many physical processes, this dependence can be described by a more general power law, i.e.,  $|x(t + \Delta t) - x(t)| \leq C \cdot \|\Delta t\|^\beta$  for some real numbers  $C$  and  $\beta \in (0, 1)$ . Such processes are a particular case of *fractals*; see, e.g., [9] (This notion is closely related with the notion of a fractal dimension: namely, the graph of the corresponding dependence  $x(t)$  has a fractal dimension  $d + (1 - \beta)$ .)

In [10], it is explained why scale invariance naturally leads to the power law – and thus, to the fractal dependence.

**Measurement errors in the case of non-smooth processes.** Let us use these formulas to estimate measurement errors for the case of non-smooth processes. We have already mentioned that if we perform (appropriately located)  $n$  measurements in a  $d$ -dimensional space, then the distance from each point  $t$  of the domain of interest to one of the points  $t^{(i)}$  in which the measurement was made does not exceed  $\rho_0 = \sqrt{d} \cdot \frac{1}{2} \cdot \frac{V^{1/d}}{n^{1/d}}$ .

In the fractal case, we can conclude that the error of approximating the desired value  $x(t)$  with the measured value  $x(t^{(i)})$  does not exceed  $C \cdot \rho^\beta$ . Thus, if we perform  $n$  measurements with a measuring device of accuracy  $\Delta$ , the resulting accuracy in reconstructing all the values of  $x(t)$  is bounded by the value

$$\Delta + C \cdot \rho_0^\beta = \Delta + C \cdot d^{\beta/2} \cdot \frac{1}{2^\beta} \cdot \frac{V^{\beta/d}}{n^{\beta/d}} = \Delta + \frac{g_\beta}{n^{\beta/d}},$$

where we denoted

$$g_\beta \stackrel{\text{def}}{=} C \cdot d^{\beta/2} \cdot \frac{1}{2^{\beta/d}} \cdot V^{\beta/d}.$$

**Trade-off problems for engineering and science: formulation and solution.** In the situation when we know the overall accuracy  $\Delta_0$ , and we want to minimize the cost of the resulting measurement, the trade-off problem takes the following form:

$$\text{Minimize } n \cdot F(\Delta) \text{ under the constraint } \Delta + \frac{g_\beta}{n^{\beta/d}} = \Delta_0. \quad (8)$$

In the situation when we are given the limit  $F_0$  on the cost, and the problem is to achieve the highest possible accuracy within this cost, we arrive at the following problem

$$\text{Minimize } \Delta + \frac{g_\beta}{n^{\beta/d}} \text{ under the constraint } n \cdot F(\Delta) = F_0. \quad (9)$$

From the mathematical viewpoint, these formulas are similar to the formulas corresponding to the smooth case, with the only difference that instead of raising  $n$  to the power  $1/d$ , we now raise  $n$  to the power  $1/d'$ , where  $d' \stackrel{\text{def}}{=} d/\beta$ .

Thus, for the basic cost model  $F(\Delta) = c/\Delta$  [11], the engineering problem has the following solution:

$$\Delta_{\text{opt}} = \frac{\beta}{d + \beta} \cdot \Delta_0; \quad n_{\text{opt}} = \left( \frac{g_\beta}{\Delta_0} \cdot \frac{d + \beta}{d} \right)^d. \quad (10)$$

For the basic cost model  $F(\Delta) = c/\Delta$ , the science problem has the following solution:

$$n_{\text{opt}} = \left( \frac{F_0}{c} \cdot \frac{g_\beta}{d} \right)^{d/(d+\beta)}; \quad \Delta_{\text{opt}} = \frac{n_{\text{opt}} \cdot c}{F_0}. \quad (11)$$

in this case too, in the optimal trade-off, both error components are of approximately the same value.

**Case of more accurate measuring instruments.** In [11], we have shown that for more accurate measuring instrument, the cost  $F(\Delta)$  of a measurement depends on its accuracy as  $F(\Delta) = c/\Delta^\alpha$ . Once we go beyond the basic cost model  $\alpha = 1$ , we get  $\alpha = 3$ , and then, as we increase accuracy, we switch to a different value  $\alpha$ .

For such a power law, in the engineering case, the optimal accuracy is  $\Delta_{\text{opt}} = \frac{\alpha}{\alpha + 2} \cdot \Delta_0$ . In particular, for  $\alpha = 3$ , we have  $\Delta_{\text{opt}} = \frac{3}{5} \cdot \Delta_0$ .

#### 4 Case study: in brief

A real-life example in which we used similar arguments to made a selection between the accuracy and the sample size is the design of radioastronomical telescope system [1, 2, 3, 4, 5, 7, 8]. As we have mentioned, for the radiotelescope of diameter  $D$ , the measurement accuracy is proportional to  $\lambda/D$ , and the cost is proportional to  $D^4$ .

The design of a large system of radiotelescopes has several objectives:

- first, we would like to solve *radioastrometry* problems, i.e., determine the location of the radiosources with as much accuracy as possible;
- second, we would like to solve the *radioimaging* problems, i.e., for each of the radiosources, we would like to know not only its location, but also its *image* – i.e., how the intensity (and polarization) of the source changes from one point of this source to the other.

In the first problem, we are interested in measuring a well-defined unchanging quantity. In the second problem, we are interested in finding the actual dependence of the measured quantity on the spatial location.

In the second problem, similar to what we discussed in the general case, the more samples we take (i.e., the more telescopes we build), the more points we will get on the image. On the other hand, within a given overall cost, if we build more telescopes, then the amount of money allocated to each telescope will be smaller, so each telescope will be small ( $D' \ll D$ ), and the resulting accuracy  $\Delta \sim 1/D$  of each of the many measurements will be not so good.

In our analysis, we have found an optimal trade-off between accuracy and sample size. This analysis was used in the design of the successful Russian network of radiotelescopes.

## 5 Conclusions

In general, if the measurement error consists of several components, then the optimal trade-off between the accuracy  $\Delta$  and the same size  $n$  occurs when these components are approximately of the same size.

In particular, if we want to achieve the overall accuracy  $\Delta_0$ , as a first approximation, it is reasonable to take  $\Delta = \Delta_0/2$  – and select the sample size for which the resulting overall error is  $\Delta_0$ .

A more accurate description of optimal selections in different situations is as follows:

- for the case when we measure a single well-defined quantity (or the average value of varying quantity), we should take  $\Delta = \frac{1}{3} \cdot \Delta_0$ ;
- for the case when are interested in reconstructing all the values  $x(t)$  of a smooth quantity  $x$  depending on  $d$  parameters  $t = (t_1, \dots, t_d)$ , we should take  $\Delta = \frac{1}{d+1} \cdot \Delta_0$ ;
- for the case when are interested in reconstructing all the values  $x(t)$  of a non-smooth quantity  $x$  depending on  $d$  parameters  $t = (t_1, \dots, t_d)$ , we should take  $\Delta = \frac{\beta}{d+\beta} \cdot \Delta_0$ , where  $\beta$  is the exponent of the power law that describes how the difference  $x(t + \Delta t) - x(t)$  changes with  $\|\Delta t\|$ .

For the case of more accurate measuring instruments, when the cost  $F(\Delta)$  of a single measurement starts growing as  $c/\Delta^3$ , we should take  $\Delta = \frac{3}{5} \cdot \Delta_0$ . In general, if  $F(\Delta) = c/\Delta^\alpha$ , we should take  $\Delta = \frac{\alpha}{\alpha+2} \cdot \Delta_0$ .

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## References

1. A. Dravskikh, A.M. Finkelstein, V. Kreinovich, “Astrometric and geodetic applications of VLBI ‘arc method’”, *Modern Astrometry, Proceedings of the IAU Colloquium No. 48*, Vienna, 1978, 143–153.
2. A. F. Dravskikh, O. M. Kosheleva, A. M. Finkelstein, P. A. Fridman, “Possibility of using a reference-object method to form a phase-stable multielement long-baseline interferometric system”, *Bulletin of the Special Astrophysical Observatory – North Caucasus*, Allerton Press, N. Y., Vol. 16, 1984, pp. 72–80.
3. A.F. Dravskikh, O.M. Kosheleva, V. Kreinovich, A.M. Finkelstein, “The method of arcs and differential astrometry,” *Soviet Astronomy Letters*, 1979, Vol. 5, No. 3, pp. 160–162.
4. A. F. Dravskikh, O. M. Kosheleva, V. Ya. Kreinovich, A. M. Finkelstein, “Optimization of the procedure for measuring arcs by radiointerferometry”, *Soviet Astronomy Letters*, 1979, Vol. 5, No. 4, pp. 227–228.
5. V. Kreinovich, A. Bernat, O. Kosheleva, A. Finkelstein, “Interval estimates for closure phase and closure amplitude imaging in radio astronomy”, *Interval Computations*, 1992, No. 2(4), pp. 51–71.
6. V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational complexity and feasibility of data processing and interval computations*, Kluwer, Dordrecht, 1997.
7. V. Kreinovich, S. A. Starks, D. Iourinski, O. Kosheleva, and A. Finkelstein, “Open-ended configurations of radio telescopes: a geometrical analysis”, *Combinatorics*, 2003, Vol. 13, No. 2, pp. 79–85.
8. V. Kreinovich, S. A. Starks, O. Kosheleva, and A. Finkelstein, “Open-ended configurations of radio telescopes: towards optimal design”, *Proceedings of the 2002 World Automation Congress WAC’2002*, Orlando, Florida, June 9–13, 2002, pp. 101–106.
9. B. B. Mandelbrot, *The fractal geometry of Nature*, Freeman, San Francisco, 1982.
10. H. T. Nguyen and V. Kreinovich, *Applications of continuous mathematics to computer science*, Kluwer, Dordrecht, 1997.
11. H. T. Nguyen and V. Kreinovich, “Trade-off between sample size and accuracy: case of dynamic measurements”, submitted to these Proceedings.
12. P. V. Novitskii and I. A. Zograph, *Estimating the Measurement Errors*, Energoatomizdat, Leningrad, 1991 (in Russian).
13. A. I. Orlov, “How often are the observations normal?”, *Industrial Laboratory*, 1991, Vol. 57. No. 7, pp. 770–772.
14. S. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer-Verlag, New York, 2005.
15. D. J. Sheskin, *Handbook of Parametric and Nonparametric Statistical Procedures*, Chapman & Hall/CRC, Boca Raton, Florida, 2004.
16. D. Stevens, “Analysis of biological systems”, *Proceedings of the NIH MARC Winter Institute on Undergraduate Education in Biology*, Santa Cruz, California, January 7–11, 2005.
17. G. L. Verschuur and K. I. Kellerman, “Galactic and Extragalactic Radio Astronomy”, Springer-Verlag, N.Y., 1988.