Algorithms For Exploration Of Advanced Electromagnetic Concepts

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ALGORITHMS FOR EXPLORATION OF ADVANCED
ELECTROMAGNETIC CONCEPTS

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Doctoral Program in Computational Science

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Stephen L. Crites, Jr., Ph.D.
Dean of the Graduate School
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by

Asad Ullah Hil Gulib

2022
Dedication

The study reported in this dissertation is related to metamaterials. While doing the literature review for the dissertation, I learned that the first scientist to conduct experiments on metamaterials was from Bangladesh. As a student from Bangladesh, I want to dedicate my dissertation to that scientist.

Sir Jagadish Chandra Bose
ALGORITHMS FOR EXPLORATION OF ADVANCED ELECTROMAGNETIC CONCEPTS

by

ASAD ULLAH HIL GULIB, M.Sc.

DISSERTATION

Presented to the Faculty of the Graduate School of
The University of Texas at El Paso
in Partial Fulfillment
of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Computational Science Program
THE UNIVERSITY OF TEXAS AT EL PASO
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Abstract

3D printing is revolutionizing the manufacturing industry and is now being considered in the electronics industry. The creation of the world’s first 3D volumetric circuit has made a way to make circuits smaller, lighter, into unconventional form factors and exploit physics like anisotropy more effectively than planar geometries can. While this is exciting, many problems must be solved to make 3D volumetric circuits more efficient. One of these problems is electromagnetic interference and mutual coupling between the circuit components that are expected to increase in high-frequency 3D circuits. Spatially variant anisotropic metamaterials (SVAMs) could be a solution to overcome this difficulty, but research in this area is not possible without an algorithm that can generate SVAMs in 3D circuits. In this dissertation, an algorithm is developed and integrated into CAD software that can generate SVAMs for 3D circuits.

There are no formulations available in the literature that consider general complex constitutive parameters in the external regions for the transfer matrix method (TMM). It is possible to handle certain special cases, such as reflection from a lossy dielectric, however, the standard TMM cannot calculate transmittance into a lossy dielectric. The TMM is combined with scattering matrices and named as transfer matrix method with scattering matrices (TMMSM) and modified to work with the complex medium. To verify the results from TMMSM, analytical equations were derived to calculate the power quantities at the interface between two complex mediums. A simpler conservation equation for transverse electric (TE) and transverse magnetic (TM) polarizations has been derived which accounts for both complex permittivity and complex permeability. The consideration of complex permeability in the formulation is important for simulating metamaterials and this is not addressed anywhere in the existing literature. The derivation also shows that the standard Fresnel coefficients are valid for a general complex medium, but the
correct sign of the wave impedance needs to be chosen to calculate the Fresnel coefficients accurately. A condition has also been found to calculate the Fresnel coefficients correctly without fully resolving the sign of wave impedance. In this dissertation, the results of TMMSM and the analytical derivation for scattering analysis at the interfaces between complex mediums were compared. Additionally, the analytical results and TMMSM results were cross-checked with results from other sources to ensure that the methods are accurate.
# Table of Contents

Dedication ........................................................................................................ iii

Acknowledgments ............................................................................................ v

Abstract ........................................................................................................... vi

Table of Contents .......................................................................................... viii

List of Tables .................................................................................................... x

List of Figures ................................................................................................... xi

Chapter 1: Introduction ..................................................................................... 1
  Purpose ........................................................................................................... 1
  Outline ......................................................................................................... 5

Chapter 2: Background theory ........................................................................ 7
  Metamaterials ............................................................................................... 7
  Maxwell’s equations in anisotropic materials .......................................... 10
  Anisotropic metamaterials .......................................................................... 14
  Space stretching ........................................................................................... 17
  Voronoi tessellation ...................................................................................... 21
  Numerical solution of Laplace’s equation ................................................ 22
  Complex angle .............................................................................................. 24

Chapter 3: Generation of spatially variant anisotropic metamaterials in 3D circuits ............................................................................................ 28
  Space stretching with negative uniaxial metamaterials ............................ 29
  Algorithm to generate SVAM .................................................................... 31
  Integrating the SVAM algorithm into Blender ........................................ 36
  Results ......................................................................................................... 38

Chapter 4: Scattering at the interface of complex mediums .......................... 39
  Generalized Fresnel equations and conservation of power ..................... 39
    Generalized Fresnel equations for TE polarization .................................. 40
    Wave expressions ...................................................................................... 40
    Boundary conditions for TE polarization ............................................... 43
    Reflection and transmission coefficients for TE polarization .............. 45
Poynting vectors for TE polarization ...............................................................48
Conservation of power: reflectance and transmittance ..................................56
Generalized Fresnel equations for TM polarization .........................................58
Wave expression ............................................................................................58
Boundary conditions ....................................................................................59
Reflection and transmission coefficients .........................................................60
Poynting vectors .............................................................................................62
Conservation of power: reflectance and transmittance .....................................63

Chapter 5: TMMSM for complex medium .......................................................66
  Numerical solution to Maxwell’s equations in LHI medium ...........................66
  Formulation of scattering matrices ...............................................................74
    Scattering matrix for a single layer .............................................................75
    Redheffer star product .............................................................................82
    Scattering matrices for external regions ....................................................83
  Calculating the source ..................................................................................85
  Calculating reflectance and transmittance ..................................................86
  Implementation of TMMSM ......................................................................89
  Results .........................................................................................................91
    Comparison of analytical results and TMMSM results .........................91
    Comparison of TMM results and TMMSM results .................................95

Chapter 6: Conclusion ....................................................................................96

References ......................................................................................................104

Vita ...............................................................................................................109
List of Tables

Table 1: Dielectric Properties of Different Anisotropic Materials [33] .......................................................... 13
Table 2. Scattering at air to complex medium (complex $\varepsilon_r$, $\text{trn}$) ................................................................. 92
Table 3. Scattering at air to complex medium (complex $\mu_r$, $\text{trn}$) ................................................................. 93
Table 4. Scattering at complex medium (complex $\varepsilon_r$, $\text{inc}$) to air ................................................................. 93
Table 5. Scattering at complex medium (complex $\mu_r$, $\text{inc}$) to air ................................................................. 93
Table 6. Scattering at complex medium (complex $\varepsilon_r$, $\text{inc}$) to complex medium (complex $\varepsilon_r$, $\text{trn}$) ........................................................................................................... 94
Table 7. Scattering at complex medium (complex $\mu_r$, $\text{inc}$) to complex medium (complex $\mu_r$, $\text{trn}$) ........................................................................................................... 94
Table 8. Scattering at complex medium (complex $\varepsilon_r$, $\text{inc}$ and $\mu_r$, $\text{inc}$) to complex medium (complex $\varepsilon_r$, $\text{trn}$ and $\mu_r$, $\text{trn}$) ........................................................................................................... 94
List of Figures

Figure 1: Classification of engineered materials [32]................................................................. 8
Figure 2: (a) Unit cell considered for the simulation. The Parameters are \( \varepsilon r1 = 4.0, \varepsilon r2 = 1.0, r = .2a, \) and \( a = 1. \) (b) The key points of symmetry for the unit cell. (c) The photonic band diagram. .......................................................................................................................... 10
Figure 3: Relation between the grating and various vector quantities involved.......................... 16
Figure 4: (a) Positive uniaxial metamaterial and (b) negative uniaxial metamaterial ............... 17
Figure 5: Two circuit components placed in close proximity...................................................... 18
Figure 6: Negative uniaxial metamaterial parameters to stretch the space between two circuit components. ...................................................................................................................... 20
Figure 7: (a) Circuit components. (b) Circuit components inside their respective Voronoi regions, ................................................................................................................................. 21
Figure 8: (a) Mesh with boundary values and unknown values before solving Laplace equation. (b) Unknown mesh values filled with numbers after solving Laplace equation...................... 23
Figure 9: Plane wave propagating from a lossless medium to a lossless medium ....................... 26
Figure 10: Simulation results for directional coupler using different materials between the waveguides. (a)-(d) Range of isotropic materials. (e) SVAM ................................................. 31
Figure 11: (a) Circuit components of a 3D circuit. (b) Circuit components with bounding region. ................................................................................................................................. 33
Figure 12: 3D circuit components with their corresponding Voronoi cells ............................... 34
Figure 13: (a) Boundary conditions to solve Laplace’s equation. (b) Gradient inside the Voronoi cell. (c) Rings formed inside Voronoic cell from the isocontours of the gradient ...................... 35
Figure 14: Cross-section of Voronoic cells showing rings formed around multiple components. 35
Figure 15: 3D printed 3D volumetric circuit with SVAM .............................................................. 38
Figure 16: (a) Material 1 of the SVAM. (b) Material 2 of the SVAM. (c) Both materials together form the overall SVAM. ............................................................................................................... 39
Figure 17: (a) Screenshot from left to right of Blender showing the python scripting interface, the SVAM panel and the 3D circuit objects. (b) Screenshot of Blender showing the SVAM panel and the SVAM formed around the 3D circuit components. ........................................................................... 38
Figure 18: 3D printed 3D volumetric circuit with SVAM .............................................................. 38
Figure 19: (a) Scattering at an interface. (b) Simplified configuration for \( \phi = 00. \) ............... 40
Figure 20: Geometry of transfer matrix method. \( \mu r \) and \( \varepsilon r \) indicate the complex relative permittivity and complex relative permeability of the respective layer. \( \theta \) and \( \phi \) are indicating the incident angle and azimuthal angle, respectively. \( kinc \) is representing the incident wave vector. \( L \) represents the thickness of respective layer. The subscript \( N \) indicates the number of layers. ..... 68
Figure 21: Representation of scattering parameters. \( S_{11} \) represents the wave reflected from port 1 and \( S_{22} \) represents wave reflected from port 2. \( S_{21} \) represents wave transmitted from port 1 to port 2 and \( S_{12} \) represents the vice versa ........................................................................................................................................ 74
Figure 22: Mathematical framework for scattering matrices. ...................................................... 76
Figure 23: Flowchart for implementation of TMMSM for TE polarization ............................... 91
Figure 24: Simulation results of Fabry-Perot type metal-insulator-metal (MIM) optical cavity device described in [72]. (a) Result obtained from TMM, and (b) result from TMMSM .......... 95
Chapter 1: Introduction

Purpose

The use of 3D printing or additive manufacturing in the manufacturing industry is revolutionizing the industry and holds great promise for the design and manufacture of electronic devices [1]. Currently, all circuits are constructed on a two-dimensional planar board, which limits the use of the third dimension. With 3D printing, the third dimension can be exploited by placing circuit components in any orientation and arbitrary position. In 2016, Carranza et al. [2] designed and 3D printed the world's first volumetric 3D circuit. The invention of the first 3D circuit opened up new possibilities for the circuit industry. A 3D circuit has many advantages over a traditional 2D circuit. 3D circuits can be made smaller, lighter, and in any shape. However, there are also several challenges to overcome. As 3D circuits can be made more compact than 2D circuits, the circuit components can be placed closer to each other, which will increase interference and mutual coupling. This issue will be even more significant in high-frequency 3D circuits. Also, thermal management, testing, troubleshooting, and lack of design tools are concerns with the 3D circuits. Therefore, these challenges must be overcome to make full use of the 3D circuits.

Rumpf et al. [3] first introduced an algorithm for generating spatially-variant lattices. In [4], it has been demonstrated how spatially variant anisotropic metamaterials (SVAMs) can be used to manipulate the near field of a microstrip transmission line. The decoupling of a metal ball from a microstrip transmission line using a SVAM was also demonstrated in [5]. These experiments show that SVAMs can be used to reduce mutual coupling and interference in 3D circuits. Since there were only one or two components in the two experiments described above, it was easy to design the SVAM. However, 3D circuits will be composed of several circuit components, making the design of the SVAM more complicated. There is no tool available that
can generate SVAM around multiple circuit components. So, research in this area is stuck due to the lack of a tool that can generate SVAM around multiple circuit components.

The first part of this dissertation focuses on the design of a 3D circuit geometry with a SVAM [6]. An algorithm is developed that can generate SVAMs around the circuit components in a 3D circuit. This algorithm is also incorporated into the computer-aided design (CAD) software Blender using its Python scripting interface. A 3D circuit with a SVAM is then designed and 3D printed to demonstrate the feasibility of the algorithm.

The transfer matrix method with scattering matrices (TMMSM) has been modified to analyze scattering at the interfaces of complex mediums [7]. The modification considers a complex medium for both external regions and internal layers. There is no formulation or implementation available for the transfer matrix method (TMM) that considers the general constitutive parameters in the external regions. The standard TMM is unable to calculate transmittance into lossy dielectrics, but some special cases can be handled, such as reflection from lossy dielectrics. To validate the formulation of the modified TMMSM, the results need to be verified. This led to the derivation of power quantities at the interface between two complex mediums.

The vast majority of research for the scattering analysis at the interface between two complex mediums considers only the complex permittivity [8, 9, 10, 11, 12, 13]. As a result, there are no generalized equations that calculate the power at the interface between two complex mediums that account for both complex permittivity and complex permeability. For optical devices, it is helpful to consider only complex permittivity, but for metamaterials, both complex permittivity and complex permeability can exist. For the study of metamaterials, it is important to consider both complex permittivity and complex permeability.
The conservation of power at the boundary of two lossless mediums states that the sum of reflectance \((R)\) and transmittance \((T)\) should equal one. The derivation presented here shows that the conservation holds in the presence of loss/gain, but the expression for \(R\) and \(T\) needs modification. Balanis [8], Stratton [14], Hayt [9], Born-Wolf [10], Towne [11], Marozas [12] have discussed the effects that occur at a lossy dielectric interface [13, 15]. Francis X. Canning claimed to have corrected the Fresnel equations for lossy materials [15] but Ioannis M. Besieris proved that Canning’s equations are identical to the standard Fresnel equations [16]. In 2014, H. Weber published a detailed derivation of the Fresnel coefficients and conservation of power equations at the interface of lossy dielectrics [13] but did not consider complex permeability or active materials. Lodenquai [17] proposed a modified conservation equation by adding a \(\Delta\) term to the conservation equation. Weber also considered the \(\Delta\) term in his derivation.

New expressions for reflectance \(R\) and transmittance \(T\) for the transverse electric (TE) and transverse magnetic (TM) polarizations have been derived in this dissertation from the analysis of power quantities at the interface of complex mediums. The derived equations are simpler than those presented in any other available literature. The present derivation is valid for both positive and negative complex refractive indices; positive and negative complex wave impedance; and active and passive mediums.

This research has developed a novel technique to calculate the correct Fresnel coefficients without fully resolving the sign of the wave impedance for all cases. Several attempts have been made to determine the sign of the refractive index and/or wave impedance in complex/lossy mediums [18, 19, 20, 21, 22, 23]. The studies either included a variety of conditions to determine the sign of the refractive index/wave impedance or admitted that it was impossible to accurately determine the sign of the refractive index/wave impedance for complex mediums. In addition,
some of these research is restricted to a particular type of material and do not offer a general solution to the sign ambiguity. So, finding a way to calculate the Fresnel coefficients without fully resolving the sign of the wave impedance is important because it was not known before. This research is also capable of analyzing scattering at an interface between any combination of linear, homogeneous, and isotropic mediums.

The TMM is a semi-analytical method that can analyze devices composed of any number of homogeneous layers [24]. Semi-analytical methods are efficient techniques to solve partial differential equations. In a semi-analytical method, part of the problem is solved analytically, and the other part is solved numerically [25]. Scattering matrices have become the most popular and efficient technique to solve the boundary conditions in semi-analytical methods [25, 26, 27, 28]. This dissertation follows the same approach for scattering matrices as described in [25]. The electromagnetic eigen-modes that exist in a layer of a layered device are calculated from Maxwell’s equations, and then boundary conditions are applied at the interfaces of the layers to propagate the mode throughout the device.

The TMMSM presented in this dissertation incorporates both complex permittivity and complex permeability for the external regions. Complex refractive index is considered for the internal layers of the layered device. TMMSM is derived for both TE and TM polarizations. It was found that the power quantities need to be calculated in terms of electric fields for the TE polarization and in terms of magnetic fields for the TM polarization. This TMMSM works for positive and negative refractive indices; and positive and negative wave impedances.
The first chapter of this dissertation summarizes the topics of the research and describes the significance of the obtained results. Additionally, it discusses the motivation and impact of this research.

In the second chapter, the fundamental electromagnetics, algorithms, and computational methods used in this research are discussed. It will help a reader to understand the following chapters. The following section of this chapter provides a detailed discussion on metamaterials. A detailed analysis of Maxwell's equation for anisotropic materials is provided. In addition, an in-depth discussion is provided on anisotropic metamaterials. The following topics are space stretching and spatially variant lattices. Following this, the discussion turns to the numerical solution of Laplace’s equation, the Voronoi tessellation, and the concept of complex angles.

The third chapter describes the SVAM algorithm for 3D circuits. A finite-difference frequency-domain (FDFD) simulation of the directional coupler is presented. The purpose of this simulation is to compare the results of filling the space between two waveguides in a directional coupler with different isotropic materials and a SVAM. The following section provides a detailed description of each step of the SVAM algorithm. Afterward, a description of the algorithm's integration with the CAD software, Blender, is provided. Finally, a 3D circuit was designed and 3D printed with a SVAM, and a picture of that 3D printed circuit with the SVAM is provided.

The fourth chapter focuses on the development of expressions for reflectance and transmittance at the interfaces between two complex mediums. As a first step, Fresnel coefficients are derived after boundary conditions are applied to electromagnetic fields. The Poynting vectors are then calculated on the incident and transmission sides to derive expressions for reflectance and
transmittance. The power quantities were first calculated for the TE polarization, and then for the TM polarization.

A detailed formulation of the TMMSM is presented in chapter five. First, a numerical solution is derived from Maxwell's equations for a linear, homogeneous, and isotropic (LHI) medium. Next, the formulation of scattering matrices is described. This is followed by the inclusion of a source into the algorithm. Next, the reflected and transmitted fields are calculated using the scattering matrices and the power is calculated. A block diagram is provided that summarizes the implementation of TMMSM. In the end, a list of benchmarking examples is provided.

Chapter six of the dissertation summarizes the research and describes the key developments that resulted from it. It also discusses the potential of 3D printed SVAM circuits and the future ways of analyzing the scattering at the interface between two complex mediums.
Chapter 2: Background theory

In this section, several topics will be discussed to assist the readers in understanding the following sections.

Metamaterials

A metamaterial is being proposed that reduces interference and mutual coupling in 3D circuit components. So, the first topic that will be discussed in this chapter is metamaterials. The study of metamaterials started in 1898 when a Bangladeshi researcher Jagadish Chandra Bose experimented with twisted elements that exhibit properties like chiral materials [29, 30]. The research in metamaterials got a boost after Pendry demonstrated negative permeability [31]. Smith published an experimental demonstration of negative index material in 2001 [32]. It is hard to find a common definition for metamaterials. An easy way to define metamaterials would be as the set of materials that are different from ordinary materials [33]. According to other sources, metamaterials are artificial structural elements designed to achieve unusual electromagnetic properties [34], engineered material that has a property not found in nature [35], artificial materials that possess unconventional material parameters [36], artificial structures that display properties not available in natural materials [37], and an engineered electromagnetic structure that may have some exotic electromagnetic properties over certain frequency bands that are normally not found in nature [38] and exhibit electromagnetic properties not found in conventional materials [39]. The most general definition that covers all the above properties is given by [29], “fabricated structures and composite materials that either mimic known material responses or qualitatively have new, physically realizable response functions that do not occur or may not be readily available in nature”.
Metamaterials do not have any specific classification [33]. The lattice spacing for typical electromagnetic metamaterials is much smaller than the wavelengths of operating frequencies [33, 39, 40, 41]. Figure 1 shows the classification of engineered materials.

![Figure 1: Classification of engineered materials [40]](image)

Most natural materials and lab synthesized materials fall into the ultra-subwavelength range. They are defined as ordinary materials. The next size scale-up is called mixtures. Mixtures are compositions of various materials, where the materials preserve their properties. Mixtures are still very subwavelength. If the composition of the engineered material is scaled up further, the next category of engineered materials is called metamaterial. Metamaterials are composite materials designed in the lab. These materials are still subwavelength enough that waves do not diffract. Metamaterials that have a periodicity of less than one-quarter of a wavelength are called non-resonant metamaterials and metamaterials that have a wavelength close to one-tenth of a wavelength are called resonant metamaterials. The last size scale of engineered materials is photonic crystals where the spacing is large enough to diffract the waves [40].

Engineered materials can also be classified according to their size relative to wavelength. Figure 2 shows all types of engineered materials that take place in the band diagram according to their properties. A common way to evaluate the electromagnetic properties of a periodic structure
is through the electromagnetic band diagram. It is not possible to completely describe the vast topic of band diagrams in this dissertation, including all the essential concepts like Wigner–Seitz primitive unit cell, Bloch wave vector, direct lattice, reciprocal lattice, $k$-space, Brillouin zone (BZ), irreducible Brillouin zone (IBZ), key points of symmetry and so on. A very good and detailed description of band diagrams can be found in [42, 43].

The electromagnetic band diagram is not complete, and it is missing some information that could be valuable [43, 44]. It is the map of the frequencies as a function of the Bloch wave vector $\vec{\beta}$. It shows the frequencies of the modes that are allowed to propagate in the periodic structure. The unit cell of the engineered material considered for the band diagram is shown in Figure 2 (a). This is a square array of dielectric cylinders in air. Figure 2 (b) shows the key points of symmetry of the IBZ. Figure 2 (c) shows the photonic band diagram of the engineered material where the key points of symmetry are shown in the $x$ axis. The $y$ axis represents the normalized frequency, $\omega_n = a/\lambda_0$, where $\lambda_0$ is the free space wavelength and $a$ is the lattice spacing. This normalization makes it very easy to design the photonic band diagram for photonic crystals [44]. The green solid line is the band line for TE polarization and the blue folded dot line is the band line for TM polarization.
10

Figure 2: (a) Unit cell considered for the simulation. The Parameters are $\varepsilon_{r1} = 4.0, \varepsilon_{r2} = 1.0, r = 0.2a$, and $a = 1$. (b) The key points of symmetry for the unit cell. (c) The photonic band diagram.

MAXWELL’S EQUATIONS IN ANISOTROPIC MATERIALS

The type of metamaterial used to wrap the 3D circuit components is anisotropic metamaterial. This section is going to discuss anisotropic materials and the next section is going to discuss anisotropic metamaterials.

The most commonly used form of Maxwell’s equations is the frequency-domain differential form [43]. The frequency-domain differential form can be written as

$$\nabla \cdot \vec{D} = \rho_v$$  \hspace{1cm} (1)

$$\nabla \times \vec{B} = 0$$  \hspace{1cm} (2)
\[ \nabla \times \vec{E} = -j\omega \vec{B} \]  
\[ \nabla \times \vec{H} = \vec{J} + j\omega \vec{D} \]  

The quantities in the above equations are the electric flux density/electric displacement field \((\vec{D})\), charge density per volume \((\rho_v)\), magnetic flux density \((\vec{B})\), electric field intensity \((\vec{E})\), magnetic field intensity \((\vec{H})\), current density \((\vec{J})\), angular frequency \((\omega)\) and \(j = \sqrt{-1}\). Equations (1)–(2) are known as divergence equations. Equation (1) is well-known as Gauss’s law for electric fields and states that the electric displacement field \(\vec{D}\) will converge into negative charges and diverge from positive charges. If there is no charge, then there will be no divergence and the \(\vec{D}\) field will form loops. Equation (2) is known as Gauss’s law for magnetic fields. There is no existence of magnetic charges in classical electromagnetics. So, the divergence of the \(\vec{B}\) field will be zero and will only form loops. Equations (3)–(4) are known as the curl equations. Equation (3) is known as Faraday’s law and states that a time-varying \(\vec{B}\) field will introduce a time-varying \(\vec{E}\) field circulating around it. Eq. (4) is known as Ampere’s circuit law and states that a time varying \(\vec{D}\) field and/or a current will introduce a time-varying \(\vec{H}\) field.

Maxwell's equations do not include the material properties. They do not directly describe the interaction between the electromagnetic fields and the materials. This information comes from the constitutive relations. The constitutive relations are

\[ \vec{D} = [\varepsilon] \vec{E} \]  
\[ \vec{B} = [\mu] \vec{H} \]  

Equation (5) describes how the \(\vec{D}\) field is related to the \(\vec{E}\) field through the permittivity tensor \([\varepsilon]\). Equation (6) describes how the \(\vec{B}\) and \(\vec{H}\) fields are related through the permeability.
The permittivity tensor $[\varepsilon]$. The permittivity is a material’s ability to store electric energy. It can also be described as the material’s ability to interact with the electric fields. The permeability indicates a material’s ability to store magnetic energy. Another way it can be said is that it measures how much the material interacts with the magnetic field.

Electromagnetic fields experience different permittivity and/or permeability when traveling in an anisotropic material depending on their polarization [43]. Charges are more easily displaced in some directions in an electrically anisotropic medium, which leads to higher permittivity in those directions [43]. Magnetic dipoles are more easily aligned in some directions than others, making the materials magnetically anisotropic. To express anisotropy mathematically, the permittivity and permeability are expressed as tensor quantities:

$$[\varepsilon] = \begin{bmatrix}
\varepsilon_{xx}(\omega) & \varepsilon_{xy}(\omega) & \varepsilon_{xz}(\omega) \\
\varepsilon_{yx}(\omega) & \varepsilon_{yy}(\omega) & \varepsilon_{yz}(\omega) \\
\varepsilon_{zx}(\omega) & \varepsilon_{zy}(\omega) & \varepsilon_{zz}(\omega)
\end{bmatrix} \quad (7)$$

$$[\mu] = \begin{bmatrix}
\mu_{xx}(\omega) & \mu_{xy}(\omega) & \mu_{xz}(\omega) \\
\mu_{yx}(\omega) & \mu_{yy}(\omega) & \mu_{yz}(\omega) \\
\mu_{zx}(\omega) & \mu_{zy}(\omega) & \mu_{zz}(\omega)
\end{bmatrix} \quad (8)$$

The real world has three dimensions, and it has three degrees of freedom. So, there are only three possible directions where the electric/magnetic field can be directed. That is why the electric/magnetic fields can experience a combination of only three different permittivity/permeability values. These three are called principal values which occur in the direction of the principal axes of the medium [44]. The constitutive tensors become diagonal matrices when they are expressed along their principal axes. When all the tensor elements are equal, they are called isotropic. If two of the tensor elements are the same and the third one is unique, then they are called uniaxial [45, 44]. Uniaxial materials have only one optical axis, that
is why they are called uniaxial. The optical axis of a crystal is the direction in which material behaves isotropic, meaning it will not exhibit birefringence [46, 43, 35]. Considering only the permittivity tensor of an uniaxial material, two of the tensor values are equal and called the ordinary permittivity $\varepsilon_o$ and the other tensor value is called the extraordinary permittivity $\varepsilon_e$. The difference between these two permittivity values is the strength of anisotropy or birefringence [47, 43]. Birefringence is formed when an electromagnetic structure shows anisotropy [43]. If $\varepsilon_e > \varepsilon_o$, then the uniaxial material shows positive birefringence. If $\varepsilon_e < \varepsilon_o$, then the uniaxial material shows negative birefringence. When the three tensor elements have unique values, then the material is called biaxial [45, 44]. Biaxial materials are named so because they have two optical axes [43]. Table 1 categorizes the different kinds of anisotropic materials.

<table>
<thead>
<tr>
<th>Anisotropy</th>
<th>Dielectric Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>$\begin{bmatrix} \varepsilon &amp; 0 &amp; 0 \ 0 &amp; \varepsilon &amp; 0 \ 0 &amp; 0 &amp; \varepsilon \end{bmatrix}$</td>
</tr>
<tr>
<td>Uniaxial</td>
<td>$\begin{bmatrix} \varepsilon_o &amp; 0 &amp; 0 \ 0 &amp; \varepsilon_o &amp; 0 \ 0 &amp; 0 &amp; \varepsilon_e \end{bmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_e &gt; \varepsilon_o$ positive birefringence</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_e &lt; \varepsilon_o$ negative birefringence</td>
</tr>
<tr>
<td>Biaxial</td>
<td>$\begin{bmatrix} \varepsilon_a &amp; 0 &amp; 0 \ 0 &amp; \varepsilon_b &amp; 0 \ 0 &amp; 0 &amp; \varepsilon_c \end{bmatrix}$</td>
</tr>
</tbody>
</table>
ANISOTROPIC METAMATERIALS

The type of metamaterial that will be generated around the 3D circuit components is an anisotropic metamaterial. This section is going to discuss what an anisotropic metamaterial is, and how different kinds of anisotropic metamaterials can be generated.

Nature does not provide strong anisotropic materials. Engineered materials provide a way to obtain strong anisotropy [48]. A periodic structure will show anisotropy when it loses its rotational symmetry [43]. If a structure looks the same after some rotation about a point, then it has rotational symmetry [49]. The anisotropic material will exhibit different electromagnetic properties for different orientations of the fields. So, the alignment of the field causes a wave to experience different material properties. The effective permittivity $\varepsilon_{\text{eff}}$ of the grating is calculated from the constitutive relations in Eq. (5) using average field values throughout the medium.

$$\varepsilon_{\text{eff}} = \frac{\bar{D}_{\text{avg}}}{\bar{E}_{\text{avg}}}$$

A subwavelength binary grating shows different material properties depending on the polarization of the field [43]. A binary grating is shown in Figure 3 which exhibits one material property in two directions and a different material property in the third direction. For perpendicular polarization, the $\bar{E}$ field is polarized parallel to the grating planes but perpendicular to the grating vector $\vec{K}$. The effective permittivity of the grating for perpendicular polarization is $\varepsilon_{\perp}$ which is estimated by Eq. (11).

For perpendicular polarization, the tangential boundary condition at the interface between two mediums requires that the electric fields should be continuous across the grating interfaces, but the displacement fields are not continuous [50, 43]. So, the electric field $\bar{E}$ is nearly constant throughout the grating as long as the device is sufficiently subwavelength [47, 43]. As the $\bar{E}$ field
is continuous throughout the grating, the average $\vec{E}$ field is $\overline{E}_{\text{avg}} = \vec{E}$. The tangential boundary condition dictates that the electric flux density $\vec{D}$ is not uniform because it is discontinuous across all the interfaces of the grating planes. The $\vec{D}$ field in each dielectric comprising the grating in Figure 3 is $\vec{D}_1 = \varepsilon_1 \overline{E}_{\text{avg}}$ and $\vec{D}_2 = \varepsilon_2 \overline{E}_{\text{avg}}$, where $\varepsilon_1$ and $\varepsilon_2$ are the permittivity values of the two materials. The average $\vec{D}$ field is calculated from the weighted average of the $\overline{E}$ field throughout the grating.

$$\overline{D}_{\text{avg}} = f\overline{D}_1 + (1 - f)\overline{D}_2 = f \varepsilon_1 \overline{E}_{\text{avg}} + (1 - f) \varepsilon_2 \overline{E}_{\text{avg}}$$

(10)

Here, $f$ if the volumetric fill factor for the fraction of grating comprising $\varepsilon_1$. It is possible to derive the expression of effective permittivity for perpendicular polarization $\varepsilon_\perp$ by dividing both sides of Eq. (10) by $\overline{E}_{\text{avg}}$.

$$\varepsilon_\perp \approx f \varepsilon_1 + (1 - f) \varepsilon_2$$

(11)

For parallel polarization, the $\overline{E}$ field is polarized perpendicular to the grating planes and parallel to the grating vector $\vec{K}$. The effective permittivity for parallel polarization is $\varepsilon_\parallel$ which is estimated by Eq. (13).

The normal boundary condition at the interface between two mediums requires that the displacement field is nearly continuous throughout the grating interfaces, but the electric field is not continuous [50, 43]. So, according to the normal boundary condition the average $\vec{D}$ field becomes $\overline{D}_{\text{avg}} = \overline{D}$, but the $\vec{E}$ field is not continuous. The $\vec{E}$ field for the dielectric with permittivity value $\varepsilon_1$ is $\vec{E}_1 = \varepsilon_1^{-1} \overline{D}_{\text{avg}}$ and for the dielectric with permittivity value $\varepsilon_2$ is $\vec{E}_2 = \varepsilon_2^{-1} \overline{D}_{\text{avg}}$.

So, the average $\overline{E}$ field throughout the grating is
\[ \bar{E}_{\text{avg}} = f\bar{E}_1 + (1 - f)\bar{E}_2 = \frac{fD_{\text{avg}}}{\varepsilon_1} + \frac{(1 - f)D_{\text{avg}}}{\varepsilon_2} \]  

(12)

The effective permittivity for parallel polarization \( \varepsilon_\parallel \) is calculated from Eq. (12) by dividing both sides \( D_{\text{avg}} \).

\[ \frac{1}{\varepsilon_\parallel} \approx \frac{f}{\varepsilon_1} + \frac{1 - f}{\varepsilon_2} \]  

(13)

The effective permittivity for perpendicular polarization is slightly higher than the effective permittivity for parallel polarization [43]. So, the effective permittivity in two directions is slightly higher than the effective permittivity in the third direction. From the two types of birefringence discussed in the previous section, it can be concluded that the subwavelength binary gratings show negative birefringence where \( \varepsilon_o = \varepsilon_\perp \) and \( \varepsilon_z = \varepsilon_\parallel \). A similar circumstance arises for the magnetic material having a grating with different permeabilities.

Figure 3: Relation between the grating and various vector quantities involved.
Metamaterials that exhibit positive birefringence are called positive uniaxial metamaterials (PUM) and metamaterials that exhibit negative birefringence are called negative uniaxial metamaterials (NUM). To achieve positive birefringence, the electric fields along two axes must be perpendicular to the interfaces of the materials, while the fields along the third must be parallel to the interfaces of the materials. An array of high permittivity cylinders and low permittivity backgrounds can work as a PUM and show positive birefringence. This combination of materials makes an extraordinary axis parallel to the cylinders and two ordinary axes perpendicular to the cylinders [48, 47]. Figure 4 (a) shows a material combination that can work as a PUM. Negative birefringence occurs when the electric fields along two axes are parallel to the interfaces of the materials while the third one is perpendicular to the interfaces of the materials. Any array of planar slabs can exhibit negative birefringence and can work as a NUM [40, 43]. So, the two ordinary axes are aligned parallel to the slabs and the extraordinary axis is perpendicular to the slabs [47]. Figure 4 (b) shows the combination of two materials that can work as a NUM.

![Figure 4](image_url)

**Figure 4:** (a) Positive uniaxial metamaterial (PUM) and (b) negative uniaxial metamaterial (NUM).

**SPACE STRETCHING**

Electrical circuits typically contain circuit components that are located close to each other, causing interference, mutual coupling, and, consequently, reducing performance. It is possible to
find engineered materials that can electrically stretch the distance between circuit components without changing their physical distance. It is expected that this engineered material will reduce interference and mutual coupling and increase the performance of the circuit. In this dissertation, the concept of space stretching has been applied to reduce mutual coupling and interference in 3D circuit components.

The process of stretching the space between two circuit components involves coordinate transformation and transformation optics. The coordinate transformation defines the parameters that transform the existing coordinate system into a stretched coordinate system. Transformation optics is used to calculate the material properties that will stretch the coordinates in the manner described by the coordinate transformation [47, 51, 52].

Figure 5: Two circuit components placed in close proximity.

Figure 5 is showing two circuit components placed in close proximity. The coordinate transformation is used to stretch the space in the z direction by a factor $a$. In this case, the $x$ and $y$ directions will be unchanged in the transformed coordinate system, and the $z$ direction gets changed. The transformed coordinate system is defined as
Now, transformation optics will be applied to calculate the permittivity and permeability values that will stretch the z axis in the prescribed manner. The permeability and permittivity tensors found from transformation optics are [47]

\[
\begin{bmatrix}
a\varepsilon & 0 & 0 \\
0 & a\varepsilon & 0 \\
0 & 0 & \varepsilon/a
\end{bmatrix}
\]

(14)

\[
\begin{bmatrix}
a\mu & 0 & 0 \\
0 & a\mu & 0 \\
0 & 0 & \mu/a
\end{bmatrix}
\]

(15)

The first two elements of the diagonal tensor are the same and the third one is different. If \(a > 1\), then the third element of the diagonal becomes smaller than the other two permittivity and permeability values. Therefore, it indicates that the material is a NUM [40, 53]. Figure 6 shows the NUM used to stretch the space between two circuit components. One of the materials that the NUM is composed of has the permittivity \(\varepsilon_1\) and another material has the permittivity \(\varepsilon_2\).
Figure 6: Negative uniaxial metamaterial parameters to stretch the space between two circuit components.

The permittivity $\varepsilon$ and the stretching factor $a$ used in Eq. (14) can be expressed in terms of the ordinary effective permittivity and extraordinary effective permittivity in the following way [47]:

\[
\varepsilon = \sqrt{\varepsilon_o \varepsilon_e} \quad \text{(16)}
\]

\[
a = \sqrt{\varepsilon_o / \varepsilon_e} \quad \text{(17)}
\]

Putting the expression of $\varepsilon_o$ from Eq. (11) and $\varepsilon_e$ from Eq. (13) ($\varepsilon_o = \varepsilon_\perp$ and $\varepsilon_e = \varepsilon_\parallel$ for a NUM) in the above two equations, the final expression for permittivity and stretching factor can be derived in terms of $f$, $\varepsilon_1$, and $\varepsilon_2$ [47, 43].

\[
\varepsilon = \frac{f \varepsilon_1 + (1-f) \varepsilon_2}{\sqrt{f \varepsilon_1^{-1} + (1-f) \varepsilon_2^{-1}}} \quad \text{(18)}
\]

\[
a = \sqrt{1 + \frac{f(1-f)(\varepsilon_1 - \varepsilon_2)^2}{\varepsilon_1 \varepsilon_2}} \quad \text{(19)}
\]
The maximum value of the stretching factor is achieved if the fill factor is 0.5 [47]. This means optimized stretching between components can be obtained without knowing the materials that the NUM will be composed of and the NUM dielectric layers will always have the largest possible feature size [40, 47].

**VORONOI TESSELLATION**

In this dissertation, the Voronoi algorithm is used to distribute the space between the circuit components where the SVAMs will be generated for each circuit component. If Voronoi tessellation is applied to some points in a space, it partitions the space into regions according to the distance between the points. Each region contains one point inside it and is defined as a Voronoi cell. The regions are allocated to the Voronoi cells in such a way that the region assigned to that Voronoi cell is closer to the inside point than any other point in that space [54]. The Voronoi diagram includes all Voronoi cells. The Voronoi algorithm is modified for the SVAM generating algorithm so that it can work on objects instead of points to create the Voronoi cells.

To illustrate the concept of Voronoi tessellation, Figure 7 (a) shows the components of a circuit. To determine the closest area to each of the components, the Voronoi algorithm is applied. Figure 7 (b) shows the circuit components with their respective Voronoi regions.

![Figure 7](image-url)

**Figure 7:** (a) Circuit components. (b) Circuit components inside their respective Voronoi regions.
NUMERICAL SOLUTION OF LAPLACE’S EQUATION

The Laplace equation is used in the SVAM generation algorithm for 3D circuits. The Laplace equation is solved inside each Voronoi cell of the 3D circuit that will be used to create the SVAMs. This section is going to discuss how the Laplace equation is solved in the Voronoi cells, but how it helps to create SVAMs is discussed in chapter 3.

CAD files are used to represent the circuit components in the SVAM algorithm. The CAD files represent the circuit components as meshes. Meshing is the discretization of the given domain of the component into simpler elements such as triangles, quadrilaterals, pentagons for 2D and tetrahedra, hexahedra, etc. for 3D [55].

Mathematically the Laplace equation is expressed as

\[ \Delta^2 u = 0 \] (20)

For Cartesian 3D coordinates, the Laplace equation is

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \] (21)

The above declaration indicates that the Laplace equation is a second-order partial differential equation. Suppose the boundary values of a function are known, then the values everywhere else need to be calculated.

The Laplace equation is set equal to a function that contains the known boundary values of the function. Everywhere else other than the boundary of the grid for which the Laplace is going to be solved is set to any value, but the most convenient is to set it to zero [56]. The solution to the Laplace equation is going to fill the grid between the two boundaries with linearly varying numbers. In other words, the Laplace equation can be used as a number filler inner between two boundary values.
The finite element method (FEM) is one of the important and useful numerical tools for scientists and engineers to find an approximate solution to differential equations [57, 58]. A detailed procedure to solve the Laplace equation using FEM can be found in [58]. In this dissertation, the geometry processing library libigl's [59] Laplace solver was used to solve the Laplace equation for meshes between two boundaries of a Voronoi cell. Figure 8 (a) shows the representation of a 2D object with quadratic meshes. The inner boundary of the object is set to 0 and the outer boundary is set to 1. The intermediate values are initialized to 0 and will be calculated using the Laplace equation. Figure 8 (b) shows the solution where the space between the two boundaries is filled with linearly varying numbers after solving the Laplace equation.

Figure 8: (a) Mesh with boundary values and unknown values before solving the Laplace equation. (b) Unknown mesh values filled with numbers after solving the Laplace equation.
The complex angle concept is used in this dissertation for the analysis of plane waves at the interface of lossy/complex mediums. Let's assume a plane wave is traveling from a lossless medium to a lossy medium at an oblique angle. This will produce a field at the interface with constant amplitude over the area of the surface. Boundary conditions at the interface dictate that the field in the lossy medium must have the same constant amplitude. However, the wave in the lossy medium must also decay. This forces the decay to manifest itself solely in the direction perpendicular to the interface, making the planes of constant amplitude parallel to the interface. At the same time, the planes of the constant phase will be aligned with the refracted angle. Refraction at the lossy interface makes the planes of constant phase to be misaligned with the planes of constant amplitude [60] and the plane wave is inhomogeneous in the lossy medium. Inhomogeneous waves can be expressed mathematically in at least two different ways. The first approach is to express the wave using a real phase vector $\vec{\beta}$ and a real attenuation vector $\vec{\alpha}$. The second approach is to express the wave using complex angle notation [61, 62] which is used in this dissertation. The first part of Eq. (22) shows the ordinary expression for a wave at a transmission angle $\tilde{\theta}_{trn}$. In this equation, $\tilde{n}_{trn}$ represents the complex refractive index of lossy medium, $k_0$ is the free space wavenumber, and $\tilde{E}_{0, trn}$ is the complex amplitude for the transmitted wave.

$$
\tilde{E}_{un} = \tilde{E}_{0, trn} \exp \left[ -jk_0 \tilde{n}_{trn} \left( x \sin \tilde{\theta}_{trn} + z \cos \tilde{\theta}_{trn} \right) \right]
$$

$$
= \tilde{E}_{0, trn} \exp \left\{ k_0 \left( x \text{Im} \left[ \tilde{n}_{trn} \sin \tilde{\theta}_{trn} \right] + z \text{Im} \left[ \tilde{n}_{trn} \cos \tilde{\theta}_{trn} \right] \right) \right\}
$$

$$
\cdot \exp \left\{ -jk_0 \left( x \text{Re} \left[ \tilde{n}_{trn} \sin \tilde{\theta}_{trn} \right] + z \text{Re} \left[ \tilde{n}_{trn} \cos \tilde{\theta}_{trn} \right] \right) \right\}
$$

(22)
The second part of Eq. (22) shows the form it takes when the real and imaginary parts are written explicitly and separately. The first exponential describes the amplitude profile of the wave while the second exponential describes the phase. This shows that complex angles are sufficient to describe inhomogeneous plane waves where the planes of constant amplitude and planes of constant phase are quantified independently.

Waves propagate in a way that the arguments of the exponential terms describing the waves are constant. So, the arguments of the exponential term describing amplitude in Eq. (22) can be written as

\[ x \text{Im}\left(\hat{n}_{im} \sin \hat{\theta}_{im}\right) + z \text{Im}\left(\hat{n}_2 \cos \hat{\theta}_{im}\right) = \text{constant} \]  

(23)

The \(x\) and \(z\)-axes are the two axes representing the opposite and adjacent legs of the right triangle formed at the interface. Using the trigonometric formula, the angle of the plane of constant amplitude is given by

\[ \psi_{\text{amp}} = \tan^{-1}\left(\frac{\text{Im}\left(\hat{n}_{im} \sin \hat{\theta}_{im}\right)}{\text{Im}\left(\hat{n}_{im} \cos \hat{\theta}_{im}\right)}\right) \]  

(24)

Similarly, the arguments of the exponential term describing phase in Eq. (22) can be written as

\[ x \text{Re}\left(\hat{n}_{im} \sin \hat{\theta}_{im}\right) + z \text{Re}\left(\hat{n}_2 \cos \hat{\theta}_{im}\right) = \text{constant} \]  

(25)

The trigonometric formula can be used to find the angle of the plane of constant phase

\[ \psi_{\text{phase}} = \tan^{-1}\left(\frac{\text{Re}\left(\hat{n}_{im} \sin \hat{\theta}_{im}\right)}{\text{Re}\left(\hat{n}_{im} \cos \hat{\theta}_{im}\right)}\right) \]  

(26)

In Figure 9, a plane wave is propagating from a lossless medium to another lossless medium. In this figure, one amplitude plane of the planes of constant amplitude is shown in broken
yellow lines, and one phase plane of the constant phase planes is shown in broken violet lines. Red arrows in the figure indicate the direction of the wave. For both mediums, the plane of constant phase and the plane of constant amplitude is aligned. As a consequence, both the incident angle and the transmission angle are real. Figure 10 shows the picture of a wave traveling from a lossless medium into a lossy medium. In the lossless medium (medium 1), the plane of constant amplitude and plane of constant phase is aligned, which is expressed mathematically with a purely real angle. In the lossy medium (medium 2), however, the plane of constant amplitude and the plane of constant phase are not aligned, requiring the angle to be complex to mathematically describe this wave.

Figure 9: Plane-wave propagating from a lossless medium to a lossless medium.
Figure 10: Plane-wave propagating from a lossless medium to a lossy medium.
Chapter 3: Generation of spatially variant anisotropic metamaterials in 3D circuits

This chapter describes the steps involved in the spatially variant anisotropic metamaterial (SVAM) generation algorithm. A prototype of the algorithm was developed in Ref. [63] using the finite-difference method (FDM). However, the staircase approximation of FDM led to errors in the lattices. In the present work, this problem was overcome by using the finite element method (FEM).

First, the space-stretching concept is discussed that is used to identify that a negative uniaxial metamaterial performs this operation. Then the simulation results were presented where SVAM was applied to decouple the two waveguides in a directional coupler. Then the SVAM generation algorithm is described step by step. The core of the algorithm is developed using the geometry processing library libigl [59] which is written in C++. The first step of the algorithm is to import the meshes of the circuit components and create a bounding region that will contain the SVAM to the vicinity of the components. The second step is to apply the Voronoi algorithm to identify regions around each component where isolated sections of SVAM will be created and eventually connected. The space allocated to each object from the Voronoi algorithm is called the Voronoic cell. Third, volumetric tetrahedral meshes are generated around each Voronoic cell. Fourth, Laplace’s equation is solved in each Voronoic cell to generate a linear gradient function inside them. This function extends from the boundary of the component to the boundary of the Voronoic cell. Fifth, isocontours are created from these gradients that become the interfaces between the alternating materials comprising the SVAM. To bound the SVAM to the final shape of the circuit, a Boolean operation is performed. This operation removes any part of the SVAM that resides outside of what is defined by the shapefile.
Next, the integration of SVAM algorithm into the Blender CAD software is discussed. Blender supports Python scripting, but the algorithm is written in C++. A library was used to integrate the algorithm into Blender. A simple and easy-to-use graphical interface is designed for ease of use. At the end, a 3D printed SVAM circuit is presented, which proves the feasibility of the algorithm.

**SPACE STRETCHING WITH NEGATIVE UNIAXIAL METAMATERIALS**

There are at least two ways that SVAMs might be used to reduce coupling and interference, space stretching and field sculpting. Field sculpting was presented in Ref. [5]. The SVAMs generated in the present work operate on the principal of space stretching, where the electrical distance between components is increased without changing the physical distance. Transformation optics (TO) was used to determine the properties of the medium that serve the purpose. The basic medium that can provide space stretching along the z axis is determined by a coordinate transformation that scales the z axis by a factor a. This coordinate transformation is applied to Maxwell’s equation. Then moving the coordinate transformation math into the constitutive parameters gives the material properties,

\[
[\mu'] = \varepsilon' = \begin{bmatrix}
\varepsilon a & 0 & 0 \\
0 & \varepsilon a & 0 \\
0 & 0 & \varepsilon/a
\end{bmatrix}
\] (27)

Equation (27) corresponds to negative uniaxial metamaterial (NUM) when \(a > 1\). In other research, it was shown that a NUM can be formed using alternating layers of two different materials with different constitutive values [47]. To optimize space stretching, the thickness of all the layers should be equal and made to be less than a quarter wavelength inside the medium [47].
To demonstrate the operation of a SVAM, a directional coupler was simulated using the finite-difference frequency-domain (FDFD) method [44, 64, 65]. The directional coupler has two waveguides on both sides of it. The waveguides exchange power periodically when they are brought close enough together to be electromagnetically coupled [44]. This three-dimensional problem was reduced to a two-dimensional simulation using the effective index method described in [44]. The length of the directional coupler was considered 1 $\mu m$. The width of the waveguides was 0.2 $\mu m$ and the gap between the waveguides was 0.16 $\mu m$. Figure 11 (a)-(d) shows the simulation results of power exchange between the waveguides considering different isotropic materials placed between them. For each simulation, the material outside the coupler has a refractive index of 1.5 and the material in the waveguides has a refractive index of 3.1. The directional coupling is weakest with air placed between the waveguides, but the directional coupling is observed in all four cases. For Figure 11 (a), the maximum power exchanged from the left waveguide to the right waveguide is 56%, for Figure 11 (b) it is 69%, for Figure 11 (c) it is 82% and for Figure 11 (d) it is 88%. An SVAM is placed between the waveguides in Figure 11 (e) and the directional coupling is suppressed dramatically. The power exchanged from the left waveguide to the right waveguide reduce to 32%. Considering Figure 11 (e), if the $\mu, \varepsilon$ values in $x, z$ direction changed to 4.5 and same values considered in $y$ direction, the power exchange from left to right waveguide reduce to 25%. The simulation was done considering TM polarization.
Figure 11: Simulation results for directional coupler using different materials between the waveguides. (a)-(d) Range of isotropic materials. (e) SVAM.

**Algorithm to Generate SVAM**

It is trivial to generate a NUM between just two objects. However, it is not as clear how to place a NUM between multiple components in a 3D circuit while keeping the overall structure of the NUM smooth, continuous, and free of defects. The orientation of the NUM must be spatially varied while being kept continuous and maintaining the baseline design of alternating layers between adjacent components. This section presents an algorithm that can generate an SVAM around any number of components in a 3D circuit.
To illustrate the algorithm, consider the 3D circuit shown in Figure 12 (a) in which an
SVAM will be generated. The black lines forming a box around the components define the final
shape of the circuit and SVAM.

First, the mesh data of circuit components are imported into the algorithm. The maximum
distance between the objects is measured, and a bounding region is created that is multiple times
larger than the maximum distance. The Voronoic cells will be generated inside the bounding
region. The bounding region restricts Voronoic cells to go to infinity.

As the algorithm progresses, isolated regions of the SVAM will be generated around each
circuit component separately. Later, these will be combined to form the overall SVAM. This
requires some volume of space around each component. These regions are calculated by applying
the Voronoi algorithm [66, 67, 68] on circuit components and the bounding region. The Voronoi
algorithm identifies the volume of space around each component that is closer to that component
than any other component. A Voronoic cell is calculated from the intersection of half-planes
created by taking the perpendicular bisector of the closest two points [69]. The CAD files used in
this algorithm are STL/OBJ, which are surface meshes. The vertices of these triangular meshes are
the point clouds of the object. The Voronoi algorithm is applied to these point clouds. Some points
at the outer surface of the circuit components do not have another point to perform Voronoi
tessellation. So, these Voronoic cells reach infinity. The bounding region is created to provide
additional points that restrict Voronoic cells from going to infinity. Figure 12 (b) shows the
bounding region with circuit components.
Figure 12: (a) Circuit components of a 3D circuit. (b) Circuit components with bounding region.

Figure 13 shows the components surrounded by their respective Voronoic cells. The Voronoic cells are only surface meshes. The next step is to calculate a gradient inside the Voronoic cells. Volumetric meshes are needed to perform the gradient operation. The tetrahedral meshing library TetWild [70] was used to convert the surface meshes to volumetric tetrahedral meshes.
Each Voronoic cell has an outer surface created from the Voronoi algorithm. It also has a hollow inner surface defined by the outer boundary of the component. The boundary conditions are applied to both inner and outer surfaces. The hollow surface inside the Voronoic cell is set to 0 and the outer surface of the Voronoic cell is set to 1. Then a gradient is generated for the meshes inside the Voronoic cell by solving the Laplace equation. Laplace’s equation for a scalar function $f$ in a 3D space is,

$$\nabla^2 f = 0$$  \hspace{1cm} (28)

Laplace’s equation fills the meshes from 0 to 1 between two surfaces throughout the entire Voronoic cell. Figure 14 shows the cross-section of a Voronoic cell and the boundary conditions used to solve Laplace’s equation.
The marching tetrahedra method [71] is applied for isocontouring the gradient solution. The idea of this method is to contour the isosurface passing through each tetrahedron. The method iterates over all tetrahedra in the mesh and stitches together the final mesh [59]. This creates rings inside the Voronoic cell. These iso-contours form the interfaces between alternating layers of the SVAM. Figure 15 shows rings generated in the space between multiple circuit components.

The desired shape of the final circuit needs to be provided in the form of a CAD file. A Boolean operation is done with the volumetric isocontours and the shapefile. It should be noted
that the length of the bounding region should be large enough to perform a Boolean operation to result in the desired shape. Figure 16 (a) and (b) show two alternating layers of SVAM after the Boolean operation. These two layers represent two different materials that form the SVAM. Figure 16 (c) shows the two alternating layers together forming the final shape of SVAM.

![Figure 16: (a) Material 1 of the SVAM. (b) Material 2 of the SVAM. (c) Both materials together form the overall SVAM.](image)

**INTEGRATING THE SVAM ALGORITHM INTO BLENDER**

The SVAM algorithm was integrated into the CAD software Blender to give it a graphical user interface (GUI) and make it more convenient to use. A user can design or import a 3D circuit in Blender and use this algorithm to fill the space between components with an SVAM. Blender is a popular, free, and versatile CAD software that allows python scripting to incorporate custom add-ons. The algorithm is written in C++ and Blender uses python scripting. So, pybind11 [72] was used to connect the C++ SVAM algorithm to Python in Blender. Screenshots of the SVAM algorithm in Blender are shown in Figure 17. The left window in Figure 17 (a) shows part of the python code that creates the SVAM panel and links the SVAM algorithm with Blender. The middle window is the SVAM panel that appears after running the python code. The right window is the Blender viewport showing circuit components where SVAM will be formed around.
After importing or designing the objects in Blender, mesh data of the objects are extracted by python and these data are sent to the algorithm written in C++ using pybind11. The algorithm creates a bounding region, performs Voronoi tessellation, solves the Laplace equation, creates isocontours around each object, and performs the final Boolean operation for the final SVAM. The C++ code returns two mesh files for two different materials which are visualized in the Blender 3D viewport. The two mesh files can be seen in green and red colors in Figure 17 (b).

The SVAM panel (middle window of Figure 17 (a) and left window of Figure 17 (b)) has three attributes. The first attribute at the top is ‘Layers’ which allows the user to select the number of alternating layers that will be generated around each component. The attribute ‘Directory’ allows users to browse the computer directory to upload the shapefile. After defining the number of layers and importing the shapefile, the ‘Create SVAM’ button is clicked to run the python and C++ code that generates the SVAM. After calculation, the final SVAM is shown along with the circuit in the 3D viewport of Blender. The right window in Figure 17 (b) shows the created SVAM layers around circuit components.
Figure 17: (a) Screenshot from left to right of Blender showing the python scripting interface, the SVAM panel, and the 3D circuit objects. (b) Screenshot of Blender showing the SVAM panel and the SVAM formed around the 3D circuit components.

**RESULTS**

The circuit demonstrated in Figure 16 (c) was 3D printed using dielectric materials and the manufactured device is shown in Figure 18.

Figure 18: 3D printed 3D volumetric circuit with SVAM.
Chapter 4: Scattering at the interface of complex mediums

This section develops the generalized Fresnel equations and calculates the conservation of energy between two complex mediums considering complex permittivity and complex permeability. First, the expressions for the electric field, magnetic field, and wave vectors are defined. In the next step, boundary conditions are applied, and transmission and reflection coefficients are derived by substituting wave expressions. In the end, the Poynting vector is used to calculate the power of incident, reflected, and transmitted waves.

GENERALIZED FRESNEL EQUATIONS AND CONSERVATION OF POWER

In this section, the generalized Fresnel equations are derived separately for TE and TM polarizations. The geometry and definitions of the polarizations are illustrated in Figure 19. The TE polarization is defined as the case where the electric field is solely perpendicular to the plane of incidence. The TM polarization is defined as the case where the electric field is solely parallel to the plane of incidence. The vectors $\vec{k}_{\text{inc}}$, $\vec{k}_{\text{ref}}$ and $\vec{k}_{\text{trn}}$ denote the complex wave vectors for the incident, reflected, and transmitted waves, respectively. The terms with the tilde symbol ~ above them indicate complex quantities. $\tilde{\theta}_{\text{inc}}$, $\tilde{\theta}_{\text{ref}}$ and $\tilde{\theta}_{\text{trn}}$ represent complex angles of the incident, reflected, and transmitted waves, respectively. The complex relative permeability and complex relative permittivity of medium 1 are denoted as $\tilde{\mu}_{r,\text{inc}}$ and $\tilde{\varepsilon}_{r,\text{inc}}$, respectively. The complex relative permeability and complex relative permittivity of medium 2 are denoted as $\tilde{\mu}_{r,\text{trn}}$ and $\tilde{\varepsilon}_{r,\text{trn}}$, respectively. The standard Cartesian axis is labeled with x, y, and z. In linear, homogeneous, and isotropic (LHI) medium, the Fresnel equations and conservation of power are independent of the angle $\tilde{\phi}$ due to symmetry. For convenience, $\tilde{\phi}$ is set to 0° and all waves reside in the xz plane. This simplified configuration is shown in Figure 19 (b) and will be used in the present analysis.
The derivation has four main steps. First, expressions for the incident, reflected, and transmitted waves are written. Second, boundary conditions are applied at the interface using the wave expressions. Third, the generalized Fresnel equations are derived from expressions coming out of the boundary conditions. Fourth, revised power conservation equations are derived considering the loss in both mediums.

Figure 19: (a) Scattering at an interface. (b) Simplified configuration for $\phi = 0^\circ$.

**Generalized Fresnel equations for TE polarization**

**Wave expressions**

Expressions for electric field components for the incident, reflected and transmitted waves for TE polarization are

$$\vec{E}_{\text{inc}} = \vec{E}_{0,\text{inc}}\alpha e^{-i\beta_{\text{inc}}\gamma}$$

(29)
\[ \vec{E}_{\text{ref}} = \hat{E}_{0,\text{ref}} \hat{a}_r e^{-j\vec{\varphi}_{\text{ref}} \cdot \vec{r}} \] (30)

\[ \vec{E}_{\text{tm}} = \hat{E}_{0,\text{tm}} \hat{a}_r e^{-j\vec{\varphi}_{\text{tm}} \cdot \vec{r}} \] (31)

\( E_{0,\text{inc}}, \ E_{0,\text{ref}} \) and \( E_{0,\text{trn}} \) are the complex amplitudes of the incident, reflected, and transmitted waves, respectively. The terms \( \hat{a}_x, \hat{a}_y, \) and \( \hat{a}_z \) are unit vectors parallel to the standard Cartesian axis. The position is denoted by the vector \( \vec{r} \). Expressions for the corresponding magnetic field components of the incident, reflected, and transmitted waves are

\[ \vec{H}_{\text{inc}} = \frac{\tilde{E}_{0,\text{inc}}}{\eta_{\text{inc}}} \left( -\cos \tilde{\theta}_{\text{inc}} \hat{a}_x + \sin \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j\vec{\varphi}_{\text{inc}} \cdot \vec{r}} \] (32)

\[ \vec{H}_{\text{ref}} = \frac{\tilde{E}_{0,\text{ref}}}{\eta_{\text{inc}}} \left( \cos \tilde{\theta}_{\text{inc}} \hat{a}_x + \sin \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j\vec{\varphi}_{\text{ref}} \cdot \vec{r}} \] (33)

\[ \vec{H}_{\text{tm}} = \frac{\tilde{E}_{0,\text{tm}}}{\eta_{\text{inc}}} \left( -\cos \tilde{\theta}_{\text{inc}} \hat{a}_x + \sin \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j\vec{\varphi}_{\text{tm}} \cdot \vec{r}} \] (34)

The complex wave vectors in these equations are defined as

\[ \vec{k}_{\text{inc}} = k_0 \tilde{n}_{\text{inc}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x + \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) \] (35)

\[ \vec{k}_{\text{ref}} = k_0 \tilde{n}_{\text{inc}} \left( \sin \tilde{\theta}_{\text{ref}} \hat{a}_x - \cos \tilde{\theta}_{\text{ref}} \hat{a}_z \right) \] (36)

\[ \vec{k}_{\text{tm}} = k_0 \tilde{n}_{\text{inc}} \left( \sin \tilde{\theta}_{\text{tm}} \hat{a}_x + \cos \tilde{\theta}_{\text{tm}} \hat{a}_z \right) \] (37)

Here, \( k_0 \) is the free space wave number and \( \tilde{\theta}_{\text{inc}}, \tilde{\theta}_{\text{ref}}, \tilde{\theta}_{\text{tm}} \) are the complex angles of incidence, reflection, and transmission, respectively. A complex angle is associated with an inhomogeneous plane wave [14]. Inhomogeneous waves can be expressed in at least two different
ways. First, the wave can be expressed using a real phase vector $\vec{\beta}$ and a real attenuation vector $\vec{\alpha}$.

Second, the wave can be expressed using complex angle notation [61, 62]. The complex angles are related through Snell’s law and the law of reflection [13]. These are

$$\vec{k}_{r,inc} = \vec{k}_{r,ref} = \vec{k}_{r,trn}$$

$$\Rightarrow k_0 \tilde{n}_{inc} \left( \sin \tilde{\theta}_{inc} \hat{a}_x \right) = k_0 \tilde{n}_{inc} \left( \sin \tilde{\theta}_{inc} \hat{a}_x \right) = k_0 \tilde{n}_{tm} \left( \sin \tilde{\theta}_{tm} \hat{a}_x \right)$$

$$\Rightarrow \tilde{n}_{inc} \sin \tilde{\theta}_{inc} = \tilde{n}_{inc} \sin \tilde{\theta}_{inc} = \tilde{n}_{tm} \sin \tilde{\theta}_{tm}$$

$$\tilde{n}_{inc} \sin \tilde{\theta}_{inc} = \tilde{n}_{tm} \sin \tilde{\theta}_{tm}$$  \hspace{1cm} (38)

$$\tilde{\theta}_{inc} = \tilde{\theta}_{tm}$$  \hspace{1cm} (39)

The complex refractive indices, $\tilde{n}_{inc}$ and $\tilde{n}_{trn}$, and the complex wave impedances, $\tilde{\eta}_{inc}$ and $\tilde{\eta}_{trn}$, are related to the complex permittivity and complex permeability. The incident wave and the reflected wave exist in medium 1 and have the same material properties. Resolving the signs of complex refractive index and complex impedance is not trivial. In fact, [21, 18] identified separate conditions to resolve the sign of complex impedance and refractive index but admitted there are still more unknown cases. For the present analysis, the easiest equations found that produce the correct signs for the most number of cases are described in [20]. Given the free space impedance $\eta_0$, these equations are

$$\tilde{n}_{inc} = \sqrt{\tilde{\mu}_{r,inc}} \sqrt{\tilde{\varepsilon}_{r,inc}}$$

$$\tilde{n}_{trn} = \sqrt{\tilde{\mu}_{r,trn}} \sqrt{\tilde{\varepsilon}_{r,trn}}$$  \hspace{1cm} (40)
It will be shown later how to deal with sign ambiguity when calculating scattering at an interface.

**Boundary conditions for TE polarization**

Assuming that there are no charges or surface currents at the interface, boundary conditions require that the tangential components $\vec{E}_T$ and $\vec{H}_T$ are continuous across the interface. The normal field components are not continuous, but the products $\varepsilon_{r,inc} \vec{E}_N$ and $\mu_{r,inc} \vec{H}_N$ are continuous. The subscripts T and N indicate tangential and normal components of the fields, respectively. The boundary conditions must hold at $z = 0$ for all values of $x$ and $y$ [13]. The boundary conditions are

$$\vec{E}_{T1}(0) = \vec{E}_{T2}(0) \quad (42)$$

$$\varepsilon_{r,inc} \vec{E}_{N1}(0) = \varepsilon_{r,rm} \vec{E}_{N2}(0) \quad (43)$$

$$\vec{H}_{T1}(0) = \vec{H}_{T2}(0) \quad (44)$$

$$\mu_{r,inc} \vec{H}_{N1}(0) = \mu_{r,rm} \vec{H}_{N2}(0) \quad (45)$$

The wave expressions in Eqs. (29)–(31) are substituted into these equations. For TE polarization, there is no normal component of the electric field, so the Eq. (43) can be eliminated. There will be only three equations left from above. Both the incident wave and reflected wave exist in medium 1 so the overall fields are expressed as.
\[
\vec{E}_1 = \vec{E}_{\text{inc}} + \vec{E}_{\text{ref}} \quad (46)
\]

\[
\vec{H}_1 = \vec{H}_{\text{inc}} + \vec{H}_{\text{ref}} \quad (47)
\]

The transmitted field has only the transmitted wave. The tangential and normal magnetic field components in Eqs. (44)–(45) are expressed in terms of the electric field components. So, the Eqs. (42), (44)–(45) are expressed as

\[
\vec{E}_{T1} (0) = \vec{E}_{T2} (0)
\]

\[
\vec{E}_{T,\text{inc}} (0) + \vec{E}_{T,\text{ref}} (0) = \vec{E}_{T,\text{tm}} (0)
\]

\[
(\vec{E}_{0,\text{inc}} \hat{a}_y) e^0 + (\vec{E}_{0,\text{ref}} \hat{a}_y) e^0 = (\vec{E}_{0,\text{tm}} \hat{a}_y) e^0
\]

\[
\vec{E}_{0,\text{inc}} + \vec{E}_{0,\text{ref}} = \vec{E}_{0,\text{tm}} \quad (48)
\]

\[
\vec{H}_{T1} (0) = \vec{H}_{T2} (0)
\]

\[
\vec{H}_{T,\text{inc}} (0) + \vec{H}_{T,\text{ref}} (0) = \vec{H}_{T,\text{tn}} (0)
\]

\[
- \frac{\vec{E}_{0,\text{inc}}}{\vec{n}_{\text{inc}}} \cos \theta_{\text{inc}} e^0 + \frac{\vec{E}_{0,\text{ref}}}{\vec{n}_{\text{inc}}} \cos \theta_{\text{inc}} e^0 = - \frac{\vec{E}_{0,\text{tm}}}{\vec{n}_{\text{tn}}} \cos \theta_{\text{tn}} e^0
\]

\[
\vec{E}_{0,\text{inc}} - \vec{E}_{0,\text{ref}} = \vec{E}_{0,\text{tm}} \frac{\vec{n}_{\text{inc}} \cos \theta_{\text{tn}}}{\vec{n}_{\text{tn}} \cos \theta_{\text{inc}}} \quad (49)
\]
Reflection and transmission coefficients for TE polarization

The reflection coefficient \( \tilde{r} \) is defined as the ratio of the amplitude of the reflected electric field \( \tilde{E}_{0,\text{ref}} \) to the amplitude of the incident electric field \( \tilde{E}_{0,\text{inc}} \). This is derived by substituting Eq. (48) into Eq. (49) to eliminate \( \tilde{E}_{0,\text{trn}} \). The new expression is solved for \( \tilde{E}_{0,\text{ref}}/\tilde{E}_{0,\text{inc}} \) which is the definition of the reflection coefficient.
\[
\frac{\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} = \frac{\tilde{E}_{0,\text{tm}} \cos \tilde{\theta}_{\text{tm}}}{\tilde{\eta}_{\text{tm}}} \\
\Rightarrow \frac{\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} = \frac{\tilde{E}_{0,\text{inc}} + \tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{tm}}} \cos \tilde{\theta}_{\text{inc}} \\
\Rightarrow \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{inc}} - \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{ref}} = \frac{1}{\tilde{\eta}_{\text{tm}}} \tilde{E}_{0,\text{inc}} + \frac{1}{\tilde{\eta}_{\text{tm}}} \tilde{E}_{0,\text{ref}} \cos \tilde{\theta}_{\text{inc}} \\
\Rightarrow \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{inc}} + \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{ref}} \cos \tilde{\theta}_{\text{inc}} = \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{inc}} - \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{ref}} \cos \tilde{\theta}_{\text{inc}} \\
\Rightarrow \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{inc}} \left( \frac{1}{\tilde{\eta}_{\text{inc}}} + \frac{1}{\tilde{\eta}_{\text{inc}}} \cos \tilde{\theta}_{\text{inc}} \right) = \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{E}_{0,\text{inc}} \left( \frac{1}{\tilde{\eta}_{\text{inc}}} - \frac{1}{\tilde{\eta}_{\text{inc}}} \cos \tilde{\theta}_{\text{inc}} \right) \\
\Rightarrow \frac{\tilde{E}_{0,\text{inc}}}{\tilde{E}_{0,\text{inc}}} = \frac{1}{\tilde{\eta}_{\text{inc}}} - \frac{1}{\tilde{\eta}_{\text{inc}}} \cos \tilde{\theta}_{\text{inc}} \\
\Rightarrow \frac{\tilde{\eta}_{\text{inc}} \tilde{\eta}_{\text{tm}} \cos \tilde{\theta}_{\text{inc}} - \tilde{\eta}_{\text{inc}} \tilde{\eta}_{\text{tm}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{inc}}} - \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{\eta}_{\text{tm}} \cos \tilde{\theta}_{\text{inc}} = \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{\eta}_{\text{tm}} \cos \tilde{\theta}_{\text{inc}} - \frac{1}{\tilde{\eta}_{\text{inc}}} \tilde{\eta}_{\text{tm}} \cos \tilde{\theta}_{\text{inc}} \\
\Rightarrow \frac{\tilde{E}_{0,\text{ref}}}{\tilde{E}_{0,\text{inc}}} = \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}} - \tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}} + \tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}} \\
\Rightarrow \tilde{r}_{\text{TE}} = \frac{\tilde{E}_{0,\text{ref}}}{\tilde{E}_{0,\text{inc}}} = \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}} - \tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}} + \tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}} 
\]

(51)

At first glance, it seems necessary to properly resolve the sign of both \(\tilde{\eta}_{\text{inc}}\) and \(\tilde{\eta}_{\text{trn}}\) to obtain a correct value for the reflection coefficient. Observe that if the sign of \(\tilde{\eta}_{\text{inc}}\) is reversed, Eq. (51) will give the same value for \(\tilde{r}_{\text{TE}}\) if it was the sign of \(\tilde{\eta}_{\text{trn}}\) that was reversed instead. This means it is not necessary to determine the correct signs of impedance to analyze scattering.
Instead, if conservation of power is not obeyed, the sign of either one of the impedances can be inverted to obtain a correct value for $\tilde{r}_{TE}$.

The transmission coefficient, $\tilde{t}$ is defined as the ratio of the amplitude of the transmitted electric field $\tilde{E}_{0,\text{trn}}$ to the amplitude of the incident electric field $\tilde{E}_{0,\text{inc}}$. Equation (51) is solved for $\tilde{E}_{0,\text{ref}}$ and this new expression is substituted into Eq. (48) and solved for $\tilde{E}_{0,\text{trn}}/\tilde{E}_{0,\text{inc}}$, which is the definition of the transmission coefficient.

\[
\tilde{E}_{0,\text{inc}} + \tilde{E}_{0,\text{ref}} = \tilde{E}_{0,\text{trn}}
\]

\[
\Rightarrow \tilde{E}_{0,\text{inc}} + \tilde{E}_{0,\text{inc}} \frac{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} - \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}}{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} + \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}} = \tilde{E}_{0,\text{trn}}
\]

\[
\Rightarrow \tilde{E}_{0,\text{inc}} \left( 1 + \frac{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} - \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}}{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} + \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}} \right) = \tilde{E}_{0,\text{trn}}
\]

\[
\Rightarrow \frac{\tilde{E}_{0,\text{trn}}}{\tilde{E}_{0,\text{inc}}} = 1 + \frac{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} - \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}}{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} + \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}}
\]

\[
\Rightarrow \frac{\tilde{E}_{0,\text{trn}}}{\tilde{E}_{0,\text{inc}}} = \frac{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} + \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}}{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} + \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}} + \frac{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} - \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}}{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} + \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}}
\]

\[
\tilde{t}_{\text{TE}} = \frac{\tilde{E}_{0,\text{trn}}}{\tilde{E}_{0,\text{inc}}} = \frac{2\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}}}{\eta_{\text{trn}} \cos \tilde{\theta}_{\text{inc}} + \eta_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}} 
\] (52)

If the sign of an impedance was inverted to obtain $\tilde{r}_{TE}$ from Eq. (51), the same impedance values should be used in Eq. (52) to obtain $\tilde{t}_{\text{TE}}$.  

47
**Poynting vectors for TE polarization**

The Poynting vector $\vec{\phi}$ describes power flow and is calculated by taking the cross product of electric field intensity and magnetic field intensity. The time average complex Poynting vector in medium 1 is

$$\tilde{\phi}_1 = \frac{1}{2} \text{Re} \left( \tilde{E}_i \times \tilde{H}_i^* \right) \quad (53)$$

Substituting these expressions for $\tilde{E}_1$ and $\tilde{H}_1$ from Eqs. (46)–(47) into Eq. (53), an expression for the Poynting vector in medium 1 is derived. After simplifying and regrouping the terms, the Poynting vector in medium 1 is written as

$$\tilde{\phi}_1 = \frac{1}{2} \text{Re} \left[ (\tilde{E}_{\text{inc}} + \tilde{E}_{\text{ref}}) \times (\tilde{H}_{\text{inc}} + \tilde{H}_{\text{ref}}) \right]$$

$$\Rightarrow \tilde{\phi}_1 = \frac{1}{2} \text{Re} \left( \tilde{E}_{\text{inc}} \times \tilde{H}_{\text{inc}}^* + \tilde{E}_{\text{inc}} \times \tilde{H}_{\text{ref}}^* + \tilde{E}_{\text{ref}} \times \tilde{H}_{\text{inc}}^* + \tilde{E}_{\text{ref}} \times \tilde{H}_{\text{ref}}^* \right)$$

$$\tilde{\phi}_1 = \frac{1}{2} \text{Re} \left( \tilde{E}_{\text{inc}} \times \tilde{H}_{\text{inc}}^* \right) + \frac{1}{2} \text{Re} \left( \tilde{E}_{\text{ref}} \times \tilde{H}_{\text{ref}}^* \right) + \frac{1}{2} \text{Re} \left( \tilde{E}_{\text{inc}} \times \tilde{H}_{\text{ref}}^* + \tilde{E}_{\text{ref}} \times \tilde{H}_{\text{inc}}^* \right) \quad (54)$$

From Eq. (54), it can be seen that the Poynting vector in medium 1 has four terms, but the two cross terms have been grouped together. The terms in this equation are referred to as the incident term, reflected term, and cross term [13]. The cross term arises due to interference between the incident and reflected fields in a complex medium [13, 17]. From Faraday’s law, the magnetic field can be written in terms of the electric field as $\vec{H} = (\vec{k} \times \vec{E})/\omega \mu$ [73]. Using this relation, Eq. (54) can be put completely in terms of the electric field.
\[ \tilde{\varphi}_1 = \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{inc}} \times \left( \frac{\tilde{k}_{\text{inc}} \times \tilde{E}_{\text{inc}}}{\omega \tilde{\mu}_{\text{inc}}} \right)^* \right] + \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{ref}} \times \left( \frac{\tilde{k}_{\text{ref}} \times \tilde{E}_{\text{ref}}}{\omega \tilde{\mu}_{\text{ref}}} \right)^* \right] + \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{ref}} \times \left( \frac{\tilde{k}_{\text{ref}} \times \tilde{E}_{\text{ref}}}{\omega \tilde{\mu}_{\text{inc}}} \right)^* \right] + \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{inc}} \times \left( \frac{\tilde{k}_{\text{inc}} \times \tilde{E}_{\text{ref}}}{\omega \tilde{\mu}_{\text{inc}}} \right)^* \right] \]

\[ \Rightarrow \tilde{\varphi}_1 = \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{inc}} \times \left( \frac{\tilde{k}_{\text{inc}} \times \tilde{E}_{\text{inc}}}{\omega \tilde{\mu}_{\text{inc}}} \right)^* \right] + \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{ref}} \times \left( \frac{\tilde{k}_{\text{ref}} \times \tilde{E}_{\text{ref}}}{\omega \tilde{\mu}_{\text{inc}}} \right)^* \right] + \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{inc}} \times \left( \frac{\tilde{k}_{\text{inc}} \times \tilde{E}_{\text{ref}}}{\omega \tilde{\mu}_{\text{inc}}} \right)^* \right] + \frac{1}{2} \text{Re} \left[ \tilde{E}_{\text{ref}} \times \left( \frac{\tilde{k}_{\text{ref}} \times \tilde{E}_{\text{ref}}}{\omega \tilde{\mu}_{\text{inc}}} \right)^* \right] \]

\[ \tilde{\varphi}_1 = \text{Re} \left\{ \frac{1}{2 \omega \tilde{\mu}_{\text{inc}}} \left( \tilde{E}_{\text{inc}} \times \left[ \tilde{k}_{\text{inc}} \times \tilde{E}_{\text{inc}}^* \right] \right) + \frac{1}{2 \omega \tilde{\mu}_{\text{inc}}} \left( \tilde{E}_{\text{ref}} \times \left[ \tilde{k}_{\text{ref}} \times \tilde{E}_{\text{ref}}^* \right] \right) \right\} \]

The Poynting vector in medium 2 only has the transmitted wave. It can be written in terms of just the electric field as

\[ \tilde{\varphi}_2 = \frac{1}{2} \text{Re} \left\{ \tilde{E}_{\text{tm}} \times \tilde{H}_{\text{tm}}^* \right\} \]

\[ \Rightarrow \tilde{\varphi}_2 = \frac{1}{2} \text{Re} \left\{ \tilde{E}_{\text{tm}} \times \left( \frac{\tilde{k}_{\text{tm}} \times \tilde{E}_{\text{tm}}}{\omega \tilde{\mu}_{\text{tm}}} \right)^* \right\} \]

\[ \Rightarrow \tilde{\varphi}_2 = \frac{1}{2} \text{Re} \left\{ \tilde{E}_{\text{tm}} \times \left( \frac{\tilde{k}_{\text{tm}} \times \tilde{E}_{\text{tm}}^*}{\omega \tilde{\mu}_{\text{tm}}^*} \right) \right\} \]

\[ \tilde{\varphi}_2 = \text{Re} \left\{ \frac{1}{2 \omega \tilde{\mu}_{\text{tm}}^*} \left( \tilde{E}_{\text{tm}} \times \left[ \tilde{k}_{\text{tm}} \times \tilde{E}_{\text{tm}}^* \right] \right) \right\} \]

Poynting vectors for the incident term, reflected term, transmitted term, and cross term are read off from Eqs. (55) and (56) to be
\[
\tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{inc}}^*} \left( \tilde{E}_{\text{inc}} \times \left[ \tilde{k}_{\text{inc}}^* \times \tilde{E}_{\text{inc}}^* \right] \right) \right\}
\]

(57)

\[
\tilde{\phi}_{\text{ref}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{inc}}^*} \left( \tilde{E}_{\text{ref}} \times \left[ \tilde{k}_{\text{ref}}^* \times \tilde{E}_{\text{ref}}^* \right] \right) \right\}
\]

(58)

\[
\tilde{\phi}_{\text{trn}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{trn}}^*} \left( \tilde{E}_{\text{trn}} \times \left[ \tilde{k}_{\text{trn}}^* \times \tilde{E}_{\text{trn}}^* \right] \right) \right\}
\]

(59)

\[
\tilde{\phi}_{c} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{inc}}^*} \left( \tilde{E}_{\text{ref}} \times \left[ \tilde{k}_{\text{inc}}^* \times \tilde{E}_{\text{inc}}^* \right] + \tilde{E}_{\text{inc}} \times \left[ \tilde{k}_{\text{ref}}^* \times \tilde{E}_{\text{ref}}^* \right] \right) \right\}
\]

(60)

Next, the double-cross products in the above four equations are expanded using the vector identity \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{B} \cdot \vec{A}) \).

\[
\tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{inc}}^*} \left[ \tilde{k}_{\text{inc}}^* \left( \tilde{E}_{\text{inc}} \cdot \tilde{E}_{\text{inc}}^* \right) - \tilde{E}_{\text{inc}} \left( \tilde{k}_{\text{inc}}^* \cdot \tilde{E}_{\text{inc}}^* \right) \right] \right\}
\]

(61)

\[
\tilde{\phi}_{\text{ref}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{inc}}^*} \left[ \tilde{k}_{\text{ref}}^* \left( \tilde{E}_{\text{ref}} \cdot \tilde{E}_{\text{ref}}^* \right) - \tilde{E}_{\text{ref}} \left( \tilde{k}_{\text{ref}}^* \cdot \tilde{E}_{\text{ref}}^* \right) \right] \right\}
\]

(62)

\[
\tilde{\phi}_{\text{trn}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{trn}}^*} \left[ \tilde{k}_{\text{trn}}^* \left( \tilde{E}_{\text{trn}} \cdot \tilde{E}_{\text{trn}}^* \right) - \tilde{E}_{\text{trn}} \left( \tilde{k}_{\text{trn}}^* \cdot \tilde{E}_{\text{trn}}^* \right) \right] \right\}
\]

(63)

\[
\tilde{\phi}_{c} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{r,\text{inc}}^*} \left[ \tilde{k}_{\text{ref}}^* \left( \tilde{E}_{\text{ref}} \cdot \tilde{E}_{\text{ref}}^* \right) - \tilde{E}_{\text{ref}} \left( \tilde{k}_{\text{ref}}^* \cdot \tilde{E}_{\text{ref}}^* \right) \right] + \right\}
\]

(64)
Here a term arises in the form of $\vec{k}_i^* \cdot \vec{E}_i$ where the subscript $i$ represents either inc, ref, or trn. In the LHI medium, the electric field, and wave vector are perpendicular, so all such terms become zero and dropped from the equations. So, the equations reduce to

\begin{align*}
\vec{\phi}_{\text{inc}} &= \text{Re} \left\{ \frac{1}{2 \omega \mu_{i, \text{inc}}} \left[ \vec{k}_{i, \text{inc}}^* \left( \vec{E}_{\text{inc}} \cdot \vec{E}_{\text{inc}}^* \right) \right] \right\} \\
\vec{\phi}_{\text{ref}} &= \text{Re} \left\{ \frac{1}{2 \omega \mu_{i, \text{inc}}} \left[ \vec{k}_{i, \text{ref}}^* \left( \vec{E}_{\text{ref}} \cdot \vec{E}_{\text{ref}}^* \right) \right] \right\} \\
\vec{\phi}_{\text{trn}} &= \text{Re} \left\{ \frac{1}{2 \omega \mu_{i, \text{trn}}} \left[ \vec{k}_{i, \text{trn}}^* \left( \vec{E}_{\text{trn}} \cdot \vec{E}_{\text{trn}}^* \right) \right] \right\} \\
\vec{\phi}_{\text{c}} &= \text{Re} \left\{ \frac{1}{2 \omega \mu_{i, \text{inc}}} \left[ \vec{k}_{i, \text{ref}}^* \left( \vec{E}_{\text{inc}} \cdot \vec{E}_{\text{ref}}^* \right) + \vec{k}_{i, \text{inc}}^* \left( \vec{E}_{\text{ref}} \cdot \vec{E}_{\text{inc}}^* \right) \right] \right\}
\end{align*}

Next, the expressions for $\vec{E}_i$ and $\vec{k}_i$ given in Eqs. (29) and (35) are substituted in Eqs. (57) –(60).
\[\tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \mu_{\text{inc}}^*} \left[ \tilde{k}_{\text{inc}}^* \left( \tilde{E}_{\text{inc}} \cdot \tilde{E}_{\text{inc}}^* \right) \right] \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \mu_{\text{inc}}^*} \left[ k_0 \tilde{n}_{\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) \right] \left[ \tilde{E}_{0,\text{inc}} \hat{a}_y e^{-j\mu_{\text{inc}}^*} \right] \left( \tilde{E}_{0,\text{inc}}^* \hat{a}_y e^{j\mu_{\text{inc}}^*} \right)^* \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \mu_{\text{inc}}^*} \left[ k_0 \tilde{n}_{\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) \right] \left[ \tilde{E}_{0,\text{inc}} \hat{a}_y e^{-j\mu_{\text{inc}}^*} \right] \left( \tilde{E}_{0,\text{inc}}^* \hat{a}_y e^{j\mu_{\text{inc}}^*} \right)^* \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{k_0 \tilde{n}_{\text{inc}}^*}{2\omega \mu_{\text{inc}}^*} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) e^{j\mu_{\text{inc}}^*} e^{-j\mu_{\text{inc}}^*} \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \left( \frac{\rho / \epsilon_0}{2 \rho \mu_{\text{inc}}^*} \right) \tilde{n}_{\text{inc}}^* \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) e^{j\mu_{\text{inc}}^*} e^{-j\mu_{\text{inc}}^*} \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{\sqrt{\mu_0 \epsilon_0}}{2 \mu_0 \mu_{\text{inc}}^*} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) e^{j\mu_{\text{inc}}^*} e^{-j\mu_{\text{inc}}^*} \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) e^{j\mu_{\text{inc}}^*} e^{-j\mu_{\text{inc}}^*} \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2} \frac{\epsilon_0}{\mu_0} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) e^{j\mu_{\text{inc}}^*} e^{-j\mu_{\text{inc}}^*} \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2\epsilon_0} \epsilon_0 \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) e^{j\mu_{\text{inc}}^*} e^{-j\mu_{\text{inc}}^*} \right\}\]

\[\Rightarrow \tilde{\phi}_{\text{inc}} = \text{Re}\left\{ \frac{1}{2\epsilon_0} \epsilon_0 \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{inc}}^* \left( \sin \tilde{\theta}_{\text{inc}}^* \hat{a}_x + \cos \tilde{\theta}_{\text{inc}}^* \hat{a}_z \right) e^{-j(\kappa_{\text{inc}} - \kappa_{\text{inc}}^*)^*} \right\}\]

So, the expressions of the Poynting vector for the incident, reflected and transmitted fields can be written as
\[
\vec{\phi}_{\text{inc}} = \text{Re} \left\{ \frac{|\vec{F}_{0,\text{inc}}|^2}{2\eta_{\text{inc}}} \left( \sin \vec{\theta}_{\text{inc}}^* \hat{\mathbf{a}}_x + \cos \vec{\theta}_{\text{inc}}^* \hat{\mathbf{a}}_z \right) e^{-j(k_{\text{inc}} - \vec{k}_{\text{inc}}) \mathbf{r}} \right\} \tag{69}
\]

\[
\vec{\phi}_{\text{ref}} = \text{Re} \left\{ \frac{|\vec{F}_{0,\text{ref}}|^2}{2\eta_{\text{inc}}} \left( \sin \vec{\theta}_{\text{ref}}^* \hat{\mathbf{a}}_x - \cos \vec{\theta}_{\text{ref}}^* \hat{\mathbf{a}}_z \right) e^{-j(k_{\text{ref}} - \vec{k}_{\text{ref}}) \mathbf{r}} \right\} \tag{70}
\]

\[
\vec{\phi}_{\text{trn}} = \text{Re} \left\{ \frac{|\vec{F}_{0,\text{trn}}|^2}{2\eta_{\text{trn}}} \left( \sin \vec{\theta}_{\text{trn}}^* \hat{\mathbf{a}}_x + \cos \vec{\theta}_{\text{trn}}^* \hat{\mathbf{a}}_z \right) e^{-j(k_{\text{trn}} - \vec{k}_{\text{trn}}) \mathbf{r}} \right\} \tag{71}
\]

The cross term can be expressed as following:

\[
\vec{\phi}_c = \text{Re} \left\{ \frac{1}{2\omega \mu_{\text{inc}}} \left[ \vec{k}_{\text{ref}}^* \left( \vec{E}_{\text{inc}} \cdot \vec{E}_{\text{ref}}^* \right) + \vec{k}_{\text{inc}}^* \left( \vec{E}_{\text{ref}} \cdot \vec{E}_{\text{inc}}^* \right) \right] \right\}
\]

\[
\Rightarrow \vec{\phi}_c = \text{Re} \left\{ \frac{1}{2\omega \mu_{\text{inc}}} \left[ \vec{k}_{\text{ref}}^* \left( \vec{E}_{\text{inc}} \cdot \vec{E}_{\text{ref}}^* \right) \right] \right\} + \text{Re} \left\{ \frac{1}{2\omega \mu_{\text{inc}}} \left[ \vec{k}_{\text{inc}}^* \left( \vec{E}_{\text{ref}} \cdot \vec{E}_{\text{inc}}^* \right) \right] \right\}
\]

\[
\Rightarrow \vec{\phi}_c = \vec{\phi}_{c,\text{ref}} + \vec{\phi}_{c,\text{inc}}
\]

where

\[
\vec{\phi}_{c,\text{ref}} = \text{Re} \left\{ \frac{1}{2\omega \mu_{\text{inc}}} \left[ \vec{k}_{\text{ref}}^* \left( \vec{E}_{\text{inc}} \cdot \vec{E}_{\text{ref}}^* \right) \right] \right\}
\]

\[
\vec{\phi}_{c,\text{inc}} = \text{Re} \left\{ \frac{1}{2\omega \mu_{\text{inc}}} \left[ \vec{k}_{\text{inc}}^* \left( \vec{E}_{\text{ref}} \cdot \vec{E}_{\text{inc}}^* \right) \right] \right\}
\]

Now, substituting the expressions for \( \vec{E}_t \) and \( \vec{k}_t \) in these equations, the expression for \( \vec{\phi}_{c,\text{ref}} \) becomes
\[ \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{1}{2\omega \tilde{\mu}_{\text{inc}}} \left[ k_0 \tilde{n}_{\text{inc}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) \right] \left[ \left( \tilde{E}_{0,\text{inc}} \hat{a_y} e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} \cdot \left( \tilde{E}_{0,\text{ref}} \hat{a_y} e^{-j \tilde{r}_{\text{ref}} \cdot \hat{n}_{\text{ref}}} \right) \right) \right] \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{1}{2\omega \tilde{\mu}_{\text{inc}}} \left[ k_0 \tilde{n}_{\text{inc}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) \right] \left[ \left( \tilde{E}_{0,\text{inc}} \hat{a_y} e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} \cdot \left( \tilde{E}_{0,\text{ref}} \hat{a_y} e^{-j \tilde{r}_{\text{ref}} \cdot \hat{n}_{\text{ref}}} \right) \right) \right] \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{1}{2\omega \tilde{\mu}_{\text{inc}}} \left[ k_0 \tilde{n}_{\text{inc}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) \right] \left[ \left( \tilde{E}_{0,\text{inc}} \hat{a_y} e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} \cdot \left( \tilde{E}_{0,\text{ref}} \hat{a_y} e^{-j \tilde{r}_{\text{ref}} \cdot \hat{n}_{\text{ref}}} \right) \right) \right] \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{1}{2\omega \tilde{\mu}_{\text{inc}}} \left[ k_0 \tilde{n}_{\text{inc}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) \right] \left[ \left( \tilde{E}_{0,\text{inc}} \hat{a_y} e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} \cdot \left( \tilde{E}_{0,\text{ref}} \hat{a_y} e^{-j \tilde{r}_{\text{ref}} \cdot \hat{n}_{\text{ref}}} \right) \right) \right] \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \left( \frac{\rho}{c_0} \right) \tilde{n}_{\text{inc}} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{ref}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} e^{j \tilde{r}_{\text{ref}} \cdot \hat{n}_{\text{ref}}} \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{\sqrt{\mu_0 \varepsilon_0} \sqrt{\mu_{r,\text{inc}} \varepsilon_{r,\text{inc}}}}{2 \mu_0 \tilde{\mu}_{\text{inc}}} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{ref}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} e^{j \tilde{r}_{\text{ref}} \cdot \hat{n}_{\text{ref}}} \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0 \mu_{r,\text{inc}}}} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{ref}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{1}{2 \eta_0} \frac{1}{\eta_{r,\text{inc}}} \tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{ref}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} \right\} \]

\[ \Rightarrow \tilde{\varphi}_{c,\text{ref}} = \text{Re} \left\{ \frac{\tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{ref}}}{2 \eta_{\text{inc}}} \left( \sin \tilde{\theta}_{\text{inc}} \hat{a}_x - \cos \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j \tilde{r}_{\text{inc}} \cdot \hat{n}_{\text{inc}}} \right\} \]

and the expression for \( \tilde{\varphi}_{c,\text{inc}} \) becomes
\[
\tilde{\varphi}_{c,\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{\text{inc}}} \left[ \tilde{k}^*_s \left( \tilde{E}_{\text{ref}} \cdot \tilde{E}^*_\text{inc} \right) \right] \right\}
\]

\[
\Rightarrow \tilde{\varphi}_{c,\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{\text{inc}}} \left[ k_0 \tilde{n}^*_\text{inc} \left( \sin \tilde{\theta}^*_\text{inc} \hat{a}_x + \cos \tilde{\theta}^*_\text{inc} \hat{a}_z \right) \right] \left[ \left( \tilde{E}_{0,\text{ref}} \hat{a}_y e^{-j\kappa_{\text{ref}}^* r} \right) \cdot \left( \tilde{E}^*_\text{inc} \hat{a}_y e^{-j\kappa_{\text{inc}}^* r} \right) \right] \right\}
\]

\[
\Rightarrow \tilde{\varphi}_{c,\text{inc}} = \text{Re}\left\{ \frac{1}{2\omega \tilde{\mu}_{\text{inc}}} \left[ k_0 \tilde{n}^*_\text{inc} \left( \sin \tilde{\theta}^*_\text{inc} \hat{a}_x + \cos \tilde{\theta}^*_\text{inc} \hat{a}_z \right) \right] \left[ \left( \tilde{E}_{0,\text{ref}} \hat{a}_y e^{-j\kappa_{\text{ref}}^* r} \right) \cdot \left( \tilde{E}^*_\text{inc} \hat{a}_y e^{j\kappa_{\text{inc}}^* r} \right) \right] \right\}
\]

\[
\Rightarrow \tilde{\varphi}_{c,\text{inc}} = \text{Re}\left\{ \frac{k_0 \tilde{n}^*_\text{inc}}{2\omega \tilde{\mu}_{\text{inc}}} \left( \tilde{E}^*_\text{inc} \tilde{E}_{0,\text{ref}} \left( \sin \tilde{\theta}^*_\text{inc} \hat{a}_x + \cos \tilde{\theta}^*_\text{inc} \hat{a}_z \right) \left( \hat{a}_y \cdot \hat{a}_y \right) e^{j\kappa_{\text{inc}}^* r} e^{-j\kappa_{\text{ref}}^* r} \right) \right\}
\]

\[
\Rightarrow \tilde{\varphi}_{c,\text{inc}} = \text{Re}\left\{ \frac{\tilde{E}^*_\text{inc} \tilde{E}_{0,\text{ref}}}{2\tilde{n}^*_\text{inc}} \left( \left( \sin \tilde{\theta}^*_\text{inc} \hat{a}_x + \cos \tilde{\theta}^*_\text{inc} \hat{a}_z \right) e^{j\kappa_{\text{inc}}^* r} e^{-j\kappa_{\text{ref}}^* r} \right) \right\}
\]

Last, it is only the \( z \) component of the Poynting vector that carries power to and from the interface. So, only the \( z \) components of the Poynting vectors are written below.

\[
\tilde{\varphi}_{z,\text{inc}} = \text{Re}\left\{ \frac{|\tilde{E}_{0,\text{inc}}|^2}{2\tilde{n}^*_\text{inc}} \cos \tilde{\theta}^*_\text{inc} \right\} \tag{72}
\]

\[
\tilde{\varphi}_{z,\text{ref}} = \text{Re}\left\{ -\frac{|\tilde{E}_{0,\text{ref}}|^2}{2\tilde{n}^*_\text{inc}} \cos \tilde{\theta}^*_\text{inc} \right\} \tag{73}
\]

\[
\tilde{\varphi}_{z,\text{trn}} = \text{Re}\left\{ \frac{|\tilde{E}_{0,\text{trn}}|^2}{2\tilde{n}^*_\text{trn}} \cos \tilde{\theta}^*_\text{trn} \right\} \tag{74}
\]

\[
\tilde{\varphi}_{z,\text{c,ref}} = \text{Re}\left\{ -\frac{|\tilde{E}_{0,\text{inc}}|^{*} \tilde{E}^*_\text{ref}}{2\tilde{n}^*_\text{inc}} \cos \tilde{\theta}^*_\text{inc} \right\} \tag{75}
\]
\[
\hat{\mathcal{S}}_{z,\text{inc}} = \operatorname{Re}\left\{ \frac{\bar{E}_{0,\text{inc}}^* \bar{E}_{0,\text{ref}}}{2\tilde{n}_{\text{inc}}^*} \cos \hat{\theta}_{\text{inc}}^* \right\}
\] (76)

**Conservation of power: reflectance and transmittance**

Conservation of power requires that the total power in medium 1 flowing to the interface is equal to the total power in medium 2 flowing away from the interface. This can be written mathematically using the Poynting vectors as

\[
\hat{\mathcal{S}}_{z,\text{inc}} + \hat{\mathcal{S}}_{z,\text{ref}} + \hat{\mathcal{S}}_{z,\text{c,ref}} + \hat{\mathcal{S}}_{z,\text{c,inc}} = \hat{\mathcal{S}}_{z,\text{tm}}
\] (77)

A more meaningful expression for conservation of power is derived by substituting Eqs. (72)-(75) into Eq. (77), dividing the new equation by \(|E_{0,\text{inc}}|^2\), simplifying, and then recognizing that some of the terms are expressions for the reflection and transmission coefficients. After doing this, the conservation of power equation for complex mediums becomes
\[
\text{Re} \left( \frac{\tilde{E}_{0,\text{inc}}^2}{2\tilde{n}_{\text{inc}}^*} \cos \tilde{\theta}_{\text{inc}}^* \right) - \text{Re} \left( \frac{\tilde{E}_{0,\text{ref}}^2}{2\tilde{n}_{\text{inc}}^*} \cos \tilde{\theta}_{\text{inc}}^* \right) - \text{Re} \left( \frac{\tilde{E}_{0,\text{inc}}^* \tilde{E}_{0,\text{ref}}^*}{2\tilde{n}_{\text{inc}}^*} \cos \tilde{\theta}_{\text{inc}}^* \right) + \\
\text{Re} \left( \frac{\tilde{E}_{0,\text{inc}}^2}{2\tilde{n}_{\text{inc}}^*} \cos \tilde{\theta}_{\text{inc}}^* \right) = \text{Re} \left( \frac{\tilde{E}_{0,\text{trn}}^2}{2\tilde{n}_{\text{trn}}^*} \cos \tilde{\theta}_{\text{trn}}^* \right)
\]

\[
\Rightarrow \left| \frac{\tilde{E}_{0,\text{inc}}}{\tilde{E}_{0,\text{inc}}^*} \right|^2 - \text{Re} \left( \frac{\tilde{E}_{0,\text{inc}}^* \tilde{E}_{0,\text{ref}}^*}{\tilde{E}_{0,\text{inc}}^*} \right) + \text{Re} \left( \frac{\tilde{E}_{0,\text{inc}}^* \tilde{E}_{0,\text{ref}}^*}{\tilde{E}_{0,\text{inc}}^*} \right) = \text{Re} \left( \frac{2\tilde{n}_{\text{inc}}^*}{\tilde{n}_{\text{trn}}^*} \frac{\tilde{E}_{0,\text{trn}}^2}{\tilde{E}_{0,\text{inc}}^2} \cos \tilde{\theta}_{\text{trn}}^* \cos \tilde{\theta}_{\text{inc}}^* \right)
\]

\[
\Rightarrow 1 - |\tilde{r}_{\text{TE}}|^2 - \text{Re} \left( \tilde{r}_{\text{TE}}^* \right) + \text{Re} \left( \tilde{r}_{\text{TE}} \right) = \text{Re} \left( \frac{\tilde{n}_{\text{inc}}^*}{\tilde{n}_{\text{trn}}^*} |\tilde{r}_{\text{TE}}|^2 \cos \tilde{\theta}_{\text{trn}}^* \cos \tilde{\theta}_{\text{inc}}^* \right)
\]

\[
\Rightarrow |\tilde{r}_{\text{TE}}|^2 + \text{Re} \left( \frac{\tilde{n}_{\text{inc}}^*}{\tilde{n}_{\text{trn}}^*} |\tilde{r}_{\text{TE}}|^2 \cos \tilde{\theta}_{\text{trn}}^* \cos \tilde{\theta}_{\text{inc}}^* \right) = 1
\]

\[
\Rightarrow |\tilde{r}_{\text{TE}}|^2 + \text{Re} \left( \frac{\tilde{n}_{\text{inc}}^*}{\tilde{n}_{\text{trn}}^*} |\tilde{r}_{\text{TE}}|^2 \cos \tilde{\theta}_{\text{trn}}^* \cos \tilde{\theta}_{\text{inc}}^* \right) - \text{Re} \left( j2 \text{Im} (\tilde{r}_{\text{TE}}) \right) = 1
\]

\[
|\tilde{r}_{\text{TE}}|^2 + \text{Re} \left( \frac{\tilde{n}_{\text{inc}}^*}{\tilde{n}_{\text{trn}}^*} \tilde{r}_{\text{TE}}^2 \cos \tilde{\theta}_{\text{trn}}^* \cos \tilde{\theta}_{\text{inc}}^* \right) - \text{Re} \left( j2 \text{Im} (\tilde{r}_{\text{TE}}) \right) = 1
\] (78)

There are three terms on the left side of Eq. (78). The first term is the reflectance \( R_{\text{TE}} \), the second term is the transmittance \( T_{\text{TE}} \). The third term occurs due to the interference between incident and reflected fields which is the cross term. Lodenquai [17] proposed a corrected conservation relation stating that \( R + T + \Delta = 1 \), where \( \Delta \) is a correction term and is small for low-loss materials. Weber [13] defined the cross term as delta \( \Delta \). Only the real part of the terms in
Eq. (78) contains power and the cross term is purely imaginary. So, the cross term $\Delta$ becomes zero and disappears entirely from the final conservation equation.

$$R_{\text{TE}} + T_{\text{TE}} = 1$$  \hspace{1cm} (79)

$$R_{\text{TE}} = |\tilde{r}_{\text{TE}}|^2$$  \hspace{1cm} (80)

$$T_{\text{TE}} = \text{Re} \left( |\tilde{r}_{\text{TE}}|^2 \frac{\tilde{\eta}_{\text{inc}}^* \cos \tilde{\theta}_{\text{trn}}^*}{\tilde{\eta}_{\text{trn}}^* \cos \tilde{\theta}_{\text{inc}}^*} \right)$$  \hspace{1cm} (81)

When performing these calculations, conservation of power in Eq. (79) will not be satisfied if the sign of an impedance is not resolved correctly. However, it was previously shown that it is necessary to determine the correct sign for both $\tilde{\eta}_{\text{inc}}$ or $\tilde{\eta}_{\text{trn}}$ in order to obtain correct values for $\tilde{\eta}_{\text{TE}}, \tilde{r}_{\text{TE}}, \tilde{R}_{\text{TE}},$ and $\tilde{T}_{\text{TE}}$. Instead, if conservation in Eq. (79) is not satisfied, the sign of either $\tilde{\eta}_{\text{inc}}$ or $\tilde{\eta}_{\text{trn}}$, but not both, can be inverted. To simplify the flow of calculations, the present study found that

$$\text{if } \text{Re} \left[ \frac{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{trn}}} \right] < 0, \text{ a sign change is needed for TE analysis}$$  \hspace{1cm} (82)

Using this condition, it is possible to identify if a sign change is necessary without having to calculate all of the terms to verify conservation.

**Generalized Fresnel equations for TM polarization**

**Wave expression**

As stated before, the electric field is parallel to the plane of incidence for TM polarization. The expression for electric field components for the incident, reflected, and transmitted waves are

$$\tilde{E}_{\text{inc}} = \tilde{E}_{0,\text{inc}} \left( \cos \tilde{\theta}_{\text{inc}} \hat{a}_x - \sin \tilde{\theta}_{\text{inc}} \hat{a}_z \right) e^{-j \tilde{\mu}_{\text{inc}} \tilde{r}}$$  \hspace{1cm} (83)
Expressions for the corresponding magnetic field components of the incident, reflected, and transmitted waves are

\[ \tilde{H}_{\text{inc}} = \frac{\tilde{E}_{0,\text{inc}}}{\tilde{\eta}_{\text{inc}}} \hat{\alpha}_y e^{-j\hat{\beta}_{\text{inc}} \cdot r} \]  \hspace{1cm} (86)

\[ \tilde{H}_{\text{ref}} = -\frac{\tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} \hat{\alpha}_y e^{-j\hat{\beta}_{\text{inc}} \cdot r} \]  \hspace{1cm} (87)

\[ \tilde{H}_{\text{trn}} = \frac{\tilde{E}_{0,\text{trn}}}{\tilde{\eta}_{\text{trn}}} \hat{\alpha}_y e^{-j\hat{\beta}_{\text{trn}} \cdot r} \]  \hspace{1cm} (88)

Complex wave vector expressions remain the same as described in Eq. (35)-(37).

**Boundary conditions**

Like TE polarization, similar boundary conditions are stated in Eqs. (42)-(45) apply for TM polarization. There is no normal component of the magnetic field for TM polarization. The boundary conditions conclude with the following equations

\[ \tilde{E}_{T1}(0) = \tilde{E}_{T2}(0) \]

\[ \Rightarrow \tilde{E}_{T,\text{inc}}(0) + \tilde{E}_{T,\text{ref}}(0) = \tilde{E}_{T,\text{trn}}(0) \]

\[ \Rightarrow \tilde{E}_{0,\text{inc}} \cos \tilde{\theta}_{\text{inc}} e^{0} + \tilde{E}_{0,\text{ref}} \cos \tilde{\theta}_{\text{inc}} e^{0} = \tilde{E}_{0,\text{trn}} \cos \tilde{\theta}_{\text{trn}} e^{0} \]

\[ \Rightarrow \tilde{E}_{0,\text{inc}} \cos \tilde{\theta}_{\text{inc}} + \tilde{E}_{0,\text{ref}} \cos \tilde{\theta}_{\text{inc}} = \tilde{E}_{0,\text{trn}} \cos \tilde{\theta}_{\text{trn}} \]
\begin{align}
\tilde{E}_{0,\text{inc}} + \tilde{E}_{0,\text{ref}} &= \tilde{E}_{0,\text{tm}} \frac{\cos \tilde{\theta}_{\text{tm}}}{\cos \tilde{\theta}_{\text{inc}}} \\
\Rightarrow \tilde{e}_{r,\text{inc}} \tilde{E}_{N1}(0) &= \tilde{e}_{r,\text{tm}} \tilde{E}_{N2}(0) \\
\Rightarrow -\tilde{e}_{r,\text{inc}} \sin \tilde{\theta}_{\text{inc}} e^0 + \tilde{e}_{r,\text{inc}} \tilde{E}_{0,\text{ref}} \sin \tilde{\theta}_{\text{inc}} e^0 &= -\tilde{e}_{r,\text{tm}} \tilde{E}_{0,\text{tm}} \sin \tilde{\theta}_{\text{tm}} e^0 \\
\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}} &= \tilde{E}_{0,\text{tm}} \frac{\sin \tilde{\theta}_{\text{tm}}}{\sin \tilde{\theta}_{\text{inc}}} \\
\Rightarrow \tilde{H}_{T1}(0) &= \tilde{H}_{T2}(0) \\
\Rightarrow \tilde{H}_{T,\text{inc}}(0) + \tilde{H}_{T,\text{ref}}(0) &= \tilde{H}_{T,\text{tm}}(0) \\
\Rightarrow \frac{\tilde{E}_{0,\text{inc}}}{\tilde{\eta}_{\text{inc}}} e^0 - \frac{\tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} e^0 &= \frac{\tilde{E}_{0,\text{tm}}}{\tilde{\eta}_{\text{tm}}} e^0 \\
\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}} &= \frac{\tilde{\eta}_{\text{inc}}}{\tilde{\eta}_{\text{tm}}} \tilde{E}_{0,\text{tm}}
\end{align}

\textit{Reflection and transmission coefficients}

The Eq. (89) is solved for \( \tilde{E}_{0,\text{tm}} \) and the expression is substituted in Eq. (91). The new expression is solved for \( \tilde{E}_{0,\text{ref}}/\tilde{E}_{0,\text{inc}} \), which is the reflection coefficient.
\[
\frac{\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} = \frac{\tilde{E}_{0,\text{trn}}}{\tilde{\eta}_{\text{trn}}}
\]

\[
\Rightarrow \frac{\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} = \frac{\cos \tilde{\theta}_{\text{inc}}}{\cos \tilde{\theta}_{\text{trn}}} \left( \tilde{E}_{0,\text{inc}} + \tilde{E}_{0,\text{ref}} \right)
\]

\[
\Rightarrow \frac{\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} = \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}}} \left( \tilde{E}_{0,\text{inc}} + \tilde{E}_{0,\text{ref}} \right)
\]

\[
\Rightarrow \frac{\tilde{E}_{0,\text{inc}} - \tilde{E}_{0,\text{ref}}}{\tilde{\eta}_{\text{inc}}} = \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}}} \left( \tilde{E}_{0,\text{inc}} + \tilde{E}_{0,\text{ref}} \right)
\]

\[
\Rightarrow \left( 1 + \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}}} \right) \tilde{E}_{0,\text{ref}} = \left( 1 - \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}}} \right) \tilde{E}_{0,\text{inc}}
\]

\[
\Rightarrow \frac{\tilde{E}_{0,\text{ref}}}{\tilde{E}_{0,\text{inc}}} = \frac{1 - \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}}}}{1 + \frac{\tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}}}}
\]

\[
\tilde{r}_{\text{TM}} = \frac{\tilde{E}_{0,\text{ref}}}{\tilde{E}_{0,\text{inc}}} = \frac{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}} - \tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}{\tilde{\eta}_{\text{trn}} \cos \tilde{\theta}_{\text{trn}} + \tilde{\eta}_{\text{inc}} \cos \tilde{\theta}_{\text{inc}}}
\]

Like Eq. (51), if conservation of power is not obeyed, the sign of either one of the impedances in Eq. (92) can be inverted to obtain a correct value for \(\tilde{r}_{\text{TM}}\).

Now Eq. (92) is solved for \(\tilde{E}_{0,\text{ref}}\), the expression substituted in Eq. (89) to eliminate \(\tilde{E}_{0,\text{ref}}\), and the new expression solved for \(\tilde{E}_{0,\text{trn}}/\tilde{E}_{0,\text{inc}}\), which is transmission coefficient.
\[ \vec{E}_{0,\text{inc}} + \vec{E}_{0,\text{ref}} = \vec{E}_{0,\text{trn}} \frac{\cos \hat{\theta}_{\text{trn}}}{\cos \hat{\theta}_{\text{inc}}} \]

\[ \Rightarrow \vec{E}_{0,\text{inc}} + \vec{E}_{0,\text{inc}} \frac{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} - \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}}{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} + \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}} = \vec{E}_{0,\text{trn}} \frac{\cos \hat{\theta}_{\text{trn}}}{\cos \hat{\theta}_{\text{inc}}} \]

\[ \Rightarrow \vec{E}_{0,\text{inc}} \left(1 + \frac{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} - \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}}{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} + \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}} \right) = \vec{E}_{0,\text{trn}} \frac{\cos \hat{\theta}_{\text{trn}}}{\cos \hat{\theta}_{\text{inc}}} \]

\[ \Rightarrow \vec{E}_{0,\text{trn}} = \frac{\cos \hat{\theta}_{\text{inc}}}{\cos \hat{\theta}_{\text{inc}}} \left(1 + \frac{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} - \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}}{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} + \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}} \right) \]

\[ \Rightarrow \vec{E}_{0,\text{trn}} = \frac{\cos \hat{\theta}_{\text{inc}}}{\cos \hat{\theta}_{\text{trn}}} \left(\frac{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} + \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}} + \hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} - \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}}{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} + \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}} \right) \]

\[ \Rightarrow \vec{E}_{0,\text{trn}} = \frac{\cos \hat{\theta}_{\text{inc}}}{\cos \hat{\theta}_{\text{trn}}} \left(\frac{2\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}}}{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} + \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}} \right) \]

\[ \tilde{\tau}_{TM} = \frac{\vec{E}_{0,\text{trn}}}{\vec{E}_{0,\text{inc}}} = \frac{2\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{inc}}}{\hat{n}_{\text{trn}} \cos \hat{\theta}_{\text{trn}} + \hat{n}_{\text{inc}} \cos \hat{\theta}_{\text{inc}}} \] (93)

If the sign of an impedance was inverted to obtain \( \tilde{\tau}_{TM} \) from Eq. (92), the same impedance values should be used in Eq. (93) to obtain \( \tilde{\tau}_{TM} \).

**Poynting vectors**

The complex Poynting vector in medium 1 is the same as in Eq. (54). For the TM polarization, the electric field terms are replaced by \( \vec{E} = -(\vec{k} \times \vec{H})/\omega \vec{e} \) to express the Poynting vectors solely in terms of magnetic field quantities. The complex Poynting vector in medium 2 includes just the transmitted wave. Following a similar sequence of steps taken for the TE polarization, the following expressions for the TM polarization are derived.
\[
\tilde{\phi}_{z,\text{inc}} = \text{Re} \left\{ \frac{1}{2} \left( \frac{\cos \tilde{\theta}_{\text{inc}}^*}{\tilde{n}_{\text{inc}}} + \frac{\cos \tilde{\theta}_{\text{inc}}^*}{\tilde{n}_{\text{inc}}} \right) \right\} 
\] (94)

\[
\tilde{\phi}_{z,\text{trn}} = \text{Re} \left\{ \frac{1}{2} \left( \frac{\cos \tilde{\theta}_{\text{trn}}^*}{\tilde{n}_{\text{trn}}} + \frac{\cos \tilde{\theta}_{\text{trn}}^*}{\tilde{n}_{\text{trn}}} \right) \right\} 
\] (95)

\[
\tilde{\phi}_{z,\text{ref}} = \text{Re} \left\{ -\frac{1}{2} \left( \frac{\cos \tilde{\theta}_{\text{ref}}^*}{\tilde{n}_{\text{inc}}} + \frac{\cos \tilde{\theta}_{\text{ref}}^*}{\tilde{n}_{\text{inc}}} \right) \right\} 
\] (96)

\[
\tilde{\phi}_{z,c,\text{inc}} = \text{Re} \left\{ -\frac{\tilde{E}_{0,\text{inc}} \tilde{E}_{0,\text{ref}}^* \cos \tilde{\theta}_{\text{inc}}}{2\tilde{n}_{\text{inc}}} \right\} 
\] (97)

\[
\tilde{\phi}_{z,c,\text{ref}} = \text{Re} \left\{ \frac{\tilde{E}_{0,\text{inc}}^* \tilde{E}_{0,\text{ref}} \cos \tilde{\theta}_{\text{inc}}^*}{2\tilde{n}_{\text{inc}}} \right\} 
\] (98)

**Conservation of power: reflectance and transmittance**

Conservation of power requires that the total power in medium 1 flowing to the interface is equal to the total power in medium 2 flowing away from the interface. The Poynting vectors can be written as

\[
\tilde{\phi}_{z,\text{inc}} + \tilde{\phi}_{z,\text{ref}} + \tilde{\phi}_{z,c,\text{ref}} + \tilde{\phi}_{z,c,\text{inc}} = \tilde{\phi}_{z,\text{trn}} 
\] (99)

A more meaningful expression for conservation of power is derived by substituting Eq. (94) into Eq. (99), dividing the new equation by \( |\tilde{E}_{0,\text{inc}}|^2 \), simplifying, and then recognizing that some of the terms are expressions for the reflection and transmission coefficients. After doing this, the conservation of power equation for complex mediums for the TM polarization becomes,
\[
\begin{align*}
\text{Re} \left\{ \frac{\tilde{E}_{0,\text{inc}}^2}{2\eta_{\text{inc}}} \cos \tilde{\theta}_{\text{inc}} \right\} - \text{Re} \left\{ \frac{\tilde{E}_{0,\text{ref}}^2}{2\eta_{\text{inc}}} \cos \tilde{\theta}_{\text{inc}} \right\} - \text{Re} \left\{ \frac{\tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{ref}}}{2\eta_{\text{inc}}} \cos \tilde{\theta}_{\text{inc}} \right\} + \\
\text{Re} \left\{ \frac{\tilde{E}_{0,\text{tron}}^2}{2\eta_{\text{tron}}} \cos \tilde{\theta}_{\text{tron}} \right\} - \text{Re} \left\{ \frac{\tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{tron}}}{2\eta_{\text{inc}}} \cos \tilde{\theta}_{\text{inc}} \right\}
\end{align*}
\]

\[
\Rightarrow \left| \tilde{E}_{0,\text{inc}} \right|^2 - \left| \tilde{E}_{0,\text{ref}} \right|^2 - \text{Re} \left\{ \tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{ref}} \right\} + \text{Re} \left\{ \tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{ref}} \right\} = \text{Re} \left\{ \frac{\tilde{E}_{0,\text{tron}}^2}{2\eta_{\text{tron}}} \cos \tilde{\theta}_{\text{tron}} \right\}
\]

\[
\Rightarrow 1 - \frac{\left| \tilde{E}_{0,\text{ref}} \right|^2}{\left| \tilde{E}_{0,\text{inc}} \right|^2} - \text{Re} \left\{ \frac{\tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{ref}}}{\tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{inc}}} \right\} + \text{Re} \left\{ \frac{\tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{ref}}}{\tilde{E}_{0,\text{inc}}^{*}\tilde{E}_{0,\text{inc}}} \right\} = \text{Re} \left\{ \frac{\tilde{n}_{\text{inc}} \tilde{n}_{\text{tron}}}{\tilde{n}_{\text{tron}} \tilde{n}_{\text{inc}}} \tilde{E}_{0,\text{tron}}^2 \cos \tilde{\theta}_{\text{tron}} \right\}
\]

\[
\Rightarrow 1 - \left| \hat{r}_{\text{TM}} \right|^2 - \text{Re} \left\{ \frac{\tilde{E}_{0,\text{ref}}^{*}}{\tilde{E}_{0,\text{inc}}^{*}} \right\} + \text{Re} \left\{ \frac{\tilde{E}_{0,\text{ref}}^{*}}{\tilde{E}_{0,\text{inc}}^{*}} \right\} = \text{Re} \left\{ \frac{\tilde{n}_{\text{inc}}}{\tilde{n}_{\text{tron}}} \tilde{r}_{\text{TM}}^2 \cos \tilde{\theta}_{\text{tron}} \right\}
\]

\[
\Rightarrow 1 - \left| \hat{r}_{\text{TM}} \right|^2 - \text{Re} \left\{ \hat{r}_{\text{TM}}^{*} \right\} + \text{Re} \left\{ \hat{r}_{\text{TM}}^{*} \right\} = \text{Re} \left\{ \frac{\tilde{n}_{\text{inc}}}{\tilde{n}_{\text{tron}}} \tilde{r}_{\text{TM}}^2 \cos \tilde{\theta}_{\text{tron}} \right\}
\]

\[
\Rightarrow \left| \hat{r}_{\text{TM}} \right|^2 + \text{Re} \left\{ \frac{\tilde{n}_{\text{inc}}}{\tilde{n}_{\text{tron}}} \tilde{r}_{\text{TM}}^2 \cos \tilde{\theta}_{\text{tron}} \right\} - \text{Re} \left\{ \hat{r}_{\text{TM}}^{*} \right\} + \text{Re} \left\{ \hat{r}_{\text{TM}}^{*} \right\} = 1
\]

\[
\Rightarrow \left| \hat{r}_{\text{TM}} \right|^2 + \text{Re} \left\{ \frac{\tilde{n}_{\text{inc}}}{\tilde{n}_{\text{tron}}} \tilde{r}_{\text{TM}}^2 \cos \tilde{\theta}_{\text{tron}} \right\} + \text{Re} \left\{ \hat{r}_{\text{TM}}^{*} - \hat{r}_{\text{TM}} \right\} = 1
\]
The first term is defined as reflectance and the second term is transmittance for TM polarization. The third term occurs due to the interference between incident and reflected waves. The third term becomes zero because it is the real part of a completely imaginary number. After all of this, Eq. (100) is written as

$$R_{TM} + T_{TM} + \Delta_{TM} = 1$$

(101)

$$R_{TM} = |\bar{r}_{TM}|^2$$

(102)

$$T_{TM} = \text{Re} \left( |\bar{r}_{TM}|^2 \frac{\bar{\eta}_{inc} \cos \bar{\phi}_{inc}}{\bar{\eta}_{inc} \cos \bar{\phi}_{inc}} \right)$$

(103)

Like the discussion around Eq. (79), if conservation in Eq. (101) is not satisfied, the sign of either $\bar{\eta}_{inc}$ or $\bar{\eta}_{trn}$ can be inverted to obtain correct values for $\bar{r}_{TM}$, $\bar{\epsilon}_{TM}$, $\bar{R}_{TM}$, and $\bar{T}_{TM}$. To simplify the flow of calculations, the present study found that

if \( \text{Re} \left[ \frac{\bar{\eta}_{im} \cos \bar{\phi}_{im}}{\bar{\eta}_{inc} \cos \bar{\phi}_{inc}} \right] < 0 \), a sign change is needed for TM analysis

(104)

Using this condition, it is possible to identify if a sign change is necessary without having to calculate all of the terms to verify conservation. This condition for TM polarization is not the same condition for TE polarization given in Eq. (82).
Chapter 5: TMMSM for complex medium

In this chapter, the modified transfer matrix method with scattering matrices (TMMSM) is presented that works for complex mediums considering external and intermediate layers of a layered device. This chapter presents a detailed formulation of TMMSM along with a block diagram summarizing its implementation. In the first step of formulating the TMMSM, Maxwell's equations are analyzed for a linear, homogeneous, and isotropic (LHI) medium to determine the electromagnetic modes that exist in a single layer of the device. The second step involves formulating the scattering matrices which will be used to combine all the layers of the device and external regions into a global scattering matrix. In the third step, the reflected and transmitted fields are calculated using the global scattering matrix, and from the fields, reflectance and transmittance are calculated for TE and TM polarization. For TE polarization, the reflectance and transmittance are calculated in terms of the electric fields and for TM polarization, it is calculated in terms of the magnetic fields. Afterward, the implementation of TMMSM is discussed, with a block diagram showing the steps involved. Finally, the feasibility of the algorithm is demonstrated by comparing the results obtained from the implementation of TMMSM with the results obtained from other sources.

Numerical solution to Maxwell’s equations in LHI medium

In this section, an equation will be derived for TMMSM that calculates the electric and magnetic fields throughout a single LHI layer. In this part of the dissertation, the tilde ~ sign on the top of a variable is going to represent the complex numbers, and the bar – sign on the top of the variable is going to represent the normalized value of that variable. In absence of charge and current, Maxwell’s curl equations can be written as

\[ \nabla \times \vec{E} = k_0 \mu_0 \mu_r \vec{H} \]  

(105)
\[ \nabla \times \vec{H} = k_0 \vec{\epsilon} \vec{E} \]  

(106)

In these equations, \( \vec{\epsilon}_r \) is the complex relative permittivity, \( \vec{\mu}_r \) is the complex relative permeability, \( k_0 \) is the free space wave number, \( \vec{E} \) is the electric field intensity and \( \vec{H} \) is magnetic field intensity. The magnetic field intensity is normalized according to \( \vec{H} = j \eta_0 \vec{h} \), where \( \eta_0 \) is the impedance of free space and \( j = \sqrt{-1} \). Each of the curl equations can be expanded into a set of three partial differential equations to get a total of six-coupled partial differential equations. These are

\[ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = k_0 \vec{\mu} \vec{H}_x \]  

(107)

\[ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = k_0 \vec{\mu} \vec{H}_y \]  

(108)

\[ \frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = k_0 \vec{\mu} \vec{H}_z \]  

(109)

\[ \frac{\partial \vec{H}_y}{\partial y} - \frac{\partial \vec{H}_y}{\partial z} = k_0 \vec{\epsilon} \vec{E}_x \]  

(110)

\[ \frac{\partial \vec{H}_x}{\partial z} - \frac{\partial \vec{H}_x}{\partial x} = k_0 \vec{\epsilon} \vec{E}_y \]  

(111)

\[ \frac{\partial \vec{H}_y}{\partial x} - \frac{\partial \vec{H}_y}{\partial y} = k_0 \vec{\epsilon} \vec{E}_z \]  

(112)

67
Figure 20: Geometry of transfer matrix method. $\tilde{\mu}_r$ and $\tilde{\varepsilon}_r$ indicate the complex relative permittivity and complex relative permeability of the respective layer. $\theta$ and $\phi$ are indicating the incident angle and azimuthal angle, respectively. $k_{inc}$ is representing the incident wave vector. $L$ represents the thickness of the respective layer. The subscript $N$ indicates the number of layers.

Plane waves have the mathematical form of $\vec{E} = \vec{E}_0 e^{j\vec{k}\cdot\vec{r}}$ and $\vec{H} = \vec{H}_0 e^{j\vec{k}\cdot\vec{r}}$, so it follows that $\partial \vec{E} / \partial m = j\vec{k}_m \vec{E}$ and $\partial \vec{H} / \partial m = j\vec{k}_m \vec{H}$, where $m = x$ or $y$. Here, $\vec{k}$ is the complex wave vector, $\vec{E}_0$ and $\vec{H}_0$ indicates the complex amplitude of electric and magnetic fields, respectively. These relations do not hold for $m = z$ because the $z$ direction is not homogeneous when multiple layers are considered. So, the partial derivative with respect to $z$ in Eqs. (107)–(112) becomes an ordinary derivative. These relations transform Eqs. (107)–(112) into
\[ jk_y E_z - \frac{dE_y}{dz} = k_0 \mu_r \bar{H}_x \] (113)

\[ \frac{dE_x}{dz} - jk_x E_z = k_0 \mu_r \bar{H}_y \] (114)

\[ jk_x E_y - jk_y E_x = k_0 \mu_r \bar{H}_z \] (115)

\[ jk_y \bar{H}_z - \frac{d\bar{H}_y}{dz} = k_0 \varepsilon_r E_x \] (116)

\[ \frac{d\bar{H}_x}{dz} - jk_x \bar{H}_z = k_0 \varepsilon_r E_y \] (117)

\[ jk_x \bar{H}_y - jk_y \bar{H}_x = k_0 \varepsilon_r E_z \] (118)

The longitudinal components of the electric and magnetic fields are eliminated by solving Eq. (115) for \( \bar{H}_z \) and Eq. (118) for \( E_z \).

\[ jk_x E_y - jk_y E_x = k_0 \mu_r \bar{H}_z \]

\[ \Rightarrow \bar{H}_z = \frac{jk_y E_x - jk_x E_y}{k_0 \mu_r} \]

\[ jk_x \bar{H}_y - jk_y \bar{H}_x = k_0 \varepsilon_r E_z \]

\[ \Rightarrow E_z = \frac{jk_y \bar{H}_x - jk_x \bar{H}_y}{k_0 \varepsilon_r} \] (120)

The \( x \) and \( y \) components of the wave vectors are the same throughout the entire problem and are defined below. The \( z \) component of the wave vector is calculated from the dispersion relation for the LHI medium and so it can be different in each layer and external medium of the
device. For this reason, the z component is written as \( \tilde{k}_{z,i} \) where \( i \) represents the layer number or external medium. The wave vector components are normalized as \( \tilde{k} = \tilde{k}/k_0 \).

\[
\tilde{k}_x = \tilde{k}_{x,\text{inc}} = \tilde{n}_r \sin \theta \cos \phi
\]  

(121)

\[
\tilde{k}_y = \tilde{k}_{y,\text{inc}} = \tilde{n}_r \sin \theta \sin \phi
\]  

(122)

\[
\tilde{k}_{z,i} = \sqrt{\tilde{\mu}_{r,i} \tilde{\varepsilon}_{r,i} - \tilde{k}_x^2 - \tilde{k}_y^2}
\]  

(123)

Here, \( \tilde{n}_r \) represents the complex refractive index of the medium. From Eq. (123), it is clear that the wave vector in z direction \( \tilde{k}_{z,i} \) can have positive or negative value. It is tough to identify if the positive or negative value will lead to the correct result. The approach followed in this dissertation is first taking the positive value of \( \tilde{k}_{z,i} \) and check if conservation is matched. For the cases conservation is not matched, the negative value of \( \tilde{k}_{z,i} \) is chosen for the transmission side of the device.

The expressions for \( \tilde{H}_z \) and \( \tilde{E}_z \) from Eq. (119) and (120) are substituted into the remaining four equations. Using the normalized wave vector components and normalizing the z coordinate according to \( z' = k_0 z \), and rearranging the terms, the four Eqs. (113), (114), (116) and (117) become

\[
\frac{dE_x}{dz'} = \frac{\tilde{k}_x \tilde{k}_y}{\tilde{\varepsilon}_r} \tilde{H}_x + \left( \tilde{\mu}_r - \frac{\tilde{k}_x^2}{\tilde{\varepsilon}_r} \right) \tilde{H}_y
\]  

(124)

\[
\frac{dE_y}{dz'} = \left( \frac{\tilde{k}_y^2}{\tilde{\varepsilon}_r} - \tilde{\mu}_r \right) \tilde{H}_x - \frac{\tilde{k}_x \tilde{k}_y}{\tilde{\varepsilon}_r} \tilde{H}_y
\]  

(125)
\[
\frac{d\vec{H}_x}{dz'} = \frac{\vec{k}_x k_y}{\hat{\mu}_t} E_x + \left( \vec{\hat{\varepsilon}}_t - \frac{\vec{k}_x^2}{\hat{\mu}_t} \right) E_y
\] (126)

\[
\frac{d\vec{H}_y}{dz'} = \left( \frac{\vec{k}_y^2}{\hat{\mu}_t} \right) E_x - \frac{\vec{k}_x \vec{k}_y}{\hat{\mu}_t} E_y
\] (127)

Equations (124) and (125) can be written as a single matrix equation, and Eqs. (126) and (127) can also be written as a single matrix equation. These are

\[
\frac{d}{dz'} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = P \begin{bmatrix} \vec{H}_x \\ \vec{H}_y \end{bmatrix}
\] (128)

\[
\frac{d}{dz'} \begin{bmatrix} \vec{H}_x \\ \vec{H}_y \end{bmatrix} = Q \begin{bmatrix} E_x \\ E_y \end{bmatrix}
\] (129)

where

\[
P = \frac{1}{\vec{\hat{\varepsilon}}_t} \begin{bmatrix} \vec{k}_x \vec{k}_y & \hat{\mu}_t \vec{\hat{\varepsilon}}_t - \vec{k}_x^2 \\ \vec{k}_y^2 - \hat{\mu}_t \vec{\hat{\varepsilon}}_t & -\vec{k}_x \vec{k}_y \end{bmatrix}
\] (130)

\[
Q = \frac{1}{\hat{\mu}_t} \begin{bmatrix} \vec{k}_x \vec{k}_y & \hat{\mu}_t \vec{\hat{\varepsilon}}_t - \vec{k}_x^2 \\ \vec{k}_y^2 - \hat{\mu}_t \vec{\hat{\varepsilon}}_t & -\vec{k}_x \vec{k}_y \end{bmatrix}
\] (131)

A wave equation is derived by differentiating Eq. (128) with respect to \(z'\) and then substituting Eq. (129) into the new expression to eliminate the magnetic fields. The result is

\[
\frac{d^2}{dz'^2} \begin{bmatrix} E_x \\ E_y \end{bmatrix} - \Omega^2 \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (132)

\[
\Omega^2 = PQ
\] (133)
Equation (132) has the following general solution that contains forward propagating waves and backward propagating waves.

\[
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix} = e^{ix \cdot \mathbf{a}^+} + e^{-ix \cdot \mathbf{a}^-}
\]  
(134)

In this solution, \( \mathbf{a}^+ \) and \( \mathbf{a}^- \) are the amplitude coefficients of the forward and backward propagating waves, respectively. Given the eigen-vector matrix \( \mathbf{W} \) and eigen-value matrix \( \lambda^2 \) of the matrix \( \Omega^2 \), the matrix exponentials can be rewritten using the Jordan canonical form [74].

\[
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix} = \mathbf{W} e^{ix \cdot \mathbf{W}^{-1} \mathbf{a}^+} + \mathbf{W} e^{-ix \cdot \mathbf{W}^{-1} \mathbf{a}^-}
\]  
(135)

The column vectors \( \mathbf{a}^+ \) and \( \mathbf{a}^- \) contain amplitude coefficients that are not yet known. The multiplication of \( \mathbf{W}^{-1} \) and the amplitude coefficient column vector give a different column vector of unknown coefficients. \( \mathbf{c}^+ \) and \( \mathbf{c}^- \) represent these new column vectors of unknown coefficients.

\[
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix} = \mathbf{W} e^{ix \cdot \mathbf{c}^+} + \mathbf{W} e^{-ix \cdot \mathbf{c}^-}
\]  
(136)

In this equation, the eigen-vector matrix \( \mathbf{W} \) describes the electric field amplitudes of the Eigen modes propagating in the medium. \( \lambda \) contains the complex propagation constants of the modes that describe how they propagate. The column vectors \( \mathbf{c}^+ \) and \( \mathbf{c}^- \) describe the complex amplitudes of the forward and backward propagating eigen-modes, respectively. The solution for the magnetic fields can be written in a similar form.

\[
\begin{bmatrix}
\vec{H}_x \\
\vec{H}_y
\end{bmatrix} = \mathbf{V} e^{ix \cdot \mathbf{c}^+} - \mathbf{V} e^{-ix \cdot \mathbf{c}^-}
\]  
(137)
Any sign can be chosen for the term with \( c^\prime \) in Eq. (137). The negative sign is chosen so that differentiating this term with respect to \( z' \) will give positive number, so the both terms in Eq. (137) becomes positive. Differentiating Eq. (137) with respect to \( z' \) and substituting in Eq. (129) and Eq. (136) leads to a relation that allows \( V \) to be calculated directly from the electric field solution.

\[
V = QW\lambda^{-1}
\]  

(138)

Here, \( V \) describes the magnetic field amplitudes of the eigen-modes. Combining Eqs. (136) and (137) into a single matrix equation gives the final form that calculates the electric and magnetic fields at any position \( z' \) throughout the medium.

\[
\psi(z') = \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = \begin{bmatrix} W & W \\ V & -V \end{bmatrix} \begin{bmatrix} e^{j\kappa z} \\ 0 \\ 0 \\ e^{-j\kappa z} \end{bmatrix} \begin{bmatrix} c^+ \\ c^- \end{bmatrix}
\]  

(139)

For an LHI medium, TMMSM can be greatly simplified. Using the dispersion relation with normalized wave vectors, the matrix equation for \( \Omega^2 \) in Eq. (133) simplifies to

\[
\Omega^2 = -\tilde{k}^2 I
\]  

(140)

In this equation, \( I \) represents the identity matrix. Since \( \Omega^2 \) becomes a diagonal matrix in the LHI medium, the eigen-vector matrix \( W \) and eigen-value matrix \( \lambda^2 \) simplify to

\[
W = I
\]  

(141)

\[
\Omega = \lambda = j\tilde{k} I
\]  

(142)
It follows that the expression to calculate eigen-vectors for the magnetic fields $\mathbf{V}$ from Eq. (138) in an LHI medium reduces to

$$\mathbf{V} = \mathbf{Q} \lambda^{-1}$$

(143)

**Formulation of Scattering Matrices**

The scattering behavior of a device can be reduced to a single scattering matrix. The parameters and the physical interpretation of the scattering matrix are illustrated in Figure 21. The $S_{11}$ parameter represents how much of a wave applied to port 1 is reflected from port 1. Similarly, the $S_{22}$ parameter represents how much of a wave applied to port 2 is reflected from port 2. The $S_{21}$ parameter quantifies how much of a wave applied at port 1 transmits through the device to port 2. Similarly, the $S_{12}$ parameter quantifies how much of an applied wave at port 2 transmits to port 1.

![Figure 21: Representation of scattering parameters. $S_{11}$ represents the wave reflected from port 1 and $S_{22}$ represents the wave reflected from port 2. $S_{21}$ represents the wave transmitted from port 1 to port 2 and $S_{12}$ represents vice versa.](image)

At first, a single scattering matrix is initialized which is named as the global scattering matrix. Separate scattering matrices are calculated for each layer by looping through the device
layers and all scattering matrices for each layer are combined into the global scattering matrix using the Redheffer star product [75]. So, every loop over a layer is a combination of two steps. The scattering matrix for a single layer is calculated and then combined into the global scattering matrix using the Redheffer star product. In the end, the scattering matrix is calculated for the reflection and transmission side and merged into the global scattering matrix using the Redheffer star product.

**Scattering matrix for a single layer**

Figure 22 shows the mathematical framework of scattering matrices for the $i$th layer of the device. Medium 1 and medium 2 on both sides of the layer represent materials outside the layer. $\psi$ are the fields within the layers represented in Eq. (139). The subscript $i$ represents the $i$th layer and 1, 2 indicate the medium outside of $i$th layer. The $+/−$ sign used in the superscripts indicate propagation direction. Positive sign means forward propagation and negative means backward propagation. The scattering matrix for the $i$th layer is defined as

\[
\begin{bmatrix}
  e_1^+ \\
e_2^+
\end{bmatrix} =
\begin{bmatrix}
  S'_{11} & S'_{12} \\
  S'_{21} & S'_{22}
\end{bmatrix}
\begin{bmatrix}
  e_1'^+ \\
e_2'^+
\end{bmatrix}
\]

Here, the $S'$ parameters are square matrices that quantify the scattering of the eigen-modes. The $e_1$ and $e_2$ terms represent the mode coefficients outside of the $i$th layer. The superscript and subscript used in this equation indicate the same meaning as described above.
Figure 22: Mathematical framework for scattering matrices.

The mode coefficients represent the fields immediately outside of the $i$th layer. Equation (139) completely describes mode coefficients when the local $z$ coordinate is zero. Away from this point in the $z$ direction, propagation is calculated by the exponential terms described in Eq. (139). The local $z$ coordinate of the left boundary of the $i$th layer in Figure 22 is assumed to be 0 and the right boundary is set to the length of the layer, $L_i$. Expressions for scattering parameters for $i$th layer can be derived by applying boundary conditions at the first and second interface as [25]

$$
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}
\begin{bmatrix}
c_{i+}^{c'} \\
c_{i-}^{c'}
\end{bmatrix}
= 
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}
\begin{bmatrix}
c_{i+}^{c} \\
c_{i-}^{c}
\end{bmatrix}
$$

Equation (145) can be rewritten to solve for the mode coefficients $[c_{i+}^{c'} \ c_{i-}^{c'}]^{T}$

$$
\begin{bmatrix}
c_{i+}^{c'} \\
c_{i-}^{c'}
\end{bmatrix}
= 
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}^{-1}
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}
\begin{bmatrix}
c_{i+}^{c} \\
c_{i-}^{c}
\end{bmatrix}
$$

Equation (147)
Placing the expression for amplitude coefficient terms in Eq. (146) and doing some simplification, the expression becomes

\[
\begin{bmatrix}
W_i & W_j \\
V_i & -V_j
\end{bmatrix}
\begin{bmatrix}
e^{\lambda_{kl}L_i} & 0 \\
e^{-\lambda_{kl}L_i} & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{c}_i^+ \\
\mathbf{c}_i^-
\end{bmatrix}
= \begin{bmatrix}
W_2 & W_2 \\
V_2 & -V_2
\end{bmatrix}
\begin{bmatrix}
\mathbf{c}_2^+ \\
\mathbf{c}_2^-
\end{bmatrix}
\]

\[
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}
\begin{bmatrix}
e^{\lambda_{kl}L_i} & 0 \\
e^{-\lambda_{kl}L_i} & 0
\end{bmatrix}
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}^{-1}
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}
\begin{bmatrix}
\mathbf{c}_i^+ \\
\mathbf{c}_i^-
\end{bmatrix}
= \begin{bmatrix}
W_2 & W_2 \\
V_2 & -V_2
\end{bmatrix}
\begin{bmatrix}
\mathbf{c}_2^+ \\
\mathbf{c}_2^-
\end{bmatrix}
\]

(148)

The inverse matrix term in the above equation can be written as

\[
\begin{bmatrix}
W_i & W_i \\
V_i & -V_i
\end{bmatrix}^{-1}
= (-W_iV_i - W_iV_i)^{-1}
\]

\[
= -\frac{1}{2}V_i^{-1}W_i^{-1}
\]

(149)

The expression of the inverse matrix is placed in Eq. (148). Then the matrices are multiplied and expressed into a simpler form.
\[
\begin{bmatrix}
X_i & 0 & W_i & W_i^{-1} & W_i & W_i^{-1} & c_i^+ \\
0 & X_i^{-1} & V_i & -V_i & V_i & -V_i & c_i^-
\end{bmatrix} = \begin{bmatrix}
W_i & W_i^{-1} & W_i & W_i^{-1} & c_i^+ \\
V_i & -V_i & V_i & -V_i & c_i^-
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_i & 0 & W_i^{-1} & V_i & W_i^{-1} & V_i & c_i^+ \\
0 & X_i^{-1} & W_i^{-1} & V_i & -V_i & V_i & c_i^-
\end{bmatrix} = \begin{bmatrix}
W_i^{-1} & W_i & V_i & W_i^{-1} & V_i & c_i^+ \\
V_i & -V_i & V_i & -V_i & c_i^-
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_i & 0 & A_{i1} & B_{i1} & c_i^+ \\
0 & X_i^{-1} & B_{i1} & A_{i1} & c_i^-
\end{bmatrix} = \begin{bmatrix}
A_{i2} & B_{i2} & c_i^+ \\
B_{i2} & A_{i2} & c_i^-
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_{A_{i1}} & X_{B_{i1}} & c_i^+ \\
B_{i1} & A_{i1} & c_i^-
\end{bmatrix} = \begin{bmatrix}
A_{i2} & B_{i2} & c_i^+ \\
X_{B_{i2}} & X_{A_{i2}} & c_i^-
\end{bmatrix}
\]

[Expression (150)]

where

\[
A_j = W_i^{-1}W_j + V_i^{-1}V_j
\]

\[
B_j = W_i^{-1}W_j - V_i^{-1}V_j
\]
Equation (151) represents general expressions for $A$ and $B$. The subscript $i$ indicates the layer number and the $j$ value can be 1 or 2. Multiplying the first row of both sides of Eq. (150) with $-X_iB_{i1}A_{i2}^{-1}$ and adding the second row, the first row is rearranged. After this manipulation, the first row becomes

$$
\begin{bmatrix}
-X_iB_{i1}A_{i2}^{-1}X_iB_{i1} + A_{i1} & X_iB_{i1}A_{i2}^{-1}A_{i2} - X_iB_{i2}
\end{bmatrix}
\begin{bmatrix}
c_i^r \\
c_i^+ 
\end{bmatrix} =
\begin{bmatrix}
X_iB_{i1}A_{i2}^{-1}X_iA_{i1} - B_{i1} & -X_iB_{i1}A_{i2}^{-1}B_{i2} + X_iA_{i2}
\end{bmatrix}
\begin{bmatrix}
c_i^t \\
c_i^+ 
\end{bmatrix}
$$

$$
\begin{bmatrix}
-X_iB_{i1}A_{i2}^{-1}X_iB_{i1} + A_{i1} & 0
\end{bmatrix}
\begin{bmatrix}
c_i^r \\
c_i^+ 
\end{bmatrix} =
\begin{bmatrix}
X_iB_{i1}A_{i2}^{-1}X_iA_{i1} - B_{i1} & X_i \left( A_{i2} - B_{i2}A_{i1}^{-1}B_{i2} \right)
\end{bmatrix}
\begin{bmatrix}
c_i^t \\
c_i^+ 
\end{bmatrix}
$$

Multiplying the second row of Eq. (150) with $X_iB_{i1}A_{i2}^{-1}$ and subtracting the first row, the second-row elements are rearranged. After this manipulation, the second row of the equation becomes

$$
\begin{bmatrix}
X_iB_{i1}A_{i2}^{-1}A_{i1} - X_iB_{i1} & -X_iB_{i1}A_{i2}^{-1}X_iB_{i2} + A_{i2}
\end{bmatrix}
\begin{bmatrix}
c_i^r \\
c_i^+ 
\end{bmatrix} =
\begin{bmatrix}
-X_iB_{i1}A_{i2}^{-1}B_{i1} + X_iA_{i1} & X_iB_{i1}A_{i2}^{-1}X_iA_{i2} - B_{i2}
\end{bmatrix}
\begin{bmatrix}
c_i^t \\
c_i^+ 
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & -X_iB_{i1}A_{i2}^{-1}X_iB_{i2} + A_{i2}
\end{bmatrix}
\begin{bmatrix}
c_i^r \\
c_i^+ 
\end{bmatrix} =
\begin{bmatrix}
X_i \left( A_{i1} - B_{i1}A_{i2}^{-1}B_{i1} \right) & X_iB_{i1}A_{i2}^{-1}X_iA_{i2} - B_{i2}
\end{bmatrix}
\begin{bmatrix}
c_i^t \\
c_i^+ 
\end{bmatrix}
$$

(154)
Equation (153) and (154) are combined into a single matrix. Then the left matrix is brought to the right side and simplified to get the final form of the scattering matrix described in Eq. (144).

\[
\begin{bmatrix}
-X_i B_{i_1} A_{i_2} - X_i B_{i_1} + A_{i_1} & 0 \\
0 & -X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_2} + A_{i_2}
\end{bmatrix}
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
= 
\begin{bmatrix}
X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1} - B_{i_1} & X_i \left( A_{i_2} - B_{i_2} A_{i_2}^{-1} B_{i_2} \right) \\
X_i \left( A_{i_1} - B_{i_1} A_{i_1}^{-1} B_{i_1} \right) & X_i B_{i_1} A_{i_2}^{-1} X_i A_{i_2} - B_{i_2}
\end{bmatrix}
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
\]

\[
\Rightarrow 
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
= 
\begin{bmatrix}
-X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1} + A_{i_1} & 0 \\
0 & -X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_2} + A_{i_2}
\end{bmatrix}^{-1}
\begin{bmatrix}
X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1} - B_{i_1} & X_i \left( A_{i_2} - B_{i_2} A_{i_2}^{-1} B_{i_2} \right) \\
X_i \left( A_{i_1} - B_{i_1} A_{i_1}^{-1} B_{i_1} \right) & X_i B_{i_1} A_{i_2}^{-1} X_i A_{i_2} - B_{i_2}
\end{bmatrix}
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
\]

\[
\Rightarrow 
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
= 
\begin{bmatrix}
(A_{i_1} - X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1})^{-1} & 0 \\
0 & (A_{i_2} - X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_2})^{-1}
\end{bmatrix}
\begin{bmatrix}
X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1} - B_{i_1} & X_i \left( A_{i_2} - B_{i_2} A_{i_2}^{-1} B_{i_2} \right) \\
X_i \left( A_{i_1} - B_{i_1} A_{i_1}^{-1} B_{i_1} \right) & X_i B_{i_1} A_{i_2}^{-1} X_i A_{i_2} - B_{i_2}
\end{bmatrix}
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
\]

\[
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
= 
\begin{bmatrix}
(A_{i_1} - X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1})^{-1} \left( X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1} - B_{i_1} \right) & \left( A_{i_1} - X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_1} \right)^{-1} X_i \left( A_{i_2} - B_{i_2} A_{i_2}^{-1} B_{i_2} \right) \\
(A_{i_2} - X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_2})^{-1} X_i \left( A_{i_1} - B_{i_1} A_{i_1}^{-1} B_{i_1} \right) & \left( A_{i_2} - X_i B_{i_1} A_{i_2}^{-1} X_i B_{i_2} \right)^{-1} \left( X_i B_{i_1} A_{i_2}^{-1} X_i A_{i_2} - B_{i_2} \right)
\end{bmatrix}
\begin{bmatrix}
c_i' \\
c_i''
\end{bmatrix}
\]

\[
(155)
\]

Comparing Eq. (144) and Eq. (155), the S coefficients can be written as
\[ S_{11}^i = \left( A_{i1} - X_i B_{i2} A_{j2}^{-1} X_j B_{j2} \right)^{-1} \left( X_j B_{j2} A_{j2}^{-1} X_i A_{i1} - B_{i1} \right) \]
\[ S_{12}^i = \left( A_{i1} - X_i B_{i2} A_{j2}^{-1} X_j B_{j2} \right)^{-1} X_j \left( A_{j1} - B_{j2} A_{j2}^{-1} B_{j2} \right) \]
\[ S_{21}^i = \left( A_{j2} - X_j B_{j1} A_{i1}^{-1} X_i B_{i1} \right)^{-1} X_i \left( A_{i1} - B_{i1} A_{i1}^{-1} B_{i1} \right) \]
\[ S_{22}^i = \left( A_{j2} - X_j B_{j1} A_{i1}^{-1} X_i B_{i1} \right)^{-1} \left( X_i B_{i1} A_{i1}^{-1} X_j A_{j2} - B_{j2} \right) \]

Scattering matrices are not only dependent on the materials of the \( i \)th layer, but also the materials outside of the layer [25]. If the same layer appears again in another part of a device, the scattering matrix from the first occurrence cannot be used when the layers are adjacent to different mediums. Recalculating the scattering matrix for the same layer is slow and inefficient. Fortunately, a novel approach was developed that overcomes this limitation [26]. To make every layer in a device have the same external mediums, a gap of zero thickness is inserted between each layer. Let \( W_g \) and \( V_g \) be the eigen-vectors for gap medium, so \( W_1 = W_2 = W_g \) and \( V_1 = V_2 = V_g \) in Eqs. (156), (151) and these equations reduce to

\[ S_{11}^i = S_{22}^i = \left( A_j - X_i B_i A_i^{-1} X_i B_i \right)^{-1} \left( X_i B_i A_i^{-1} X_j A_j - B_j \right) \]

\[ S_{12}^i = S_{21}^i = \left( A_j - X_i B_i A_i^{-1} X_i B_i \right)^{-1} X_i \left( A_j - B_i A_i^{-1} B_i \right) \]

\[ A_j = W_j^{-1} W_g + V_j^{-1} V_g \]

\[ B_j = W_j^{-1} W_g - V_j^{-1} V_g \]

For the gap medium, Eq. (123) can be expressed as

\[ \vec{k}_z = \sqrt{\mu_{t,g} \vec{k}_{t,g} - \vec{k}_x^2 - \vec{k}_y^2} \]
Here, $\tilde{\mu}_{r,g}$ and $\tilde{\varepsilon}_{r,g}$ represents the permeability and permittivity of the gap medium, respectively. It is possible to choose any value for $\tilde{\mu}_{r,g}$ and $\tilde{\varepsilon}_{r,g}$, but it is convenient to choose $\tilde{\mu}_{r,g} = 1.0$ and $\tilde{\varepsilon}_{r,g} = 1 + \bar{k}_x^2 + \bar{k}_y^2$ so that $\bar{k}_z$ is never equal to zero in Eq. (159) as this causes numerical instabilities.

Given this choice of material, the gap medium parameters become

$$Q_g = \begin{bmatrix} \bar{k}_x \bar{k}_y & 1 + \bar{k}_y^2 \\ -\left(1 + \bar{k}_z^2\right) & -\bar{k}_x \bar{k}_y \end{bmatrix}$$  \hspace{1cm} (160)

$$W_g = I$$  \hspace{1cm} (161)

$$V_g = -jQ_g$$  \hspace{1cm} (162)

$W_g$ and $W_i$ become a diagonal matrix for the LHI medium. So, the scattering matrix parameters $A_i$ and $B_i$ in Eq. (158) reduce to

$$A_i = I + V_i^{-1}V_g$$  \hspace{1cm} (163)

$$B_i = I - V_i^{-1}V_g$$

**Redheffer star product**

Given the scattering matrices for all individual layers, it is necessary to combine them into a single scattering matrix. The Redheffer star product is used to combine all scattering matrices into a single scattering matrix [25, 27, 75, 76]. To understand the Redheffer star product, let's assume two scattering matrices $S^A$ and $S^B$ are describing two adjacent layers. The combined scattering matrix $S^{AB}$ using the Redheffer star product can be written as

$$\begin{bmatrix} S_{11}^{AB} & S_{12}^{AB} \\ S_{21}^{AB} & S_{22}^{AB} \end{bmatrix} = \begin{bmatrix} S_{11}^A & S_{12}^A \\ S_{21}^A & S_{22}^A \end{bmatrix} \star \begin{bmatrix} S_{11}^B & S_{12}^B \\ S_{21}^B & S_{22}^B \end{bmatrix}$$  \hspace{1cm} (164)
Where [25]

\[
S_{11}^{AB} = S_{11}^A + S_{12}^A \left[ I - S_{11}^B S_{22}^A \right]^{-1} S_{11}^B S_{21}^A \\
S_{12}^{AB} = S_{12}^A \left[ I - S_{11}^B S_{22}^A \right]^{-1} S_{12}^B \\
S_{21}^{AB} = S_{21}^B \left[ I - S_{22}^A S_{11}^B \right]^{-1} S_{21}^A \\
S_{22}^{AB} = S_{22}^B + S_{21}^B \left[ I - S_{22}^A S_{11}^B \right]^{-1} S_{22}^A S_{12}^B
\]  

(165)

**Scattering matrices for external regions**

After combining all scattering matrices representing the device layers into the global scattering matrix via Redheffer star products, it is needed to add the gap medium and the external regions. If the gap medium is the same as the external regions of the device, no extra steps are needed to calculate the final scattering matrix. However, usually, the gap medium is different than the physical external regions and extra steps are needed to calculate the final global scattering matrix. This is done by calculating two additional scattering matrices. The reflection-side scattering matrix \( S_{\text{ref}} \) converts the gap medium on the reflection side to the physical medium on the reflection side of the device. Similarly, the transmission-side scattering matrix \( S_{\text{trn}} \) converts the gap medium on the transmission side to the physical medium on the transmission side of the device. Altogether, the final global scattering matrix is calculated using the Redheffer star product as

\[
S_{\text{global}}^\text{global} = S_{\text{global}} \otimes S_{\text{global}} \otimes \cdots \otimes S_{\text{global}}^N
\]

(166)

\[
S_{\text{global}}^\text{global} = S_{\text{trn}} S_{\text{global}}
\]

(167)

\[
S_{\text{global}}^\text{global} = S_{\text{global}} S_{\text{trn}}
\]

(168)
Expressions for the reflection-side scattering matrix parameters can be calculated from Eqs. (151) and (156) by setting $L_i = 0$, $V_1 = V_i = V_{\text{ref}}$, and $V_2 = V_g$.

\[
S_{11}^{\text{ref}} = B_{\text{ref}}^{-1} A_{\text{ref}}^{-1}
\]

\[
S_{12}^{\text{ref}} = 0.5(A_{\text{ref}}^{-1} B_{\text{ref}}^{-1} - B_{\text{ref}} A_{\text{ref}}^{-1})
\]

\[
S_{21}^{\text{ref}} = 2A_{\text{ref}}^{-1}
\]

\[
S_{22}^{\text{ref}} = -A_{\text{ref}}^{-1} B_{\text{ref}}
\]

\[
A_{\text{ref}} = W_{\text{ref}}^{-1} W_{g} + V_{\text{ref}}^{-1} V_{g}
\]

\[
B_{\text{ref}} = W_{\text{ref}}^{-1} W_{g} - V_{\text{ref}}^{-1} V_{g}
\]

Expressions for the transmission-side scattering parameters can calculated from Eqs. (151) and (156) by setting $L_i = 0$, $V_1 = V_g$, and $V_2 = V_i = V_{\text{trn}}$.

\[
S_{11}^{\text{trn}} = -A_{\text{trn}}^{-1} B_{\text{trn}}
\]

\[
S_{12}^{\text{trn}} = 2A_{\text{trn}}^{-1}
\]

\[
S_{21}^{\text{trn}} = 0.5(A_{\text{trn}}^{-1} B_{\text{trn}}^{-1} - B_{\text{trn}} A_{\text{trn}}^{-1})
\]

\[
S_{22}^{\text{trn}} = B_{\text{trn}} A_{\text{trn}}^{-1}
\]

\[
A_{\text{trn}} = W_{\text{trn}}^{-1} W_{g} + V_{\text{trn}}^{-1} V_{g}
\]

\[
B_{\text{trn}} = W_{\text{trn}}^{-1} W_{g} - V_{\text{trn}}^{-1} V_{g}
\]

For an LHI medium, reflection side scattering matrix coefficients $A_{\text{ref}}$ and $B_{\text{ref}}$ and the transmission side scattering matrix coefficients $A_{\text{trn}}$ and $B_{\text{trn}}$ can be written in form of Eq. (163).
CALCULATING THE SOURCE

The source is defined through the mode coefficients of the incident wave vector, \( c_{\text{inc}} \) and the polarization vector \( \vec{P} \).

\[
c_{\text{inc}} = \begin{bmatrix} P_x \\ P_y \end{bmatrix}
\]  \hspace{1cm} (173)

\[
P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}
\]  \hspace{1cm} (174)

The composite polarization vector is defined as

\[
\vec{P} = p_{\text{TE}} \hat{a}_{\text{TE}} + p_{\text{TM}} \hat{a}_{\text{TM}}
\]  \hspace{1cm} (175)

Here, \( \hat{a}_{\text{TE}} \) and \( \hat{a}_{\text{TM}} \) are the unit vectors in the direction of TE and TM polarizations, respectively. For TE polarization, \( p_{\text{TE}} \) is set to 1 and \( p_{\text{TM}} \) is set to 0 in Eq. (175). For TM polarization, \( p_{\text{TE}} \) is set to 0 and \( p_{\text{TM}} \) is set to 1 in Eq. (175). \( \hat{a}_{\text{TE}} \) and \( \hat{a}_{\text{TM}} \) are calculated from the cross product of the surface normal \( \hat{n} \) and the incident wave vector \( \vec{k}_{\text{inc}} \). To get a unit amplitude, that cross product is divided by the magnitude of the cross product. In the case of normal incidence, the surface normal and the incident wave vector are in the same direction, so their cross product becomes zero. For this special case, the unit vector for TE polarization is defined either in \( x \) or \( y \) direction. The unit vector for TM polarization is defined in terms of the unit vector of TE polarization. So, it is not needed to define the special case for TM polarization. The expressions for \( \hat{a}_{\text{TE}} \) and \( \hat{a}_{\text{TM}} \) are [56]
The incident wave vector $\mathbf{k}_{\text{inc}}$ is defined as

$$\mathbf{k}_{\text{inc}} = k_{\text{inc}} \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} \tag{178}$$

Here $\theta$ is the incident wave angle with the plane of incidence and $\phi$ is the azimuthal angle.

**Calculating Reflectance and Transmittance**

The mode coefficients of the reflected waves $c_{\text{ref}}$ and the transmitted waves $c_{\text{trn}}$ are related to the mode coefficients of the incident waves $c_{\text{inc}}$ through global scattering matrix $S_{\text{global}}^{\text{global}}$ [25]. If it is assumed that there is no incident wave on the transmission side, then Eq. (144) can be written as

$$\begin{bmatrix} c_{\text{ref}} \\ c_{\text{trn}} \end{bmatrix} = \begin{bmatrix} S_{11}^{\text{global}} & S_{12}^{\text{global}} \\ S_{21}^{\text{global}} & S_{22}^{\text{global}} \end{bmatrix} \begin{bmatrix} c_{\text{inc}} \\ 0 \end{bmatrix} \tag{179}$$

From this matrix equation, the expressions to calculate $c_{\text{ref}}$ and $c_{\text{trn}}$ are

$$c_{\text{ref}} = S_{11}^{\text{global}} c_{\text{inc}} \tag{180}$$

$$c_{\text{trn}} = S_{21}^{\text{global}} c_{\text{inc}} \tag{181}$$
Equation (136) is used to get the expression for the incident, reflected and transmitted electric fields

\[
\begin{bmatrix}
E_{x,\text{inc}} \\
E_{y,\text{inc}}
\end{bmatrix} = W_{\text{ref}} c_{\text{inc}}
\]  
(182)

\[
\begin{bmatrix}
E_{x,\text{ref}} \\
E_{y,\text{ref}}
\end{bmatrix} = W_{\text{ref}} c_{\text{ref}}
\]  
(183)

\[
\begin{bmatrix}
E_{x,\text{trn}} \\
E_{y,\text{trn}}
\end{bmatrix} = W_{\text{tm}} c_{\text{tm}}
\]  
(184)

Solving Eq. (182) for \(c_{\text{inc}}\) and substituting that expression into Eqs. (180) and (181) gives new expressions for the mode coefficient \(c_{\text{ref}}\) and \(c_{\text{tm}}\). Substituting these mode coefficient expressions in Eqs. (183) and (184) gives the generalized expression for the reflected and the transmitted fields. Using the LHI medium condition in Eq. (141), the field expressions are simplified to

\[
\begin{bmatrix}
E_{x,\text{ref}} \\
E_{y,\text{ref}}
\end{bmatrix} = S_{11}^{\text{global}} \begin{bmatrix}
E_{x,\text{inc}} \\
E_{y,\text{inc}}
\end{bmatrix}
\]  
(185)

\[
\begin{bmatrix}
E_{x,\text{trn}} \\
E_{y,\text{trn}}
\end{bmatrix} = S_{21}^{\text{global}} \begin{bmatrix}
E_{x,\text{inc}} \\
E_{y,\text{inc}}
\end{bmatrix}
\]  
(186)

To calculate the total reflected and transmitted power, the longitudinal field components must be calculated on both the reflection and transmission sides. The longitudinal field components are calculated from the \(x\) and \(y\) components using Maxwell’s divergence equation \(\nabla \cdot \vec{E} = 0\). Maxwell’s divergence equation is expanded into Cartesian coordinates and then solved only for the \(z\) component of the electric field.
As mentioned in the introduction, the transmittance and reflectance for the TE polarization are calculated in terms of the electric fields. Given all components of the electric field, the reflectance $R_{TE}$ and transmittance $T_{TE}$ is [77]

\[ R_{TE} = \frac{|\vec{E}_{\text{ref}}|^2}{|\vec{E}_{\text{inc}}|^2} \]  

\[ T_{TE} = \frac{|\vec{E}_{\text{trn}}|^2}{|\vec{E}_{\text{inc}}|^2} \text{Re} \left[ \frac{\vec{k}_{z,\text{trn}} / \vec{\mu}_{r,\text{trn}}}{\vec{k}_{z,\text{inc}} / \vec{\mu}_{r,\text{inc}}} \right] \]  

where, $|\vec{E}|^2 = |E_x|^2 + |E_y|^2 + |E_z|^2$ and the subscript TE indicates TE polarization.

According to Stratton [14] and Gallant [78], sometimes it is convenient to calculate Fresnel reflection and transmission coefficients in terms of magnetic fields for TM polarization. Here, the reflectance and transmittance for TM polarization are calculated in terms of the magnetic fields. From Faraday’s law, the relation between the magnetic and electric field can be written as [73]

\[ \vec{H} = \frac{\vec{k} \times \vec{E}}{\omega \vec{\mu}} \]  

For the normalized magnetic field, components of the incident, reflected and transmitted fields can be written in terms of the electric field components as

\[ E_{z,\text{ref}} = -\frac{\vec{k}_x E_{x,\text{ref}} + \vec{k}_y E_{y,\text{ref}}}{\vec{k}_{z,\text{ref}}} \]  

\[ E_{z,\text{trn}} = -\frac{\vec{k}_x E_{x,\text{trn}} + \vec{k}_y E_{y,\text{trn}}}{\vec{k}_{z,\text{trn}}} \]
\[ \vec{H}_{\text{inc}} = \frac{\vec{k}_{\text{inc}} \times \vec{E}_{\text{inc}}}{\sqrt{k_0 n_0 \mu_{\text{r,inc}}}} \] (192)

\[ \vec{H}_{\text{ref}} = \frac{\vec{k}_{\text{ref}} \times \vec{E}_{\text{ref}}}{\sqrt{k_0 n_0 \mu_{\text{r,ref}}}} \] (193)

\[ \vec{H}_{\text{tm}} = \frac{\vec{k}_{\text{tm}} \times \vec{E}_{\text{tm}}}{\sqrt{k_0 n_0 \mu_{\text{r,tm}}}} \] (194)

So, the reflectance and transmittance for TM polarization can be written in terms of the magnetic field as

\[ R_{\text{TM}} = \frac{|\vec{H}_{\text{ref}}|^2}{|\vec{H}_{\text{inc}}|^2} \] (195)

\[ T_{\text{TM}} = \frac{|\vec{H}_{\text{tm}}|^2}{|\vec{H}_{\text{inc}}|^2} \text{Re} \left[ \frac{\vec{k}_{z,\text{tm}} / \vec{\varepsilon}_{r,\text{tm}}}{\vec{k}_{z,\text{inc}} / \vec{\varepsilon}_{r,\text{inc}}} \right] \] (196)

Here, the subscript TM indicates the TM polarization.

**IMPLEMENTATION OF TMMSM**

Before the TMMSM can simulate a device, the problem must be described. This includes identifying the number of layers, layer thicknesses, material properties of each layer, material properties of reflection and transmission region, angle of incidence, the polarization of the source, etc.
First, the transverse wave vector components are calculated in the $x$ and $y$ direction. The gap medium parameters are calculated before calculating the scattering matrix parameters for each layer. Then the global scattering matrix is initialized as

$$S_{11}^{\text{global}} = 0, \quad S_{12}^{\text{global}} = I, \quad S_{21}^{\text{global}} = I, \quad \text{and} \quad S_{22}^{\text{global}} = 0,$$

(197)

The TMMSM algorithm then enters the layers of the device and calculates the wave vector component along the $z$ axis for the first layer. The derivative/differential matrices $Q$, $\Omega$, and $V$ are calculated for that layer. Using these and the gap medium matrices, the scattering matrix parameters are calculated, and the global scattering matrix is updated. This process is done for each layer, and the global scattering matrix is calculated that represents the entire device. Then the scattering matrix is calculated for the reflection and transmission region and combined with the global scattering matrix.

The source needs to be introduced to analyze the device. The source is calculated from the incident wave vector and polarization vector. The calculation of the source is described in the ‘calculating the source’ section. Using the global scattering matrix parameters and the source, the reflected and transmitted fields are calculated. The wave vector component in the $z$ direction for reflection and transmission region is calculated to calculate the longitudinal field components. In the end, the transmittance and reflectance are calculated for the entire device. The process of calculating the reflected/transmitted fields and reflectance/transmittance is described in the ‘calculating reflectance and transmittance’ section. If the transmittance is negative, then the sign of refractive index of the transmission/reflection side is reversed. It should be noted that the sign of the refractive index is changed for either the reflection side or the transmission side, not for both sides. A block diagram of the TMMSM algorithm is provided in the flowchart in Figure 23.
RESULTS

The results for this chapter are presented in two parts. In the first part, the results obtained from the analytical equation obtained in the previous chapter are compared with the numerical results obtained from this section. This part only considers external regions without any intermediate layers. In the second part, intermediate layers with complex refractive index are considered, and the results are compared to the results published in [79].

Comparison of analytical results and TMMSM results

The analytical equations derived in Chapter 4 were implemented in MATLAB and simulated for different combinations of complex and non-complex mediums. The derived equations are valid for a single interface between two semi-infinite mediums. Therefore, multiple interfaces cannot be handled without additional analysis and derivations.
The TMMSM presented in this chapter was also implemented in MATLAB. Since the analytical equations are only capable of handling a single interface, only the external regions will be considered without any intermediate layers in this portion of results for TMMSM.

Three different material combinations were considered for this single interface: complex medium to non-complex medium, non-complex medium to complex medium and complex medium to complex medium. The results are compiled in seven different tables with different material properties and different angle of incident. The polarization column indicates the type of polarization considered for the corresponding set of material combinations. $\varepsilon_r$ and $\mu_r$ denote complex relative permittivity and complex relative permeability of the respective medium. The subscripts ‘inc’ and ‘trn’ represent the incident (medium 1) and transmission (medium 2) regions, respectively. $R$ and $T$ denote reflectance and transmittance. The ‘Analytical’ column represents the results from the analytical equations derived in Chapter 4. The ‘TMMSM’ column indicates the results got from TMMSM.

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Polarization</th>
<th>Medium 1</th>
<th>Medium 2</th>
<th>Angle</th>
<th>Analytical</th>
<th>TMMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon_{r,inc}$</td>
<td>$\mu_{r,inc}$</td>
<td>$\varepsilon_{r,trn}$</td>
<td>$\mu_{r,trn}$</td>
<td>$\theta_{inc}$</td>
<td>$R$</td>
</tr>
<tr>
<td>TE</td>
<td>1.0</td>
<td>1.0</td>
<td>5.1 + i2.3</td>
<td>1.0</td>
<td>30$^\circ$</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>2.9 − i2.7</td>
<td>1.0</td>
<td>57$^\circ$</td>
<td>0.35</td>
</tr>
<tr>
<td>TM</td>
<td>1.0</td>
<td>1.0</td>
<td>1.8 + i4.9</td>
<td>1.0</td>
<td>73$^\circ$</td>
<td>0.11</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>−2.5 − i1.9</td>
<td>1.0</td>
<td>37$^\circ$</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 2. Scattering at the air to complex medium (complex $\varepsilon_{r,trn}$)
Table 3. Scattering at the air to complex medium (complex $\tilde{\mu}_{r,\text{trn}}$)

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Medium 1</th>
<th>Medium 2</th>
<th>Angle</th>
<th>Analytical</th>
<th>TMMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tilde{\varepsilon}_{r,\text{inc}}$</td>
<td>$\tilde{\mu}_{r,\text{inc}}$</td>
<td>$\tilde{\varepsilon}_{r,\text{trn}}$</td>
<td>$\tilde{\mu}_{r,\text{trn}}$</td>
<td>$\theta_{\text{inc}}$</td>
</tr>
<tr>
<td>TE</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0 + i2.0</td>
<td>66°</td>
<td>0.06</td>
</tr>
<tr>
<td>TM</td>
<td>1.0</td>
<td>1.0</td>
<td>2.5 − i2.3</td>
<td>37°</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>1.4 + i3.7</td>
<td>84°</td>
<td>0.85</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>−2.5 − i1.9</td>
<td>55°</td>
<td>0.75</td>
</tr>
</tbody>
</table>

Table 4. Scattering at complex medium (complex $\tilde{\varepsilon}_{r,\text{inc}}$) to air

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Medium 1</th>
<th>Medium 2</th>
<th>Angle</th>
<th>Analytical</th>
<th>TMMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tilde{\varepsilon}_{r,\text{inc}}$</td>
<td>$\tilde{\mu}_{r,\text{inc}}$</td>
<td>$\tilde{\varepsilon}_{r,\text{trn}}$</td>
<td>$\tilde{\mu}_{r,\text{trn}}$</td>
<td>$\theta_{\text{inc}}$</td>
</tr>
<tr>
<td>TE</td>
<td>−3.6 + i4.7</td>
<td>1.0</td>
<td>1.0</td>
<td>75°</td>
<td>0.94</td>
</tr>
<tr>
<td>TM</td>
<td>2.5 − i5.5</td>
<td>1.0</td>
<td>1.0</td>
<td>46°</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td>−7.6 − i4.0</td>
<td>1.0</td>
<td>1.0</td>
<td>39°</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>3.5 + i6.9</td>
<td>1.0</td>
<td>1.0</td>
<td>22°</td>
<td>0.27</td>
</tr>
</tbody>
</table>

Table 5. Scattering at complex medium (complex $\tilde{\mu}_{r,\text{inc}}$) to air

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Medium 1</th>
<th>Medium 2</th>
<th>Angle</th>
<th>Analytical</th>
<th>TMMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tilde{\varepsilon}_{r,\text{inc}}$</td>
<td>$\tilde{\mu}_{r,\text{inc}}$</td>
<td>$\tilde{\varepsilon}_{r,\text{trn}}$</td>
<td>$\tilde{\mu}_{r,\text{trn}}$</td>
<td>$\theta_{\text{inc}}$</td>
</tr>
<tr>
<td>TE</td>
<td>1.0</td>
<td>1.5 + i2.9</td>
<td>1.0</td>
<td>1.0</td>
<td>29°</td>
</tr>
<tr>
<td>TM</td>
<td>1.0</td>
<td>3.5 − i2.3</td>
<td>1.0</td>
<td>1.0</td>
<td>76°</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>−2.2 + i4.7</td>
<td>1.0</td>
<td>1.0</td>
<td>54°</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.8 − i3.4</td>
<td>1.0</td>
<td>1.0</td>
<td>36°</td>
</tr>
</tbody>
</table>
Table 6. Scattering at complex medium (complex $\varepsilon_{r,\text{inc}}$) to complex medium (complex $\varepsilon_{r,\text{trn}}$)

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Medium 1</th>
<th>Medium 2</th>
<th>Angle</th>
<th>Analytical</th>
<th>TMMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon_{r,\text{inc}}$</td>
<td>$\mu_{r,\text{inc}}$</td>
<td>$\varepsilon_{r,\text{trn}}$</td>
<td>$\mu_{r,\text{trn}}$</td>
<td>$\theta_{\text{inc}}$</td>
</tr>
<tr>
<td>TE</td>
<td>2.4 $- i4.7$</td>
<td>1.0</td>
<td>$-1.6 + i4.7$</td>
<td>1.0</td>
<td>53°</td>
</tr>
<tr>
<td></td>
<td>$-1.8 + i5.8$</td>
<td>1.0</td>
<td>$3.8 - i2.6$</td>
<td>1.0</td>
<td>67°</td>
</tr>
<tr>
<td>TM</td>
<td>4.6 $+ i5.9$</td>
<td>1.0</td>
<td>$-6.6 - i8.0$</td>
<td>1.0</td>
<td>28°</td>
</tr>
<tr>
<td></td>
<td>7.8 $- i2.8$</td>
<td>1.0</td>
<td>$-3.1 + i7.4$</td>
<td>1.0</td>
<td>77°</td>
</tr>
</tbody>
</table>

Table 7. Scattering at complex medium (complex $\mu_{r,\text{inc}}$) to complex medium (complex $\mu_{r,\text{trn}}$)

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Medium 1</th>
<th>Medium 2</th>
<th>Angle</th>
<th>Analytical</th>
<th>TMMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon_{r,\text{inc}}$</td>
<td>$\mu_{r,\text{inc}}$</td>
<td>$\varepsilon_{r,\text{trn}}$</td>
<td>$\mu_{r,\text{trn}}$</td>
<td>$\theta_{\text{inc}}$</td>
</tr>
<tr>
<td>TE</td>
<td>1.0</td>
<td>$5.3 + i4.0$</td>
<td>1.0</td>
<td>$7.5 + i2.7$</td>
<td>05°</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>$3.5 - i2.3$</td>
<td>1.0</td>
<td>$-4.5 - i7.3$</td>
<td>25°</td>
</tr>
<tr>
<td>TM</td>
<td>1.0</td>
<td>$-2.2 + i4.7$</td>
<td>1.0</td>
<td>$4.7 + i2.2$</td>
<td>36°</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>$1.8 - i3.4$</td>
<td>1.0</td>
<td>$9.2 - i1.5$</td>
<td>22°</td>
</tr>
</tbody>
</table>

Table 8. Scattering at complex medium (complex $\varepsilon_{r,\text{inc}}$ and $\mu_{r,\text{inc}}$) to complex medium (complex $\varepsilon_{r,\text{trn}}$ and $\mu_{r,\text{trn}}$)

<table>
<thead>
<tr>
<th>Polarization</th>
<th>Medium 1</th>
<th>Medium 2</th>
<th>Angle</th>
<th>Analytical</th>
<th>TMMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\varepsilon_{r,\text{inc}}$</td>
<td>$\mu_{r,\text{inc}}$</td>
<td>$\varepsilon_{r,\text{trn}}$</td>
<td>$\mu_{r,\text{trn}}$</td>
<td>$\theta_{\text{inc}}$</td>
</tr>
<tr>
<td>TE</td>
<td>2.5 $+ i4.7$</td>
<td>1.9 $+ i3.7$</td>
<td>1.2 $+ i3.5$</td>
<td>5.1 $+ i7.8$</td>
<td>43°</td>
</tr>
<tr>
<td></td>
<td>2.1 $- i4.1$</td>
<td>3.5 $+ i5.3$</td>
<td>7.1 $- i4.3$</td>
<td>6.3 $+ i3.7$</td>
<td>15°</td>
</tr>
<tr>
<td></td>
<td>3.7 $+ i6.5$</td>
<td>2.5 $- i5.1$</td>
<td>4.5 $+ i7.7$</td>
<td>4.8 $- i3.1$</td>
<td>65°</td>
</tr>
<tr>
<td>TM</td>
<td>2.3 $- i4.1$</td>
<td>$-6.7 - i5.3$</td>
<td>$-1.6 - i7.3$</td>
<td>$-4.5 - i2.9$</td>
<td>50°</td>
</tr>
<tr>
<td></td>
<td>8.2 $+ i3.7$</td>
<td>3.5 $+ i2.8$</td>
<td>7.3 $+ i5.8$</td>
<td>6.3 $+ i1.6$</td>
<td>38°</td>
</tr>
<tr>
<td></td>
<td>2.3 $+ i4.9$</td>
<td>3.7 $- i2.8$</td>
<td>4.4 $+ i6.3$</td>
<td>$-5.3 - i7.2$</td>
<td>75°</td>
</tr>
<tr>
<td></td>
<td>$-1.5 - i3.8$</td>
<td>2.8 $+ i4.7$</td>
<td>3.9 $- i5.3$</td>
<td>7.4 $+ i6.5$</td>
<td>40°</td>
</tr>
</tbody>
</table>
Comparison of TMM results and TMMSM results

This section will consider intermediate layers of complex mediums. An alternative TMM described in [79] was implemented to simulate a Fabry-Perot type metal-insulator-metal (MIM) optical cavity device for an incident angle of 40°. The device has three layers. The first layer is 6 nm thick chromium (Cr), the second layer is 590 nm thick poly methyl methacrylate (PMMA), and the third layer is 100 nm thick lossy gold (Au). Sapphire was used as the substrate. The refractive index for PMMA and Sapphire was set to constant values, but complex refractive indices were considered for Cr and Au from [80] which varies with wavelength. Spline interpolation is used to interpolate the refractive index at unknown wavelengths. The material properties used for the simulation in [79] were not clearly stated and so it was not possible to replicate their results exactly, but the present work got very close using the spline interpolation. The result found after implementing the TMM method in [79] matched exactly the TMMSM presented here. These results are provided in Figure 24.

Figure 24: Simulation results of Fabry-Perot type metal-insulator-metal (MIM) optical cavity device described in [79]. (a) Result obtained from TMM, and (b) result from TMMSM.
Chapter 6: Conclusion

The three main deliverables from this dissertation are: an algorithm developed and implemented that generates spatially variant anisotropic metamaterials (SVAMs) in 3D volumetric circuits, derived analytical equations that calculate the power at the interface between two complex mediums, and a modified transfer matrix method with scattering matrices (TMMSM) that works for complex mediums in both external regions and intermediate layers of a layered device. The following sections summarize the achievements and potential future of this research.

The third chapter of this dissertation describes the SVAM generating algorithm and its implementation. The algorithm has been integrated into CAD software and its capabilities have been demonstrated by 3D printing a 3D circuit with a SVAM. It has been shown by the 3D printed 3D circuit with a SVAM that the algorithm can spatially vary the layers of two materials in a defect-free manner to produce the SVAM. Furthermore, it proves that the mesh generated by this algorithm was good enough to be 3D printed. As a result of this algorithm, it is possible to investigate the incorporation of SVAMs into various types of 3D circuits and structures. Simulation results of the directional coupler confirm the SVAM's ability to decouple waveguides. With the SVAM generating algorithm, research can be conducted to determine the best way to design and implement SVAMs.

As the 3D volumetric circuit with a SVAM is going to be 3D printed, the resolution of the 3D printer is an important factor that needs to be considered when designing the SVAMs. The thickness of each layer of the SVAM should be thick enough so that the 3D printers will be able to print it. The layer thickness is dependent on the number of SVAM layers and the distance between the circuit components. The thickness of a single SVAM layer between two circuit components can be found by dividing the distance between the circuit components by the number
of SVAM layers. If the thickness of a layer of the SVAM is less than the resolution of the 3D printer, then it will not be possible to 3D print it. Future studies should also focus on finding the sufficient number of layers needed for the SVAM to exhibit the target constitutive values. A single layer is not going to work as a SVAM as it will not form a NUM. The optimum layer numbers should be determined by further research.

Other research can be done on determining the materials for the SVAM. Ansys HFSS can be used to simulate SVAM structures and identify the optimal material combination that will reduce interference and decouple the circuit components. Then the model can be 3D printed using that specific material combination to form the SVAM and the performance of the SVAM can be evaluated. It is necessary to do further studies to identify which components need to be decoupled. In particular, it is not clear to what degree the SVAM should be applied around the electrical interconnects. Additional research is required to determine the appropriate way to incorporate SVAMs into a circuit and to find the other functions that SVAMs may enable. The results generated in this dissertation will enable this research to happen.

Chapter four presents generalized Fresnel coefficients and conservation equations at the interface between two complex mediums. The derivation shows that the extra delta term added in some literature to the conservation equation of complex mediums becomes zero. The delta term arises due to the interaction between reflected and incident waves at the interface and was claimed to be significant in complex mediums [13, 17]. Eliminating the delta term from the conservation equation results in a more elegant and simpler formulation than any previous work. The derived equations were benchmarked with the examples given in the Weber paper [13] and the same results were achieved. There was one case where Weber had a verified mistake. The mistake was with the transmittance values in the gold-to-silver example, which was verified with the author. As Weber
only considered complex permittivity for lossy mediums, these findings demonstrate that the results obtained by the simpler analytical derivation presented here are correct for lossy-to-lossy interfaces considering complex permittivity. The analytical equations derived in this dissertation calculate the power quantities at the interface between two complex mediums which is valid only for a single interface. Future studies can be extended to multiple interfaces considering complex mediums.

A modified TMMSM is presented in chapter five that works for the external regions and the intermediate layers with complex mediums. Analytical equations derived in chapter four and the TMMSM were simulated considering the same scenario (polarization, material properties, and angle of incidence) and identical results were obtained. A single interface between two complex mediums was considered for that simulation, so there were no intermediate layers included. Simulations were conducted for both the TE and TM polarizations, as well as various incident angles. This comparison demonstrates that the modified TMMSM works for external regions with complex mediums. To confirm that the TMMSM is valid for intermediate layers with complex mediums, it was simulated using a Fabry-Perot type metal-insulator-metal (MIM) optical cavity device described in [79], and the results obtained were identical. These results demonstrate that the TMMSM presented here can also handle intermediate layers with complex mediums.

Some simulation results have been provided at the end of chapter five that consider complex mediums considering complex permittivity and complex permeability at both sides of the interface to compare the analytical derivation and TMMSM. There are some special cases when the transmittance becomes negative, so the reflectance becomes greater than unity to match the conservation. Weber's derivation also has negative transmittance for some material combinations.
Weber did not notice negative transmittance instances and it is more frequent when complex permeability is considered.

There are at least two possible explanations for negative transmittance. The first one is, if the transmittance is negative, it indicates that gain or loss is manifested [19, 81]. Another explanation is the sign of the wave impedance/refractive index should be altered if the transmittance is negative [21, 18]. Simulations were performed with FDFD to analyze scattering at the interface between two complex mediums when the transmittance becomes negative. The simulation shows negative refraction confirming the sign change of the refractive index.

The negative transmittance occurs due to the incorrect choice of the sign of the complex wave impedance/complex refractive index (depends on how the power quantities are derived, Weber’s derivation depends on refractive index and this derivation depends on wave impedance), which calculates Fresnel coefficients incorrectly. Research in this study presents a novel technique that allows the Fresnel coefficients to be calculated correctly without resolving the sign of the complex wave impedance completely. It is not entirely known to the scientific community how to resolve the sign of the complex wave impedance. This research does not completely provide the right condition to choose the sign of the wave impedance but shows a way to correctly calculate the Fresnel coefficients, which could lead to solving the sign of the wave impedance in the future.

A TMMSM simulation was also conducted for layered devices with only one intermediate layer. The material properties of the layer were same as the transmission region. Since the layer and transmission region have the same material, the scenario is identical to one without the intermediate layer. For the analytical simulation, only the transmission side material was considered, ignoring the intermediate layer. The incident side material did not change in either the analytical or layered TMMSM simulations. So, the analytical simulation and the TMMSM have
the same scenario and should have the same results. It was observed that the gain/loss became significant for TMMSM as the layer thickness was increased. When the layer thickness is very thin, layered TMMSM produces the same result as an analytical derivation.

Based on the results above, it can be concluded that the TMMSM calculations were performed immediately at the interface between the two complex mediums. Gain or loss does not affect the fields immediately at the interface. Therefore, conservation was always achieved. Calculations performed away from the interface might not have achieved conservation. When the intermediate layer thickness was not very thin, the results of layered TMMSM did not match the analytical derivation results. It is because the gain or loss manifests as the field travels away from the interface. The results from numerical methods may differ from the results presented here since numerical methods calculate power away from the interface.

The total internal reflection and Brewster’s angle are two special phenomena that occur at the interface between two mediums. Both these phenomena are dependent on the incident wave angle and the material properties of both sides of the interface. In the case of complex mediums, the material properties are complex, as well as the angles. It is not clear how total internal reflection and Brewster’s angle can be explained in terms of complex angles and complex material properties.

An angle sweep was performed with the analytical equations for the incident angle vs. the power quantities. In Figure 25, the simulation result is shown considering the interface of two complex mediums. The material properties considered for the simulation were \(\varepsilon_{r1} = 2.8\), \(\mu_{r1} = 1.0\), \(\varepsilon_{r2} = 1.0 - 0.71i\) and \(\mu_{r2} = 1.0 + 1.73i\), where the subscript 1 indicates the incident side and subscript 2 indicates the transmission side.
Figure 25 (a) shows that after approximately $73^\circ$ angle, the transmittance becomes negative. The previous discussion implies that the sign change is necessary for the wave impedance/refractive index after $73^\circ$. Since the analytical derivation depends on the wave impedance, the transmittance becomes positive after changing the sign of the wave impedance of one medium. The result after changing the sign of the wave impedance is shown in Figure 25 (b).

![Figure 25](image.png)

Figure 25: (a) Power quantities vs. incident angle sweep without changing the sign of the wave impedance. (b) power quantities vs. incident angle sweep changing the sign of the wave impedance at $75^\circ$.

It is clear from the above results that the sign change of the wave impedance depends on the angle of incidence. The angle at which the sign of the wave impedance needs to be changed also depends on the material combinations of the incident and transmitted region. The relationship between the sign change of the wave impedance, incident wave angle, and the material properties of the two regions needs to be further studied.

The same result in Figure 25 was achieved with the TMMSM, but the TMMSM does not depend on complex wave impedance. It depends on the wave vector of the wave which is
dependent on the complex refractive index. So, for the TMMSM, Figure 25 implies that the sign change is necessary for the complex refractive index. A study needs to be conducted to find the relation between the sign change of the refractive index, incident angle, and material properties of the incident and transmission side.

From the discussion in the previous two sections, it can be seen that the sign of the wave impedance needs to be changed for some scenarios considering the analytical derivation of scattering analysis at complex mediums. The sign of the refractive index needs to be changed for some scenarios considering scattering analysis using the TMMSM. Figure 25 is a case where the sign of both the complex wave impedance and the complex refractive index needs to be changed for the material combination used in the incident and transmission region. The definition of the refractive index and the wave impedance is different, although they depend on the same material properties for a particular material. Taking the same material properties and incident angle into account, it is not always necessary to change the sign for the wave impedance and the refractive index. There are cases when the sign change is needed either for refractive index or wave impedance.

The result of the TMMSM for the layered device provided at the end of chapter 5 considers the complex materials with complex refractive indices as the intermediate layers. However, external regions with complex mediums were not considered in that case. Currently, no benchmarking examples were found for scattering analysis that considers complex materials (both complex permittivity and complex permeability) for both the external region and the intermediate layers of a layered device. Future research could be focused on the scattering analysis of a layered device with complex mediums considered both in external regions and intermediate layers. The rigorous coupled-wave analysis (RCWA) method can be modified to perform the scattering
analysis that can handle complex mediums considered in both the intermediate layers and external regions of a layered device. As a first step, RCWA can be tested using the known examples that are provided in this dissertation to verify that the method works for complex external regions and also for complex intermediate layers. Then the RCWA and TMMSM can be simulated with complex mediums (considering complex permittivity and complex permeability) in external and intermediate layers and the results can be cross-examined. Future research on the study of scattering at the interface of complex mediums can utilize other computational electromagnetic methods such as the finite-difference time-domain (FDTD) method and the finite-difference frequency-domain (FDFD) method. The benchmarking examples provided here will be helpful to verify the results from these methods.

Analytical and semi-analytical analysis of scattering at the interface of complex mediums presented in this study is unique because it can handle any combination of complex permittivity and complex permeability; positive and negative complex refractive indices; positive and negative complex wave impedances; and active and passive mediums. In this study, both TE and TM polarizations were considered, as well as different incident angles of the wave.
References


Vita

Asad Gulib was born in Dhaka, Bangladesh, and is the youngest son in his family. He earned his bachelor's degree in Electrical & Electronic Engineering at Ahsanullah University of Science & Technology. After graduating, he worked as an instructor at North South University and Ahsanullah University of Science & Technology. He came to UTEP for higher education in 2014 and completed a master's degree in Computational Science in 2016. After that, he concentrated on his Ph.D. and worked with EMLab on generating 3D circuit geometry. Furthermore, he generalized the Fresnel equations to calculate the power quantities at the interface of two complex medium. He specializes in electromagnetics, computational electromagnetics, numerical methods, computational geometry, and metamaterials. In addition to a Ph.D. degree in Computational Science, he have an M.Sc. in Electrical and Computer Engineering.

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