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Recursive Forms For Determinant Of K-Tridiagonal Toeplitz Matrices

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RECURSIVE FORMS FOR DETERMINANT OF

K-TRIDIAGONAL TOEPLITZ MATRICES

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Dedication

To my

FAMILY and ADVISOR Dr. Dogan

with love

RECURSIVE FORMS FOR DETERMINANT OF K-TRIDIAGONAL TOEPLITZ MATRICES

by

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THESIS

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Abstract

Toeplitz matrices have garnered renewed interest in recent years due to their practical applications in engineering and computational sciences. Additionally, research has shown their connection to other matrices and their significance in matrix theory. For example, one study demonstrated that any matrix can be expressed as the product of Toeplitz matrices (Ye and Lim, 2016), while another showed that any square matrix is similar to a Toeplitz matrix (Mackey et al., 1999).

Numerous studies have examined various properties of Toeplitz matrices, including ideals of lower triangular Toeplitz matrices (Dogan et al., 2018), matrix power computation with band Toeplitz structures (Dogan and Suarez, 2017), and norms of Toeplitz matrices. Moreover, the use of Lucas and Fibonacci numbers has been employed to describe Toeplitz matrix norms (Akbulak and Bozkurt, 2008). With their spectral properties, Toeplitz matrices are crucial to physics, statistics, and signal processing. Furthermore, they aid in the modeling of problems such as computing spline functions, signal and image processing, and polynomial and power series computations (Bini, 1995).

This study investigates recursive forms for the determinants of $k-$tridiagonal Toeplitz matrices. The aim is to extend the known recursions for 1 and 2-tridiagonal Toeplitz matrices. The current research has led to a conjecture on recursive forms for determinants of all $k-$tridiagonal Toeplitz matrices, $k > 2$. The study gives a finding of recursions in two forms: one applying Binomial expansion and the other applying LU-decomposition of matrices.The LU-Decomposition is considered, in the Literature, for $k-$tridiagonal of any matrix but not for Toeplitz matrices. This thesis focused on Toeplitz matrices.

Table of Contents

Chapter 1

Introduction

1.1 Significance and Motivation

1.1.1 Significance

Due to recent advances and discoveries, Toeplitz matrices have become increasingly relevant in the fields of science, engineering, and computerized algorithms.One recent discovery,is that every $n \times n$ matrix can be expressed as a product of $\lfloor \frac{n}{2} \rfloor + 1$ Toeplitz matrices (Ye and Lim, 2016). Here, $\lfloor \frac{n}{2} \rfloor$ stands for the floor function of $\frac{n}{2}$. Additionally, it was shown in the same article that any $n \times n$ matrix can be expressed as a product of at most $2n + 5$ Toeplitz matrices. The authors demonstrated that $\lfloor \frac{n}{2} \rfloor + 1$ is the minimum number of r-Toeplitz matrices required to express any generic $n \times n$ matrix (for more information on r-Toeplitz matrices, see Ye and Lim (2016). Based on this finding, the authors further proved that every $n \times n$ matrix can be expressed as a product of $4r + 1$ Toeplitz matrices, where $r = \lfloor \frac{n}{2} \rfloor$. It is important to note that, unless stated otherwise, this proposal considers complex valued matrices.

Mackey et al. (1999) presented another significant work in this field. These authors proved that every $n \times n$ complex matrix with $n \leq 4$ can be transformed into a Toeplitz matrix by means of a similarity transformation. Specifically, they showed that every $n \times n$ complex nonderogatory matrix can be transformed into a unique upper Hessenberg Toeplitz matrix. Here, nonderogatory means that the matrix has only one linearly independent eigenvector for each eigenvalue or the eigenvalue has a geometric multiplicity of one. The authors utilized the concept of Jordan Canonical form to achieve this. Moreover, they established that any $n \times n$

matrix A with a canonical form and $n \leq 4$ is either nonderogatory or diagonalizable, implying that such a matrix can be transformed into a Toeplitz matrix via similarity transformation.

The presence of Toeplitz matrices in computerized algorithms highlights their significant role in modeling various problems, including the computation of spline functions, statistics, parallel computing, signal and image processing, numerical solutions of differential equations, boundary value problems, interpolation problems, physics, and polynomial and power series computations. As a result, there has been a renewed interest in Toeplitz matrices over the years, and various properties have been studied. For example, studies have been conducted on the ideals of lower triangular Toeplitz matrices (Dogan et al., 2018). Other areas of research include Matrix Power Computation Band Toeplitz Structure (Dogan and Suarez, 2017), norms of Toeplitz matrices, and the use of Lucas and Fibonacci numbers to describe Toeplitz Matrix norms (Akbulak and Bozkurt, 2008). The most pertinent Literature to my thesis includes studies on the recursive forms of determinants for 1 and 2-tridiagonal Toeplitz matrices, (Borowska et al., 2012; Borowska and Łacińska, 2015), respectively. Additionally, Gover's work (1994) on tridiagonal 2-Toeplitz matrices is also relevant to my research.

In the thesis, I investigated recursive forms of the determinant for k-tridiagonal Toeplitz matrices,$k > 2$, building upon the existing work on 1 and 2-tridiagonal Toeplitz matrices.

Toeplitz matrices play a crucial role in STEM fields as well. Various properties of these matrices are studied extensively, including determinants, eigenvalues, eigen vectors, condition numbers, norms, and singular values. Mukherjee and Maiti (1988) noted that positive definite Toeplitz matrices and their spectral properties have applications in econometrics, psychometrics, structural engineering, seismology, and statistics. These authors asserted that Toeplitz matrices can give rise to different matrix structures, such as flip matrix, centrosymmetric matrix, symmetric Toeplitz matrix, and Hankel matrix. Böttcher and Grudsky (2005) studied the spectral properties of band Toeplitz matrices, although their results were asymptotic in nature, and mainly considered larger sized matrices.

The location of zeros associated with the eigenpolynomials in relation to a Hermitian Toeplitz matrix is of high interest to signal processing (Trench, 1994). Toeplitz matrices are also used in various modeling problems (Bini, 1995). These problems include the numerical solution of certain differential and integral equations, computation of spline functions, time series analysis, signal and image processing, Markov chains and queueing theory, and polynomial and power series computations. In short, Toeplitz matrices are essential in scientific research, and any new findings will add significant value to the field. This thesis is adding new information to the field, discussing recursive forms of determinant of $k-$ tridiagonal Toeplitz matrices for $k > 2$.

1.1.2 Motivation-Driving Force

Before I began my thesis,I was experimenting with different determinant forms for various $k-$ tridiagonal Toeplitz matrices. As a result, I identified 4 different patterns listed below.

Conjecture 1.1.1. *Given* $T_n^{(k)}, k > 2$ *and* $n = km + s$ *for all* $0 \leq s < k$.

Then, in the expression:

$$W_n^{(k)} = W_{\left[\frac{n}{k}\right]}^{(1)} W_{n-\left[\frac{n}{k}\right]}^{(k-1)} \tag{1.1}$$

- Case 1

  When $k \in \mathbb{O}$ and $n \in 2\mathbb{N}$. Then,

  $$\left[\frac{n}{k}\right]$$

  is the nearest even integer.

- Case 2

  When $k, n \in \mathbb{O}$. Then,

  $$\left[\frac{n}{k}\right]$$

  is the nearest odd integer.

- Case 3

  When $k \in 2\mathbb{N}$ and $n \in \mathbb{O}$. Then,

  $$\left[\frac{n}{k}\right]$$

is the nearest even integer.

- Case 4 (Special Case)

  Let $T_n^{(k)}, \quad k > 2$ and $n, k \in 2\mathbb{N}$. Then for $n = km + s, \quad 0 \leq s < k$.

$$W_n^{(k)} = \left(W_{\frac{n}{2}}^{(\frac{k}{2})}\right)^2$$

0nce I recognized these patterns among some Toeplitz matrices, I decided to take on the challenge of proving them.Thus, my thesis is the result of this challenge

1.2 Terminology and Symbols

To ensure clarity, let us first define symbols and terminologies utilized throughout the thesis.

- k, n : a natural number.
- M : Any square Matrix.
- M_{ij}: represents the ij entry of any matrix M.
- T_n: $n \times n$ Toeplitz Matrix.
- $M^{(k)}$: $k-$ tridiagonal matrix, $k > 0$.
- $T_n^{(k)}$: $k-$ tridiagonal Toeplitz matrix.
- W_n or $W_n^{(1)}$: determinant of $T_n^{(1)}$. That is $\det T_n^{(1)} = W_n$.
- $W_n^{(k)}$: determinant of $T_n^{(k)}, k > 1$. That is, $\det T_n^{(k)} = W_n^{(k)}, k > 1$.
- R_j: jth row of a matrix.
- $\mathbb{N}$: Set of Natural Numbers. Here, we exclude 0.
- $\mathbb{O}$: Set of odd natural numbers.
- $2\mathbb{N}$: Set of even natural numbers.
- $\mathbb{C}$: Set of Complex Numbers.
- $\left[\frac{n}{k}\right]^e$: The nearest even integer
- $\left[\frac{n}{k}\right]^o$:The nearest odd integer

It is worth noting here that certain theorems discussed in the later sections of the thesis may employ different symbols and terminologies than the ones in section 1.2 . If so, it will be explicitly highlighted.

1.3 Basic Definitions

In this section,fundamental concepts and main ideas are pivoted. These concepts encompass Toeplitz Matrix, Tridiagonal Matrix, and Tridiagonal Toeplitz Matrix. One can find information on determinants and cofactor expansion in Shah and Thakkar (2020, p. 21).

Definition 1.3.1. *A square matrix is considered a* $1-$ *tridiagonal matrix, denoted as* $M^{(1)}$, *when it satisfies the following conditions:*

$$M_{ij}^{(1)} = \begin{cases} a_i & ; \quad i = j \\ b_i & ; \quad j - i = 1 \\ c_j & ; \quad i - j = 1 \\ 0 & otherwise \end{cases} ,$$

for $i, j = 1, 2, \cdots, n$ *and* $a_i, b_i, c_j \in \mathbb{C}$.

A tridiagonal matrix generally exhibits the following structure:

$$M^{(1)} = \begin{bmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & 0 & \\ & c_2 & a_3 & b_3 & & \\ & & c_3 & a_4 & \ddots & \\ & 0 & & \ddots & \ddots & b_{n-1} \\ & & & & c_{n-1} & a_n \end{bmatrix} . \tag{1.2}$$

Example 1.3.2. *This example gives a* 4×4 *tridiagonal matrix.*

$$M^{(1)} = \begin{bmatrix} 1 & 5 & 0 & 0 \\ i & 2 & e & 0 \\ 0 & 4 & 7 & -i \\ 0 & 0 & 1 & \pi \end{bmatrix} .$$

Definition 1.3.3. *An $n \times n$ matrix, denoted as $M_1^{(k)}$, is referred to as a $k-$tridiagonal matrix when it satisfies the following conditions:*

$$M_{ij}^{(k)} = \begin{cases} a_i & ; \quad i = j \\ b_i & ; \quad j - i = k \\ c_j & ; \quad i - j = k \\ 0 & otherwise \end{cases} ,$$

for $i, j = 1, 2, \cdots, n$ and $a_i, b_i, c_j \in \mathbb{C}$. Here, $k \in \mathbb{N}$.

The following is a typical structure of a $k-$ tridiagonal Matrix.

$$M^{(k)} = \begin{bmatrix} a_1 & 0 & \dots & b_1 & & 0 \\ 0 & a_2 & & 0 & \ddots & \vdots \\ \vdots & & & \vdots & & b_{n-k} \\ c_1 & 0 & \dots & a_{k+1} & & \\ & \ddots & & & \ddots & \vdots \\ 0 & \cdots & c_{n-k} & 0 & \cdots & a_n \end{bmatrix} , \tag{1.3}$$

where $1 \leq k < n$.

Remark 1.3.4. *$k = 1$ gives the definition of a Tridiagonal Matrix.*

Example 1.3.5. *This is a 7×7 matrix that shows an example of Definition1.3.3 when $k = 3$.*

$$M^{(3)} = \begin{bmatrix} 1 & 0 & 0 & -\pi & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 3 & 0 \\ i & 0 & 0 & 5 & 0 & 0 & -i \\ 0 & \pi & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -10 & 0 & 0 & 7 \end{bmatrix} .$$

Definition 1.3.6. *A Toeplitz matrix is a matrix T_n with*

$$T_{ij} = T_{i+1,j+1}, \quad \forall i, j = 1, 2, \cdots, n.$$

Or $T_n = [t_{i,j}]$ where $t_{i,j} = t_{i-j}$.

According to Kırklar and Yılmaz (2015), a common structure of a Toeplitz matrix can be represented using the form below.

$$T_n = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & \cdots & a_n \\ a_{-1} & a_0 & a_1 & \ddots & & \vdots \\ a_{-2} & a_{-1} & \ddots & \ddots & & \vdots \\ \ddots & \ddots & \ddots & \ddots & a_1 & a_2 \\ \vdots & & \ddots & \ddots & \ddots & a_1 \\ a_{-n} & \cdots & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix} . \tag{1.4}$$

Example 1.3.7. *This example gives a 3×3 Toeplitz matrix.*

$$T^{(1)} = \begin{bmatrix} 3 & 1 & \pi \\ 2 & 3 & 1 \\ i & 2 & 3 \end{bmatrix}$$

Note that, definitions given next are special cases of definitions (1.3.1) and (1.3.3).

Definition 1.3.8. *A $1-$ tridiagonal Toeplitz matrix, $T_n^{(1)}$ is a $1-$ tridiagonal matrix where $a = a_i, b = b_i, \forall i,$ and $c = c_j, \forall j$. That is,*

$$T_{ij}^{(1)} = \begin{cases} a & ; \quad i = j \\ b & ; \quad j - i = 1 \\ c & ; \quad i - j = 1 \\ 0 & \textit{otherwise} \end{cases} ,$$

for $i, j = 1, 2, \cdots, n$ and $a, b, c \in \mathbb{C}$. Here, $a, b, c \neq 0$.

The following describes a typical form of Definition 1.3.8.

$$T_n^{(1)} = \begin{bmatrix} a & b & & & & \\ c & a & b & & 0 & \\ & c & a & b & & \\ & & \ddots & \ddots & \ddots & \\ & & 0 & \ddots & \ddots & b \\ & & & & c & a \end{bmatrix} . \tag{1.5}$$

Definition 1.3.9. *A k-tridiagonal Toeplitz matrix, $T_n^{(k)}$, is a $k-$ tridiagonal matrix where $a = a_i, b = b_i, \forall i$ and $c = c_j, \forall j$. That is*

$$T_{ij}^{(k)} = \begin{cases} a & ; \quad i = j \\ b & ; \quad j - i = k \\ c & ; \quad i - j = k \\ 0 & \text{otherwise} \end{cases} ,$$

for $i, j = 1, 2, \cdots, n$ and $a, b, c \in \mathbb{C}$. Here, $a, b, c \neq 0$.

The following describes a typical structure of (1.3.9)

$$T_n^{(k)} = \begin{bmatrix} a & 0 & \dots & b & & 0 \\ 0 & a & & 0 & \ddots & \vdots \\ \vdots & & \ddots & \vdots & & b \\ c & 0 & \dots & a & & \\ & \ddots & & & \ddots & \vdots \\ 0 & \cdots & c & 0 & \cdots & a \end{bmatrix} . \tag{1.6}$$

Remark 1.3.10. *$k = 1$ gives the definition of a tridiagonal Toeplitz Matrix.*

Remark 1.3.11. *All the work presented in this thesis is based upon the premise that $W_n^{(k)} \neq 0$ for all $k \geq 1$.*

1.3.1 Organization of the study

The following is a conceptual breakdown of the remaining chapters:

- Chapter 2 delves into the foundational mathematical concepts essential to understanding the subsequent chapters. Here,we explore existing lemmas and theorems related to determinants of Toeplitz matrices.

- Chapter 3 discusses the new findings via lemmas, corollaries, and theorems. Specifically, new ideas are presented on previously established theorems on determinants of Toeplitz matrices. These ideas are considered in the proof of the main conjecture, which ignited my Thesis work.

- In Chapter 4, we present a proof of the main conjecture concerning the recursive forms of the determinant of $k-$ Tridiagonal Toeplitz matrices, $k \geq 2$.

- Finally, in Chapter 5, we discuss potential avenues for future research. We explore possible directions that can build upon the current work and highlight areas that warrant further investigations.

Chapter 2

Preliminaries

This section provides a brief overview of concepts and theories from relevant literature that are closely linked to my study. Specifically, I explore various theorems that furnish the necessary mathematical foundation for my research.

2.1 1-Tridiagonal Toeplitz Matrices

In this section, we delve into several concepts uncovered in articles pertaining to $1-$ tiadiagonal Toeplitz matrices. For instance, Bergum and Hoggatt Jr (1978) elaborated on "A family of Tridiagonal Matrices" and presented a recursive sequence of determinants.

Theorem 2.1.1. *(Bergum and Hoggatt Jr, 1978) For $k = 1$, determinants of T_n are recursively given by:*

$$W_{n+2} = aW_{n+1} - bcW_n$$

for $n \geq 1$, with initial values, $W_0 = 1$ and $W_1 = a$.

Proof. Let us briefly discuss the ideas that were utilized in proving the theorem. Verifying a few terms of the closed form of the sequence, $\{W_n\}_1^\infty$,

$$W_2 = a^2 - bc, \quad W_3 = a^3 - 2abc, \quad W_4 = a^4 - 3a^2bc + b^2c^2.$$

Inductively, one shows,

$$W_{n+2} = aW_{n+1} - bcW_n \quad , \qquad \text{for} \quad n \geq 1.$$

□

Another related work on the topic is presented by Zhang (2011). Zang established recursions for the determinants of 1− Tridiagonal Toeplitz as follows.

Theorem 2.1.2. *(Zhang, 2011, p.133) Let $T_n^{(1)}$ be as given in definition 1.3.8. Then, its determinant is:*

$$W_n = \begin{cases} a^n & if \quad bc = 0 \\ (n+1)(\frac{a}{2})^n & if \quad a^2 = 4bc \\ \frac{(\alpha^{n+1}-\beta^{n+1})}{\alpha-\beta} & if \quad a^2 \neq 4bc \end{cases}$$

where $\alpha = \frac{a+\sqrt{a^2-4bc}}{2}$ *and* $\beta = \frac{a-\sqrt{a^2-4bc}}{2}$

Proof. A brief discussion of the ideas employed in the theorem is given here.

Considering the equation,

$$W_n = aW_{n-1} - bcW_{n-2}. \tag{2.1}$$

The case, $bc = 0$, will give $W_n = a^n$.

For the case $bc \neq 0$, we suppose that α and β are the roots of the monic polynomial

$$x^2 - ax + bc = 0.$$

Algebraically, we can infer that $\alpha + \beta = a$ and $\alpha\beta = bc$ for the roots α and β. Utilizing the discriminant $a^2 - 4bc$, one obtains

$$a^2 - 4bc = (\alpha - \beta)^2. \tag{2.2}$$

Then,

$$W_n - \alpha W_{n-1} = \beta(W_{n-1} - \alpha W_{n-2})$$

and

$$W_n - \beta W_{n-1} = \alpha(W_{n-1} - \beta W_{n-2}).$$

Let's now consider

$$d_n = W_n - \alpha W_{n-1}$$

and

$$h_n = W_n - \beta W_{n-1}.$$

Then, $d_n = \beta^n$ and $h_n = \alpha^n$. It follows that,

$$\beta^n = W_n - \alpha W_{n-1} \tag{2.3}$$

and

$$\alpha^n = W_n - \beta W_{n-1}. \tag{2.4}$$

Subtracting (2.3) from (2.4),one obtains,

$$W_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

for $\alpha \neq \beta$.

For the case, $\alpha = \beta$, one obtains,

$$W_n = (n+1)\left(\frac{a}{2}\right)^n.$$

□

2.2 2-Tridiagonal Toeplitz Matrices

Here, we discuss briefly some known results on 2− Tridiagonal Toeplitz matrices.

Theorem 2.2.1. *(Bergum and Hoggatt Jr, 1978) For $k = 2$ and $n \geq 5$,*

$$W_n^{(2)} = aW_{n-1}^{(2)} - abcW_{n-3}^{(2)} + b^2c^2W_{n-4}^{(2)}.$$

Proof. A brief discussion of the ideas employed in proving the Theorem. The first 3−terms are:

$$W_1^{(2)} = a, \quad W_2^{(2)} = a^2, \quad W_3^{(2)} = a^3 - abc.$$

Inductively, one shows

$$W_n^{(2)} = aW_{n-1}^{(2)} - abcW_{n-3}^{(2)} + b^2c^2W_{n-4}^{(2)} \quad , \quad n \geq 5.$$

□

In the following Theorem, Borowska and Łacińska (2015) presented an additional recursive formula for the determinant of the 2-tridiagonal Toeplitz matrix.

Theorem 2.2.2. *(Borowska and Łacińska, 2015)*

For $T_n^{(k)}$,

$$W_n^{(2)} = \begin{cases} \left(W_{\frac{n}{2}}\right)^2, & n \;\; is \;\; even \\ W_{\frac{n-1}{2}}W_{\frac{n+1}{2}}, & n \;\; is \;\; odd \end{cases}$$

Proof. A brief overview of the ideas is as follows. A proof of theorem 2.2.2 can be found in Borowska et al. (2013). First, we shall introduce the notations used by the Authors in their proof.The authors use the symbol **P** to represent 2-tridiagonal Toeplitz matrices and denote the determinant as F_n. That is,

$$F_n = \det \mathbf{P}.$$

Additionally, the authors use the notation x_i to represent the diagonal entries of U, an upper triangular matrix. The proof employs the idea of $LU-$ Decomposition,giving $F_n = \prod_{i=1}^{n} x_i$, where $x_i \neq 0$ and given as :

$$x_i = \begin{cases} a, & i = 1, 2 \\ a - \frac{bc}{x_{i-2}}, & i = 3, \cdots, n \end{cases}$$

In the paper, $x_{2k-1} = \frac{W_k}{W_{k-1}}$. Specifically, for the case, $n = 2k$, $x_{2k-1} = x_{2k}$. Therefore, considering our terminology for the determinant of $1-$ tridiagonal Toeplitz matrices, one writes,

$$\begin{aligned} F_n &= \left(\frac{W_1}{W_0}\right)^2 \left(\frac{W_2}{W_1}\right)^2 \times \cdots \times \left(\frac{W_{k-1}}{W_{k-2}}\right)^2 \times \left(\frac{W_k}{W_{k-1}}\right)^2 \quad . \\ &= (W_k)^2 \end{aligned} \tag{2.5}$$

Hence,

$$W_n^{(2)} = \left(W_{\frac{n}{2}}\right)^2 \quad \text{for } \; n = 2k.$$

For the case $n = 2k + 1$, $k = \frac{n+1}{2}$. Following similar logic, one arrives at:

$$\begin{aligned} F_n &= \left(\frac{W_1}{W_0}\right)^2 \left(\frac{W_2}{W_1}\right)^2 \times \cdots \times \left(\frac{W_{k-1}}{W_{k-2}}\right)^2 \times \left(\frac{W_k}{W_{k-1}}\right) \quad . \\ &= W_{k-1} W_k \end{aligned} \tag{2.6}$$

Hence,

$$W_n^{(2)} = W_{\frac{n-1}{2}} W_{\frac{n+1}{2}} \quad .$$

□

2.3 LU Factorizations of Tridiagonal Matrices

In this section, we explore basic concepts related to the LU factorization (Lu, 2021), and share literature work on $k-$ tridiagonal matrices, $M^{(k)}$.

2.3.1 Framework of LU-Decompositions

Within this section, we aim to examine the composition of LU-Decomposition. Specifically, we will focus on the visual representation of the Lower triangular Matrix, denoted as L, and the Upper triangular Matrix, denoted as U, concerning a k-Triangular Toeplitz matrix. We denote this matrix as $T_n^{(k)}$, where $n = km + s$ and $0 \leq s < k$.

To establish the visual representation, we employ the relationship $v_{l+1} = \frac{W_{l+1}}{W_l}$, where $j = lk + 1, \cdots, (l+1)k$, with $l = 0, \cdots, m-1$ for the case when $n = km$. The proof for this relationship can be found in Theorem 3.1.7. Additionally, for the remaining s rows, we utilize $\frac{W_{m+1}}{W_m}$, where $j = km + 1, \cdots, km + s$ and $l = m$. The proof of this can be found in Corollary 3.1.7.1.

2.3.1.1 1-Tridiagonal Toeplitz Matrix

Remark 2.3.1. *The following describes structures of L and U for $1-$ tridigonal Topelitz matrix when $n = km$. Here, k=1.*

$$L = \begin{bmatrix} 1 & & & & & \\ \frac{c}{v_1} & 1 & & & 0 & \\ & \frac{c}{v_2} & 1 & & & \\ & & \frac{c}{v_3} & \ddots & & \\ & & & \ddots & \ddots & \\ & 0 & & & \frac{c}{v_{n-1}} & 1 \end{bmatrix}, \tag{2.7}$$

and

$$U = \begin{bmatrix} v_1 & b & & & & \\ & v_2 & b & & 0 & \\ & & v_3 & b & & \\ & & & v_4 & \ddots & \\ & 0 & & & \ddots & b \\ & & & & & v_n \end{bmatrix} \quad . \tag{2.8}$$

2.3.1.2 2-Tridiagonal Toeplitz Matrix

Remark 2.3.2. *The following describes structures of L and U for $2-$ tridiagonal Toeplitz matrix when $n = km$. Here, $k = 2$.*

$$L = \begin{bmatrix} 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & \\ \frac{c}{v_1} & \ddots & 1 & & & 0 & & & & \\ & \frac{c}{v_1} & \ddots & 1 & & & & & & \\ & & \frac{c}{v_2} & \ddots & \ddots & & & & & \\ & & & \frac{c}{v_2} & \ddots & \ddots & & & & \\ & & & & \ddots & & & & & \\ & 0 & & & & \frac{c}{v_{m-1}} & \ddots & \ddots & \\ & & & & & & \frac{c}{v_{m-1}} & 0 & 1 \end{bmatrix} \quad , \tag{2.9}$$

and

$$U = \begin{bmatrix} v_1 & 0 & b & & & & \\ & v_1 & \ddots & b & & 0 & \\ & & v_2 & \ddots & b & & \\ & & & v_2 & \ddots & \ddots & \\ & 0 & & & \ddots & \ddots & b \\ & & & & & v_m & 0 \\ & & & & & & v_m \end{bmatrix} . \tag{2.10}$$

Remark 2.3.3. *The following describes structures of L and U for $2-$ tridiagonal Toeplitz matrix when $n = km + s$ for $s = 1$. Here, $k = 2$.*

$$L = \begin{bmatrix} 1 & & & & & & & & & \\ 0 & 1 & & & & & & & & \\ \frac{c}{v_1} & \ddots & 1 & & & 0 & & & & \\ & \frac{c}{v_1} & \ddots & 1 & & & & & & \\ & & \frac{c}{v_2} & \ddots & \ddots & & & & & \\ & & & \frac{c}{v_2} & \ddots & \ddots & & & & \\ & & & & \ddots & & & & & \\ & 0 & & & & \frac{c}{v_{m-1}} & \ddots & \ddots & & \\ & & & & & & \frac{c}{v_{m-1}} & 0 & 1 & \\ & & & & & & & \frac{c}{v_m} & 0 & 1 \end{bmatrix} , \tag{2.11}$$

and

$$U = \begin{bmatrix} v_1 & 0 & b & & & & & \\ & v_1 & \ddots & b & & 0 & & \\ & & v_2 & \ddots & b & & & \\ & & & v_2 & \ddots & \ddots & & \\ & 0 & & & \ddots & \ddots & \ddots & \\ & & & & & v_m & \ddots & b \\ & & & & & & v_m & 0 \\ & & & & & & & v_{m+1} \end{bmatrix} . \tag{2.12}$$

2.3.1.3 k-Tridiagonal Toeplitz Matrix

Remark 2.3.4. *The following describes structures of L and U for $k-$ tridiagonal Toeplitz matrix when $n = km$. Here, $k \geq 3$.*

$$L = \begin{bmatrix} 1 & & & & & & & & & & & \\ \vdots & 1 & & & & & & & & & & \\ \frac{c}{v_1} & \vdots & 1 & & & & & & & & & \\ & \ddots & \vdots & \ddots & & & & & 0 & & & \\ & & \frac{c}{v_1} & \cdots & \ddots & & & & & & & \\ & & & \frac{c}{v_2} & & 1 & & & & & & \\ & & & & \ddots & & \ddots & & & & & \\ & & & & & \frac{c}{v_2} & & & & & & \\ & & 0 & & & & \ddots & & \ddots & & & \\ & & & & & & & \frac{c}{v_{m-1}} & \cdots & 1 & & \\ & & & & & & & & \ddots & \cdots & \ddots & \\ & & & & & & & & & \frac{c}{v_{m-1}} & \cdots & 1 \end{bmatrix}, \tag{2.13}$$

and

$$
U = \begin{bmatrix}
v_1 & \cdots & b & & & & & & & & & & \\
& \ddots & \cdots & \ddots & & & & & & & & & \\
& & v_1 & \cdots & \ddots & & & & & & & & \\
& & & v_2 & \cdots & b & & & & 0 & & & \\
& & & & \ddots & \cdots & b & & & & & & \\
& & & & & v_2 & \cdots & b & & & & & \\
& & & & & & \ddots & \cdots & \ddots & & & & \\
& & & & & & & \ddots & \cdots & \ddots & & & \\
& 0 & & & & & & & \ddots & \cdots & \ddots & & \\
& & & & & & & & & \ddots & \cdots & b & \\
& & & & & & & & & & v_m & \cdots & b \\
& & & & & & & & & & & \ddots & \vdots \\
& & & & & & & & & & & & v_m
\end{bmatrix}.
$$

Remark 2.3.5. *The following describes structures of L and U for $k-$ tridiagonal Toeplitz matrix when $n = km + s$. Here, $k \geq 3$.*

$$L = \begin{bmatrix}
1 & & & & & & & & & & & & \\
\vdots & \ddots & & & & & & & & & & & \\
\frac{c}{v_1} & & 1 & & & & & & & & & & \\
 & \ddots & & 1 & & & & & & 0 & & & \\
 & & \frac{c}{v_1} & & \ddots & & & & & & & & \\
 & & & \ddots & & 1 & & & & & & & \\
 & & & & \ddots & & \ddots & & & & & & \\
 & & & & & \frac{c}{v_{m-1}} & & 1 & & & & & \\
 & 0 & & & & & \ddots & & \ddots & & & & \\
 & & & & & & & \frac{c}{v_{m-1}} & & 1 & & & \\
 & & & & & & & & \frac{c}{v_m} & & 1 & & \\
 & & & & & & & & & \ddots & & \ddots & \\
 & & & & & & & & & & \frac{c}{v_m} & \cdots & 1
\end{bmatrix},$$

and

$$U = \begin{bmatrix} v_1 & \cdots & b & & & & & & & & & & \\ & \ddots & \cdots & \ddots & & & & & & & & & \\ & & v_1 & \cdots & \ddots & & & & & & & & \\ & & & v_2 & \cdots & b & & & & 0 & & & \\ & & & & \ddots & \cdots & b & & & & & & \\ & & & & & v_2 & \cdots & b & & & & & \\ & & & & & & \ddots & \cdots & \ddots & & & & \\ & & & & & & & v_m & \cdots & \ddots & & & \\ & 0 & & & & & & & \ddots & \cdots & \ddots & & \\ & & & & & & & & & v_m & \cdots & b & \\ & & & & & & & & & & v_{m+1} & \cdots & b \\ & & & & & & & & & & & \ddots & \vdots \\ & & & & & & & & & & & & v_{m+1} \end{bmatrix}.$$

Let's now take a look at a few works from Literature making use of LU-decomposition in their proofs.

Theorem 2.3.6. *(Yalçiner, 2011)*

(a). Given a $k-$ tridiagonal matrix $M^{(k)}$ and $n = ks + m$, for $r = 1, 2, \cdots, k$; $i = 1, 2, \cdots, s$ with $M^{(k)} = LU$. And, let $t_i^{(r)}$ be the determinant of a 1-tridiagonal matrix with dimensions $i \times i$ where r indicates the row in which this determinant is located within the matrix along the diagonal.

Then,

$$t_{i+1}^{(r)} = a_{ki+r} t_i^{(r)} - b_{ki-k+r} c_{k_{i-k+r}} t_{i-1}^{(r)}. \tag{2.14}$$

(b). Given a k-tridiagonal matrix and $n = ks + m$ for $0 \leq m < k$. Then

$$\det M^{(k)} = \begin{cases} \prod_{r=1}^{k} t_s^{(r)}, & m = 0 \\ \prod_{r=1}^{m} t_{s+1}^{(r)} \prod_{r=m+1}^{k} t_s^{(r)}, & 1 \leq m < k \end{cases} \tag{2.15}$$

Proof. We offer a brief overview of the concepts employed in the proof. The author furnishes a comprehensive proof for (2.15), but omits the proof for (2.14). To prove the former, the author utilizes the concept of elementary row operations.

The first step in the proof is to process by employing the first elementary row operation.That is, multiplying the first row by $-\frac{c_1}{a_1}$ and adding it to the $(k+1)^{\text{th}}$ row. This process eliminates c_1 and modifies the corresponding diagonal entry accordingly as:

$$\frac{t_2^{(1)}}{t_1^{(1)}}.$$

Here, $t_1^{(1)}$ is the determinant of 1×1 matrix on R_1. That is,

$$\begin{bmatrix} a_1 \end{bmatrix},$$

and $t_2^{(1)}$ is the determinant of a 2×2 matrix on R_2. That is,

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & a_2 \end{bmatrix} \quad .$$

Repeating this process results in,

$$a_{r+(s-1)k} - \frac{b_{r+(s-2)k} c_{r+(s-2)k} t_{s-2}^{(r)}}{t_{s-1}^{(r)}} = \frac{t_s{}^{(r)}}{t_{s-1}{}^{(r)}}, 1 \leq r \leq k.$$

For the case $m = 0$, one obtains:

$$\begin{aligned} \det M^{(k)} &= t_1^{(1)} t_1^{(2)} \cdots t_1^{(k)} \frac{t_2{}^{(1)}}{t_1{}^{(1)}} \cdots \frac{t_2{}^{(k)}}{t_1{}^{(k)}} \cdots \frac{t_{s-1}{}^{(k)}}{t_{s-2}{}^{(k)}} \frac{t_s{}^{(1)}}{t_{s-1}{}^{(1)}} \cdots \frac{t_s{}^{(k)}}{t_{s-1}{}^{(k)}} \\ &= t_s^{(1)} \cdots t_s^{(k)} \\ &= \prod_{r=1}^{k} t_s^{(r)}. \end{aligned} \tag{2.16}$$

Similarly for the case of $m \neq 0$, one obtains:

$$\det M^{(k)} = \prod_{r=m+1}^{k} t_s^{(r)} \prod_{r=1}^{m} t_{s+1}^{(r)} \tag{2.17}$$

□

We shall note here that, a recursion for $\det T_n^{(k)}$ is briefly mentioned without a proof in Yalçiner (2011). A proof is included in the thesis.

Chapter 3

New Contributions To the Field

In this section, we first highlight new mathematical ideas with proofs. These are used in the proof of the main theorem.

3.1 Description and Proofs

In this section, we share two separate ideas for the determinant of $k-$ tridiagonal Toeplitz matrices. One is using **Binomial Expansion** and the others are applying **LU-Decomposition**.

3.1.1 Binomial Coefficients

Let's first take a look at two Lemmas regarding a few properties of Binomial coefficients (Lipschutz, 1981, p.19).

Lemma 3.1.1. *Given* $\binom{n-1}{m-1}, \binom{n-1}{m}$ *for* $n, m \in \mathbb{N}$, $n \geq m$. *Then,*

$$\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}.$$

Proof. This is a standard proof in Lipschutz (1981, p.27). □

Note: Lemma 3.1.1 is also the same as

$$\binom{(n+1-m)-1}{m} + \binom{(n+1-m)-1}{m-1} = \binom{n+1-m}{m} \tag{3.1}$$

where $n^* = n + 1 - m$.

Lemma 3.1.2. *For* $n, m \in \mathbb{N}$ *with* $m \leq n$ *and* $n = 2m + 1$,

$$\binom{n-m}{m+1} = \binom{n-m-1}{m}$$

Proof. Let's consider,

$$\binom{n-m}{m+1} - \binom{n-m-1}{m} \tag{3.2}$$

where $n = 2m + 1$.

By Binomial Coefficients,

$$\begin{aligned}\binom{n-m-1}{m} &= \frac{(n-m-1)!}{(n-2m-1)!m!} \\ &= \frac{(n-m-1)!(m+1)}{(n-2m-1)!m!(m+1)} \\ &= \frac{(n-m-1)!(m+1)}{(n-2m-1)!(m+1)!} \end{aligned} \tag{3.3}$$

and

$$\binom{n-m}{m+1} = \frac{(n-m)!}{(n-2m-1)!(m+1)!} \tag{3.4}$$

Then, one obtains:

$$\begin{aligned}\binom{n-m}{m+1} - \binom{n-m-1}{m} &= \frac{(n-m-1)!\left[(m+1)-(n-m)\right]}{(m+1)!(n-2m-1)!} \\ &= \frac{(n-m-1)!(2m+1-n)}{(m+1)!(n-2m-1)!} \\ &= 0 \quad \text{for} \;\; n = 2m+1. \end{aligned} \tag{3.5}$$

□

Now, let's turn our attention to a theorem determining $W_n^{(1)}$ of $T_n^{(1)}$ in terms of Binomial Coefficients.

Theorem 3.1.3 (New Contribution). *Let $n \geq 2$ and the largest $m \in \mathbb{N}$ with $2m \leq n$.*

Then,

$$W_n^{(1)} = \sum_{i=0}^{m} (-1)^i \binom{n-i}{i} a^{n-2i}(bc)^i \tag{3.6}$$

Proof. We prove the Theorem inductively in two cases.

Case 1: n=2m

Let us first verify $W_2^{(1)}$ and $W_3^{(1)}$.

$$\begin{aligned} W_2^{(1)} &= \sum_{i=0}^{1}(-1)^i \binom{2-i}{i} a^{2-2i}(bc)^i \\ &= a^2 - \binom{2-1}{1} a^{2-2}(bc)^1 \\ &= a^2 - bc \end{aligned}$$

and

$$\begin{aligned} W_3^{(1)} &= \sum_{i=0}^{1}(-1)^i \binom{3-i}{i} a^{3-2i}(bc)^i \\ &= a^3 - \binom{3-1}{1} a^{3-2}(bc)^1 \\ &= a^3 - 2abc \end{aligned}$$

Now, assume (3.6) for $k = n-1$ and $k = n$.

Then by Theorem 2.1.1, for

$$W_n^{(1)} = \sum_{i=0}^{m} (-1)^i \binom{n-i}{i} a^{n-2i}(bc)^i \tag{3.7}$$

and

$$W_{n-1}^{(1)} = \sum_{i=0}^{m-1} (-1)^i \binom{n-1-i}{i} a^{n-1-2i} (bc)^i \tag{3.8}$$

we write

$$\begin{aligned} W_{n+1} &= aW_n^{(1)} - bcW_{n-1}^{(1)} \\ &= a\sum_{i=0}^{m} (-1)^i \binom{n-i}{i} a^{n-2i} (bc)^i - bc\sum_{i=0}^{m-1} (-1)^i \binom{n-1-i}{i} a^{n-1-2i} (bc)^i \end{aligned} \tag{3.9}$$

Let's now show that each Binomial term of $W_{n+1}^{(1)}$ is coming from 3.9

Considering $i = 0$ for $W_{n+1}^{(1)}$,

$$(-1)^0 \binom{n+1-0}{0} a^{n+1-0} (bc)^0 = a^{n+1}$$

we see that it is the first term of $W_n^{(1)}$ multiplied by "a".

That is,

$$\begin{aligned} a\left((-1)^0 \binom{n-0}{0} a^{n-0} (bc)^0\right) &= a(a^n) \\ &= a^{n+1} \end{aligned}$$

Now, let's consider any ith term of W_{n+1} for $0 < i < m$:

$$(-1)^i \binom{n+1-i}{i} a^{n+1-2i} (bc)^i \tag{3.10}$$

which is just the same as "a" times the ith term of W_n and "$-bc$" times $(i-1)th$ term of W_{n-1}

That is:

$$
\begin{aligned}
& a\left((-1)^i\binom{n-i}{i}a^{n-2i}(bc)^i\right) - bc\left((-1)^{i-1}\binom{n-1-i+1}{i-1}a^{n-1-2(i-1)}(bc)^{i-1}\right) \\
& = \left((-1)^i\binom{n-i}{i}a^{n+1-2i}(bc)^i\right) + \left((-1)^i\binom{n-1-i+1}{i-1}a^{n+1-2i)}(bc)^i\right) \\
& = (-1)^i a^{n+1-2i}(bc)^i\left[\binom{n-i}{i} + \binom{n-i}{i-1}\right] \\
& = (-1)^i a^{n+1-2i}(bc)^i\left[\binom{n+1-i-1}{i} + \binom{n+1-i-1}{i-1}\right] \qquad (3.11)
\end{aligned}
$$

By Lemma 3.1.1, we get,

$$(-1)^i a^{n+1-2i}(bc)^i\binom{n+1-i}{i}.$$

This completes this part of the proof.

Now, for the last term $i = m$ of W_{n+1}, one follows a similar process. That is:

$$
\begin{aligned}
& (-1)^m\binom{n+1-m}{m}a^{n+1-2m}(bc)^m \\
& = a\left((-1)^m\binom{n-m}{m}a^{n-2m}(bc)^m\right) - bc\left((-1)^{m-1}\binom{n-1-m+1}{m-1}a^{n-1-2(m-1)}(bc)^{m-1}\right)
\end{aligned}
$$

Case 2: n=2m+1

This case follows the steps similar to ones in **Case 1**. The only difference is in the verification of the $(m+1)th$ term of W_{n+1}.

Let's verify this step. Here, one considers: $n+1 = 2m+2$ and $n-1 = 2m$.

The last term of W_{n+1} comes from $i = m+1$. That is:

$$(-1)^{m+1}\binom{n+1-(m+1)}{m+1}a^{n+1-2(m+1)}(bc)^{m+1}, \qquad (3.12)$$

and the last term of W_{n-1} comes from $i = m$ for $n = 2m+1$. That is,

$$(-1)^m\binom{n-1-m}{m}a^{n-1-2m}(bc)^m \qquad (3.13)$$

This could clearly be seen that, 3.12 is obtained by multiplying the 3.13 by $-bc$.

That's :

$$(-1)^m \binom{n-1-m}{m} a^{n-1-2m}(bc)^m \times (-bc). \tag{3.14}$$

Hence, by Lemma 3.1.2, we recognize 3.14 is the same as 3.12.

Remark 3.1.4. *This completes the case by case aspect of proof.*

In summary, given $n = 2m$ then $n-1 = 2m-1 = 2t+1$ where $t = m-1$.

One obtains:

$$W_{n+1}^{(1)} = a\sum_{i=0}^{m}(-1)^i\binom{n-i}{i}a^{n-2i}(bc)^i - bc\sum_{i=0}^{t}(-1)^i\binom{n-1-i}{i}a^{n-1-2i}(bc)^i$$

Therefore, writing out the various terms, one obtains:

$$\begin{aligned}
aW_n^{(1)} &= a\sum_{i=0}^{m}(-1)^i\binom{n-i}{i}a^{n-2i}(bc)^i \\
&= a\binom{n}{0}a^n - a\binom{n-1}{1}a^{n-2}(bc) + a\binom{n-2}{2}a^{n-4}(bc)^2 - \cdots + a(-1)^m\binom{n-m}{m}a^{n-2m}(bc)^m \\
&= \binom{n+1}{0}a^{n+1} - \binom{n-1}{1}a^{n-1}(bc) + \cdots + (-1)^m\binom{n-m}{m}a^{n+1-2m}(bc)^m \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
-bcW_{n-1}^{(1)} &= -bc\sum_{i=0}^{t}(-1)^i\binom{n-1-i}{i}a^{n-1-2i}(bc)^i \\
&= -bc\binom{n-1}{0}a^{n-1} + bc\binom{n-2}{1}a^{n-3}(bc) - \cdots - bc(-1)^t\binom{n-1-t}{t}a^{n-1-2t}(bc)^t \\
&= -\binom{n-1}{0}a^{n-1}bc + \binom{n-2}{1}a^{n-3}(bc)^2 - \cdots - (-1)^t\binom{n-1-t}{t}a^{n-1-2t}(bc)^{t+1} \\
&= -a^{n-1}bc + \binom{n-2}{1}a^{n-3}(bc)^2 - \cdots - (-1)^{m-1}\binom{n-m}{m-1}a^{n+1-2m}(bc)^m \qquad (3.16)
\end{aligned}$$

Thus,with Lemma 3.1.2 together with 3.15 and 3.16,

$$\begin{aligned}
W_{n+1}^{(1)} &= \binom{n+1}{0}a^{n+1} - \binom{n}{1}a^{n-1}(bc) + \binom{n-1}{2}a^{n-3}(bc)^2 - \cdots + (-1)^m\binom{n+1-m}{m}a^{n+1-2m}(bc)^m \\
&= \sum_{i=0}^{m}(-1)^i\binom{n+1-i}{i}a^{n+1-2i}(bc)^i \qquad (3.17)
\end{aligned}$$

□

We believe that this new Theorem can be useful in computational environments,especially in improving complexity of algorithms applying determinants of $k-$ tridiagonal Toeplitz matrices, since it is recursive in nature and using basic arithmetic.

3.1.2 LU-Decomposition

This section will discuss basics on determinants in the context of LU Decomposition (Lu, 2021, page 22), and a new perspective to the determinant of $T_n^{(k)}, k > 2$.

It is well known that for $T_n^{(1)} = LU$,

$$W_n = \det \quad L \times \det \quad \text{U} \qquad \text{(Shah and Thakkar, 2020, p.26)}$$

Using this idea, one can easily verify that if L is a lower triangular matrix with diagonals 1, then

$$W_n = \det \quad \text{U}$$

For the remainder of the thesis, we consider L as a lower triangular matrix with diagonals 1 as described in subsection 2.3.1.

Now, let's take a look at the Upper triangular matrix ,U, closely.

Theorem 3.1.5 (**New Contribution**). *Let $T_n^{(1)} = LU$ with the condition $W_0 = 1$. Then,*

$$U_{jj} = s_j = \frac{W_j}{W_{j-1}}$$

where $W_j = \det\left(T_j^{(1)}\right)$ for $j \geq 1$. Additionally,

$$s_j = \begin{cases} a, & j = 1 \\ a - \frac{bcW_{j-2}}{W_{j-1}}, & j = 2, 3, \cdots, n \end{cases} \tag{3.18}$$

Ideas used in proving this theorem reported in Borowska et al. (2013) for $k = 2$ and Yalçiner (2011) for $k = 1$. We see no reason to provide another proof. We refer readers to the proof of the two papers included in an earlier section.

Corollary 3.1.5.1. *Let* $T_n^{(1)} = LU$. *Then,*

$$W_n = \prod_{j=1}^{n} s_j$$

Proof. By Theorem 3.1.5 and Remark 2.3.1 and $W_n = \det U$, one obtains,

$$\begin{aligned} W_n &= s_1 \times \cdots \times s_n \\ &= \prod_{j=1}^{n} s_j. \end{aligned} \tag{3.19}$$

□

3.1.2.1 Determinant of $T_n^{(k)} = W_n^{(k)}$ for $k > 2$.

The following Lemma uses the Least Integer function denoted by $\lceil \frac{j}{k} \rceil$ for fractions $\frac{j}{k}$.

Lemma 3.1.6. *For* $n = km$

$$t[j] = \left\lceil \frac{j}{k} \right\rceil = l + 1$$

$\forall j = lk + 1, \cdots, (l+1)k$ *with* $l = 0, \cdots, m-1$.

Proof. The proof is obvious and straightforward. □

Now, we are ready to introduce $LU-$ Decomposition to the determinant of $k-$ tridiagonal Toeplitz matrices with $k > 2$.

Theorem 3.1.7 (**New Contribution**)**.** *Given* $n = km$, $T_n^{(k)} = LU$, $k > 2$ *and the condition* $W_0 = 1$.

Then,

$$U_{jj} = v_{t[j]} = \frac{W_{l+1}}{W_l} \tag{3.20}$$

where $j = lk + 1, \cdots, (l+1)k$ *with* $l = 0, \cdots, m-1$.

Proof. Let $U_{jj} = v_{t[j]}$. The proof considers ideas from Theorem 3.1.5 for each $k-$ cluster of $k-$tridiagonal Toeplitz matrix. That is, each $k-$ cluster of U has identical diagonal entries in the $LU-$ Decomposition of the matrix.

This repetition is determined by $t\lceil j \rceil$ in Lemma 3.1.6.Thus, for all $j = lk+1, \cdots, (l+1)k$ and each $l = 0, \cdots, m-1$, one gets same diagonal value in U which is:

$$U_{jj} = v_{t[j]} = \frac{W_{l+1}}{W_l}$$

This completes the proof. □

Corollary 3.1.7.1 (**New Contribution**). *Given $T_n^{(k)} = LU$, for $n = km + s, s \neq 0$, then*

$$U_{jj} = \frac{W_{m+1}}{W_m} \tag{3.21}$$

where $j = km + 1, \cdots, km + s$ and $l = m$.

Proof. In addition to the first $km-$rows, this corollary looks at the last $s-$ rows of $T_n^{(k)}$ where $n = km + s$. That is, this is a direct result of Theorem 3.1.7 for the case $l = m$ $\forall j = km + 1, \cdots, km + s$. □

Theorem 3.1.8 (**New Contribution**). *Let $T_n^{(k)} = LU$ with $n = km$.*

Then,

$$W_n^{(k)} = \left(\prod_{l=0}^{m-1} v_{l+1} \right)^k,$$

where $v_{l+1} = v_{t[j]}$, $\forall j = lk + 1, \cdots, (l+1)k$ and $l = 0. \cdots, m-1$.

Proof. By Lemma 3.1.6, $t[j] = l + 1, \quad \forall j = lk + 1, \cdots, (l + 1)k$.

Thus, for $l = 0, \cdots, m - 1$,

$$\begin{aligned} W_n^{(k)} &= \overbrace{v_1 \times \cdots \times v_1}^{k-many} \times \cdots \times \overbrace{v_m \times \cdots \times v_m}^{k-many} \\ &= (v_1)^k \times \ (v_2)^k \times \cdots \times (v_m)^k \\ &= \left(\prod_{l=0}^{m-1} v_{l+1} \right)^k \end{aligned} \tag{3.22}$$

□

Remark 3.1.9. *Since $n = km$, then an interesting observation of Theorem 3.1.8 is that:*

$$W_{km}^{(k)} = (v_1)^k \times (v_2)^k \times \cdots \times (v_m)^k$$

Corollary 3.1.9.1 (New Contribution). *Let $T_n^{(k)} = LU$ with $n = km$. Then,*

$$W_n^{(k)} = (W_m)^k .$$

Proof. From Theorems 3.1.7 and 3.1.8,

$$\begin{aligned} W_n^{(k)} &= (v_1)^k \times (v_2)^k \times \cdots \times (v_m)^k \\ &= \left(\frac{W_1}{W_0}\right)^k \times \left(\frac{W_2}{W_1}\right)^k \times \cdots \times \left(\frac{W_{m-1}}{W_{m-2}}\right)^k \times \left(\frac{W_m}{W_{m-1}}\right)^k \\ &= (W_m)^k \end{aligned} \tag{3.23}$$

□

We shall note that Corollary 3.1.9.1 was shortly mentioned in Yalçiner (2011) but the author did not provide a proof.

Corollary 3.1.9.2 (New Contribution). *Given that, $T_n^{(k)} = LU$ with $n = km + s$.*

Then,

$$W_n^{(k)} = \left(\prod_{l=0}^{m-1} v_{l+1}\right)^k (v_{m+1})^s$$

Proof. The idea to this proof follows from Theorem 3.1.8. That is,

$$\begin{aligned} W_n^{(k)} &= \overbrace{v_1 \times \cdots \times v_1}^{k-many} \times \cdots \times \overbrace{v_m \times \cdots \times v_m}^{k-many} \times \overbrace{v_{m+1} \cdots \times v_{m+1}}^{s-many} \\ &= (v_1)^k \times (v_2)^k \times \cdots \times (v_m)^k \times (v_{m+1})^s \\ &= \left(\prod_{l=1}^{m-1} v_{l+1}\right)^k \times (v_{m+1})^s \end{aligned} \tag{3.24}$$

This completes the proof. □

Remark 3.1.10. *Since $n = km + s$, then an interesting observation of Corollary 3.1.9.2 is that:*

$$W_{km+s}^{(k)} = (v_1)^k \times (v_2)^k \times \cdots \times (v_m)^k \times (v_{m+1})^s$$

Corollary 3.1.10.1 (New Contribution). *Let* $T_n^{(k)} = LU$ *with* $n = km + s$*. Then,*

$$W_n^{(k)} = (W_m)^{k-s} \times (W_{m+1})^s .$$

Proof. The proof to this follows from Corollary(3.1.9.1) and Corollary(3.1.9.2).

$$\begin{aligned} W_n^{(k)} &= (v_1)^k \times \ (v_2)^k \times \cdots \times (v_m)^k \times (v_{m+1})^s \\ &= (W_m)^k \times (v_{m+1})^s \\ &= (W_m)^k \times \left(\frac{W_{m+1}}{W_m}\right)^s \\ &= (W_m)^{k-s} \times (W_{m+1})^s \end{aligned} \tag{3.25}$$

□

Remark 3.1.11. *Since* $n = km + s$*, then an interesting observation of Corollary 3.1.10.1 is that:*

$$W_{km+s}^{(k)} = (W_m)^{k-s} \times (W_{m+1})^s .$$

Chapter 4

Recursive Forms of $W_n^{(k)}, \quad k > 2$

The objective of this chapter is to highlight the primary theorems that were conjectured at the start of the thesis.

4.1 Theorems and Proofs

Within this section, our focus lies on the main theorems along with their accompanying proofs. In this context, we have identified two key theorems pertaining to the determinant of $k-$ Tridiagonal Toeplitz matrices, $k > 2$.

Let's start with a Lemma.

Lemma 4.1.1 (New Contribution). *Let $T_n^{(k)} = LU$ with $n = km + s, s \neq 0$.*

Then,

$$W^{(k-1)}_{n-\lceil \frac{n}{k} \rceil} = (W_m)^{k-s}(W_{m+1})^{s-1} \tag{4.1}$$

Proof. For $n = km + s$,

$$\begin{aligned} \left\lceil \frac{n}{k} \right\rceil &= \left\lceil \frac{mk+s}{k} \right\rceil \\ &= m+1 \end{aligned} \tag{4.2}$$

Then,

$$n - \left\lceil \frac{n}{k} \right\rceil = mk + s - (m+1) = m(k-1) + (s-1).$$

By Corollary 3.1.10.1, it follows that

$$\mathrm{W}^{(k-1)}_{n-\lceil \frac{n}{k} \rceil} = W^{(k-1)}_{m(k-1)+(s-1)}$$

$$= (W_m)^{k-1-(s-1)}(W_{m+1})^{s-1} \text{ by Remark 3.1.11}$$

$$= (W_m)^{k-s}(W_{m+1})^{s-1} \tag{4.3}$$

□

Theorem 4.1.2 (New Contribution). *Let $T_n^{(k)} = LU$ and $n = km$.*

Then,

$$W_n^{(k)} = W^{(1)}_{\lceil \frac{n}{k} \rceil} W^{(k-1)}_{n-\lceil \frac{n}{k} \rceil} \tag{4.4}$$

Proof. Let L and U be given as shown in Remark 2.3.4.

Since $\lceil \frac{n}{k} \rceil = m$, the proof is a direct result of Theorem 3.1.8.

Here, we separate one v_i from each $k-$ cluster $m-$ many times. That is,

$$\begin{aligned}
\mathrm{W}_n^{(k)} &= \overbrace{v_1 \times \cdots \times v_1}^{k-many} \times \cdots \times \overbrace{v_m \times \cdots \times v_m}^{k-many} \text{ by Theorem 3.1.8} \\
&= \underbrace{v_1 \times v_2 \times \cdots \times v_m}_{W_m} \times \underbrace{(v_1)^{k-1} \times \cdots \times (v_m)^{k-1}}_{\mathrm{W}^{(k-1)}_{m\times(k-1)}} \\
&= \mathrm{W}_m \mathrm{W}^{(k-1)}_{m\times(k-1)} \text{ by Remark 3.1.9} \\
&= \mathrm{W}_m \mathrm{W}^{(k-1)}_{mk-m} \\
&= \mathrm{W}_{\lceil \frac{n}{k} \rceil} \mathrm{W}^{(k-1)}_{n-\lceil \frac{n}{k} \rceil} && (4.5)
\end{aligned}$$

$$\implies \mathrm{W}_n^{(k)} = \mathrm{W}_{\lceil \frac{n}{k} \rceil} \mathrm{W}^{(k-1)}_{n-\lceil \frac{n}{k} \rceil}$$

□

Corollary 4.1.2.1 (New Contribution). *Let $T_n^{(k)} = LU$ with $n = km + s, \quad s \neq 0$.*

Then,

$$W_n^{(k)} = W_{\lceil \frac{n}{k} \rceil} W^{(k-1)}_{n-\lceil \frac{n}{k} \rceil} \tag{4.6}$$

Proof. Let L and U be given as shown in Remark 2.3.5

Then, by Corollary 3.1.9.2, and separating one v_i from each $k-$ cluster and $s-$ cluster, one obtains:

$$\begin{aligned}
W_n^{(k)} &= \overbrace{v_1 \times \cdots \times v_1}^{k-many} \times \cdots \times \overbrace{v_m \times \cdots \times v_m}^{k-many} \times \overbrace{v_{m+1} \cdots \times v_{m+1}}^{s-many} \\
&= \underbrace{v_1 \times v_2 \times \cdots \times v_{m+1}}_{W_{m+1}} \times \underbrace{(v_1)^{k-1} \times \cdots \times (v_m)^{k-1}}_{(\mathrm{W}_m)^{k-1}} \times \underbrace{(v_{m+1})^{s-1}}_{\left(\frac{W_{m+1}}{W_m}\right)^{s-1}} \\
&= \mathrm{W}_{m+1} \times (\mathrm{W}_m)^{k-1} \times \left(\frac{W_{m+1}}{W_m}\right)^{s-1} \\
&= \mathrm{W}_{m+1} \times \underbrace{(\mathrm{W}_m)^{k-1-(s-1)} \times (\mathrm{W}_{m+1})^{s-1}}_{\mathrm{W}^{(k-1)}_{n-\left\lceil \frac{n}{k} \right\rceil}} \qquad \text{by Lemma 4.1.1} \\
&= \mathrm{W}_{\left\lceil \frac{n}{k} \right\rceil} \mathrm{W}^{(k-1)}_{n-\left\lceil \frac{n}{k} \right\rceil} \qquad (4.7)
\end{aligned}$$

$$\implies W_n^{(k)} = \mathrm{W}_{\left\lceil \frac{n}{k} \right\rceil} \mathrm{W}^{(k-1)}_{n-\left\lceil \frac{n}{k} \right\rceil}$$

□

Theorem 4.1.3 (Initial Conjecture). *Let* $T_n^{(k)}$, $k > 2$ *and* $n = km + s$, $0 \le s < k$.

Then in the expression (4.8),

$$W_n^{(k)} = W_{\left[\frac{n}{k}\right]}^{(1)} W_{n-\left[\frac{n}{k}\right]}^{(k-1)} \tag{4.8}$$

- Case 1

  When $k \in \mathbb{O}$ and $n \in 2\mathbb{N}$. Then,

  $$\left[\frac{n}{k}\right]$$

  is the nearest even integer.

- Case 2

  When $k, n \in \mathbb{O}$. Then,

  $$\left[\frac{n}{k}\right]$$

  is the nearest odd integer.

- Case 3

  When $k \in 2\mathbb{N}$ and $n \in \mathbb{O}$. Then,

  $$\left[\frac{n}{k}\right]$$

  is the nearest even integer.

Remark 4.1.4. *In this context, the term **"nearest"** refers to the closest or most immediate. That's, either the greatest or least. Here,* $\left[\frac{n}{k}\right] \in \mathbb{N}$. *Recall that,* $\left[\frac{n}{k}\right]$ *is either* m *or* $m+1$ *for* $n = mk + s$, $0 \le s < k$.

Lastly, since $\left[\frac{n}{k}\right]$ *in Theorem 4.1.3 is nearest even or odd integer, we shall represent*

$$\left[\frac{n}{k}\right]^o$$

and

$$\left[\frac{n}{k}\right]^e$$

to be the nearest odd and even integers respectively.

Before giving a proof for Theorem 4.1.3, let's take a look at a few lemmas to be used in the proof.

Lemma 4.1.5 (**New Contribution**). *Let* $m = \left[\frac{n}{k}\right]^e$. *For* $T_n^{(k)}$ $, k > 2$ *and* $n = km + s$ *where* $0 < s < k$. *Then,*

$$W_{n-\left[\frac{n}{k}\right]^e}^{(k-1)} = (W_m)^{k-1-s}(W_{m+1})^s \tag{4.9}$$

Proof.

The proof is quite trivial. Since $n = km + s$ and $m = \left[\frac{n}{k}\right]^e$.

Then,

$$n - \left[\frac{n}{k}\right]^e = km + s - m = m(k-1) + s$$

Thus, $W_{n-\left[\frac{n}{k}\right]^e}^{(k-1)} = W_{m(k-1)+s}^{(k-1)}$.

By Corollary 3.1.10.1 and Remark 3.1.11, one obtains:

$$W_{m(k-1)+s}^{(k-1)} = (W_m)^{k-1-s}\,(W_{m+1})^s \tag{4.10}$$

□

Lemma 4.1.6 (**New Contribution**). *Let* $m = \left[\frac{n}{k}\right]^o$. *For* $T_n^{(k)}$, $k > 2$ *and* $n = km + s$ *where* $0 < s < k$. *Then,*

$$W_{n-\left[\frac{n}{k}\right]^o}^{(k-1)} = (W_m)^{k-1-s}(W_{m+1})^s \tag{4.11}$$

Proof.

The proof is quite trivial. Since $n = km + s$ and $m = \left[\frac{n}{k}\right]^o$.

Then,

$$\begin{aligned} n - \left[\frac{n}{k}\right]^o &= km + s - m \\ &= m(k-1) + s \end{aligned} \tag{4.12}$$

Thus, $W_{n-\left[\frac{n}{k}\right]^o}^{(k-1)} = W_{m(k-1)+s}^{(k-1)}$.

By Corollary 3.1.10.1 and Remark 3.1.11, one obtains:

$$W^{(k-1)}_{m(k-1)+s} = (W_m)^{k-1-s}\,(W_{m+1})^{s} \tag{4.13}$$

□

let's now turn our attention to the proof of the theorem.

Proof.
Let $T_n^{(k)}, \quad k > 2$. The proof of Theorem 4.1.3 is a direct result of Theorem 4.1.2 and Corollary 4.1.2.1 needing only the verification of each case. In each case, we shall consider various scenarios of n. That's, $n = km$ or $n = km + s$.

Case 1 : $k \in \mathbb{O}$ and $n \in 2\mathbb{N}$.

In this case, we shall consider both $n = km$ and $n = km + s$.

- Step 1 : $n = km$

Let $n \in 2\mathbb{N}$ and $k \in \mathbb{O}$. Recall that, $\left\lceil \frac{n}{k} \right\rceil = m$ in this step. Then, for $n = km$, $m \in 2\mathbb{N}$. Hence, by ideas from Theorem 4.1.2, we select one v_i from each $k-$ cluster to form $\left\lceil \frac{n}{k} \right\rceil = m$. Since $m \in 2\mathbb{N}$, then we conclude that $\left\lceil \frac{n}{k} \right\rceil = \left[\frac{n}{k} \right]^e = m$.

Thus, by Theorem 4.1.3

$$W_n^{(k)} = W_{\left[\frac{n}{k}\right]^e} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^e}.$$

- Step 2 : $n = km + s, s \neq 0$.

Here, we have two scenarios. m is even or odd as in the case of Example 4.1.7.

Since our emphasis in **Case 1** is that $\left[\frac{n}{k} \right] \in 2\mathbb{N}$. If $m \in 2\mathbb{N}$, then $\left\lceil \frac{n}{k} \right\rceil$ is odd. We only select v_i's from $k-$ clusters. That's one obtains,

$$W_n^{(k)} = (W_m)^{k-s} \times (W_{m+1})^{s} \quad \text{by Corollary } 3.1.10.1$$

$$= W_m \times \underbrace{(W_m)^{k-s-1} \times (W_{m+1})^s}_{\mathrm{W}^{(k-1)}_{n-\left[\frac{n}{k}\right]^e}} \tag{4.14}$$

$$= W_m \times \mathrm{W}^{(k-1)}_{n-\left[\frac{n}{k}\right]^e} \quad \text{by Lemma 4.1.5} \tag{4.15}$$

$$= W_{\left[\frac{n}{k}\right]^e} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^e} \tag{4.16}$$

In addition, if $m \in \mathbb{O}$, then, by Corollary 4.1.2.1, we select one from each $k-$ cluster and $s-$ cluster. Then, $m+1 \in 2\mathbb{N}$. That's $\left\lceil \frac{n}{k} \right\rceil = \left[\frac{n}{k}\right]^e = m+1$. Hence, by Corollary 4.1.2.1

$$W^{(k)}_n = W_{\left[\frac{n}{k}\right]^e} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^e}$$

Thus, we conclude that, for **Case 1**,

$$W^{(k)}_n = W_{\left[\frac{n}{k}\right]^e} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^e}$$

is verified.

Case 2 : $k, n \in \mathbb{O}$.

In this case, we shall consider both $n = km$ and $n = km + s$.

- Step 1 : $n = km$

Let $n, k \in \mathbb{O}$. Recall that, $\left\lceil \frac{n}{k} \right\rceil = m$ in this step. Therefore, for $n = km$ where both $n, k \in \mathbb{O}$, $m \in \mathbb{O}$. Hence, by ideas from Theorem 4.1.2, we select one from each $k-$ cluster to form $\left\lceil \frac{n}{k} \right\rceil = m$. Since $m \in \mathbb{O}$, then we conclude that $\left\lceil \frac{n}{k} \right\rceil = \left[\frac{n}{k}\right]^o = m$.

Thus, by Theorem 4.1.2

$$W^{(k)}_n = W_{\left[\frac{n}{k}\right]^o} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^o}.$$

- Step 2 : $n = km + s, s \neq 0$.

Here, we have two scenarios. m is even or odd as in the case of Example 4.1.8.

Since our emphasis is $\left[\frac{n}{k}\right] \in \mathbb{O}$ in this case. Therefore, If $m \in \mathbb{O}$, then $\lceil \frac{n}{k} \rceil$ is even. We only select v_i's from $k-$ clusters. Hence, one obtains:

$$
\begin{aligned}
W_n^{(k)} &= (W_m)^{k-s} \times (W_{m+1})^s \quad \text{by Corollary } 3.1.10.1 \\
&= W_m \times \underbrace{(W_m)^{k-s-1} \times (W_{m+1})^s}_{\mathrm{W}^{(k-1)}_{n-\left[\frac{n}{k}\right]^o}} \\
&= W_m \times \mathrm{W}^{(k-1)}_{n-\left[\frac{n}{k}\right]^o} \quad \text{by Lemma } 4.1.6 \qquad (4.17) \\
&= W_{\left[\frac{n}{k}\right]^o} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^o} \qquad (4.18)
\end{aligned}
$$

In addition, if $m \in 2\mathbb{N}$, then, by Corollary 4.1.2.1, we select one from each $k-$ cluster and $s-$ cluster. Then, $m+1 \in \mathbb{O}$. That's, $\lceil \frac{n}{k} \rceil = \left[\frac{n}{k}\right]^o = m+1$.

Hence, by Corollary 4.1.2.1

$$W_n^{(k)} = W_{\left[\frac{n}{k}\right]^o} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^o}$$

Thus, we conclude that, for **Case 2**,

$$W_n^{(k)} = W_{\left[\frac{n}{k}\right]^o} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^o}$$

is verified.

Case 3 : $n \in \mathbb{O}$ and $k \in 2\mathbb{N}$.

In this case, n could only be $km+s$, $s \neq 0$. However, we have 2 scenarios since m could be even or odd. Consider Example 4.1.9. Our emphasis in this case is $\left[\frac{n}{k}\right] \in 2\mathbb{N}$. If $m \in 2\mathbb{N}$, $\left\lceil\frac{n}{k}\right\rceil \in \mathbb{O}$. That's, we select v_i's from $k-$ clusters and obtain:

$$\begin{aligned}
W_n^{(k)} &= (W_m)^{k-s} \times (W_{m+1})^s \quad \text{by Corollary } 3.1.10.1 \\
&= W_m \times \underbrace{(W_m)^{k-s-1} \times (W_{m+1})^s}_{\mathrm{W}^{(k-1)}_{n-\left[\frac{n}{k}\right]^e}} \qquad (4.19) \\
&= W_m \times \mathrm{W}^{(k-1)}_{n-\left[\frac{n}{k}\right]^e} \quad \text{by Lemma } 4.1.5 \qquad (4.20) \\
&= W_{\left[\frac{n}{k}\right]^e} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^e} \qquad (4.21)
\end{aligned}$$

On the other hand, if $m \in \mathbb{O}$, then, by Corollary 4.1.2.1, we select one from each $k-$ cluster and $s-$ cluster obtaining $m+1$ many v_i's, and $m+1 \in 2\mathbb{N}$. That's, $m+1 = \left[\frac{n}{k}\right]^e$.

Hence,

$$W_n^{(k)} = W_{\left[\frac{n}{k}\right]^e} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^e}$$

Thus, we conclude that, for **Case 3**,

$$W_n^{(k)} = W_{\left[\frac{n}{k}\right]^e} W^{(k-1)}_{n-\left[\frac{n}{k}\right]^e}$$

is verified.

Example 4.1.7.

- *Let* $n = 18$ *and* $k = 7$, *then* $m = 2$.
- *Let* $n = 16$ *and* $k = 3$, *then* $m = 5$.

Example 4.1.8.

- *Let* $n = 19$ *and* $k = 3$, *then* $m = 6$.
- *Let* $n = 17$ *and* $k = 5$, *then* $m = 3$.

Example 4.1.9.

- $n = 21$ *and* $k = 4$, *then* $m = 5$.
- $n = 25$ *and* $k = 6$, *then* $m = 4$.

This completes the proof of Theorem 4.1.3. □

Corollary 4.1.9.1 (Special case of Initial Conjecture). *Let* $T_n^{(k)}, \quad k > 2$ *and* $n, k \in 2\mathbb{N}$. *Given* $n = km + s, \quad 0 \le s < k$.

Then,

$$W_n^{(k)} = \left(W_{\frac{n}{2}}^{(\frac{k}{2})}\right)^2$$

Proof.

The proof of Corollary 4.1.9.1 is a direct result of Theorem 4.1.2 and Corollary 4.1.2.1.

Scenario 1 : $n = km$.

Let $T_n^{(k)}, \quad k > 2$ and $n = km$. Here, m could be even or odd. Considering,

$$\begin{aligned}
W_n^{(k)} &= \overbrace{v_1 \times \cdots \times v_1}^{k-many} \times \cdots \times \overbrace{v_m \times \cdots \times v_m}^{k-many} \quad \text{by Theorem } 3.1.8 \\
&= (v_1)^k \times (v_2)^k \times \cdots \times (v_m)^k \\
&= \underbrace{(v_1)^{\frac{k}{2}} \times (v_2)^{\frac{k}{2}} \times \cdots \times (v_m)^{\frac{k}{2}}} \times \underbrace{(v_1)^{\frac{k}{2}} \times (v_2)^{\frac{k}{2}} \times \cdots \times (v_m)^{\frac{k}{2}}} \\
&= \underbrace{(v_1)^p \times (v_2)^p \times \cdots \times (v_m)^p}_{\mathrm{W}_{mp}^{(p)}} \times \underbrace{(v_1)^p \times (v_2)^p \times \cdots \times (v_m)^p}_{\mathrm{W}_{mp}^{(p)}}, \quad \text{where } \frac{k}{2} = p \\
&= \mathrm{W}_{mp}^{(p)} \times \mathrm{W}_{mp}^{(p)} \quad \text{by Remark } 3.1.9 \\
&= \left(\mathrm{W}_{mp}^{(p)}\right)^2 \\
&= \left(\mathrm{W}_{\frac{n}{2}}^{(\frac{k}{2})}\right)^2 \quad \text{since } k = 2p, \quad \frac{n}{2} = mp.
\end{aligned} \tag{4.22}$$

This shows that, regardless of whether $m \in \mathbb{O}$ or $m \in 2\mathbb{N}$, a selection of v_i's $\frac{k}{2}$ many from each $k-$ cluster is made $m-$ many times. This will give us half- many entries of $T_n^{(k)}$. Thus, we split $T_n^{(k)}$ into two identical $T_{\frac{n}{2}}^{(\frac{k}{2})}$ giving $W_n^{(k)} = \left(W_{\frac{n}{2}}^{(\frac{k}{2})}\right)^2$

Scenario 2 : $n = km + s, \quad s \in 2\mathbb{N}$.

Let $T_n^{(k)}, \quad k > 2$ and $n = km + s, \quad s \in 2\mathbb{N}$. Here, m could be even or odd. Regardless of whether $m \in \mathbb{O}$ or $m \in 2\mathbb{N}$, a selection of v_i's $\frac{k}{2}$ many from each $k-$ cluster and $\frac{s}{2}$ many from $s-$ cluster results in ;

$$W_n^{(k)} = \overbrace{v_1 \times \cdots \times v_1}^{k-many} \times \cdots \times \overbrace{v_m \times \cdots \times v_m}^{k-many} \times \overbrace{v_{m+1} \cdots \times v_{m+1}}^{s-many} \quad \text{by Corollary } 3.1.9.2$$

$$
\begin{aligned}
&= (v_1)^k \times \ (v_2)^k \times \cdots \times (v_m)^k \times (v_{m+1})^s \\
&= \underbrace{(v_1)^{\frac{k}{2}} \times (v_2)^{\frac{k}{2}} \times \cdots \times (v_m)^{\frac{k}{2}} \times (v_{m+1})^{\frac{s}{2}}} \times \underbrace{(v_1)^{\frac{k}{2}} \times (v_2)^{\frac{k}{2}} \times \cdots \times (v_m)^{\frac{k}{2}} \times (v_{m+1})^{\frac{s}{2}}} \\
&= \underbrace{(v_1)^p \times (v_2)^p \times \cdots \times (v_m)^p \times (v_{m+1})^t}_{\mathrm{W}^{(p)}_{mp+t}} \times \underbrace{(v_1)^p \times (v_2)^p \times \cdots \times (v_m)^p \times (v_{m+1})^t}_{\mathrm{W}^{(p)}_{mp+t}}, \quad \text{let} \ \ \frac{k}{2} = p, \frac{s}{2} = t \\
&= \mathrm{W}^{(p)}_{mp+t} \times \mathrm{W}^{(p)}_{mp+t} \quad \text{by Remark} \ \ 3.1.10 \\
&= \left(\mathrm{W}^{(p)}_{mp+t}\right)^2 \\
&= \left(\mathrm{W}^{(\frac{k}{2})}_{\frac{n}{2}}\right)^2 \quad \text{since} \ \ k = 2p, \quad \frac{n}{2} = mp + t.
\end{aligned} \tag{4.23}
$$

This completes the proof.

Remark 4.1.10. *If you recall, Borowska et al. (2013) proved a special case of Corollary 4.1.9.1 in Theorem 2.2.2 where* $k = 2$. *They used a different approach in proving.*

Infact, Theorem 2.2.2 is one example of Corollary 4.1.9.1. In comparison to the proof ideas they used, our approach is more powerful and simpler covering all cases not only $k = 2$.

□

Chapter 5

Conclusion

Within this segment, I will provide a succinct overview of the accomplished tasks and outline the anticipated future work related to our research.

5.1 Summary

To provide a brief overview of my project's advancement, I would like to state that we have successfully showcased how the determinant of a k-Tridiagonal Toeplitz matrix can be computed based two main Theorems. That's, Theorems 4.1.2 together with Corollary 4.1.2.1 and Theorem 4.1.3 together with Corollary 4.1.9.1. The methodology utilized to validate our findings involves the incorporation of principles from LU decomposition.

5.2 Future Glance

Within this subsection, I draw attention to potential future work related to my research findings. While my research focus was on identifying determinants of $k-$ Tridiagonal Toeplitz matrices, I could broaden our work to encompass $k-$ Tridiagonal $k-$ Toeplitz matrices. Additionally, I could explore the eigenvalues and eigenvectors associated with these $k-$ Tridiagonal Toeplitz matrices and examine cases where $W_n^{(k)} = 0$ is permitted.

References

Mehmet Akbulak and Durmus Bozkurt. On the norms of toeplitz matrices involving fibonacci and lucas numbers. *Hacettepe Journal of Mathematics and Statistics*, 37(2):89–95, 2008.

G E Bergum and V E Hoggatt Jr. The first few terms p. *Knights Tour Revisited*, pages 285–288, 1978.

Dario Bini. Toeplitz matrices, algorithms and applications. *http://www. ercim. org/publication/Ercim_News/enw22/toeplitz. html*, 1995.

Jolanta Borowska and Lena Łacińska. Eigenvalues of 2-tridiagonal toeplitz matrix. *Journal of Applied Mathematics and Computational Mechanics*, 14(4), 2015.

Jolanta Borowska, Lena Łacińska, and Jowita Rychlewska. Application of difference equation to certain tridiagonal matrices. *Scientific Research of the Institute of Mathematics and Computer Science*, 3(11):15–20, 2012.

Jolanta Borowska, Lena Łacińska, and Jowita Rychlewska. On determinant of certain pentadiagonal matrix. *Journal of Applied Mathematics and Computational Mechanics*, 12(3): 21–26, 2013.

Albrecht Böttcher and Sergei M Grudsky. *Spectral properties of banded Toeplitz matrices.* SIAM, 2005.

Hamide Dogan and Luis R Suarez. Matrix power computation band toeplitz structure. *International Journal of Computing Algorithm*, 6(01), 2017.

Hamide Dogan, Kachmar Khalil, and Luis R Suarez. Some results on the ideals of real-valued lower triangular toeplitz matrices. *Turkish Journal of Mathematics and Computer Science*, 9:50–54, 2018.

Michael JC Gover. The eigenproblem of a tridiagonal 2-toeplitz matrix. *Linear Algebra and its Applications*, 197:63–78, 1994.

Emrullah Kırklar and Fatih Yılmaz. A note on k-tridiagonal k-toeplitz matrices. *Alabama Journal of Mathematics*, 39, 2015.

Seymour Lipschutz. *Theory and problems of probability*. McGraw-Hill, 1981.

Jun Lu. Numerical matrix decomposition and its modern applications: A rigorous first course. *arXiv preprint arXiv:2107.02579*, 2021.

D Steven Mackey, Niloufer Mackey, and Srdjan Petrovic. Is every matrix similar to a toeplitz matrix? *Linear algebra and its applications*, 297(1-3):87–105, 1999.

Bishwa Nath Mukherjee and Sadhan Samar Maiti. On some properties of positive definite toeplitz matrices and their possible applications. *Linear algebra and its applications*, 102: 211–240, 1988.

Nita H Shah and Foram A Thakkar. *Matrix and Determinant: Fundamentals and Applications*. CRC Press, 2020.

William F Trench. Some spectral properties of hermitian toeplitz matrices. *SIAM Journal on Matrix Analysis and Applications*, 15(3):938–942, 1994.

Aynur Yalçiner. The lu factorizations and determinants of the k-tridiagonal matrices. *Asian-European Journal of Mathematics*, 4(01):187–197, 2011.

Ke Ye and Lek-Heng Lim. Every matrix is a product of toeplitz matrices. *Foundations of Computational Mathematics*, 16(3):577–598, 2016.

Fuzhen Zhang. *Matrix theory: basic results and techniques*. Springer, 2011.

Curriculum Vitae

Eugene Agyei-Kodie entered the world on October 4, 1994. He accomplished his secondary education at T.I. Ahmadiyya Senior High School in Ghana in 2013. Eugene obtained a Bachelor of Education (B.Ed) degree in Mathematics from the University of Cape Coast (UCC), where he achieved recognition as the top Mathematics student and was honored with the Dean's Award in 2019.

During his time of national service, Eugene served as a Teaching Assistant at the Mathematics and Statistics Department at UCC. He actively engaged in various initiatives such as the National Mathematics Camp, Mathematics Sensitization to High Schools, and Seminars. These programs aimed to enhance students' interest and knowledge in Mathematics, while also fostering collaboration among Mathematicians from diverse backgrounds. Through these workshops, Eugene refined his interpersonal skills, creativity, and analytical thinking.

In the autumn of 2020, Eugene commenced his graduate studies at the African Institute for Mathematical Sciences in Ghana. Later on, in the fall of 2021, he gained admission to the University of Texas at El Paso. Throughout his Master's program, Eugene held the position of Graduate Teaching Assistant at the Department of Mathematical Sciences. In his second year of graduate school, he embarked on his thesis work titled "Recursive Forms for Determinants of k-Tridiagonal Toeplitz Matrices," under the guidance of Dr. Hamide Dogan-Dunlap.

After the successful completion of his Master's degree, Eugene gained admission to the Mathematics Program at Michigan State University, where he will be pursuing his doctoral degree. For any communication, he can be reached via email at eagyeikodi@miners.utep.edu.