Selecting the Best Location for a Meteorological Tower: A Case Study of Multi-Objective Constraint Optimization

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Abstract. Using the problem of selecting the best location for a meteorological tower as an example, we show that in multi-objective optimization under constraints, the traditional weighted average approach is often inadequate. We also show that natural invariance requirements lead to a more adequate approach – a generalization of Nash's bargaining solution.

Case study. We want to select the best location of a sophisticated multi-sensor meteorological tower. We have several criteria to satisfy.

For example, the station should not be located too close to a road, so that the gas flux generated by the cars do not influence our measurements of atmospheric fluxes; in other words, the distance x_1 to the road should be larger than a certain threshold t_1: $x_1 > t_1$, or $y_1 \stackrel{\text{def}}{=} x_1 - t_1 > 0$.

Also, the inclination x_2 at the should be smaller than a corresponding threshold t_2, because otherwise, the flux will be mostly determined by this inclination and will not be reflective of the atmospheric processes: $x_2 < t_2$, or $y_2 \stackrel{\text{def}}{=} t_2 - x_2 > 0$.

General case. In general, we have several such differences $y_1, \ldots, y_n$ all of which have to be non-negative. For each of the differences y_i, the larger its value, the better.

Multi-criteria optimization. Our problem is a typical setting for *multi-criteria optimization*; see, e.g., [1, 4, 5].

Weighted average. A most widely used approach to multi-criteria optimization is *weighted average*, where we assign weights $w_1, \ldots, w_n > 0$ to different criteria y_i and select an alternative for which the weighted average $w_1 \cdot y_1 + \ldots + w_n \cdot y_n$ attains the largest possible value.

Additional requirement. In our problem, we have an additional requirement – that all the values y_i must be positive. Thus, we must only compare solutions with $y_i > 0$ when selecting an alternative with the largest possible value of the weighted average.

Limitations of the weighted average approach. In general, the weighted average approach often leads to reasonable solutions of the multi-criteria optimization problem. However, as we will show, in the presence of the additional positivity requirement, the weighted average approach is not fully satisfactory.

A practical multi-criteria optimization must take into account that measurements are not absolutely accurate. Indeed, the values y_i come from measurements, and measurements are never absolutely accurate. The results $\widetilde{y}_i$ of the measurements are close to the actual (unknown) values y_i of the measured quantities, but they are not exactly equal to these values. If

- we measure the values y_i with higher and higher accuracy and,
- based on the resulting measurement results $\widetilde{y}_i$, we conclude that the alternative $y = (y_1, \ldots, y_n)$ is better than some other alternative $y' = (y'_1, \ldots, y'_n)$,

then we expect that the actual alternative y is indeed either better than y' or at least of the same quality as y'. Otherwise, if we do not make this assumption, we will not be able to make any meaningful conclusions based on real-life (approximate) measurements.

The above natural requirement is not always satisfied for weighted average. Let us show that for the weighted average, this "continuity" requirement is not satisfied even in the simplest case when we have only two criteria y_1 and y_2. Indeed, let $w_1 > 0$ and $w_2 > 0$ be the weights corresponding to these two criteria. Then, the resulting strict preference relation $\succ$ has the following properties:

- if $y_1 > 0$, $y_2 > 0$, $y'_1 > 0$, and $y'_2 > 0$, and $w_1 \cdot y_1 + w_2 \cdot y_2 > w_1 \cdot y'_1 + w_2 \cdot y'_2$, then
$$y = (y_1, y_2) \succ y' = (y'_1, y'_2);$$
- if $y_1 > 0$, $y_2 > 0$, and at least one of the values y'_1 and y'_2 is non-positive, then
$$y = (y_1, y_2) \succ y' = (y'_1, y'_2).$$

Let us consider, for every $\varepsilon > 0$, the tuple $y(\varepsilon) \stackrel{\text{def}}{=} \left(\varepsilon, 1 + \dfrac{w_1}{w_2}\right)$, with $y_1(\varepsilon) = \varepsilon$ and $y_2(\varepsilon) = 1 + \dfrac{w_1}{w_2}$, and also the comparison tuple $y' = (1, 1)$. In this case, for every $\varepsilon > 0$, we have

$$w_1 \cdot y_1(\varepsilon) + w_2 \cdot y_2(\varepsilon) = w_1 \cdot \varepsilon + w_2 + w_2 \cdot \frac{w_1}{w_2} = w_1 \cdot (1 + \varepsilon) + w_2$$

and

$$w_1 \cdot y'_1 + w_2 \cdot y'_2 = w_1 + w_2,$$

hence $y(\varepsilon) \succ y'$. However, in the limit $\varepsilon \to 0$, we have $y(0) = \left(0, 1 + \dfrac{w_1}{w_2}\right)$, with $y(0)_1 = 0$ and thus, $y(0) \prec y'$.

What we want: a precise description. We want to be able to compare different alternatives.

Each alternative is characterized by a tuple of n values $y = (y_1, \ldots, y_n)$, and only alternatives for which all the values y_i are positive are allowed. Thus, from the mathematical viewpoint, the set of all alternatives is the set $(R^+)^n$ of all the tuples of positive numbers.

For each two alternatives y and y', we want to tell whether y is better than y' (we will denote it by $y \succ y'$ or $y' \prec y$), or y' is better than y ($y' \succ y$), or y and y' are equally good ($y' \sim y$). These relations must satisfy natural properties. For example, if y is better than y' and y' is better than y'', then y is better than y''. In other words, the relation $\succ$ must be transitive. Similarly, the relation $\sim$ must be transitive, symmetric, and reflexive ($y \sim y$), i.e., in mathematical terms, an *equivalence relation.*

So, we want to define a pair of relations $\succ$ and $\sim$ such that $\succ$ is transitive, $\sim$ is transitive, $\sim$ is an equivalence relation, and for every y and y', one and only one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.

It is also reasonable to require that if each criterion is better, then the alternative is better as well, i.e., that if $y_i > y'_i$ for all i, then $y \succ y'$.

Comment. Pairs of relations of the above type can be alternatively characterized by a *pre-ordering* relation

$$a \succeq b \Leftrightarrow (a \succ b \vee a \sim b).$$

This relation must be transitive and – in our case – total (i.e., for every y and y', we have $y \succeq y' \vee y' \succeq y$. Once we know the pre-ordering relation $\succeq$, we can reconstruct $\succ$ and $\sim$ as follows:

$$y \succ y' \Leftrightarrow (y \succeq y' \,\&\, y' \not\succeq y);$$

$$y \sim y' \Leftrightarrow (y \succeq y' \,\&\, y' \succeq y).$$

Scale invariance: motivation. The quantities y_i describe completely different physical notions, measured in completely different units. In our meteorological case, some of these values are wind velocities measured in meters per second, or in kilometers per hour, or miles per hour. Other values are elevations described in meters, kilometers, or feet, etc. Each of these quantities can be described in many different units. A priori, we do not know which units match each other, so it is reasonable to assume that the units used for measuring different quantities may not be exactly matched.

It is therefore reasonable to require that the relations $\succ$ and $\sim$ between the two alternatives $y = (y_1, \ldots, y_n)$ and $y' = (y'_1, \ldots, y'_n)$ do not change if we simply change the units in which we measure each of the corresponding n quantities.

Scale invariance: towards a precise description. When we replace a unit in which we measure a certain quantity q by a new measuring unit which is $\lambda > 0$ times smaller, then the numerical values of this quantity increase by a factor of λ: $q \to \lambda \cdot q$. For example, 1 cm is $\lambda = 100$ times smaller than 1 m, so the length $q = 2$ m, when measured in cm, becomes $\lambda \cdot q = 2 \cdot 100 = 200$ cm.

Let λ_i denote the ratio of the old to the new units corresponding to the i-th quantity. Then, the quantity that had the value y_i in the old units will be described by a numerical value $\lambda_i \cdot y_i$ in the new unit. Therefore, scale-invariance means that for all $y, y \in (R^+)^n$ and for all $\lambda_i > 0$, we have

$$y = (y_1, \ldots, y_n) \succ y' = (y'_1, \ldots, y'_n) \Rightarrow (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \succ (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n)$$

and

$$y = (y_1, \ldots, y_n) \sim y' = (y'_1, \ldots, y'_n) \Rightarrow (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n) \sim (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n).$$

Continuity. As we have mentioned in the previous section, we also want to require that the relations $\succ$ and $\sim$ are *continuous* in the following sense: if $y(\varepsilon) \succeq y'(\varepsilon)$ for every ε, then in the limit, when $y(\varepsilon) \to y(0)$ and $y'(\varepsilon) \to y'(0)$ (in the sense of normal convergence in R^n), we should have $y(0) \succeq y'(0)$.

Let us now describe our requirements in precise terms.

Definition 1. *By a* total pre-ordering relation *on a set Y, we mean a pair of a transitive relation $\succ$ and an equivalence relation $\sim$ for which, for every $y, y' \in Y$, one and only one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.*

Comment. We will denote $y \succeq y' \stackrel{\text{def}}{=} (y \succ y' \vee y \sim y')$.

Definition 2. *We say that a total pre-ordering is* non-trivial *if there exist y and y' for which $y \succ y'$.*

Comment. This definition excludes the trivial pre-ordering in which every two tuples are equivalent to each other.

Definition 3. *We say that a total pre-ordering relation on the set $(R^+)^n$ is:*

- monotonic *if $y'_i > y_i$ for all i implies $y' \succ y$;*
- scale-invariant *if for all $\lambda_i > 0$:*
  - *$(y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n)$ implies $(\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n) \succ (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n)$, and*
  - *$(y'_1, \ldots, y'_n) \sim y = (y_1, \ldots, y_n)$ implies $(\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n) \sim (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n)$.*
- continuous *if whenever we have a sequence $y^{(k)}$ of tuples for which $y^{(k)} \succeq y'$ for some tuple y', and the sequence $y^{(k)}$ tends to a limit y, then $y \succeq y'$.*

Theorem. *Every non-trivial monotonic scale-invariant continuous total pre-ordering relation on $(R^+)^n$ has the following form:*

$$y' = (y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} > \prod_{i=1}^{n} y_i^{\alpha_i};$$

$$y' = (y'_1, \ldots, y'_n) \sim y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} = \prod_{i=1}^{n} y_i^{\alpha_i},$$

for some constants $\alpha_i > 0$.

Comment. In other words, for every non-trivial monotonic scale-invariant continuous total pre-ordering relation on $(R^+)^n$, there exist values $\alpha_1 > 0$, ..., $\alpha_n > 0$ for which the above equivalence hold. Vice versa, for each set of values $\alpha_1 > 0$, ..., $\alpha_n > 0$, the above formulas define a monotonic scale-invariant continuous pre-ordering relation on $(R^+)^n$.

It is worth mentioning that the resulting relation coincides with the asymmetric version [3] of the bargaining solution proposed by the Nobelist John Nash in 1953 [2].

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References

1. Ehrgott, M., Gandibleux X. (eds.): *Multiple Criteria Optimization: State of the Art Annotated Bibliographic Surveys*, Springer Verlag, Berlin-Heidelberg-New York, 2002.
2. Nash, J.: "Two-Person Cooperative Games," *Econometrica*, 21:128–140, 1953
3. Roth, A.: *Axiomatic Models of Bargaining*, Springer-Verlag, Berlin, 1979.
4. Sawaragi, Y., Nakayama, H., Tanino, T.: *Theory of Multiobjective Optimization*, Academic Press, Orlando, Florida, 1985.
5. Steuer, E. E.: *Multiple Criteria Optimization: Theory, Computations, and Application*, John Wiley & Sons, New York, 1986