2021-05-01

Multiplicative And Additive Arithmetic Functions And Formal Power Series

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MULTIPLICATIVE AND ADDITIVE ARITHMETIC FUNCTIONS AND FORMAL POWER SERIES

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MULTIPLICATIVE AND ADDITIVE ARITHMETIC FUNCTIONS AND FORMAL POWER SERIES

by

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THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Department of Mathematical Sciences

THE UNIVERSITY OF TEXAS AT EL PASO

May 2021

Abstract

The theory of arithmetic functions and the theory of formal power series are classical and active parts of mathematics. Algebraic operations on sets of arithmetic functions, called convolutions, have an important place in the theory of arithmetic functions. The theory of formal power series also has its place firmly anchored in abstract algebra.

A first goal of this thesis will be to present a parallelism of known characterizations of the concepts of multiplicative and additive for arithmetic functions (Theorems 2.1.2 and 2.2.3) on the one hand and for formal power series on the other (Theorems 3.4.3 and 3.4.4). Therefore, in Chapter 1 and in the first part of Chapter 3 are listed notions and properties that make possible the transposition from arithmetic functions to formal power series. Further, an approach in which formal power series brought to the fore (Sections 3.1 and 3.2) will add new elements in our study on multiplicative arithmetic functions (Section 3.3). So, if mainly, our presentation of Section 3.3 follows P.J. McCarthy's book [6], the proofs of some main results on completely and specially multiplicative functions has been replaced with new proofs (Theorems 3.2.6, 3.2.7, 3.3.3, 3.3.4) using Bell series. This was a second goal of giving new proofs using Bell series, and so we bring the two topics (arithmetic functions and formal power series) closer together. If in the achievement of the first goal a significant role was played by the embedding of the ring of formal power series in the unitary ring of arithmetic functions, in the case of the second goal, Theorem 2.24 and 2.25 of T.M. Apostol's book [1] influenced me to use the Bell series in proofs.

Contents

Chapter 1

Arithmetic Functions and Convolutions

1.1 Arithmetic Functions

The theory of Arithmetic Functions has always been a vital part of Number Theory. An arithmetic, arithmetical or number-theoretic function is any function defined on the set of positive integers (natural numbers) $\mathbb{N} = \{1, 2, 3, ...\}$ with values in the set of complex numbers $\mathbb{C}$. We will focus on the ring of arithmetic functions with the standard addition of functions and the Dirichlet convolution or unitary convolution as the multiplicative operation. The following definitions and arithmetic functions of this chapter can be found in Sivaramakrishnan [12], Burton [2], McCarthy [6] and Niven [8]. However, notations may be different.

Definition 1.1.1. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be an ***arithmetic function***.

Notation 1.1.2. The set of all arithmetic functions will be denoted by

$$\mathcal{A} = \{f : \mathbb{N} \to \mathbb{C}\}.$$

We give some examples of the arithmetic functions that will be used and discussed throughout this paper.

- $\tau(n) = \sum_{d|n} 1$, for all $n \in \mathbb{N}$, is the number of positive divisors of n
- $\sigma(n) = \sum_{d|n} d$, for all $n \in \mathbb{N}$, is the sum of all positive divisors of n

- $\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if there exists a prime such that} \quad p^2 | n \\ (-1)^k \quad \text{if} \quad n = p_1 \cdot p_2 \cdots p_k & \text{with distinct primes} \end{cases}$
  (The Möbius Function)

- $o(n) = 0$ for all $n \in \mathbb{N}$

  Lastly, let us define Euler's totient function

- $\phi(n)$ is the number of positive integers less than or equal to a natural number n and relatively prime to n (for all $n \in \mathbb{N}$)

Example of the first few positive integer values are the following

$$\phi(1) = 1 \quad \phi(2) = 1 \quad \phi(4) = 2 \quad \phi(12) = 4$$

$$\phi(5) = 4 \quad \phi(6) = 2 \quad \phi(7) = 6$$

When p is a prime number we have:

$$\phi(p) = p - 1$$

With this brief introduction of arithmetic functions, we can introduce one of the fundamental structures which will be used in the thesis: the ring of arithmetic functions.

1.2 Dirichlet Convolution and Unitary Convolution

Definition 1.2.1. Let f and g be arithmetic functions, then the ***Dirichlet Convolution*** is defined as

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) \quad \forall n \in \mathbb{N}$$

.

Before we discuss the properties of the structure $(\mathcal{A}, *)$, it would be quite helpful to illustrate this operation through example. So, if we take two arithmetic functions f, g and the integer 12, we obtain:

$$(f * g)(12) = f(1)g(12) + f(2)g(6) + f(3)g(4) + f(4)g(3) + f(6)g(2) + f(12)g(1)$$

More concretely, let us take τ, σ and the integer 12, we obtain

$$\begin{aligned}(\tau * \sigma)(12) &= \tau(1)\sigma(12) + \tau(2)\sigma(6) + \tau(3)\sigma(4) + \tau(4)\sigma(3) + \tau(6)\sigma(2) + \tau(12)\sigma(1) \\ &= 1 \cdot 28 + 2 \cdot 9 + 2 \cdot 7 + 3 \cdot 4 + 4 \cdot 3 + 12 \cdot 1 = 96\end{aligned}$$

Also, we can notice

$$\tau(1) = 1 \quad \text{and} \quad \sigma(1) = 1$$

We will discuss the significance of this result later. Now, let us examine the properties of the structure $(\mathcal{A}, *)$.

Theorem 1.2.2. *The structure $(\mathcal{A}, *)$, is a commutative monoid.*

Proof. It needs to be shown that the operation is commutative, associative, and has identity. To show that this structure is commutative, we need to verify this property

$$f * g = g * f$$

for all arithmetic functions f, g in $\mathcal{A}$. So

$$\begin{aligned}(f*g)(n) &= \sum_{d|n} f(d) g\left(\frac{n}{d}\right) = \sum_{d_1 d_2 = n} f(d_1) g(d_2) \\ &= \sum_{d_1 d_2 = n} g(d_2) f(d_1) = \sum_{d|n} g(d) f\left(\frac{n}{d}\right) = (g*f)(n)\end{aligned}$$

Now we will verify the associative property

$$(f*g)*h = f*(g*h) \text{ for any } f, g, h \in \mathcal{A}$$

So

$$\begin{aligned}[(f*g)*h](n) &= \sum_{d d_3 = n} [(f*g)(d)]\, h(d_3) \\ &= \sum_{d d_3 = n} \left[\sum_{d_1 d_2 = d} f(d_1) g(d_2)\right] h(d_3) \\ &= \sum_{d_1 d_2 d_3 = n} f(d_1) g(d_2) h(d_3)\end{aligned}$$

By a similar calculation, it can be shown

$$[f*(g*h)](n) = \sum_{d_1 d_2 d_3 = n} f(d_1) g(d_2) h(d_3)$$

which implies this structure is associative. To determine the identity element, we need to identify $e \in \mathcal{A}$ with the property

$$f*e = e*f = f \text{ for all } f \in \mathcal{A}$$

To do this, consider the arithmetic function

$$e(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1 \end{cases}$$

Then it follows

$$(f*e)(n) = \sum_{d|n} f(d) e\left(\frac{n}{d}\right) = f(1)e(n) + ... + f(n)e(1) = 0 + 0 + ... + 0 + f(n) = f(n)$$

This implies that the arithmetic function e is the Dirichlet identity. With this, we have shown that $(\mathcal{A}, *)$ is a commutative monoid. □

Now that the properties of this structure have been identified, it would be beneficial to determine what it's inverses are.

Notation 1.2.3. The set of units/ invertible elements of the structure $(\mathcal{A}, *)$ will be denoted by

$$\mathcal{U}(\mathcal{A}) = \{f \in \mathcal{A} \mid f \text{ is invertible}\}.$$

The question then becomes, "Which are these elements?"

Theorem 1.2.4. *The invertible elements of the structure $(\mathcal{A}, *)$ are exactly those arithmetic functions with the property $f(1) \neq 0$, or*

$$\mathcal{U}(\mathcal{A}) = \{f \in \mathcal{A} \mid f(1) \neq 0\}.$$

Proof. To prove this, it must be shown that $f \in \mathcal{U}(\mathcal{A})$ implies and is implied by $f(1) \neq 0$. So, let $f \in \mathcal{U}(\mathcal{A})$. This implies that there exists an arithmetic function $\widetilde{f} \in \mathcal{A}$ with the property

$$f * \widetilde{f} = e$$

Recall that

$$e(1) = 1$$

Then

$$e(1) = (f * \widetilde{f})(1) = f(1)\widetilde{f}(1) = 1$$

This implies $f(1) \neq 0$.

Conversely, let us assume $f(1) \neq 0$. Now, we will define the following arithmetic function recursively such that

$$\widetilde{f}(n) = \begin{cases} \frac{1}{f(1)} & \text{if } n = 1 \\ -\frac{1}{f(1)} \sum\limits_{d|n} f(d)\widetilde{f}\left(\frac{n}{d}\right) & \text{if } n > 1 \end{cases}$$

So, for $n = 1$ we have

$$(f * \widetilde{f})(1) = \sum_{d|1} f(d)\widetilde{f}\left(\frac{1}{d}\right) = f(1)\widetilde{f}(1) = f(1) \cdot \frac{1}{f(1)} = 1 = e(1)$$

For $n > 1$ we have

$$(f * \widetilde{f})(n) = \sum_{d|n} f(d)\widetilde{f}\left(\frac{n}{d}\right) = f(1)\widetilde{f}(n) + \sum_{\substack{d|n \\ n>1}} f(d)\widetilde{f}\left(\frac{n}{d}\right)$$

Notice

$$\widetilde{f}(n) = -\frac{1}{f(1)} \sum_{\substack{d|n \\ n>1}} f(d)\widetilde{f}\left(\frac{n}{d}\right)$$

Therefore

$$\sum_{d|n} f(d)\widetilde{f}\left(\frac{n}{d}\right) = -\widetilde{f}(n)f(1)$$

Then this is what follows

$$f(1)\widetilde{f}(n) + \sum_{\substack{d|n \\ n>1}} f(d)\widetilde{f}\left(\frac{n}{d}\right) = f(1)\widetilde{f}(n) - f(1)\widetilde{f}(n) = 0 = e(n)$$

So, we can say

$$(f * \widetilde{f})(n) = e(n)$$

for all natural numbers n. This implies, $\widetilde{f}$ is the inverse of f. Therefore, arithmetic functions with the property, $f(1) \neq 0$, are inverse elements of $(\mathcal{A}, *)$ $\square$

It is of consequence to note that $\mathcal{U}(\mathcal{A})$ is a subset of $\mathcal{A}$. Moreover, $\mathcal{U}(\mathcal{A})$ is a subgroup of $\mathcal{A}$. This implies that $(\mathcal{U}(\mathcal{A}), *)$ is an abelian group.

Definition 1.2.5. Let f be an arithmetic function, then f is called ***multiplicative*** if

$$f(mn) = f(m)f(n)$$

when $(m, n) = 1$.

Notation 1.2.6. The set of all non-zero multiplicative arithmetic functions will be denoted by

$$\mathcal{M} = \{f \in \mathcal{A} - \{o\} \mid f \text{ is multiplicative}\}.$$

In what follows, we will show that structure formed by this set and the Dirichlet Convolution is an abelian group.

Theorem 1.2.7. *The structure* $(\mathcal{M}, *)$, *is an abelian group.*

Proof. It is sufficient to prove that $\mathcal{M}$ is a subgroup of $\mathcal{U}(\mathcal{A})$, since $(\mathcal{U}(\mathcal{A}), *)$ is an abelian group. It needs to be shown that

1. The set $\mathcal{M}$ is a nonempty subset of $\mathcal{U}(\mathcal{A})$
2. If arithmetic functions f and g are multiplicative, then their convolution, $f * g$, is a multiplicative arithmetic function
3. If f is a multiplicative arithmetic function, then its inverse, $\widetilde{f}$, is a multiplicative arithmetic function

1. Assume $f \in \mathcal{M}$. Then, it follows that f is not the zero function. This implies that there exists a positive integer k, with the property

$$f(k) \neq 0$$

So

$$f(k) = f(1 \cdot k) = f(1) \cdot f(k)$$

This implies

$$f(1) = 1 \neq 0$$

Therefore, this multiplicative function, f, is an element of $\mathcal{U}(\mathcal{A})$. It can be concluded that $\mathcal{M}$ is a subset of $\mathcal{U}(\mathcal{A})$. Let us digress for a moment and recognize a consequence of the result we have just proved.

- If f is a multiplicative arithmetic function, then $f(1) = 1$
- The Dirichlet identity, e, is a multiplicative arithmetic function

2. Next, let $f, g \in \mathcal{M}$ and $(m, n) = 1$, then

$$(f * g)(nm) = \sum_{d|nm} f(d) g\left(\frac{nm}{d}\right)$$

It should be noted if $d|nm$ and $(n, m) = 1$ then

$$d = d_1 \cdot d_2$$

such that $d_1|n$ and $d_2|m$. Also

$$(d_1, d_2) = 1 \quad \text{and} \quad \left(\frac{n}{d_1}, \frac{m}{d_2}\right) = 1$$

With this, it can be said

$$\sum_{d|nm} f(d) g\left(\frac{nm}{d}\right) = \sum_{\substack{d_1|n \\ d_2|m}} f(d_1 d_2) g\left(\frac{n}{d_1}\frac{m}{d_2}\right) = \sum_{\substack{d_1|n \\ d_2|m}} f(d_1) f(d_2) g\left(\frac{n}{d_1}\right) g\left(\frac{m}{d_2}\right)$$

$$= \left[\sum_{d_1|n} f(d_1) g\left(\frac{n}{d_1}\right)\right] \left[\sum_{d_2|m} f(d_2) g\left(\frac{n}{d_2}\right)\right] = (f * g)(n)(f * g)(m)$$

This implies, $f * g \in \mathcal{M}$

3. Finally, we will show if f is a multiplicative function, then its inverse $\widetilde{f}$, is as well. We will do this by letting the arithmetic function g be defined as

$$g(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ \widetilde{f}(p_1^{\alpha_1}) \cdot \widetilde{f}(p_2^{\alpha_2}) \cdots \widetilde{f}(p_k^{\alpha_k}) & \text{if} \quad n > 1 \end{cases}$$

Where $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the factorization of n into powers of distinct primes.

It will be shown that g is multiplicative and $g = \widetilde{f}$. Take $(m, n) = 1$ and $g(m \cdot n)$ such that $m, n > 1$. Now

$$m = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdots q_s^{\beta_s} \quad \text{and} \quad n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

Then we can say, since $(m, n = 1)$, that $q_1, ..., q_s, p_1, ..., p_k$ are distinct primes. Therefore

$$g(m \cdot n) = \widetilde{f}(q_1^{\beta_1}) \cdot \widetilde{f}(q_2^{\beta_2}) \cdots \widetilde{f}(q_s^{\beta_s}) \cdot \widetilde{f}(p_1^{\alpha_1}) \cdots \widetilde{f}(p_k^{\alpha_k})$$

$$= \prod_{i=1}^{s} \widetilde{f}(q_i^{\beta_i}) \cdot \prod_{i=1}^{k} \widetilde{f}(p_i^{\alpha_i}) = g(m) \cdot g(n)$$

This means, $g \in \mathcal{M}$. Now, it must be shown that, for all natural numbers n

$$(f * g)(n) = e(n)$$

For $n = 1$ we can clearly see that this holds. We will verify that this is true for an arbitrary prime, p, with power $\alpha \geq 1$. It is clear to see that the set

$$D(p^\alpha) = \{1, p, p^2, ..., p^\alpha\}$$

contains all of the positive divisors of p^α. This means if $d | p^\alpha$ then

$$d = p^i; \quad i = 0, ..., \alpha$$

which implies

$$\frac{p^\alpha}{d} = p^{\alpha - i}$$

So

$$(f * g)(p^\alpha) = \sum_{i=0}^{\alpha} f(p^\alpha) g(p^{\alpha - i}) = \sum_{i=0}^{\alpha} f(p^\alpha) \widetilde{f}(p^{\alpha - i}) = (f * \widetilde{f})(p^\alpha) = e(p^\alpha)$$

Now for the prime factorization, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, we have

$$\begin{aligned}(f * g)(n) &= (f * g)(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = (f * g)(p_1^{\alpha_1}) \cdot (f * g)(p_2^{\alpha_2}) \cdots (f * g)(p_k^{\alpha_k}) \\ &= e(p_1^{\alpha_1}) \cdot e(p_2^{\alpha_2}) \cdots e(p_k^{\alpha_k}) = e(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = e(n)\end{aligned}$$

This implies

$$(f * g)(n) = e(n) \text{ for all } n \in \mathbb{N}$$

We can then say $g = \widetilde{f}$, which means this inverse function $\widetilde{f}$ is a member of $\mathcal{M}$. Therefore, $\mathcal{M}$ is a subgroup of $\mathcal{U}(\mathcal{A})$. With the above satisfied, we can conclude that $(\mathcal{M}, *)$ is an abelian group. $\square$

Definition 1.2.8. Let f and g be arithmetic functions, then

$$(f + g)(n) = f(n) + g(n) \quad \forall f, g \in \mathcal{A}$$

Theorem 1.2.9. *The algebraic structure* $(\mathcal{A}, +, *)$ *is an integral domain.*

Proof. It is trivial to verify that $(\mathcal{A}, +)$ is an abelian group. $(\mathcal{A}, +)$ is associative and commutative. The arithmetic function $o(n) = 0$ is the additive identity. For all arithmetical functions f, the additive inverse is $-f$. We also know $(\mathcal{A}, *)$ is a commutative monoid. What must be shown is that the distributive property holds, and there are no zero divisors. For the distributive property we have

$$\begin{aligned}[f * (g + h)](n) &= \sum_{d|n} f(d)(g + h)\left(\frac{n}{d}\right) = \sum_{d|n} f(d)\left[g\left(\frac{n}{d}\right) + h\left(\frac{n}{d}\right)\right] \\ &= \sum_{d|n} f(d)g\left(\frac{n}{d}\right) + \sum_{d|n} f(d)h\left(\frac{n}{d}\right) = (f * g)(n) + (f * h)(n)\end{aligned}$$

So the distributive property holds. Finally, it must be shown that this structure has no zero divisors. Let

$$f, g \neq o$$

then we must show

$$f * g \neq o$$

Let us take $f, g \neq o$. Let m be the smallest value for which $f(m) \neq 0$ and let n be the smallest value for which $g(n) \neq 0$. Then we have

$$\begin{aligned}(f * g)(mn) &= \sum_{d|mn} f(d) g\left(\frac{mn}{d}\right) \\ &= \sum_{\substack{d|mn \\ d<m}} 0 \cdot g\left(\frac{mn}{d}\right) + f(m)g(n) + \sum_{\substack{d|mn \\ d<n}} f(d) \cdot 0 \\ &= f(m)g(n) \neq 0\end{aligned}$$

So, $f * g \neq o$ which implies that $(\mathcal{A}, +, *)$ has no zero divisors. Therefore, $(\mathcal{A}, +, *)$ is an integral domain. □

So far, our concern has been with the Dirichlet convolution, which takes the sum over all divisors of a particular n. But what kind of structure can be made if we take the sum of only the divisors of n which have the property $\left(d, \frac{n}{d}\right) = 1$? We will briefly discus this type of structure, however, let us first define this property.

Definition 1.2.10. Let n be a positive integer. Then d, a divisor of n, with the property

$$\left(d, \frac{n}{d}\right) = 1$$

is called a **unitary divisor** of n.

Definition 1.2.11. Let f and g be arithmetic functions, then the **Unitary Convolution** is defined as

$$(f \oplus g) = \sum_{d||n} f(d) g\left(\frac{n}{d}\right) \quad \forall n \in \mathbb{N}$$

where $d||n$ means that d runs through the unitary divisors of n.

Theorem 1.2.12. $(\mathcal{A}, +, \oplus)$ *is a commutative ring with unity.*

Proof. We know $(\mathcal{A}, +)$ is an abelian group and it can be shown that $(\mathcal{A}, \oplus)$ is associative and commutative by similar means to the Dirichlet convolution and $\oplus$ distributes over $+$. Also, e, the Dirichlet identity, is the unitary convolution identity. □

We introduce this structure, because we will be using it later in Chapter 3. However, let us now discuss some applications to the theorems we have introduced

1.3 More on σ, τ and ϕ

First, let's introduce a few more arithmetic functions.

- $\zeta(n) = 1 \; \forall n \in \mathbb{N}$
- $i(n) = n \; \forall n \in \mathbb{N}$

It is important to note that:

- $\zeta(mn) = 1 = 1 \cdot 1 = \zeta(m)\zeta(n)$
- $i(mn) = m \cdot n = i(m)i(n)$

This implies that the arithmetic functions ζ and i are both multiplicative. We introduce these functions here because they have a special relationship with some of the arithmetic functions we have already discussed. Before we explore that relationship, we should introduce this concept.

Definition 1.3.1. Let f be an arithmetic function, then

$$F(n) = \sum_{d|n} f(d)$$

is called the **summation** of f.

This summation function will allow us to verify some important properties concerning the arithmetic functions we have discussed. One of those properties is determining multiplicative functions. The following results of this section are classical in the field of Number Theory and in some proofs we use the properties of the group $(\mathcal{M}, *)$.

Theorem 1.3.2. *If f is a multiplicative arithmetic function, then the summation of f is a multiplicative arithmetic function.*

Proof. Let $f \in \mathcal{M}$ and let F be the summation of f, then

$$F(n) = \sum_{d|n} f(d) = \sum_{d|n} f(d) \cdot 1 = \sum_{d|n} f(d)\zeta\left(\frac{n}{d}\right)$$

So

$$F = f * \zeta$$

We should note that f and ζ are both multiplicative arithmetic functions. This implies F is a multiplicative arithmetic function, because, as we have shown, $(\mathcal{M}, *)$ is a closed structure $\square$

Theorem 1.3.3. *We have*

$$\sum_{d|n} \mu(d) = e(n)$$

for all $n \geq 1$.

Proof. Let $n = 1$, then

$$\sum_{d|1} \mu(d) = 1 = e(1)$$

Now, let $n = p$ be prime. Therefore

$$\sum_{d|p} \mu(d) = \mu(1) + \mu(p) = 1 - 1 = 0 = e(p)$$

If $n = p^\alpha$ where $\alpha \geq 2$, then

$$\sum_{d|p^\alpha} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + ... + \mu(p^\alpha) = 1 - 1 + 0 + ... + 0 = 0 = e(p^\alpha)$$

So for the prime factorization, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, we have

$$\sum_{d|n} \mu(d) = 0 = e(n)$$

Therefore for any $n \in \mathbb{N}$ we have

$$\sum_{d|n} \mu(d) = e(n).$$

$\square$

With this we come to a nice corollary.

Corollary 1.3.4. *μ is the Dirichlet inverse of ζ.*

Proof. Let the above equation hold, then for any natural number n we have

$$e(n) = \sum_{d|n} \mu(d) = \sum_{d|n} \mu(d) \cdot 1 = \sum_{d|n} \mu(d)\zeta\left(\frac{n}{d}\right) = (\mu * \zeta)(n)$$

This implies

$$\mu = \widetilde{\zeta}.$$

□

This corollary gives us the following theorem which is called the Möbius Inversion Formula

Theorem 1.3.5. *Let f be an arithmetic function, then*

$$F(n) = \sum_{d|n} f(d) \text{ if and only if } f(n) = \sum_{d|n} F(d)\mu\left(\frac{n}{d}\right)$$

.

Proof. Let $F = f * \zeta \iff F * \widetilde{\zeta} = f * \zeta * \widetilde{\zeta} \iff F * \widetilde{\zeta} = f * e \iff f = F * \mu$. □

This result lead to the converse of Theorem 1.3.2.

Corollary 1.3.6. *An arithmetic function, f, is multiplicative if and only if the summation of f is multiplicative.*

The next theorem points out the summation function of the Euler function which was originally noticed by Gauss (see [2] pg 141).

Theorem 1.3.7. *We have*

$$\sum_{d|n}\phi(d) = n$$

for all $n \geq 1$.

Proof. Let us take the set of divisors d over n and denote it by

$$S_d = \{m : 1 \leq m \leq n \quad and \quad (m, n) = d\}$$

Now $(m, n) = d$ which implies $\left(\frac{m}{d}, \frac{n}{d}\right) = 1$, giving us this equality

$$|S_d| = \phi\left(\frac{n}{d}\right)$$

So, we can say

$$n = \sum_{d|n}|S_d| = \sum_{d|n}\phi\left(\frac{n}{d}\right) = \sum_{d|n}\phi(d).$$

□

With Theorem 1.3.7 and Corollary 1.3.6, we see the following result

Corollary 1.3.8. ϕ, τ *and* σ *are multiplicative arithmetic functions.*

Proof. Let the equation from Theorem 1.3.7 hold, then

$$n = i(n) = \sum_{d|n}\phi(d)\zeta\left(\frac{n}{d}\right) = (\phi * \zeta)(n)$$

and

$$\phi(n) = (i * \widetilde{\zeta})(n) = (i * \mu)(n)$$

Which shows us that ϕ is multiplicative. Now let's look at the sum of divisors function

$$\tau(n) = \sum_{d|n}1 = \sum_{d|n}1 \cdot 1 = \sum_{d|n}\zeta(d)\zeta\left(\frac{n}{d}\right) = (\zeta * \zeta)(n)$$

This implies that τ is a multiplicative function

Also

$$\sigma(n) = \sum_{d|n}d = \sum_{d|n}d \cdot 1 = \sum_{d|n}i(d)\zeta\left(\frac{n}{d}\right) = (i * \zeta)(n)$$

Similarly, σ can be said to be a multiplicative function. □

Showing that τ, ϕ and σ are a multiplicative function, we can find the formulas of τ, ϕ and σ.

Theorem 1.3.9. *If $n > 1$ with the prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then*

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1).$$

Proof. Let p be prime and $\alpha \geq 1$. The set

$$D(p^\alpha) = \{1, p, p^2, ..., p^\alpha\}$$

is the set of all positive divisors of p^α. Therefore

$$\tau(p^\alpha) = \alpha + 1$$

We will now consider the prime factorization, $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Since, we have just shown that τ is multiplicative, it follows that

$$\tau(n) = \tau(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \tau(p_1^{\alpha_1}) \cdots \tau(p_k^{\alpha_k}) = (\alpha_1 + 1) \cdots (\alpha_k + 1) = \prod_{i=1}^{k} (\alpha_i + 1).$$

□

Theorem 1.3.10. *If $n > 1$ with the prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then*

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

Proof. Let $n = p$ be prime. Then

$$\phi(p) = p - 1 = p\left(1 - \frac{1}{p}\right)$$

Now, let $n = p^\alpha$ where $\alpha \geq 1$. We desire those integers who are relatively prime to p^k. It can be seen that the integers who are not relatively prime are those of the form

$$p, 2p, 3p, ..., p^{\alpha-1} \cdot p = p^\alpha.$$

Therefore, there are $p^{\alpha-1}$ integers who are not relatively prime to p^α, so we can say

$$\phi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha\left(1 - \frac{1}{p}\right)$$

If we let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, the prime factorization of n, then it follows from ϕ being multiplicative that

$$\begin{aligned}\phi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) &= \phi\left(p_1^{\alpha_1}\right) \cdots \phi\left(p_k^{\alpha_k}\right) = p_1^{\alpha_1}\left(1 - \frac{1}{p_1}\right) \cdots p_k^{\alpha_k}\left(1 - \frac{1}{p_k}\right) \\ &= n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right) = n\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right).\end{aligned}$$

□

Theorem 1.3.11. *If $n > 1$ with the prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then*

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

Proof. Let $n = p^\alpha$ where p is prime and $\alpha \geq 1$, then

$$\sigma(p^\alpha) = 1 + p + p^2 + ... + p^\alpha = \frac{p^{\alpha+1} - 1}{p - 1}$$

Therefore, for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, the prime factorization of n, we have

$$\sigma(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \sigma(p_1^{\alpha_1}) \cdots \sigma(p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}.$$

□

Theorem 1.3.12. *If f and g are multiplicative arithmetic functions with positive values and $n > 1$, then n is prime if and only if*

$$(f * g)(n) = (f + g)(n).$$

Proof. Let n be prime, then

$$(f*g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = f(1)g(n) + f(n)g(1) = 1 \cdot g(n) + f(n) \cdot 1 = f(n) + g(n) = (f+g)(n)$$

Conversely, let us suppose $(f * g)(n) = (f + g)(n)$ and n is not prime. Then

$$\sum_{d|n} f(d)g\left(\frac{n}{d}\right) = f(n) + g(n)$$

This implies

$$\sum_{\substack{d|n \\ d\neq 1,n}} f(d)g\left(\frac{n}{d}\right) + f(1)g(n) + f(n)g(1) = f(n) + g(n)$$

Thus we can conclude

$$\sum_{\substack{d|n \\ d\neq 1,n}} f(d)g\left(\frac{n}{d}\right) = 0$$

This leads to a contradiction, since it is assumed that $f, g > 0$ for any positive integer n. So, we can conclude that n must be prime. $\square$

This theorem leads to nice characterizations of primes with τ, ϕ and σ.

Corollary 1.3.13. *Let $n > 1$, then n is prime if and only if*

$$\sigma(n) + \phi(n) = n \cdot \tau(n).$$

Proof. Let n be prime. We also know

$$(\sigma * \phi)(n) = (\sigma + \phi)(n)$$

So, all that needs to be shown is

$$(\sigma * \phi)(n) = n \cdot \tau(n)$$

Notice

$$\sigma * \phi = (i * \zeta) * \phi = i * (\zeta * \phi) = i * i$$

Now

$$(i * i)(n) = \sum_{d|n} i(d)i\left(\frac{n}{d}\right) = \sum_{d|n} d \cdot \frac{n}{d} = n \cdot \sum_{d|n} 1 = n \cdot \tau(n)$$

$\square$

Corollary 1.3.14. *Let $n > 1$, then n is prime if and only if*

$$\tau(n) + \phi(n) = \sigma(n).$$

Proof. Let n be prime, then

$$(\tau + \phi)(n) = (\tau * \phi)(n)$$

Therefore

$$\tau * \phi = (\zeta * \zeta) * \phi = \zeta * (\zeta * \phi) = \zeta * i = \sigma.$$

□

Chapter 2

Characterization of Completely Multiplicative and Additive Arithmetic Functions

As of now we have only seen arithmetic functions and multiplicative arithmetic functions. This chapter will discuss new concepts of arithmetic functions, those of which were studied by Carlitz and Niederreiter [3], Lambek [4], and Schwab [10].

2.1 Completely Multiplicative Functions

In the previous chapter we discussed the concept of multiplicative functions. However, our previous definition was only concerned with relatively prime elements of non-negative integers. Now we will expand this property to any two non-negative integers.

Definition 2.1.1. An arithmetic function, f, is said to be **completely multiplicative** if

$$f(n \cdot m) = f(n) \cdot f(m)$$

for all n, m positive integers.

With this, we can show some properties that these types of functions will possess. Recall, that we are working with the structure $(\mathcal{A}, *)$, and we have the following characterizations of completely multiplicative arithmetic functions from Lambeck and Carlitz.

Theorem 2.1.2. *If f is an arithmetic function then the following statements are equivalent*

1. *f is completely multiplicative*
2. *$f(g * h) = fg * fh$ for all arithmetic functions g and h*
3. *$f(g * g) = fg * fg$ for all arithmetic functions g*
4. *$f\tau = f * f$.*

Proof. (1) $\implies$ (2)

Let f be completely multiplicative, then

$$[f(g * h)](n) = f(n)\left[\sum_{d|n} g(d)h\left(\frac{n}{d}\right)\right] = \sum_{d|n} f(n)g(d)h\left(\frac{n}{d}\right) = \sum_{d|n} f\left(d \cdot \frac{n}{d}\right) g(d)h\left(\frac{n}{d}\right)$$
$$= \sum_{d|n} f(d)f\left(\frac{n}{d}\right) g(d)h\left(\frac{n}{d}\right) = \sum_{d|n} [f(d)g(d)][f\left(\frac{n}{d}\right) h\left(\frac{n}{d}\right)] = [fg * fh](n).$$

(2) $\implies$ (3)

Assume

$$f(g * h) = fg * fh$$

for all $g, h \in \mathcal{A}$

Then, it immediately follows that

$$f(g * g) = fg * fg.$$

(3) $\implies$ (4)

Assume

$$f(g * g) = fg * fg$$

Then, for all g

$$f\tau = f(\zeta * \zeta) = f\zeta * f\zeta = f \cdot 1 * f \cdot 1 = f * f.$$

(4) $\implies$ (1) [3]

Suppose $f * f = \tau f$. We will show inductively that f is completely multiplicative. Now take $n = 1$, then

$$(f * f)(1) = f(1)f(1) = \tau(1)f(1) = 1 \cdot f(1)$$

Therefore, $f(1) = 1$ or $f(1) = 0$.

Now take $n \geq 2$ and let $n = p_1^{e_1} \cdots p_m^{e_m}$, which is the canonical factorization of n, and let $\alpha(n) = e_1 + ... + e_m$. Then, it is enough to show

$$f(n) = f(1)f(p_1)^{e_1} \cdots f(p_m)^{e_m}$$

So, let $\alpha(n) = 1$, then n is prime, say $n = p$, which implies

$$2f(p) = \tau(p)f(p) = f(1)f(p) + f(p)f(1) = 2f(1)f(p)$$

Suppose this is true for all n with $\alpha(n) \leq k$ and $k \geq 1$. Then, we can take an n with $\alpha(n) = k + 1$ which gives

$$\tau(n)f(n) = \sum_{d|n} f(d)f\left(\frac{n}{d}\right) = 2f(1)f(n) + \sum_{d|n, d\neq 1, n} f(d)f\left(\frac{n}{d}\right)$$

Now, let $d = d_1$ and $\frac{n}{d} = d_2$, so $d_1 \cdot d_2 = n$. Also, $\alpha(d_1), \alpha(d_2) \leq k$. Then

$$\tau(n)f(n) = 2f(1)f(n) + \sum_{d|n, d\neq 1, n} f(d_1)f(d_2)$$

Now, this fulfils the inductive step, so

$$\tau(n)f(n) = 2f(1)f(n) + (\tau(n) - 2)f(1)^2 f(p_1)^{e_1} \cdots f(p_m)^{e_m}$$

Since n is not prime, it is clear to see that $\tau(n) > 2$. So, for both $f(1) = 1$ and $f(1) = 0$ we get the desired result. □

2.2 Completely Additive Functions

Now, we will introduce a set of functions which have a similar property to the multiplicative functions, however, the functions are not split by multiplication, but by addition.

Definition 2.2.1. An arithmetic function, f, is said to be **completely additive** if

$$f(n \cdot m) = f(n) + f(m)$$

for all n, m positive integers.

A familiar example of a completely additive function is the logarithmic function, as it is well known that

$$\log(n \cdot m) = \log(n) + \log(m) \quad \forall n, m \in \mathbb{N}$$

Also, an immediate consequence of this property is, if $f \in \mathcal{S}$ then

$$f(1) = f(1 \cdot 1) = f(1) + f(1)$$

This implies, $f(1) = 0$.

Another example comes from the following function.

- $\Omega(n) = \sum\limits_{p^\alpha || n} \alpha$ is the sum of prime powers α where p^α exactly divides n

Some examples of this function are

$$\Omega(12) = \Omega(2^2 \cdot 3) = 2 + 1 = 3, \quad \Omega(30) = \Omega(2 \cdot 3 \cdot 5) = 1 + 1 + 1 = 3$$

We should also notice that when $n = 1$ we get

$$\Omega(1) = 0$$

because, 1 has no prime divisors.

If we take arbitrary $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $m = q_1^{\beta_1} \cdots q_l^{\beta_l}$ both being the canonical factorization of natural numbers n, m then we have

$$\Omega(n \cdot m) = \Omega(p_1^{\alpha_1} \cdots p_k^{\alpha_k} \cdot q_1^{\beta_1} \cdots q_l^{\beta_l}) = \alpha_1 + ... + \alpha_k + \beta_1 + ... + \beta_l = \Omega(n) + \Omega(m)$$

This means that Ω is completely additive. Now with this we can define a function which we can show to be completely multiplicative.

- $\lambda(n) = (-1)^{\Omega(n)}$

Following the fact that $\Omega(1) = 0$ we see that $\lambda(1) = 1$ and since Ω is completely additive we get for any natural number n, m we get

$$\lambda(n \cdot m) = (-1)^{\Omega(n \cdot m)} = (-1)^{\Omega(n)+\Omega(m)} = (-1)^{\Omega(n)} \cdot (-1)^{\Omega(m)} = \lambda(n) \cdot \lambda(m)$$

This function is known as the Louiville Lambda function. Before we discuss additional properties of this function, it would be beneficial to introduce another arithmetic function, but first we must add a restriction to our definition of the completely additive arithmetic function.

Definition 2.2.2. An arithmetic function, f, is said to be **additive** if

$$f(n \cdot m) = f(n) + f(m)$$

when $(n, m) = 1$.

Additive arithmetic functions, much like multiplicative arithmetic functions, only satisfy this "splitting" property for relatively prime natural numbers. The following function gives an example of this property.

- $\omega(n) = \sum_{p'|n} 1$ is the number of distinct primes, p', which divide n

It is important to note the different values of ω. Let p be prime, then

$$\omega(1) = \sum_{p'|1} 1 = 0, \quad \omega(p) = \sum_{p'|p} 1 = 1$$

It follows that when $\alpha \geq 1$, we have

$$\omega(p^\alpha) = \sum_{p'|p^\alpha} 1 = 1$$

This implies that when $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, the prime factorization of n, that

$$\omega(n) = \omega(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \sum_{p'|p_1^{\alpha_1} \cdots p_k^{\alpha_k}} 1 = k$$

Notice, if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $m = q_1^{\beta_1} \cdots q_l^{\beta_l}$, the prime factorization of n and m, where $(n, m) = 1$ then

$$\omega(n) + \omega(m) = k + l = \omega(n \cdot m)$$

This implies that ω is additive. We will now discuss some of the properties of completely additive functions.

Theorem 2.2.3. *If f is an arithmetic function, then the following statements are equivalent*

1. *f is completely additive*
2. *$f(g * h) = fg * h + g * fh$ for all arithmetic functions g and h*
3. *$f(g * g) = 2(fg * g)$ for all arithmetic functions g*
4. *$f\tau = 2(f * \zeta)$.*

The proof of this theorem is found in the article written by Schwab [10].

Proof. (1) $\implies$ (2)

Let f be completely additive, then

$$\begin{aligned}
[f(g*h)](n) = f(n)\left[\sum_{d|n} g(d)h\left(\frac{n}{d}\right)\right] &= \sum_{d|n} f(n)g(d)h\left(\frac{n}{d}\right) \\
&= \sum_{d|n} f\left(d \cdot \frac{n}{d}\right) g(d)h\left(\frac{n}{d}\right) = \sum_{d|n}\left[f(d) + f\left(\frac{n}{d}\right)\right] g(d)h\left(\frac{n}{d}\right) \\
&= \sum_{d|n} f(d)g(d)h\left(\frac{n}{d}\right) + \sum_{d|n} f\left(\frac{n}{d}\right) g(d)h\left(\frac{n}{d}\right) = [fg * h + g * fh](n)
\end{aligned}$$

$(2) \implies (3)$

Let

$$f(g * h) = fg * h + g * fh$$

for all $g, h \in \mathcal{A}$. Then we have

$$f(g * g) = fg * g + g * fg = fg * g + fg * g = 2(fg * g).$$

$(3) \implies (4)$

Let $f(g * g) = 2(fg * g)$

then, $f\tau = f(\zeta * \zeta) = 2(f\zeta * \zeta) = 2(f \cdot 1 * \zeta) = 2(f * \zeta)$.

$(4) \implies (1)$ [10]

Suppose $f\tau = 2(f * \zeta)$ and let $n = p$. Then,

$$f(p)\tau(p) = 2f(p) = 2(f * \zeta) \implies f(p) = f(1) + f(p) \implies f(1) = 0$$

Now, let $n \in \mathbb{N}, n > 1$ and $n = p_1^{k_1} \cdots p_t^{k_t}$. Then, it will be shown, when $m = k_1 + ... + k_t$, that

$$f(n) = k_1 f(p_1) + ... + k_t f(p_t)$$

So, if $M_i = \{0, 1, 2, ..., k_i\}$ for $i = 1, 2, ..., t$ and $M = M_1 \times M_2 \times ... \times M_t$, then

$$\frac{1}{2} f(n)\tau(n) = \sum_{(i_1, ... i_t) \in M} f(p_1^{i_1} \cdots p_t^{i_t}) = f(n) + \sum_{\substack{(i_1, ... i_t) \in M \\ i_1 + ... + i_t \neq m}} f(p_1^{i_1} \cdots p_t^{i_t})$$

by induction

$$\frac{1}{2} f(n)\tau(n) = f(n) + \sum_{\substack{(i_1, ... i_t) \in M \\ i_1 + ... + i_t \neq m}} [i_1 f(p_1) + ... + i_t f(p_t)]$$

Now,

$$\sum_{\substack{(i_1,...i_t)\in M\\ i_1+...+i_t\neq m}} [i_1 f(p_1) + ... + i_t f(p_t)] = \frac{1}{2}\left[\prod_{i=1}^{t}(k_i+1)\right]\left[\sum_{i=1}^{t} k_i f(p_i)\right] - \sum_{i=1}^{t} k_i f(p_i)$$

This implies

$$f(n) = \sum_{i=1}^{t} k_i f(p_i).$$

□

Chapter 3

Multiplicative and Additive Power Series

In this chapter we will discus a relationship between formal power series and arithmetic functions. However, we will need to first define the concept of formal power series.

3.1 The Formal Power Series

We will take a classical approach to defining the formal power series.

Definition 3.1.1. Let R be a commutative ring with unity and $\mathbb{N}_0 = \{0, 1, 2, ...\}$ with

$$f : \mathbb{N}_0 \to R \quad \text{such that } f = (a_0, a_1, a_2, ..., a_i, ...) = (a_i)_{i\in\mathbb{N}_0} \text{ and } a_i \in R.$$

Then define $R' = \{f \mid f = (a_i)_{i\in\mathbb{N}_0}\}$ and

$$f = (a_i)_{i\in\mathbb{N}_0}, \quad g = (b_i)_{i\in\mathbb{N}_0}$$

with the properties

- $f + g = (a_i + b_i)_{i\in\mathbb{N}_0}$
- $f \cdot g = (c_k)_{k\in\mathbb{N}_0}; \quad c_k = \sum_{i+j=k} a_i b_j.$

We will show that R' with addition and multiplication forms a commutative ring with unity, see [9].

Theorem 3.1.2. $(R', +, \cdot)$ *is a commutative ring with unity.*

Proof. We will first show that $(R', +)$ is an abelian group and will start with the associative property. To show it is associative we have

$$f + (g + h) = (a_0, a_1, a_2, ...) + [(b_0 + c_0, b_1 + c_1, +b_2 + c_2, ...)] = (a_0 + b_0 + c_0, a_1 + b_1 + c_1, ...)$$
$$= [(a_0 + b_0, a_1 + b_1, +a_2 + b_2, ...)] + (c_0, c_1, c_2, ...) = (f + g) + h$$

To show this structure is commutative

$$f + g = (a_i + b_i)_{i \in \mathbb{N}_0} = (b_i + a_i)_{i \in \mathbb{N}_0} = g + f$$

The additive identity of R' is the function $o \in R'$ with the property

$$o : \mathbb{N}_0 \to R \text{ such that } o = (0, 0, 0, ...)$$

Where 0_R is the additive identity of R. And finally, the additive inverse is shown by

$$-f = (-a_i)_{i \in \mathbb{N}_0}$$

Now, to show $(R', \cdot)$ is commutative monoid, we first show the commutative property holds. So

$$f \cdot g = \left(\sum_{i+j=k} a_i b_j \right)_{k \in \mathbb{N}_0} = \left(\sum_{i+j=k} b_j a_i \right)_{k \in \mathbb{N}_0} = g \cdot f$$

Then the associative property is shown by

$$f \cdot (g \cdot h) = (a_i)_{i \in \mathbb{N}_0} \cdot \left(\sum_{n+m=j} b_n c_m \right)_{j \in \mathbb{N}_0} = \left(\sum_{i+j=k} a_i \cdot \sum_{n+m=j} b_n c_m \right)_{k \in \mathbb{N}_0} = \left(\sum_{i+n+m=k} a_i b_n c_m \right)_{k \in \mathbb{N}_0}$$

Then likewise

$$(f \cdot g) \cdot h = \left(\sum_{i+n=j} a_i b_n \right)_{j \in \mathbb{N}_0} \cdot (c_m)_{m \in \mathbb{N}_0} = \left(\sum_{i+n+m=k} a_i b_n c_m \right)_{k \in \mathbb{N}_0}$$

The multiplicative identity of R' is:

$$i : \mathbb{N}_0 \to R \text{ such that } i = (1, 0, 0, ...)$$

Where 1_R is the multiplicative identity in R. With this all we have to show now is that the distributive property holds. So

$$f(g+h) = (a_i)_{i\in\mathbb{N}_0} \cdot (b_j + c_j)_{j\in\mathbb{N}_0} = \left(\sum_{i+j=k} a_i \cdot (b_j + c_j)\right)_{k\in\mathbb{N}_0} = \left(\sum_{i+j=k} a_i \cdot b_j + \sum_{i+j=k} a_i \cdot c_j\right)_{k\in\mathbb{N}_0}$$
$$= \left(\sum_{i+j=k} a_i \cdot b_j\right)_{k\in\mathbb{N}_0} + \left(\sum_{i+j=k} a_i \cdot c_j\right)_{k\in\mathbb{N}_0} = fg + fh$$

Therefore, $(R', +, \cdot)$ is a commutative ring with unity. □

Let us formalize this concept by introducing some notation.

$$(1, 0, 0, 0, ...) = x^0$$

$$(0, 1, 0, 0, ...) = x^1$$

Also,

$$(0, 0, 0, ..., 1, ...) = x^k$$

where there are k many terms before 1. For example,

$$(a, b, 0, 0, ...) = ax^0 + bx^1$$

We will call these $a, b \in R$ coefficients of x. With this we can say

$$R' \overset{\text{notaion}}{=} R[[x]] = \left\{ f = \sum_{k=0}^{\infty} a_k x^k \right\}$$

Now we can define the following.

Definition 3.1.3. The ring R' is called the **formal power series** in x with coefficients in R is denoted by $R[[x]]$. The elements of $R[[x]]$ are infinite expressions of the form

$$f(x) = a_0x^0 + a_1x^1 + ... + a_kx^k + ... = \sum_{k=0}^{\infty} a_kx^k$$

and addition and multiplication are defined as

$$\sum_{k=0}^{\infty} a_kx^k + \sum_{k=0}^{\infty} b_kx^k = \sum_{k=0}^{\infty}(a_k + b_k)x^k$$
$$\sum_{k=0}^{\infty} a_kx^k \cdot \sum_{k=0}^{\infty} b_kx^k = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_ib_j\right) x^k$$

A few well known examples of formal power series include the geometric series

$$S(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

and

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{and} \quad \log\left(\frac{1}{1-x}\right) = \sum_{k=0}^{\infty} \frac{x^k}{k}$$

where $\exp(x)$ represents the traditional exponential function and $k! = k \cdot (k-1) \cdots 1$. An interesting observation can be made regarding the power series of $\exp(x)$. A known property of the exponential function is sort of a reverse additive property

$$\exp(z + w) = \exp(z) \cdot \exp(w) \quad \forall z, w \in \mathbb{C}$$

So, if we have $exp(ax)$, where $a \in \mathbb{N}_0$, then we obtain

$$\sum_{k=0}^{\infty} \frac{(ax)^k}{k!} = \exp(ax) = \exp(x + ... + x) = \exp(x) \cdot \exp(x) \cdots \exp(x)$$
$$= \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdots \sum_{k=0}^{\infty} \frac{x^k}{k!} = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)^a.$$

Let us introduce a type of power series which gives an immediate connection to arithmetic functions.

3.2 Bell Series

In this section and the following, we will be addressing the known results found in Apostol [1] regarding the concept of Bell Series and their connection to arithmetic functions. Then, we will come to results proposed by McCarthy [6]. However, we will verify them using Bell Series.

Definition 3.2.1. Let f be an arithmetic function and p be a prime. Then the formal power series

$$f_p(x) = \sum_{k=0}^{\infty} f(p^k)x^k$$

is called the **Bell Series** of f modulo p.

This concept was first studies by E. T. Bell in order to observe multiplicative properties of arithmetic functions with power series. To illustrate an example of this type of series, recall the Möbius function. It can be observed that

$$\mu_p(x) = \sum_{k=0}^{\infty} \mu(p^k)x^k$$

Remember, the Möobius function is defined as follows

$$\mu(n) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if there exists a prime such that} \quad p^2|n \\ (-1)^k & \text{if} \quad n = p_1 \cdot p_2 \cdots p_k \quad \text{with distinct primes} \end{cases}$$

Therefore it is clear to see

$$\mu_p(x) = \sum_{k=0}^{\infty} \mu(p^k)x^k = 1 \cdot x^0 + (-1)x^1 + 0 \cdot x^2 + ... + 0 \cdot x^k + ...$$
$$= 1 - x$$

Also, we can see the Bell series representation of the Dirichlet identity function by

$$e_p(x) = \sum_{k=0}^{\infty} e(p^k)x^k = 1 + 0 + ... = 1$$

This gives us a good representation of the mobius and identity function, but how would we define the other arithmetic functions we have discussed? Let us recall the completely multiplicative function, then this result and proof from Apostol [1] follows immediately.

Theorem 3.2.2. *If f is a completely multiplicative arithmetic function, then*

$$f_p(x) = \frac{1}{1 - f(p)x}.$$

Proof. Let f be completely multiplicative and p prime with $k \geq 1$, then

$$f(p^k) = f(p) \cdots f(p) = f(p)^k$$

So,

$$f_p(x) = \sum_{k=0}^{\infty} f(p^k)x^k = \sum_{k=0}^{\infty} f(p)^k x^k = \sum_{k=0}^{\infty} 1 \cdot (f(p)x)^k$$

Note, that the above expression yields a geometric power series, meaning

$$\sum_{k=0}^{\infty} 1 \cdot (f(p)x)^k = \frac{1}{1 - f(p)x}.$$

□

We have studied quite a few completely multiplicative functions in this paper so their power series representations are the following [1].

- $\zeta_p(x) = \frac{1}{1-\zeta(p)x} = \frac{1}{1-x}$
- $i_p(x) = \frac{1}{1-i(p)x} = \frac{1}{1-p \cdot x}$
- $i_p^{\alpha}(x) = \frac{1}{1-i(p^{\alpha})x} = \frac{1}{1-p^{\alpha} \cdot x}$
- $\lambda_p(x) = \frac{1}{1-\lambda(p)x} = \frac{1}{1+x}$

Before we continue it is important to discuss the following theorem.

Theorem 3.2.3. *If f and g are multiplicative arithmetic functions, then $f = g$ if and only if*

$$f_p(x) = g_p(x)$$

for all primes p.

The proof can be found in [1].

Proof. First, let us assume that $f = g$. Then we see that

$$f(p^k) = g(p^k) \quad \text{for any prime, } p \text{, and } k \geq 1$$

Therefore, it is clear to see

$$f_p(x) = g_p(x) \quad \text{for all } p$$

Conversely, let $f_p(x) = g_p(x)$ for all p. Then

$$\sum_{k=0}^{\infty} f(p^k)x^k = \sum_{k=0}^{\infty} g(p^k)x^k$$

This means, $f(p^k) = g(p^k)$ for any power k. Also, f and g are assumed to multiplicative, therefore, we can say for any prime p

$$f = g$$

□

As we move on, it is interesting to note that

$$\zeta_p(x) \cdot \mu_p(x) = e_p(x)$$

This comes from the fact that μ is the Dirichlet inverse of ζ, and this revelation leads us to our next theorem from Apostol [1].

Theorem 3.2.4. *If f, g and h are arithmetic functions and $h = f * g$, then*

$$h_p(x) = f_p(x)g_p(x).$$

Proof. Let p be prime and $k \geq 1$. Recall that the divisors of p^k are

$$D = \{1, p, p^2, ..., p^k\}$$

So,

$$h(p^k) = (f * g)(p^k) = \sum_{d|p^k} f(d) \cdot g\left(\frac{p^k}{d}\right) = \sum_{i+j=k} f(p^i)g(p^j)$$

Then, following our definition of formal power series multiplication, we can say

$$h_p(x) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} f(p^i)g(p^j)\right)x^k = f_p(x) \cdot g_p(x).$$

□

With this result we can determine the Bell series representation of some arithmetic functions.

Application 3.2.5. Recall

$$\phi = i * \mu$$

Therefore, we can say

$$\phi_p(x) = i_p(x) \cdot \mu_p(x) = \frac{1}{1 - p \cdot x} \cdot (1 - x) = \frac{1 - x}{1 - p \cdot x}$$

This is quite significant, since with this we can determine the formula representation of the Euler totient function.

$$\phi_p(x) = \frac{1 - x}{1 - p \cdot x} = (1 - x) \cdot \sum_{k=0}^{\infty} p^k x^k = \sum_{k=0}^{\infty} p^k x^k - x\sum_{k=0}^{\infty} p^k x^k$$
$$= 1 + \sum_{k=1}^{\infty} p^k x^k - \sum_{k=1}^{\infty} p^{k-1} x^k = 1 + \sum_{k=1}^{\infty}(p^k - p^{k-1})x^k$$

Thus, we can say for $k \geq 1$

$$\phi(p^k) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right)$$

Extending this to any natural number $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and recalling the fact that ϕ is multiplicative we obtain

$$\phi(n) = \phi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \phi(p_1^{\alpha_1}) \cdots \phi(p_k^{\alpha_k}) = p_1^{\alpha_1}(1 - \frac{1}{p_1}) \cdots p_k^{\alpha_k}(1 - \frac{1}{p_k}) = n \prod_{i=1}^{k}(1 - \frac{1}{p_i})$$

This gives us a new way of determining the formula for the totient function, using what was learned from Apostol [1]. This gives us reason to believe that there are potentially more formulas of arithmetic functions which can be derived using Bell Series. Another application of this theorem comes from the arithmetic function, $\sigma_\alpha = i^\alpha * \zeta$.

$$\sigma_{\alpha_p}(x) = i_p^\alpha(x) \cdot \zeta_p(x) = \frac{1}{1 - p^\alpha x} \cdot \frac{1}{1 - x} = \frac{1}{(1 - p^\alpha x)(1 - x)}$$

Using this theorem we can construct this table of arithmetic functions.

Bell Series of Arithmetic Functions	
Arithmetic function	Bell Series Representation
μ	$\mu_p(x) = 1 - x$
e	$e_p(x) = 1$
ζ	$\zeta_p(x) = \frac{1}{1-x}$
i	$i_p(x) = \frac{1}{1-px}$
λ	$\lambda_p(x) = \frac{1}{1+x}$
$\phi = \mu * i$	$\phi_p(x) = \frac{1-x}{1-px}$
$\sigma = \zeta * i$	$\sigma_p(x) = \frac{1}{(1-x)(1-px)}$
$\tau = \zeta * \zeta$	$\tau_p(x) = \frac{1}{(1-x)^2}$

The following theorem, which comes from McCarthy [6], has significant application in regards to Bell Series.

Theorem 3.2.6. *If f is a multiplicative arithmetic function, then f is completely multiplicative if and only if*

$$\widetilde{f} = \mu f.$$

McCarthy verifies this proof rigorously using the Dirichlet Convolution [6]. However, as we will see, the proof becomes much simpler using Bell Series.

Proof. First we will assume that f is completely multiplicative. Then, we can say

$$f_p(x) = \frac{1}{1 - f(p)x}$$

Now

$$(\mu f)_p(x) = \sum_{k=0}^{\infty} (\mu f)(p^k)x^k = \sum_{k=0}^{\infty} \mu(p^k)f(p^k)x^k$$

Recalling a multiplicative property, we can say $f(1) = 1$. Therefore

$$\sum_{k=0}^{\infty} \mu(p^k)f(p^k)x^k = 1 - f(p)x$$

Also, we can clearly see

$$e_p(x) = (\mu f)_p(x) \cdot f_p(x)$$

This is only the case if μf is Dirichlet inverse of f. Conversely, assume $\mu f = \widetilde{f}$, then we have

$$\widetilde{f}_p(x) = (\mu f)_p(x) = 1 - f(p)x$$

If μf is the inverse, it must be that

$$\mu f * f = e$$

which implies

$$1 = (1 - f(p)x) \cdot f_p(x)$$

This implies

$$f_p(x) = \frac{1}{1 - f(p)x}$$

meaning, f must be completely multiplicative. □

This immediately leads us to a nice result from McCarthy [6].

Theorem 3.2.7. *If f is a multiplicative arithmetic function, then f is completely multiplicative if and only if*

$$\widetilde{f}(p^\alpha) = 0 \quad \textit{for all } \alpha \geq 2$$

By using Bell Series, the proof of this result becomes quite obvious. Observing the proof of the previous theorem and using the fact that $\mu(p^\alpha) = 0$ when $\alpha \geq 2$, this result is immediately apparent.

The following applications are found in Apostol [1]. However, in the first of the proceeding applications, we use a different approach also using Bell Series.

Application 3.2.8. Since

$$\widetilde{\lambda} = \mu\lambda$$

Then we have

$$\widetilde{\lambda}_p(x) = (\mu\lambda)_p(x) = \sum_{k=0}^{\infty} \mu(p^k)\lambda(p^k)x^k = 1 + x = \sum_{k=0}^{\infty} \mu(p^k)\mu(p^k)x^k = \mu_p^2(x)$$

So we can see that

$$\widetilde{\lambda} = \mu^2$$

Application 3.2.9. Let $f(n) = 2^{\omega(n)}$. Then

$$f_p(x) = \sum_{k=0}^{\infty} 2^{\omega(p^k)}x^k = 1 + \sum_{k=1}^{\infty} 2x^k = 1 + \frac{2x}{1+x} = \frac{1+x}{1-x}$$

Therefore, we have

$$f_p(x) = \mu_p^2(x) \cdot \zeta_p(x)$$

Meaning we have a formalization for this function

$$2^{\omega(n)} = \mu^2 * \zeta = \sum_{d|n} \mu^2(d)$$

Therefore, the arithmetic function 2^{ω} is the summation of μ^2.

3.3 Bell Series and Specially Multiplicative Functions

In this section we will use Bell Series to verify results shown in the work of McCarthy [6], in regards to the concept of specially multiplicative arithmetic functions.

Definition 3.3.1. Let f be a multiplicative arithmetic function, then f is said to be **specially multiplicative** if

$$f = g * h$$

where g and h are completely multiplicative arithmetic functions.

The following result comes from McCarthy [6].

Theorem 3.3.2. *If f is a multiplicative arithmetic function, then it is specially multiplicative if and only if*

$$f_p(x) = \frac{1}{1 - bx + cx^2}$$

where $b, c \in \mathbb{C}$

Proof. Let f be specially multiplicative and p be prime, then we have the following

$$f = g * h$$

where g, h are completely multiplicative. Also we have

$$g_p(x) = \frac{1}{1 - g(p)x} \quad \text{and} \quad h_p(x) = \frac{1}{1 - h(p)x}$$

It is also known, that

$$f_p(x) = \frac{1}{1 - g(p)x} \cdot \frac{1}{1 - h(p)x} = \frac{1}{1 - [g(p) + h(p)]x + [g(p)h(p)]x^2}$$

Notice that $[g(p) + h(p)]$ and $[g(p)h(p)]$ are elements of $\mathbb{C}$, so we can see the condition is satisfied.

Conversely, assume

$$f_p(x) = \frac{1}{1 - bx + cx^2}$$

where $b, c \in \mathbb{C}$. Then we have

$$f_p(x) = \frac{1}{1 - bx + cx^2} = \frac{1}{1 - a_1 \cdot x} \cdot \frac{1}{1 - a_2 \cdot x}$$

with a_1 and a_2 being the roots of quadratic equation,

$$x^2 - bx + c = 0.$$

Now, we can say there exists two arithmetic functions g and h where $g(p) = a_1$ and $h(p) = a_2$. Therefore

$$g_p(x) = \frac{1}{1 - g(p)x} \quad \text{and} \quad h_p(x) = \frac{1}{1 - h(p)x}$$

meaning, g and h are completely multiplicative. With this, we can conclude

$$f_p(x) = g_p(x)h_p(x)$$

which implies

$$f = g * h$$

therefore, f is specially multiplicative. □

As an illustration of this type of function consider

$$g(n) = 2^{\Omega(n)} \quad \text{and} \quad h(n) = 3^{\Omega(n)}$$

Both functions are completely multiplicative, so the function

$$f(n) = 2^{\Omega(n)} * 3^{\Omega(n)}$$

is a specially multiplicative function. Also, recall ζ and i are completely multiplicative, and

$$\tau = \zeta * \zeta \quad \text{and} \quad \sigma = \zeta * i.$$

Therefore we can say τ and σ are specially multiplicative.

Now we should recall the property of completely multiplicative functions, that being if f is completely multiplicative then

$$\widetilde{f}(p^\alpha) = 0 \quad \text{for all } \alpha \geq 2$$

We can observe a similar result from McCarthy [6] for specially multiplicative functions.

Theorem 3.3.3. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if*

$$\widetilde{f}(p^\alpha) = 0 \quad \textit{for all } \alpha \geq 3$$

Bell Series will be used once again to make a quite rigorous proof much simpler.

Proof. Let f be specially multiplicative. Then we have

$$f = g * h$$

where g, h are completely multiplicative. Then, it is the case that

$$\widetilde{f} = \widetilde{g} * \widetilde{h} = \mu g * \mu h$$

which gives us

$$\widetilde{f}_p(x) = \left(\sum_{k=0}^{\infty} \mu(p^k) g(p^k) x^k\right)\left(\sum_{k=0}^{\infty} \mu(p^k) h(p^k) x^k\right) = (1 - g(p)x)(1 - h(p)x)$$

Now, this implies $\widetilde{f}(p^\alpha) = 0$ for all $\alpha \geq 3$. Conversely, we will say

$$\widetilde{f}(p^\alpha) = 0 \quad \text{for all } \alpha \geq 3$$

then it follows that

$$\widetilde{f}_p(x) = \sum_{k=0}^{\infty} \widetilde{f}(p^k) x^k = 1 + \widetilde{f}(p)x + \widetilde{f}(p^2)x^2.$$

Therefore, we have

$$f_p(x) = \frac{1}{1 + \widetilde{f}(p)x + \widetilde{f}(p^2)x^2} = \frac{1}{1 - (-\widetilde{f}(p))x + \widetilde{f}(p^2)x^2}.$$

So by the previous theorem we see that it must be the case that f is specially multiplicative.

□

Specially multiplicative functions have a variety of interesting properties.

Theorem 3.3.4. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if*

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

for all primes, p, and for all $\alpha \geq 1$.

This theorem comes from McCarthy [6]. However, we will verify it differently here.

Proof. Let f be specially multiplicative, then

$$f = g * h$$

where g and h are completely multiplicative. Then we have

$$f_p(x) = g_p(x)h_p(x).$$

This gives us,

$$\sum_{k=0}^{\infty} f(p^k)x^k = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} g(p^i)h(p^j) \right) x^k$$

So for $k = 1$ we obtain

$$f(p)x = [g(p)h(1) + g(1)h(p)]x = [g(p) + h(p)]x$$

and for $k = 2$ we obtain

$$f(p^2)x^2 = \sum_{i+j=2} g(p^i)h(p^j)x^2 = [h(p^2) + g(p)h(p) + g(p^2)]x^2$$

Notice, since g and h are completely multiplicative we see

$$f(p^2) - f(p)^2 = [h(p^2) + g(p)h(p) + g(p^2)] - [g(p^2) + 2g(p)h(p) + h(p^2) = -g(p)h(p)$$

Then

$$\begin{aligned}
& f(p)f(p^\alpha) + f(p^{\alpha-1})[f(p^2) - f(p)^2] = \\
&= [g(p) + h(p)] \left[\sum_{i+j=\alpha} g(p^i)h(p^j) \right] + \left[\sum_{i+j=\alpha-1} g(p^i)h(p^j) \right] [-g(p)h(p)] \\
&= \sum_{i+j=\alpha} g(p^{i+1})h(p^j) + \sum_{i+j=\alpha} g(p^i)h(p^{j+1}) - \sum_{i+j=\alpha-1} g(p^{i+1})h(p^{j+1}) \\
&= \left[\sum_{i+j=\alpha} g(p^{i+1})h(p^j) - \sum_{i+j=\alpha} g(p^i)h(p^j) \right] + \sum_{i+j=\alpha} g(p^i)h(p^{j+1}) \\
&= g(p^{i+1})h(1) + \sum_{i+j=\alpha} g(p^i)h(p^{j+1}) = \sum_{i+j=\alpha} g(p^{i+1})h(p^{j+1}) \\
&= \sum_{i+j=\alpha+1} g(p^i)h(p^j) = f(p^{\alpha+1})
\end{aligned}$$

Conversely, let us assume that, for all $\alpha \geq 1$, we have

$$f(p^{\alpha+1}) = f(p)f(p^\alpha) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

Now, $f \in \mathcal{M}$, so $f(1) = 1$. Also, for any prime p we have

$$0 = e(p) = (f * \widetilde{f})(p) = f(1)\widetilde{f}(p) + f(p)\widetilde{f}(1)$$

This gives us

$$\widetilde{f}(p) = -f(p)$$

Following for p^2 we obtain

$$0 = e(p^2) = (f * \widetilde{f})(p^2) = f(1)\widetilde{f}(p^2) + f(p)\widetilde{f}(p) + f(p^2)\widetilde{f}(1)$$

which implies

$$\widetilde{f}(p^2) = f(p)^2 - f(p^2)$$

Also, for p^3 we obtain

$$0 = e(p^3) = (f * \widetilde{f})(p^3) = f(1)\widetilde{f}(p^3) + f(p)\widetilde{f}(p^2) + f(p^2)\widetilde{f}(p) + f(p^3)\widetilde{f}(1)$$

This implies the following

$$\begin{aligned}\widetilde{f}(p^3) &= -f(p)\widetilde{f}(p^2) - f(p^2)\widetilde{f}(p) - f(p^3)\\ &= -f(p)(f(p)^2 - f(p^2)) - f(p^2)(-f(p)) - f(p^3)\\ &= -f(p)^3 + 2f(p)f(p^2) - f(p^3)\end{aligned}$$

Recall

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha}) + f(p^{\alpha-1})[f(p^2) - f(p)^2] \quad \text{for all } \alpha \geq 1$$

Then we can say for $\alpha = 2$

$$f(p^3) = f(p)f(p^2) + f(p)[f(p^2) - f(p)^2] = 2f(p)f(p^2) - f(p)^3$$

So

$$\begin{aligned}\widetilde{f}(p^3) &= -f(p)^3 + 2f(p)f(p^2) - f(p^3)\\ &= -f(p)^3 + 2f(p)f(p^2) - [2f(p)f(p^2) - f(p)^3] = 0\end{aligned}$$

Next, let us assume that it is the case

$$\widetilde{f}(p^{\alpha}) = 0 \quad \text{when} \quad 3 \leq \alpha \leq n$$

If this is true, then what follows is

$$\begin{aligned}0 = e(p^{n+1}) = (f * \widetilde{f})(p^{n+1}) = (\widetilde{f} * f)(p^{n+1}) &= \sum_{i=0}^{n+1} \widetilde{f}(p^i)f(p^{n+1-i})\\ &= \widetilde{f}(1)f(p^{n+1}) + \widetilde{f}(p)f(p^n) + \widetilde{f}(p^2)f(p^{n-1}) + \widetilde{f}(p^{n+1})f(1)\\ &= f(p^{n+1}) - f(p)f(p^n) + [f(p)^2 - f(p^2)]f(p^{n-1}) + \widetilde{f}(p^{n+1})\end{aligned}$$

Therefore

$$\begin{aligned}\widetilde{f}(p^{n+1}) &= -f(p^{n+1}) + f(p)f(p^n) - [f(p)^2 - f(p^2)]f(p^{n-1}) \\ &= -(f(p)f(p^n) + f(p^{n-1})[f(p^2) - f(p)^2]) + f(p)f(p^n) - [f(p)^2 - f(p^2)]f(p^{n-1}) = 0\end{aligned}$$

So f must be specially multiplicative. □

The following results and proofs come from McCarthy [6].

Theorem 3.3.5. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if there exists a multiplicative function, F, such that for all m and n*

$$f(mn) = \sum_{d|(m,n)} f\left(\frac{m}{d}\right) f\left(\frac{n}{d}\right) F(d).$$

Proof. Let us assume f is specially multiplicative. If $(mn, m'n') = 1$, then $((m, n), (m', n')) = 1$ and $(mm', nn') = (m, n)(m', n')$. It must be shown that there exists some multiplicative function F which satisfies

$$f(p^{\alpha+\beta}) = \sum_{i=0}^{min(\alpha,\beta)} f(p^{\alpha-i})f(p^{\beta-i})F(p^i) \quad \text{for all } \alpha, \beta \geq 1$$

Now, let $F = \mu G$ where G is a completely multiplicative function and for each prime p

$$G(p) = f(p)^2 - f(p^2)$$

Then, for $\beta \leq \alpha$ and $\beta = 1$ we have since, f is specially multiplicative

$$\begin{aligned}f(p^{\alpha+1}) &= f(p)f(p^\alpha) + f(p^{\alpha-1})[f(p^2) - f(p)^2] = f(p)f(p^\alpha)G(1) - f(p^{\alpha-1})G(p) \\ &= f(p)f(p^\alpha)\mu(1)G(1) + f(p^{\alpha-1})\mu(p)G(p) = f(p)f(p^\alpha)F(1) - f(p^{\alpha-1})F(p)\end{aligned}$$

which satisfies the sum. Now assume for $\beta > 1$ that the equation holds for $\beta - 1$ for all $\alpha \geq \beta - 1$. Also, by Theorem 3.2.7

$$F(p^2) = F(p^3) = ... = 0$$

Therefore we obtain the following

$$\begin{aligned} f(p^{\alpha+\beta}) &= f(p^{\alpha+1+\beta-1}) = f(p^{\alpha+1})f(p^{\beta-1}) + f(p^{\alpha})f(p^{\beta-2})F(p) \\ &= [f(p)f(p^{\alpha}) - f(p^{\alpha-1})G(p)]f(p^{\beta-1}) + f(p^{\alpha})f(p^{\beta-2})\mu(p)G(p) \\ &= f(p^{\alpha})[f(p)f(p^{\beta-1}) - f(p^{\beta-2})G(p)] - f(p^{\alpha-1})f(p^{\beta-1})G(p) \\ &= f(p^{\alpha})f(p^{\beta})F(1) + f(p^{\alpha-1})f(p^{\beta-1})F(p) \end{aligned}$$

This is what we needed to show. Conversely assume the equation defined above holds. Let p be a prime with $m = p^{\alpha}$ and $n = p$ where $\alpha \geq 1$. Then

$$f(p^{\alpha+1}) = f(p)f(p^{\alpha})F(1) + f(1)f(p^{\alpha-1})F(p)$$

If we let $\alpha = 1$, then

$$f(p^2) = f(p)^2 + F(p)$$

which implies

$$F(p) = f(p^2) - f(p)^2$$

With the equation from Theorem 3.3.4 satisfied, we can say f is specially multiplicative. □

Theorem 3.3.6. *If f is a multiplicative arithmetic function, then f is specially multiplicative if and only if there exists a completely multiplicative function, G, such that for all m and n*

$$f(m)f(n) = \sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right) G(d).$$

Proof. Assume f is specially multiplicative. Then the equation from Theorem 3.3.5 holds. So

$$\sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right) G'(d) = \sum_{d|(m,n)} f\left(\frac{m}{d} \cdot \frac{n}{d}\right) G'(d)$$

$$= \sum_{d|(m,n)} \sum_{D|\left(\frac{m}{d},\frac{n}{d}\right)} f\left(\frac{\frac{m}{d}}{D}\right) f\left(\frac{\frac{n}{d}}{D}\right) \mu(D)G'(D)B(d)$$

Continuing we obtain

$$\sum_{d|(m,n)} \sum_{\substack{k|(m,n)\\ d|k}} f\left(\frac{m}{k}\right) f\left(\frac{n}{k}\right) \mu\left(\frac{k}{d}\right) G'\left(\frac{k}{d}\right) G'(d) = \sum_{d|(m,n)} \sum_{\substack{k|(m,n)\\ d|k}} f\left(\frac{m}{k}\right) f\left(\frac{n}{k}\right) \mu\left(\frac{k}{d}\right) G'(k)$$

$$= \sum_{k|(m,n)} f\left(\frac{m}{k}\right) f\left(\frac{n}{k}\right) G'(k) \sum_{d|k} \mu\left(\frac{k}{d}\right) = f(m)f(n)$$

Conversely, assume the above equation holds. Let p be a prime and $p = m = n$, then

$$f(m)f(n) = f(p)^2 = f(p^2) + G(p)$$

Therefore,

$$G(p) = f(p)^2 - f(p^2)$$

If $m = p^\alpha$ and $n = p$ with $\alpha \geq 1$, we obtain

$$f(p^\alpha)f(p) = \sum_{d|(p^\alpha,p)} f\left(\frac{p^{\alpha+1}}{d^2}\right) G(d) = f(p^{\alpha+1}) + f(p^{\alpha-1})[f(p)^2 - f(p^2)]$$

Therefore

$$f(p^{\alpha+1}) = f(p^\alpha)f(p) + f(p^{\alpha-1})[f(p^2) - f(p)^2]$$

This means, by Theorem 3.3.4, f is specially multiplicative □

3.4 Multiplicative and Additive Power Series

This section will discus the embedding of the formal power into the unitary ring of arithmetic functions. This work can be found in [11]. Let us recall that $(\mathcal{A}, +, \oplus)$ is the unitary ring and let us consider the formal power series ring $\mathbb{C}[[x]]$. Then this result can be observed

Theorem 3.4.1. *The ring* $\mathbb{C}[[x]]$ *can be embedded in the unitary ring of arithmetic functions*

The proof of this result can be found in [11].

Proof. Consider the map $\eta : \mathbb{C}[[x]] \to \mathcal{A}$ such that

$$\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(n) = \omega(n)! a_{\omega(n)}$$

Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1, ..., p_k$ are distinct primes and $\alpha \geq 1$, then

$$\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(n) + \eta\left(\sum_{k=0}^{\infty} b_k x^k\right)(n) = \omega(n)! a_{\omega(n)} + \omega(n)! b_{\omega(n)} = k! a_k + k! b_k$$

$$= k!(a_k + b_k) = \eta\left(\sum_{k=0}^{\infty}(a_k + b_k)x^k\right)(n) = \eta\left(\sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k\right)(n)$$

Also

$$\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(n) \oplus \eta\left(\sum_{k=0}^{\infty} b_k X^k\right)(n) = \omega(n)! a_{\omega(n)} \oplus \omega(n)! b_{\omega(n)} = k! a_k \oplus k! b_k$$

$$= \sum_{d||k}(d)! a_d \cdot \left(\frac{k}{d}\right)! b_{\frac{k}{d}} = k! \sum_{d||k} a_d b_{\frac{k}{d}} = \eta\left(\sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i b_j\right) x^k\right)(n) = \eta\left(\sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} b_k x^k\right)(n)$$

Therefore, η is a homomorphism. We can also show that this mapping is injective.

Let

$$\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(n) = \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(m)$$

This implies

$$\omega(n)!a_{\omega(n)} = \omega(m)!a_{\omega(m)}$$

Now this is only the case when $\omega(n) = \omega(m)$. Therefore, n and m must a product of primes, both with k factors, meaning η is injective. So $\mathbb{C}[[X]]$ can be embedded in $\mathcal{A}$. $\square$

This is a powerful result. Now, we can determine the characteristics of a formal power series in $\mathbb{C}[[x]]$ as if it were an arithmetic function.

Definition 3.4.2. A formal power series $\sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}[[x]]$ is called **multiplicative** if the arithmetic function $\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)$ is multiplicative. A formal power series $\sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}[[x]]$ is called **additive** if the arithmetic function $\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)$ is additive.

The binary operation

$$\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} k! a_k b_k x^k$$

will give us the opportunity to create analogues for the properties studied in chapter 2 with multiplicative and additive formal power series. The results and proofs of the following results can be found in [11].

Theorem 3.4.3. *Let* $\sum_{k=0}^{\infty} a_k x^k \in \mathbb{C}[[x]]$ *be a non-zero power series. Then the following are equivalent*

1. $\sum_{k=0}^{\infty} a_k x^k$ *is multiplicative*
2. $a_k = \frac{a_1^k}{k!}$ *for all* $n \in \mathbb{N}$
3. $$\sum_{k=0}^{\infty} a_k x^k \odot \left(\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k \right) = \left(\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} b_k x^k \right) \left(\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} c_k x^k \right)$$
   for all $\sum_{k=0}^{\infty} b_k x^k, \sum_{k=0}^{\infty} c_k x^k \in \mathbb{C}[[x]]$
4. $\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} a_k x^k$

Proof. (1) $\implies$ (2)

Let $\sum_{k=0}^{\infty} a_k x^k$ be multiplicative. Then

$$1 = \eta \left(\sum_{k=0}^{\infty} a_k x^k \right) (1) = \omega(1)! a_{\omega(1)} = a_0$$
$$a_1 = \omega(p^\alpha)! a_{\omega(p^\alpha)} = \eta \left(\sum_{k=0}^{\infty} a_k x^k \right) (p^\alpha)$$

So we can say

$$a_1^k = \eta \left(\sum_{k=0}^{\infty} a_k x^k \right) (p_1^{\alpha_1}) \cdots \eta \left(\sum_{k=0}^{\infty} a_k x^k \right) (p_k^{\alpha_k})$$
$$= \eta \left(\sum_{k=0}^{\infty} a_k x^k)(p_1^{\alpha_1} \cdots p_k^{\alpha_k} \right) = k! a_k$$

(2) $\implies$ (1)

Let $a_k = \frac{a_1^k}{k!}$ for all $n \in \mathbb{N}$, and let $m, n \in \mathbb{N}$ such that $(m, n) = 1$. Then

$$\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(m \cdot n) = \eta\left(\sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k\right)(m \cdot n) = a_1^{\omega(m \cdot n)}$$

$$= a_1^{\omega(m)+\omega(n)} = a_1^{\omega(m)} \cdot a_1^{\omega(n)} = \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(m) \cdot \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(n)$$

(2) $\implies$ (3)

Let $a_k = \frac{a_1^k}{k!}$ for all $n \in \mathbb{N}$. Then

$$\left(\sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot \sum_{k=0}^{\infty} b_k x^k\right)\left(\sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot \sum_{k=0}^{\infty} c_k x^k\right) = \left(\sum_{k=0}^{\infty} k! \frac{a_1^k}{k!} b_k x^k\right)\left(\sum_{k=0}^{\infty} k! \frac{a_1^k}{k!} c_k x^k\right)$$

$$= \left(\sum_{k=0}^{\infty} a_1^k b_k x^k\right)\left(\sum_{k=0}^{\infty} a_1^k c_k x^k\right) = \sum_{k=0}^{\infty} a_1^k \left(\sum_{i+j=k} b_i c_j\right) x^k$$

$$= \sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot \sum_{k=0}^{\infty} \left(\sum_{i+j=k} b_i c_j\right) x^k = \sum_{k=0}^{\infty} \frac{a_1^k}{k!} x^k \odot \left(\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k\right)$$

(3) $\implies$ (4)

Let the distributive property hold. Then

$$\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k$$

Recall

$$\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

This would imply

$$\sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{k!} (2x)^k = \exp(2x) = \exp(x) \cdot \exp(x) = \left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k\right)\left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k\right)$$

So

$$\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = \sum_{k=0}^{\infty} a_k x^k \odot \left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k \cdot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right)$$
$$= \left(\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right) \left(\sum_{k=0}^{\infty} a_k x^k \odot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right) = \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} a_k x^k$$

(4) $\implies$ (2)

This immediately follows by induction. □

Theorem 3.4.4. *Let* $\sum\limits_{k=0}^{\infty} a_k x^k \in \mathbb{C}[[x]]$ *be a non-zero power series. Then the following are equivalent*

1. $\sum\limits_{k=0}^{\infty} a_k x^k$ *is additive*
2. $a_0 = 0$ *and* $a_k = \frac{a_1}{(k-1)!}$ *for all* $n \in \mathbb{N}$
3. $\sum\limits_{k=0}^{\infty} a_k x^k \odot \left(\sum\limits_{k=0}^{\infty} b_k x^k \cdot \sum\limits_{k=0}^{\infty} c_k x^k \right) =$

   $= \left[\left(\sum\limits_{k=0}^{\infty} a_k x^k \odot \sum\limits_{k=0}^{\infty} b_k x^k \right) \cdot \sum\limits_{k=0}^{\infty} c_k x^k \right] + \left[\left(\sum\limits_{k=0}^{\infty} a_k x^k \odot \sum\limits_{k=0}^{\infty} c_k x^k \right) \cdot \sum\limits_{k=0}^{\infty} b_k x^k \right]$

   for all $\sum\limits_{k=0}^{\infty} b_k x^k, \sum\limits_{k=0}^{\infty} c_k x^k \in \mathbb{C}[[x]]$
4. $\sum\limits_{k=0}^{\infty} 2^k a_k x^k = \sum\limits_{k=0}^{\infty} \frac{2}{k!} x^k \cdot \sum\limits_{k=0}^{\infty} a_k x^k$

Proof. (1) $\implies$ (2)

Let $\sum\limits_{k=0}^{\infty} a_k x^k$ be additive. Then

$$0 = \eta \left(\sum_{k=0}^{\infty} a_k x^k \right) (1) = \omega(1)! a_{\omega(1)} = a_0$$
$$a_1 = \omega(p^\alpha)! a_{\omega(p^\alpha)} = \eta \left(\sum_{k=0}^{\infty} a_k x^k \right) (p^\alpha)$$

Then we can say

$$ka_1 = \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(p_1^{\alpha_1}) + ... + \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(p_k^{\alpha_k})$$
$$= \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(p_1^{\alpha_1}\cdots p_k^{\alpha_k}) = k!a_k$$

Therefore

$$a_k = \frac{a_1}{(k-1)!}$$

(2) $\implies$ (1)

Let $a_k = \frac{a_1}{(k-1)!}$ for all $n \in \mathbb{N}$, and let $m, n \in \mathbb{N}$ such that $(m, n) = 1$. Then

$$\eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(m\cdot n) = \eta\left(\sum_{k=0}^{\infty} \frac{k\cdot a_1}{k!} x^k\right)(m\cdot n) = \omega(m\cdot n)a_1$$
$$= a_1\omega(m) + a_1\omega(n) = \eta\left(\sum_{k=0}^{\infty} \frac{k\cdot a_1}{k!} x^k\right)(m) + \eta\left(\sum_{k=0}^{\infty} \frac{k\cdot a_1}{k!} x^k\right)(n)$$
$$= \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(m) + \eta\left(\sum_{k=0}^{\infty} a_k x^k\right)(n)$$

(2) $\implies$ (3)

Let $a_k = \frac{a_1}{(k-1)!}$ for all $n \in \mathbb{N}$. Then

$$\left[\left(\sum_{k=0}^{\infty} \frac{k\cdot a_1}{k!} x^k \odot \sum_{k=0}^{\infty} b_k x^k\right)\cdot \sum_{k=0}^{\infty} c_k x^k\right] + \left[\left(\sum_{k=0}^{\infty} \frac{k\cdot a_1}{k!} x^k \odot \sum_{k=0}^{\infty} c_k x^k\right)\cdot \sum_{k=0}^{\infty} b_k x^k\right]$$
$$= \sum_{k=0}^{\infty} ka_1 b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k + \sum_{k=0}^{\infty} ka_1 c_k x^k \cdot \sum_{k=0}^{\infty} b_k x^k$$
$$= \sum_{k=0}^{\infty}\left(\sum_{i+j=k} ia_1 b_i c_j\right)x^k + \sum_{k=0}^{\infty}\left(\sum_{i+j=k} ia_1 c_i b_j\right)x^k = \sum_{k=0}^{\infty}\left[\sum_{i+j=k} ia_1(b_i c_j + c_i b_j)\right]x^k$$
$$= \sum_{k=0}^{\infty}\left[ka_1 \sum_{i+j=k}(b_i c_j)\right]x^k = \sum_{k=0}^{\infty} \frac{k\cdot a_1^k}{k!} x^k \odot \left(\sum_{k=0}^{\infty} b_k c_k x^k\right) = \sum_{k=0}^{\infty} \frac{k\cdot a_1^k}{k!} x^k \odot \left(\sum_{k=0}^{\infty} b_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k\right)$$

(3) $\implies$ (4)

Let the distributive property hold. Then

$$\sum_{k=0}^{\infty} 2^k a_k x^k = \sum_{k=0}^{\infty} a_k \odot \sum_{k=0}^{\infty} \frac{2^k}{k!} x^k = \sum_{k=0}^{\infty} a_k \odot \left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k \cdot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right)$$
$$= \left[\left(\sum_{k=0}^{\infty} a_k \odot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right) \cdot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right] + \left[\left(\sum_{k=0}^{\infty} a_k \odot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right) \cdot \sum_{k=0}^{\infty} \frac{1}{k!} x^k \right]$$
$$= \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} \frac{2}{k!} x^k$$

(4) $\implies$ (2)

This immediately follows by induction. □

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Curriculum Vitae

John Byron Snell was born on November 10, 1994. He is the second son Steven Lee Snell and Kathryn Rose Snell. He entered The University of Texas at El Paso in the summer of 2015. While working on his undergraduate degree, he was a mathematics instructor for the YWCA's LYFT program and was a high school mathematics teacher for Key Stone Christian Schools. In the spring of 2019 he graduated from The University of Texas at El Paso with a Bachelor's in Mathematics. In the summer of 2019, he entered the Graduate School of The University of Texas at El Paso. While working towards his Master's degree in Mathematical Sciences he worked as a Teaching Assistant at the Department of Mathematical Sciences at the University of Texas at El Paso.