Mathematical Modeling, Optimal Control And Stability Analysis For Dynamic Supply Chains

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MATHEMATICAL MODELING, OPTIMAL CONTROL AND STABILITY ANALYSIS FOR DYNAMIC SUPPLY CHAINS

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Dedication

To my parents, Alberto and Beatriz for their support and guidance during my life. Also, to my brothers: Pável, Anuar, and Alberto for their company and sponsorship during the Master process.

I would like to express my sincere gratitude to my advisor, Doctor Heidi Taboada, and my co-advisor Doctor José F. Espíritu, the conceiver minds of this project. Their insights and observations have made this research experience a continuous learning process.

I also want to express my appreciation to Doctor Jaime Sánchez-Leal, Doctor Eric D. Smith, and Doctor Oswaldo Aguirre for their advice and patience.
MATHEMATICAL MODELING, OPTIMAL CONTROL AND STABILITY ANALYSIS FOR
DYNAMIC SUPPLY CHAINS

by

YASSER ALBERTO DAVIZON CASTILLO, PhD

THESIS

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Abstract

Dynamic supply chains (SC) are important to reduce inventory, to enable the flow of materials, and maximize profits. By this, dynamic SC require proper decision-making with effective performance and synchronization. In operations and production planning processes, decision-makers develop convenient and suitable decisions, to validate their hypothesis about which decisions incorporate more profit while reducing costs on enterprise operations. Based on this, inventory management plays a crucial point in the supply chain analysis. In general, SC processes raw material, cash, and information flows, taking into account the demand profile of the system. This research work presents the mathematical modeling, optimal control, and stability analysis for dynamic SC. Novel mathematical models are developed to incorporate the use of compartmental analysis in the context of SC modeling, for forward and closed-loop SC. Optimal control (OC) formulations are developed in the context of Pontryagin maximum principle, for energy-based OC and present-value Hamiltonian OC, with proper stability analysis for each SC mathematical model addressed.
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Chapter 1: Introduction to systems theory and control for supply chains

This chapter presents a systems theory and control, introductory perspective, for supply chains. In the first section, supply chain definitions are defined, considering the forward material flows and closed-loop supply chains. At second stage, the strategic, tactical, and operational decision-making along the supply chain are present. Mathematical modeling, control, and stability analysis are developed in the supply chain context, are analyzed in the rest of the chapter.

1.1. SUPPLY CHAIN MANAGEMENT: DEFINITIONS

Supply chain management (SCM) refers to the cooperation process management of materials and information flows between supply chain partners (Sucky, 2005). In a general, the principal components of supply chains are suppliers, manufacturers, customers, logistics and distribution networks, and retailing centers (Kamble, et al 2020). See Figure 1.1, for a four echelons supply chain.

![Figure 1.1: Four echelons supply chain](image)

Typically, three flows are considered in supply chain dynamics: Raw material, cash and information. Supply chains are complex dynamic systems set off by customer demands (Marufuzzaman and Deif, 2010). As a formal definition: “A supply chain is a network of suppliers that produce goods, both, for one another and for generic customers” (Daganzo, 2003).
Based on this context, this research work intends to present the mathematical modeling of dynamic supply chains, from a model-based control theory and stability analysis context. Our interest is to analyze the supply chains from systems and processes perspectives.

### 1.2. DECISION LEVEL MAKING IN SUPPLY CHAINS

There are three decision level making in the supply chain, such as strategical, tactical and operational.

#### 1.2.1 Strategic level

In supply chain the strategic level impacts the long term planning. In general, a taxonomy for supply chains presents 1) Forward material flows and 2) Closed-loop supply chain (CLSC). In general, strategic planning involves the selection of suppliers and distributors, location, and capacity planning of manufacturing and servicing units (Chauhan et al, 2004). In (Das, 2011) a strategic global supply chain (GSC) operates for multi-layered, multi-location-based suppliers, and manufacturing plants by the integration quality based supply management. In (Seuring, 2009) presents five strategic decision levels in supply chain management such as products and services, partners and partnerships, plants and stocks, processes and planning and control. A CLSC network consists of various conflicting decisions of forward and reverse facilities, in (Özceylan and Paksoy, 2014) the CLSC proposed model integrates the strategic and tactical decisions to avoid separated design in both chain networks. In (Bhattacharyya, 2017) an optimization model is present for CLSC network with a multi-echelon inventory, multi-period planning, and multi-product scenario. In the case of green supply chains, in (Dey et al, 2018) strategic inventory on the development-intensive and marginal-cost intensive green product types are present, while in (Kuiti, 2019) manufacturer-retailer channel analyzes strategic decisions in a green supply chain.
1.2.2 Tactical level

Tactical decision level, in supply chains, presents a mid-term planning cycle. This level determines a preliminary plan of the regular operations (Pereira, et al, 2020), usually made at an aggregate level, it is more related to demand, inventory, and supply planning. In (Ivanov, 2010) presents planning decisions as an adaptive process to be interrelated at all the decision-making levels, in the supply chain. In (Tian, et al 2011) the problems of tactical safety stock placement and tactical production planning are analyzed via an iterative approach. Considering that is based on physical parameters such as stock level, demand satisfaction, etc., in (Comelli et al, 2008) proposes an approach to evaluate tactical production planning in supply chains, considering financial evaluation. In (Fahimnia, et al 2014) present a tactical planning model that integrates economic and carbon emission for a green supply chain.

1.2.3 Operational level

In supply chains, at operational level, the planning horizon is a short term (between days and weeks). In (Schutz and Tomasmgard, 2011), volume, delivery, and operational flexibility effects, in supply chain planning under uncertain demand, are discussed. To pursue an appropriate performance at the operational level, the supply chain design must develop on this operational level performance (Mansoornejad, et al, 2013). The decision, at operational planning level, covers two main areas (Monthatipkul and Kawtummacha, 2007): inventory area (related with inventory replenishment order to satisfy demand) and transportation area (physical movement of goods between different geographical points). In (Zhang, et al, 2020), presents operational decisions and financing strategies in a CLSC by a manufacturer and a retailer.
1.3. MATHEMATICAL MODELING IN DYNAMIC SUPPLY CHAINS

Inventory management (IM) presents an important role in the operations and management sciences. In this research work, I present a systemic analysis for IM for a class of manufacturing systems, applying systems theory and model-based control approaches. The main goal is to explore the dynamic nature of production systems via the development of ordinary differential equations (linear and nonlinear), also a data-driven control approaches via system identification is addressed. Proper mathematical models and simulations are conducted. By definition, inventory management is part of the SCM that plans, implements, and controls the forward and reverse flow of goods and services (as well as storage), between the point of origin and the point of consumption to meet customer requirements. (Singh and Verma, 2017).

Inventories, in nature, are present all along the supply chain (SC) and inventory control is a crucial activity by a company’s management (Bieniek, 2019). To present appropriate inventory levels is crucial task for a company (Duan and Ventura, 2019), considering that quick and positive response to customers is related to high inventory levels (which increase the cost), while low inventory levels might cause scarcity. In general, for manufacturing firms, inventory usually represents from 20% to 60% of the total assets. (Giannoccaro, 2003). Coordination of inventory policies in IM for SCM between suppliers, manufactures, and distributors is a major task. As well, to smooth material flow and minimize costs while reaches customer demand. (Giannoccaro and Pontrandolfo, 2002).

The main considerations for IM policies along the SCM are (Ryu, et al, 2013) (1) type of optimization (local or global); (2) control type (centralized or distributed); (3) inventory levels nature (periodic, continuous or hybrid); (4) demand function type (stochastic or deterministic) and (5) inventory control responsibility (self-managed or vendor-managed).
1.4. MODEL-BASED CONTROL SYSTEMS: A GENERAL TAXONOMY

Control theory relates the core idea to maintain equilibrium and stability state with uncertainty and disturbance (Wiener, 1948). In general, a classical control system has its roots in the feedback systems concept, see Figure 1.2. A general goal in control systems is to minimize the level of error in the system, in Figure 1.2, the reference signal r(t), error signal e(t), manipulated variable signal u(t), controlled variable y(t), and output sensor signal y*(t), are the basis of a general feedback system classical control.

![Figure 1.2: Feedback system classical control](image)

Modeling and analysis of control systems present the following taxonomy: Model-based control and data-driven control. In general, model-based control has the following sub-classification: Optimal control, robust control, adaptive control, and intelligent control. Our aim, in this research work, is to develop optimal controls, which are analyzed via the Pontryagin maximum principle. Figure 1.3, presents a conceptual map for systems and control theory approaches.
From Figure 1.3, a general taxonomy is present for model-based control systems, which incorporates: Optimal control, robust control, intelligent control and adaptive control. The most common model-based control systems strategies, nowadays are control structures which have been applied in oil, aerospace, automotive and manufacturing industries, in general.
1.5. CONTROL ORIENTED APPROACHES FOR SUPPLY CHAINS

Decision-makers are facing with greater amounts of data and business decisions. There is a growing interest in using real-time information to improve, support, and validate business decision-making. (Sourirajan, et al 2008). Today’s enterprises work towards flexible, reliable, and responsive business operations. Therefore, they need to implement systematic decision-making processes. (Perea, et al 2000).

In production and logistics systems, SC and Industry 4.0 networks: Uncertainty, feedback cycles, and system dynamics are mandatory goals (Dolgui, et al, 2018). Since 1960, control theory (CT) has gained attention in the modeling and analysis for operational production and logistics systems. (Ivanov, et al 2018). Considering the dynamic nature of production-inventory systems, CT is a suitable approach to handle time-varying phenomena. (Ortega and Lin, 2004).

A supply chain is a network of facilities and distribution entities such as: suppliers, manufacturers, distributors, retailers. (Sarimveis, et al, 2008). A supply chain is characterized by three flows: materials and cash (forward flow) and information (backward flow).

1.5.1 Model-based optimal control theory

Nowadays, the interest to investigate the dynamic behavior of management control system, has been growing (Connors, 1967). The mathematical description of dynamical system is often in terms of difference or differential equations. A general theory is present that permits us to determine the optimal control of such a system according to some suitable performance criterion. Necessary and sufficient conditions of optimality are derived for a class of optimal control problems. Our interest is to present the conventional optimal control problems, for linear time-invariant dynamical systems. Applications of optimal control in management sciences and operations research are (1) pricing; (2) scheduling; (3) logistics networks; (4) optimal transfer of
technology, and (5) optimal remanufacturing and recycling in closed-loop supply chains, among others. Decision-makers in management sciences and operations research use optimal control theory to map optimal control methods towards optimization methods based on the nature of the complexity of the manufacturing system.

An optimal control (OC) is defined as an admissible control, which minimizes a functional objective. Based on this, given a dynamical system with initial condition $x_0$, which evolves in time according to the state space equation $\dot{x} = f(x, u, t)$, to find an admissible control and make the functional objective to achieve its maximum.

A general mathematical form for an optimal control problem is:

$$\min_{u} J(u) = \frac{1}{2} \int_{t_0}^{t_f} F(x, u, t) \, dt + S[x(t_f)]$$

s.t.

$$\dot{x} = f(x, u, t)$$
$$x(t_0) = x_0, x \in X, u \in U$$

In general, optimal control methods require:

1) A performance index
2) Dynamical systems to optimize
3) Constraints in states, inputs, and outputs

### 1.5.2 Minimum time-optimal control

Consider the following optimal control problem for a linear system.
\[
\min_u f(u) = \int_{t_0}^{t_f} 1 \, dt
\]
\[\text{s.t.}\]
\[
\dot{x} = Ax(t) + Bu(t)
\]
\[
x(0) = x_0, \quad x(T) = x_f \text{ (fixed)}
\]
\[
u \in U
\]

This is called, a minimum time optimal control problem based on the context that the objective is to transfer the state \(x\) from a fixed initial point \(x_i\) to a fixed final point \(x_f\) in a minimum time, which implies a time-optimal way.

**1.5.3 Minimum fuel optimal control**

The general form of the minimum fuel performance index is:

\[
\min_u f(u) = \int_0^T |u| \, dt
\]
\[\text{s.t.}\]
\[
\dot{x} = Ax(t) + Bu(t)
\]
\[
u \in U
\]

**1.5.4 Minimum energy optimal control**

For a minimum energy optimal control problem, the performance index is:

\[
\min_u f(u) = \int_0^T \frac{1}{2} u^2 \, dt
\]
\[\text{s.t.}\]
\[
\dot{x} = Ax(t) + Bu(t)
\]
\[
u \in U
\]
1.6. STABILITY ANALYSIS FOR SUPPLY CHAINS

A supply chain as a dynamical system is subject to proper stability analysis using the following main characteristics: 1) Routh-Hurwitz stability analysis, and 2) Lyapunov direct and indirect methods. In general, the Routh-Hurwitz stability analysis is present for single-input single-output (SISO) dynamical systems in continuous linear time domain. While Lyapunov direct and indirect methods provide stability results more in general for nonlinear systems.

Our aim in this section is to present, the most recent works related to stability analysis for supply chains. A continuous time version beer game model and its stability and robust stability properties are investigated in (Riddalls and Bennett, 2002). In (Nagatani and Helbing, 2004) presents conceivable production strategies to stabilize supply chains, controlling the production speed with dependence to the stock levels. A continuous time Automatic Pipeline, Inventory and Order Based Production Control System (APIOBPCS) investigation for stability boundary via Padé approximation and the Routh Hurwitz criteria, is present in (Warburton et al, 2004). In (Rabelo et al, 2008) presents a methodology for manufacturing supply chains, to predict changes due to endogenous and/or exogenous influences in the short and long-term horizons. A supply chain can perform sudden and intense change in demand, in (Cannella and Ciancimino, 2010) presents an analysis for collaboration and smoothing replenishment rules on supply chain using differential equations. In (Sipahi and Delice, 2010) stability of inventory dynamics controlled by the APIOBPCS is, with delays present in; lead time, transportation, and decision-making.

Based on this, Table 1.1 summarizes research works for stability for supply chains from 2011 to 2020.
<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Stability approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kastrian, and Monnigmann, 2011</td>
<td>This paper addresses the steady-state optimization of a supply chain for the vendor managed inventory, automatic pipeline, inventory, and order based production control systems (VMI-APIOBPCS).</td>
<td>Robust stability</td>
</tr>
<tr>
<td>Makuschewitz et al, 2011</td>
<td>The problem to investigate input to state stability in a proper mathematical modeling approach for a supply chain network is present.</td>
<td>Input to state stability</td>
</tr>
<tr>
<td>Wang et al, 2012</td>
<td>This paper analyses the stability of a constrained production and inventory system with a non-negative order rate via a piecewise linear model, an eigenvalue analysis.</td>
<td>Lyapunov stability</td>
</tr>
<tr>
<td>Palsule-Desai et al, 2013</td>
<td>A non-cooperative game-theoretic is present to analyze the network stability in a two-tier supply chain.</td>
<td>Network stability</td>
</tr>
<tr>
<td>Luo et al, 2015</td>
<td>A supply chain coalition stability is analyzed considering that the retailer ally with each other without restraints.</td>
<td>Coalition stability</td>
</tr>
<tr>
<td>Klug, 2016</td>
<td>In this paper, bullwhip and backlash effects in supply chains are analyzed. A number of dynamic and transient behaviours are present, such as supply chain stability, compactness, and bullwhip/backlash trade-offs.</td>
<td>Phase space trajectory</td>
</tr>
</tbody>
</table>
Chapter 2: Literature review and problem definition

In operations and production planning processes, decision-makers require to achieve fast and suitable decisions to validate their hypothesis about which decision incorporates more profitable and cost down impacts in the enterprise operations. Based on this, inventory management plays a crucial point in the supply chain analysis. Each supply chain processes raw material, cash, and information flow, taking into account a demand forecast of the system. This chapter presents a KBES to validate the literature review for control of production-inventory systems. Based on, the rapid growth process to apply control and systems theory approaches for supply chains from 2000 to 2018. A KBES is proposed considering the mathematical approach of possibility theory (PT). Control and systems theory to explore the dynamic nature of the supply chains. In this literature review, our goal is to analyze the evolution of control theory in the analysis of inventory management for supply chains in a time horizon of almost twenty years.

In section 2.1, a KBES and PT-based background theory to validate the literature review is present; in section 2.2, a literature review is conducted for control theory and supply chain analysis is conducted. In section 2.3, the KBES methodology is presented. Finally, proper problem definition is addressed in section 2.4.

2.1 BACKGROUND THEORY

In this section a background theory is developed for PT and KBES, taking into account the potential opportunities as tools which enable the decision-making process.

2.1.1 Possibility theory

Decision making involves the handling of the information available about uncertainty in a decision. (Yager, 1979). In some situations, the information on uncertainty does not have the character of a probability. Possibility theory (PT) is a mathematical theory for dealing with certain
types of uncertainty. PT is an uncertainty theory to work with incomplete information. (Agarwal and Nayal, 2015). The importance of possibility theory is that much of the information on which human decisions are based is possibilistic, in nature, rather than probabilistic (Zadeh, 1999). PT expresses the facility in which an event can occur, or belong to a set (Khoury, et al. 2008).

<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Level of application</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cayrac, et al, 1996</td>
<td>This paper proposes allowing a representation of the available knowledge, through the introduction of an appropriate representation of uncertainty and incompleteness based on Zadeh’s possibility theory and fuzzy sets.</td>
<td>Satellite fault diagnosis</td>
</tr>
<tr>
<td>Mauris, et al, 2000</td>
<td>This paper proposes to build a fuzzy model of data acquired from physical sensors by applying a truncated triangular probability-possibility transformation.</td>
<td>Physical sensors</td>
</tr>
<tr>
<td>Denguir-Rekik, et al., 2006</td>
<td>This paper analyzes some aspects of uncertainty evaluations in multicriteria decision-making in the framework of e-commerce website recommendation.</td>
<td>E-commerce website choice support</td>
</tr>
<tr>
<td>Mauris, 2007</td>
<td>This paper explores the possibility expression of measurement uncertainty applied to situations where only very limited knowledge is available.</td>
<td>Measurement uncertainty in a very limited knowledge context</td>
</tr>
<tr>
<td>Dewalle-Vignion, et al 2011</td>
<td>This paper presents a designed and evaluated a new, nearly automatic, and operator-independent segmentation approach. This incorporating possibility theory, to take into account the uncertainty and inaccuracy inherent in the image.</td>
<td>Medical imaging</td>
</tr>
<tr>
<td>Guo, 2011</td>
<td>This paper focuses on one-shot decision problems, which concern the situations where a decision is experienced only once, which are commonly encountered in business, economics, and social systems.</td>
<td>Decision theory</td>
</tr>
<tr>
<td>Romero, et al, 2011</td>
<td>In this study a methodology for harmonic load-flow calculation based on the possibility theory is presented. Possibility distributions instead of probabilities are the input used to describe the uncertainty in the magnitude and composition of the loads.</td>
<td>Power systems</td>
</tr>
<tr>
<td>Li and Ma, 2015</td>
<td>This research proposes the generalized possibilistic computation tree logic model checking in this, which is an extension of possibilistic computation logic model checking.</td>
<td>Possibilistic computation tree logic</td>
</tr>
</tbody>
</table>
This paper proposes an approach referred as the iterative refinement of possibility distributions by learning for pixel-based image classification, which is based on the use of possibilistic reasoning concepts exploiting expert knowledge sources.

Ren, et al., 2016

This paper, considering the insufficient uncertainty data, proposes a possibility-based optimal design algorithm to get a robust and reliable optimal design of electromagnetic devices.

Mei, 2019

This paper intends to formulate fuzzy control in the framework of possibility theory by regarding the fuzzification procedure of fuzzy control.

### 2.1.2 Knowledge-based expert systems

In terms of knowledge transfer, knowledge-based systems (KBS) are categorized as expert systems and knowledge-sharing systems (Niwa, 1990). A KBES is a computer program which use a knowledge base to solve complex problems in a particular field and achieves some reasoning (López-Santana and Méndez-Giraldo, 2016). KBES uses human knowledge to solve problems that commonly require human intelligence (Tripathi, 2011).

<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Level of application</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kusiak, 1988</td>
<td>In this paper a knowledge-based system for solving the generalized group technology which involves constraints related to machine capacity, material handling system capabilities, machine cell dimensions and technological requirements.</td>
<td>Cellular manufacturing</td>
</tr>
<tr>
<td>Willis, et al, 1989</td>
<td>This paper presents the development of a process planning system using knowledge-based techniques with the derivation of component geometric information from a three-dimensional solid modeler.</td>
<td>Manufacturing process planning</td>
</tr>
<tr>
<td>Parlar, 1989</td>
<td>This paper develops a knowledge-based expert system which identify and recommend up to 30 production-inventory models.</td>
<td>Production-inventory systems</td>
</tr>
<tr>
<td>Hsu and Chen, 1990</td>
<td>This research paper presents a knowledge-based expert system for distribution system planning.</td>
<td>Power systems (distribution planning)</td>
</tr>
</tbody>
</table>
2.2 LITERATURE REVIEW FOR CONTROL THEORY IN SUPPLY CHAINS

This section presents a literature review for control approaches in supply chains from 2000 to 2021, considering the control-oriented characteristics such as:

1) Model-based optimal control
2) Robust control
3) Control approaches based on computational intelligence

To provide the current state of the art in control theory in supply chains, this review considers 70 journal papers.

2.2.1 Model-based optimal control

Supply chains present a dynamical behavior, which from model-based optimal control perspective, the following approaches are present:

1) Classical control theory
2) Optimal control theory
3) Model predictive control

Classical control theory:

In the literature, classical control theory is used to control supply chains. In (Perea, et al 2000), present a dynamic supply chain management approach based on a dynamic framework to model supply chains. An infinite horizon two-echelon supply chain inventory problem is present
in (Hosoda and Disney, 2006) and shows a sequence of the optimum ordering policies. In (Schwaninger and Vrhovec, 2006) address a distributed control via the regulation of supply systems from production to the customer and vice versa, applying system dynamics modeling and simulation. The use of proportional-integral and proportional-derivative control theoretic principles to manage the inventory replenishment process in a supply chain is present in (Sourirajan et al, 2008). In (Yuan and Ashayeri, 2009), proposes system dynamics loops and control theory simulations to analyze the impacts on capacity expansion strategies within a supply chain. The Integral of the Time Absolute Error (ITAE), is present as, an appropriate control engineering measure of resilience for inventory levels and shipment rates in (Spiegler et al, 2012). In (Janamanchi and Burns, 2013) presents control theory concepts to develop a performance functional that optimizes a retail supply chain, via system dynamics modeling methodology.

Table 2.3, presents a review for research papers between 2015 to 2021 with the classical control theory developed.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Control structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spiegler et al, 2015</td>
<td>A dynamic analysis of supply chain resilience is present by the use of nonlinear control theory.</td>
<td>Nonlinear control</td>
</tr>
<tr>
<td>Qing et al, 2016</td>
<td>This research work presents a dynamic performance and optimization of a discrete production control system subject supply disruption and demand uncertainty.</td>
<td>Classical control APIOBPCS</td>
</tr>
<tr>
<td>Lin et al, 2017</td>
<td>By the application of control engineering approach, a feedback forecasting in the make to order element plays a major role in the degree of the bullwhip effect, and the customer order decoupling point impacts both the bullwhip effect and the inventory variance in the make to stock.</td>
<td>Classical control IOBPCS</td>
</tr>
<tr>
<td>Zhao and Wang, 2018</td>
<td>In this research work, a multiproduct three-echelon inventory control system is analyzed for the hybrid supply chain.</td>
<td>Classical control</td>
</tr>
</tbody>
</table>
**Optimal control theory:**

In general, optimal control theory requires a mathematical model to apply Pontryagin Maximum principle techniques. In relation to optimal integrated ordering and production control in a supply chain for finite capacitated warehouses is analyzed in (Song, 2009). In (Kogan et al, 2010), address a differential game where the effect of information asymmetry is present under stochastic demand. Considering the dynamic nature of goods flows process, efficient inventory management in production–inventory systems is present in (Ignaciuk and Bartoszewicz, 2010). In (Ignaciuk and Bartoszewicz, 2012), a control theory approach, is present, for the problem of inventory control in systems with perishable goods. The applicability of optimal control theory for supply chain management is analyzed in (Ivanov et al, 2012), this is based on the fundamentals of control and systems theory.

Table 2.4, presents a review for research papers between 2015 to 2021 with the optimal control theory developed.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Control structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ivanov et al, 2016</td>
<td>This paper addresses the problem of supply chain coordination by a robust schedule coordination approach, applying optimal control theory.</td>
<td>Optimal control</td>
</tr>
<tr>
<td>Ivanov et al, 2018</td>
<td>This research work extends a previous survey in the Annual Reviews in Control (Ivanov et al. 2012) by identifying two new directions of control theory applications (ripple effect analysis in the supply chains and scheduling in Industry 4.0) and the digital technology use in control-theoretic models.</td>
<td>Optimal control</td>
</tr>
<tr>
<td>Dolgui et al, 2018</td>
<td>A survey on optimal control applications to scheduling in production, supply chain, and Industry 4.0 systems via deterministic maximum principle is present.</td>
<td>Optimal control</td>
</tr>
<tr>
<td>Authors</td>
<td>Title</td>
<td></td>
</tr>
<tr>
<td>------------------</td>
<td>----------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>Zu, et al, 2019</td>
<td>This paper presents a method to study the Stackelberg differential game between a manufacturer and a supplier.</td>
<td></td>
</tr>
<tr>
<td>Wu, 2019</td>
<td>This paper presents a differential game for a single manufacturer and single retailer in a supply chain under the consignment contract.</td>
<td></td>
</tr>
<tr>
<td>Wu and Chen, 2019</td>
<td>The cooperative advertising problem is addressed, applying optimal control theory for a supply chain under the consignment contract in a competitive environment.</td>
<td></td>
</tr>
<tr>
<td>Yu, et al, 2020</td>
<td>An optimal control model is present, for a class of low-carbon supply chain system. A linear differential equation describes the dynamics of emission reduction level.</td>
<td></td>
</tr>
<tr>
<td>Rarita, et al, 2021</td>
<td>This paper considers supply chains modeled by partial and ordinary differential equations.</td>
<td></td>
</tr>
</tbody>
</table>

**Model predictive control:**

In recent years, the use and application of Model Predictive Control (MPC) as an advanced control structure has gained interest in the processes industries. In the context of supply chains, in (Perea-López et al, 2003) presents a model predictive control strategy, for multiproduct, multi-echelon distribution networks, to find the optimal decision variables to maximize profit in supply chains. The presentation of the supply chain problem into a formulation to MPC application is analyzed for a single-product, the two-node example in (Braun et al, 2003). In (Seferlis and Giannelos, 2004), is present a two-layered optimization-based control structure for multi-product, multi-echelon supply chain networks. Decision policies for inventory management in supply chains are present in (Schwartz et al, 2006), via internal model control (IMC) and model predictive control. In (Dunbar, 2007), addresses the problem of distributed control of dynamically coupled nonlinear systems, such as supply chain systems, that are subject to decoupled constraints. The implementation of model predictive control, to semiconductor manufacturing supply chain management, is discussed in (Wang, et al, 2007). In (Wang and Rivera, 2008), a model predictive
control algorithm is addressed for tactical decision-making on semiconductor manufacturing supply chains. Semiconductor manufacturing requires simulation modeling of discrete processes combined with control policies such as: Discrete Event System Specification and model predictive control, which are developed in (Huang et al, 2009). In (Li and Marlin, 2009), presents a robust Model Predictive Control approach for supply chain optimization under uncertainties. Inventory control for multi-item, multi-echelon distribution management in supply chains via model predictive control is present in (Alessandri, et al, 2011). In (Subramanian, et al, 2013), proposes the application of distributed model predictive control for inventory management in supply chain optimization. A hybrid model predictive control formulation for inventory management in supply chains subject to limited capacity is present in (Nandola and Rivera, 2013). In (Subramanian, et al, 2014), an economic model predictive control with the closed-loop structure for inventory management in supply chain optimization is discussed. A robust MPC optimization-based decision-making for supply chain management is present in (Mastragostino et al, 2014). In (Fu, et al, 2014), a four-echelon supply chain is present for centralized and decentralized model predictive control strategies, to control inventory, and to reduce the bullwhip effect. A perspective for demand modeling in a production-inventory system is discussed in (Schwartz and Rivera, 2014), which forecasts inventory management policies based on IMC or MPC.

Table 2.5, presents a review for research papers between 2015 to 2021 with the model predictive control developed.
Table 2.5: Model Predictive Control literature review from 2015 to 2021

<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Control structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fu, et al, 2016</td>
<td>This paper presents an inventory control problem for a class of supply chain, which is solved using the extended prediction self-adaptive control approach to MPC.</td>
<td>MPC</td>
</tr>
<tr>
<td>Pinho, et al, 2017</td>
<td>This research work proposes a multilayer model predictive control strategy to improve the performance of the biomass supply chain operational level.</td>
<td>MPC</td>
</tr>
<tr>
<td>Dias and Ierapetritou, 2017</td>
<td>This paper presents an integrated decision-making strategies and review recent advances from process control to supply chain management.</td>
<td>Process control</td>
</tr>
<tr>
<td>Hsiao, et al, 2018</td>
<td>This work proposes an MPC framework for a non-stationary semiconductor manufacturing supply chain network from the integrated device manufacturer perspective.</td>
<td>MPC</td>
</tr>
<tr>
<td>Fu, et al, 2019</td>
<td>This paper presents a distributed model predictive control approach, with minimal information exchange and communication, for supply chain operations and management.</td>
<td>Distributed MPC</td>
</tr>
<tr>
<td>Fu, et al, 2019</td>
<td>This paper presents a distributed model predictive control scheme for the inventory management problem in supply chain networks.</td>
<td>Distributed MPC</td>
</tr>
<tr>
<td>Hipólito, et al, 2020</td>
<td>A centralized model predictive control of perishable goods supply chains for logistics management is present.</td>
<td>MPC</td>
</tr>
</tbody>
</table>

### 2.2.2 Robust control

Considering that supply chains are dynamical systems subject to variability and uncertainty. Several approaches have been addressed in the context of robust control such as:

1) H∞ robust control
2) Sliding mode control
3) Adaptive control
In (Huang et al, 2007) develops models for multi-echelon supply chain systems applying $H_\infty$ control methods, while in (Krishnamurthy et al, 2008) consider an inventory control technique for large-scale supply chains via adaptive control. $H_\infty$ control technique is addressed for a class of supply chain model linearized about nominal operating conditions in (Boccadoro et al, 2008). Considering that closed-loop supply chains (CLSC) present novel topologies, in (Huang et al, 2009) robust $H_\infty$ control strategies constrain all uncertainties for this CLSC. Another robust control technique is sliding mode control (SMC) which in (Ignaciuk and Bartoszewicz, 2010) SMC for inventory policy is proposed, which assurance that the demand is satisfied. Considering that SC systems are modeled as networked systems in (Luo and Yang, 2014) a $H_\infty$ control is present for a discrete SC.

Table 2.6, presents a review for research papers between 2015 to 2021 with the robust control structure developed.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Control structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ignaciuk, 2015</td>
<td>This paper presents the problem of establishing a robust inventory management policy for production-inventory systems with time delays.</td>
<td>Robust control LQ suboptimal</td>
</tr>
<tr>
<td>Li et al, 2018</td>
<td>This research work addresses a two-time-scale production-inventory system as a switching system, subjected to a Markovian process.</td>
<td>$H_\infty$ control</td>
</tr>
<tr>
<td>Lesniewski and Bartoszewicz, 2020</td>
<td>This article presents the case of a logistic system with a single warehouse, in which a model reference sliding mode controller is developed for a perishable inventory system.</td>
<td>Sliding mode control</td>
</tr>
<tr>
<td>Xu et al, 2020</td>
<td>This paper presents a sliding mode control structure to manage chaotic supply chain system this control synthesis with dynamical analysis is useful for strategic decision-makers in supply chain management.</td>
<td>Adaptive sliding mode control</td>
</tr>
</tbody>
</table>
2.2.3 Control approaches based on computational intelligence

Control techniques based on computational intelligence have been developed for supply chains, such as:

1) Fuzzy control
2) Neural networks control
3) Chaos control
4) Petri nets

In (Anne et al, 2009) investigates three-echelon supply chain dynamics subjected to uncertainties which can exhibit saturation and chaos. Hybrid Petri nets are present in (Dotoli et al, 2009) where it proposes a modular model to describe material, financial, and information flow of supply chains. In (Hong et al, 2010) an on-line neural network controller optimize a three-stage supply chain is present.

Table 2.7: Control based computational intelligence literature review from 2015 to 2021

<table>
<thead>
<tr>
<th>Paper</th>
<th>Contribution</th>
<th>Control structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Göksu, et al, 2015</td>
<td>This paper presents a mathematical model for the synchronization and control of a chaotic supply chain management system.</td>
<td>Chaos control</td>
</tr>
<tr>
<td>Zhang and Zhao, 2015</td>
<td>This research work presents a dynamic discrete switched dual-channel closed-loop supply chain with a time delay in remanufacturing.</td>
<td>Fuzzy control</td>
</tr>
<tr>
<td>Kocamaz, et al. 2016</td>
<td>This paper addresses the control and synchronization of chaotic supply chain compensated with Artificial Neural Network based controllers.</td>
<td>Artificial Neural Network</td>
</tr>
<tr>
<td>Zhang, et al, 2017</td>
<td>A Takagi-Sugeno fuzzy robust control technique for the nonlinear supply chain system subject to lead times is present.</td>
<td>Fuzzy control</td>
</tr>
<tr>
<td>Zhang, et al, 2017</td>
<td>This paper presents a closed-loop supply chain subject to uncertainties,</td>
<td>Fuzzy control</td>
</tr>
</tbody>
</table>
with a fuzzy control model which restrain the bullwhip effect.

Pavlov, et al, 2018  This research work is based on a hybrid fuzzy-probabilistic approach in which the supply chain resilience assessment is extended by incorporating ripple effect and structure reconfiguration.

Zhang, et al, 2018  This paper presents a fuzzy robust strategy for the robust operation of the supply chain network subject to production lead times under the uncertain customer demand.

Zhang and Zhang, 2020  This paper addresses a fuzzy robust control for a class of closed-loop supply chain with lead times, to mitigate the bullwhip effect.

Yan, et al 2021  This research work presents a mathematical model for a supply chain with computer-aided digital manufacturing process for control and synchronization.

2.3 KNOWLEDGE-BASED EXPERT SYSTEM FORMULATION

In this section a KBES is present as a rule-based system, from the development of nine questions survey in a Likert scale, considering the level of application of optimal control, robust control, and control-based computational intelligence.

The proposed methodology has been adapted from (Espino-Román, et al 2020), and summarized such as:

1) Survey design

To measure the level of application of optimal control in SC the following questions are part of the survey:

Q1: Considering the dynamic nature of supply chains, do you consider suitable the use via model-based optimal control theory in the decision-making process of SC?
Q2: Do you consider useful the use of Pontryagin Maximum Principle in the design via optimal controls for SC?

Q3: How often do you apply Model Predictive Control techniques in the operation of SC?

For robust control:

Q4: Assuming the measure of uncertainty, do you consider convenient apply robust control techniques in the design of SC?

Q5: Do you consider important the design of $H_{\infty}$ robust controller for SC?

Q6: Sliding Mode Control, as a robust control technique, how often do you it in the design of SC controllers?

For computer-based computational intelligence:

Q7: Do you consider convenient the development of fuzzy control techniques for SC?

Q8: In the context that SC requires data manipulation, how convenient do you consider the application of Neural networks in SC?

Q9: Assuming SC networks are complex and dynamic nature, do you consider suitable the application of chaos control for SC?

2) Calculate the level of uncertainty (LU) for each survey question such as:

$$LU = \sum_{i=1}^{5} \pi_i (f_i)^i$$

(1)

Where $\pi_i$ = possibility level and $f_i$ = frequency per question.

3) Calculate the certainty factor (CF) for each question

$$CF = 1 - LU$$

(2)

4) Develop and calculate the CF for each category

$$R_q = 1 - \prod_{j=1}^{m} LU_j$$

(3)
5) Construct the production rules for the AND/OR decision tree

6) For the categories: Optimal control (OC), Robust control (RC) and Computational intelligence (CI), define the tree with higher CF.

2.4 PROBLEM DEFINITION

Considering the dynamic nature of supply chains, and the importance of inventory management optimization in the SC operations. To the best of our knowledge, the following SC problems are mandatory to address, in the context of:

1) Mathematical modeling: To present mathematical models using differential equations, which in this context the application of compartmental analysis (which has been applied in engineering and economics), becomes a novel mathematical approach for SC inventory optimization.

2) Control oriented approaches: To develop control-oriented approaches which enable to optimized the SC operation by the description of differential equations applying optimal control theory via Pontryagin´s maximum principle.

3) Stability analysis: To analyze the stability of the SC by the application of proper linear and nonlinear systems approaches, considering Lyapunov stability approaches for the proposed mathematical models.
Chapter 3: Optimal control for capacity-inventory management in serial supply chains

This chapter presents capacity-inventory management modeling via system dynamics for a serial supply chain (SC) applying model-based optimal control techniques. For mathematical modeling purposes, a set of coupled first-order ordinary differential equations, with an analogy from the mixing problem is addressed to relate capacity and inventory levels, taking into account a production rate in each node of interaction. The mathematical model is present in a linear time-invariant (LTI) state-space formulation. Stability analysis for the dynamic serial SC is present, and also a sensitivity analysis is conducted for the capacity and production rate parameters, and considers the effects of variations in parameters the SC. An energy-based optimal control is developed with proper simulations.

3.1 INTRODUCTION

Traditional manufacturing converts raw material into end products, within a demand profile. However, these manufacturing systems face problems such as long downtime, labor inefficiency, not accurate scheduling, and weak energy consumption (Umble, 1992). Recently, SC market forecasts in profitability are increasing. Based on this, nowadays SC has gained special attention, within industry practitioners of automation science and control systems. Applications of SC in industry integrates potential solutions such as Automotive, aerospace, biotechnology, etc. Considering that manufacturing systems are dynamic and complex (Eyers, 2018), proper design of control systems requires flexibility and should be adaptive to process disturbances. In this context, new trends are required to present from systems theory and control theory perspective to support accurate, agile, and flexible solutions for SC towards applications. Industrial production systems require competitive and adaptive solutions, to ensure profitability in markets.

In manufacturing systems, decision-makers require to achieve and approximate, online decisions. A manufacturing SC is a system composed of suppliers, manufacturers, distributors,
and customers, which serve customer requirements (Orji and Liu, 2018). SC integrates solutions to decrease operations management costs such as production, logistics, and quality. In this context, dynamic approaches for production-inventory systems are valuable to maximize profits and minimize costs over the SC system. Recently, in SC the focus moves from the factory level management to enterprise level (Akyuz and Erkan, 2018).

Inventory management presents a crucial role through the performance analysis of a supply chain. By definition, inventory management is “the continuing process of planning, organizing and controlling inventory while balancing supply and demand” (Singha and Verma, 2018). In this research work, our main interest is to provide proper modeling, analysis, and model-based optimal control for a serial SC, from a systems theory perspective. Designing an inventory management system is directly related to sales, finance, production, and procurement (De Vries, 2007). Moreover, the goal of inventory management is to minimize the average cost per unit of time experienced for the inventory system, in the long run (Fiestras-Janeiro, et al, 2011). Considering that decision-variables are related to inventory levels the SC, and fixed parameters are production rate and capacity levels, our interest is to present the mixing problem from which approximate an analogy for a four-echelon supply chain.

The rest of this chapter presents the mixing problem background theory in section 2. A mathematical modeling description of the problem, sensitivity, and stability analysis for the system are present in section 3; in section 4, a model-based optimal control for a composed energy performance index is develop, finally, conclusions and future work are developed in Section 5.

3.2 MATHEMATICAL MODELING

3.2.1 The mixing problem and notation

In systems science, by definition, a system is a group of components that interacts towards keeping a collection of relationships with the sum of the systems themselves to other entities (Badillo, et al, 2011). A mathematical model is a mathematical description of the behavior of some
real-life systems. Such as physical, sociological, economics, etc. (Zill, 2017). Production and supply chain dynamic modeling, present diverse mathematical approaches at distinct scales (Herty and Ringhofer, 2011). Therefore, here we are interested to model the SC as mixing problems (MPs), which is addressed with the interest to present an analogy for the capacity-inventory management, taking into account as decision variables (the inventory level), with production rate and capacity levels as constant parameters (which are provided), and considering the conservation of mass from the mixing problem.

MPs, also known as “compartment analysis” in chemistry consists in creating a mixture of two or more substances to determine some concentration of the resulting mixture (Martinez-Luaces, 2018). MPs mathematical modeling has been used in science and engineering applications such as: Biological systems (Bassingthwaighte, et al, 2012) and biomedical engineering (Rahimian, et al, 2018). The MPs incorporate the rate change in concentration for a solute, which follows the following first-order differential equation:

\[ \frac{dS}{dt} = R_{in} - R_{out} \]  

(1)

Where: S(t) denotes the amount of solute in the tank at time t, R_{in} corresponds to solute input rate and R_{out} is the output rate of solute.

In general, Equation 1 is expressed as:

\[ \frac{dM}{dt} = \phi_i - \phi_o \frac{M}{V} \]  

(2)

For inventory management, a first order differential equation describes the dynamics:

\[ \frac{dI}{dt} = D_r - \phi_o \frac{I}{C} \]  

(3)
Where: Dr corresponds to the demand rate, \( \phi_o \) (production rate); I (inventory level), C (Capacity). In terms of inventory management, to approximate a proper analogy for the MPs system, for two nodes interaction, the first-order differential equation is:

\[
\frac{dI_i}{dt} = \phi_i \frac{I_i}{C_i} - \phi_f \frac{I_f}{C_f}
\]  

\hspace{1cm} (4)

3.2.2 System dynamics

In this section, to propose an LTI system with the general form:

\[\dot{x} = Ax + Bu\] 
\[y = Cx + Du\]  

\hspace{1cm} (5)
\hspace{1cm} (6)

For the set of first-order differential equations, which have been mentioned in Section 2. A general structure of the four echelons serial SC, which is integrated by factory (F), distributors (D), wholesalers (W), and retailers (R), is present in Figure 3.1.

![Figure 3.1: Generic four echelon serial SC realization](image)

The MPs conditions for the SC dynamics presents two cases: (1) when the inflow is equal to outflow (\( \phi_i = \phi_o \)), and (2) when inflow is bigger than outflow (\( \phi_i > \phi_o \)).

Case 1: \( \phi_i = \phi_o \) In case 1, in which there is a balance between the production rate at each node, which implies \( \phi_i = \phi_o \). In general, a single-input single-output (SISO) mathematical model for each node of the form:

\[
\frac{dI_0}{dt} = \phi_i - \phi_o \frac{I_o}{C_o}
\]  

\hspace{1cm} (7)
Expanding Equation (7), for each echelon of the four nodes SC we have:

\[
\begin{align*}
\frac{dl_1}{dt} &= \varphi_i - \varphi_a \frac{l_1}{c_1} \\
\frac{dl_2}{dt} &= \varphi_1 \frac{l_1}{c_1} - \varphi_2 \frac{l_2}{c_2} \\
\frac{dl_3}{dt} &= \varphi_2 \frac{l_2}{c_2} - \varphi_3 \frac{l_3}{c_3} \\
\frac{dl_4}{dt} &= \varphi_3 \frac{l_3}{c_3} - \varphi_4 \frac{l_4}{c_4}
\end{align*}
\]

(8) \hspace{1cm} (9) \hspace{1cm} (10) \hspace{1cm} (11)

Case 2: \(\varphi_i > \varphi_o\). When production rates at each node follows the condition \(\varphi_i > \varphi_o\), capacity control presents the following relation:

\[ R_{out} = q(t) r_{out} \]

(12)

Considering that:

\[ q(t) = \frac{l}{c_o + (\varphi_i - \varphi_o) t} \]

(13)

Analyzing each node, with condition \(\varphi_i > \varphi_o\), we have:

\[
\begin{align*}
\frac{dl_1}{dt} &= \varphi_i - \varphi_d \frac{l_1}{c_1 + (\varphi_i - \varphi_o) t} \\
\frac{dl_2}{dt} &= \varphi_1 \frac{l_1}{c_1} - \varphi_2 \frac{l_2}{c_2 + (\varphi_i - \varphi_2) t} \\
\frac{dl_3}{dt} &= \varphi_2 \frac{l_2}{c_2} - \varphi_3 \frac{l_3}{c_3 + (\varphi_2 - \varphi_3) t} \\
\frac{dl_4}{dt} &= \varphi_3 \frac{l_3}{c_3} - \varphi_4 \frac{l_4}{c_4 + (\varphi_3 - \varphi_4) t}
\end{align*}
\]

(14) \hspace{1cm} (15) \hspace{1cm} (16) \hspace{1cm} (17)

3.2.3 System dynamics with lead time

To incorporate, the lead time, for each echelon of the four nodes SC the set of coupled ordinary differential equations are:

\[
\begin{align*}
\frac{dl_1}{dt} &= \varphi_i - \varphi_1 \frac{l_1}{c_1} (t - \hat{\theta}_1) \\
\frac{dl_2}{dt} &= \varphi_1 \frac{l_1}{c_1} - \varphi_2 \frac{l_2}{c_2} (t - \hat{\theta}_2) \\
\frac{dl_3}{dt} &= \varphi_2 \frac{l_2}{c_2} - \varphi_3 \frac{l_3}{c_3} (t - \hat{\theta}_3) \\
\frac{dl_4}{dt} &= \varphi_3 \frac{l_3}{c_3} - \varphi_4 \frac{l_4}{c_4} (t - \hat{\theta}_4)
\end{align*}
\]

(18) \hspace{1cm} (19) \hspace{1cm} (20) \hspace{1cm} (21)
Considering the fractional lead time as:

\[ \hat{\theta}_j = \frac{\theta_j}{\sum \theta_j} \]  

(22)

Expanding the delayed-state inventory level \( I_j(t - \hat{\theta}_j) \) in terms of a Taylor series:

\[ I_j(t - \hat{\theta}_j) = I_j - \hat{\theta}_j \dot{I}_j \]  

(23)

Applying equation (23) to Eqs. (18-21):

\[
\frac{dI_1}{dt} = \left(1 - \varphi_1 \hat{\theta}_1 \right)^{-1} \left\{ \varphi_1 - \varphi_1 \frac{l_1}{c_1} \right\} \\
\frac{dI_2}{dt} = \left(1 - \varphi_2 \hat{\theta}_2 \right)^{-1} \left\{ \varphi_1 \frac{l_1}{c_1} - \varphi_2 \frac{l_2}{c_2} \right\} \\
\frac{dI_3}{dt} = \left(1 - \varphi_3 \hat{\theta}_3 \right)^{-1} \left\{ \varphi_2 \frac{l_2}{c_2} - \varphi_3 \frac{l_3}{c_3} \right\} \\
\frac{dI_4}{dt} = \left(1 - \varphi_4 \hat{\theta}_4 \right)^{-1} \left\{ \varphi_3 \frac{l_3}{c_3} - \varphi_4 \frac{l_4}{c_4} \right\}
\]  

(24-27)

### 3.2.4 Sensitivity analysis

Considering the level of variation in capacity and production rate for the SC, a sensitivity analysis is present, which incorporates in the state space for the LTI with proper dynamics the SC. For production rates sensitivity analysis: Considering equations (8)-(11), and derivate respect to \( \varphi_1, \varphi_2, \varphi_3 \) and, \( \varphi_4 \), respectively it produces:

\[
\frac{dI_k}{d\varphi_k} = -\frac{l_k}{c_k} \\
\]  

(28)

With \( k = 1,2,3,4 \).

Analyzing the capacity control sensitivity analysis.

\[
\frac{dI_l}{dC_l} = \varphi l \frac{l_l}{c_l^2} \\
\]  

(29)

With \( l = 1,2,3,4 \).
To relate capacity level with production rate, to work with Equations (28) and (29), from this we have:

$$\frac{\partial I_1}{\partial C_1} = -\frac{\varphi_1}{C_1} \frac{\partial I_1}{\partial \varphi_1}$$  \hspace{1cm} (30)

After some mathematical analysis and integrating once, Equation (30) can be expressed as:

$$C_1 \frac{\partial I_1}{\partial C_1} = -\varphi_1 \frac{\partial I_1}{\partial \varphi_1}$$  \hspace{1cm} (31)

Equation (31), relates capacity level with production rate, based on this:

$$\frac{dC_1}{C_1} = -\frac{d\varphi_1}{\varphi_1}$$  \hspace{1cm} (32)

Integrating equation (32), and solving we have: $ln(C_1 \varphi_a) = K$. Generalizing for each node, and considering the LTI system:

$$C_1 = \frac{e^K}{\varphi_1}$$  \hspace{1cm} (33)

Assuming $K=0$, Equation (33) reduces to: $C_1 = \frac{1}{\varphi_1}$.

3.2.5 Stability analysis

Theorem 1. An equilibrium point $x^*$ of the dynamical system, in equations (8)-(11), is stable if all the eigenvalues of $J^*$, the Jacobian evaluated at $x^*$, have negative real parts such as:

$$\lambda_1 < -\frac{\varphi_1}{C_1}, \lambda_2 < -\frac{\varphi_2}{C_2}, \lambda_3 < -\frac{\varphi_3}{C_3}, \lambda_4 < -\frac{\varphi_4}{C_4}.$$  

To prove theorem 1, the Jacobian for the dynamical system is:
\[
J = \begin{bmatrix}
-\varphi_1 & 0 & 0 & 0 \\
\varphi_1 & -\varphi_2 & 0 & 0 \\
0 & \varphi_2 & -\varphi_3 & 0 \\
0 & 0 & \varphi_3 & -\varphi_4 \\
\end{bmatrix}
\] (34)

Applying the determinant to:

\[
|\lambda I - J| = 0
\] (35)

From (35) the eigenvalues are:

\[
\lambda_1 < -\frac{\varphi_1}{c_1}, \lambda_2 < -\frac{\varphi_2}{c_2}, \lambda_3 < -\frac{\varphi_3}{c_3}, \lambda_4 < -\frac{\varphi_4}{c_4}
\] (36)

Considering that production rates and capacities are positive numbers, we can conclude that all the eigenvalues of the Jacobian have negative real parts.

3.3 RESULTS

3.3.1 Optimal control

An energy-based optimal control is solved for:

\[
\min J = \frac{1}{2} \int_0^T u^2 d\tau
\] (37)

s.t.

\[
\frac{di_1}{dt} = u - \varphi_1 \frac{i_1}{c_1} \\
\frac{di_2}{dt} = \varphi_1 \frac{i_1}{c_1} - \varphi_2 \frac{i_2}{c_2} \\
\frac{di_3}{dt} = \varphi_2 \frac{i_2}{c_2} - \varphi_3 \frac{i_3}{c_3} \\
\frac{di_4}{dt} = \varphi_3 \frac{i_3}{c_3} - \varphi_4 \frac{i_4}{c_4}
\] (38)

Applying the Pontryagin maximum principle (PMP), we achieve the following associated Hamiltonian for the problem:
\[ H(l_i, u, \lambda_i, t) = \frac{1}{2} u^2 + \lambda_1 (u - \varphi_1 \frac{l_1}{c_1}) + \lambda_2 (\varphi_1 \frac{l_1}{c_1} - \varphi_2 \frac{l_2}{c_2}) + \lambda_3 (\varphi_2 \frac{l_2}{c_2} - \varphi_3 \frac{l_3}{c_3}) + \\
\lambda_4 (\varphi_3 \frac{l_3}{c_3} - \varphi_4 \frac{l_4}{c_4}) \tag{39} \]

In order to develop the PMP, the conditions for co-states are:

\[ \dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \tag{40} \]

And also, the condition:

\[ \frac{\partial H}{\partial u} = 0 \tag{41} \]

By the application of equations (40) and (41), to the Hamiltonian presented in equation (39), we proceed to solve the energy-based optimal control, for the serial supply chain.

### 3.3.2 Case study simulations

A serial supply chain has been modeled via the application of an energy-based optimal control (as performance index), with demand rate as the optimized variable (input for the system).

![Factory and Distributors inventory level](image)

**Figure 3.2: Factory and Distributors inventory level**

In Figure 3.2, the inventory level for factory and distributors is present, for a time horizon of 20 arbitrary units (au). It can be seen that factory inventory starts from a higher inventory level compared to distributors.
In Figure 3.3, the analysis for wholesalers and retailers inventory levels is present, considering a higher inventory level for wholesalers compared to retailers, in a time horizon of 20 au. In Figure 3.4, a demand rate graph for a time horizon of 20 au, is present, considering the time evolves to the final time horizon, the demand of the serial supply chain tends to zero.

Figure 3.3: Wholesalers and Retailers inventory level

Figure 3.4: Demand rate
The relation inventory level at each stage of the SC versus demand presents a negative trend with stable demand as is proposed in (Fogarty, 1991). The demand profile takes into account that is an independent demand, which implies that demand can be affected by trends and seasonal patterns. For an independent demand the product usage pattern is uniform and gradual.

A serial supply chain is present, which incorporates the mathematical modeling via the mixing problem, for inventory management in the factory, distributors, wholesalers, and retailers. Also, sensitivity and stability analysis are present. Finally, an energy-based optimal control is developed for the system, with proper simulations. In future work, our goal is to work the mixing problem mathematical modeling and model predictive control for dynamic divergent supply chains.
Chapter 4: Mathematical modeling and stability analysis of closed-loop supply chains

Closed-loop supply chains (CLSC) integrate forward and reverse logistics, towards the main interest to process finished goods in a re-manufacturing analysis, to achieve sustainable solutions for enterprises. This chapter proposes a capacitated control analysis in inventory management for a class of CLSC. To present two mathematical models for CLSC exploring the dynamic nature of the system via compartmental analysis. Proper stability analysis for each CLSC system is present. Considering that decision-makers require mathematical models maximize profits and minimize costs along the CLSC. This work aim is to approximate a capacity control for the production levels of CLSC applying the compartmental analysis with a set of coupled ordinary differential equations. The general structure of the proposed mathematical modeling considers production level, throughput, and capacity for each node along of each CLSC system.

4.1 INTRODUCTION

The network of forward and reverse logistics synergetic analysis integrates, closed-loop supply chains (CLSC) (Govidan, et al, 2015), for process in which remanufactures finished goods into the manufacturers in a closed-loop context. Most of the major work of product recovery is developed by the manufacturer (He, et al, 2019). Recalling that, in the forward supply chain, raw materials are manufactured into new products, while in the reverse supply chain, used products are remanufactured into as new products, also called: remanufactured products (Yuan and Gao, 2010). Industry practitioners, considers in general processes between 2 and 5% of scrap, for a proper sustainable process. According to the Remanufacturing Institute, manufacturers of remanufactured products save 85% of energy use, 86% of water use, and 85% of material use compared to manufacturing a new product (Kumar, et al, 2017). This research work proposes a general framework for a CLSC, which can approximate solutions for non-perishable products.
Decision-makers require mathematical models that maximize profits and minimize costs along the CLSC. Based on this, to approximate a capacity control for inventory management of CLSC applying the problem of mixtures in fluids with coupled ordinary differential equations.

A mathematical model is a mathematical description of the behavior of some real-life system or phenomenon. Such as physical, sociological, economics, etc. (Zill, 2017). The general structure of the proposed mathematical modeling considers throughput, inventory level, production level, and capacity for each node along the CLSC. Compartmental analysis is present in this research work, to develop proper mathematical modeling for the two CLSC systems. Table 4.1, presents a literature review of mathematical decision-making approaches for CLSC.

<table>
<thead>
<tr>
<th>Article</th>
<th>Paper scope</th>
<th>Network structure</th>
<th>Mathematical decision making approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zhao, et al, 2019</td>
<td>A game theory is present for benefits allocation, in the CLSC management scheme.</td>
<td>Closed-loop</td>
<td>Game theory</td>
</tr>
<tr>
<td>Wang and Shai, 2019</td>
<td>The practical problem describes a MNIP model with multi-objective and multi-constraint, for spare parts optimization, in a multi-period closed-loop logistics.</td>
<td>Closed-loop</td>
<td>Multiobjective optimization</td>
</tr>
<tr>
<td>Su and Sun, 2019</td>
<td>This paper investigates a CLSC network that includes forward-reverse logistics and the uncertainty of demand.</td>
<td>Closed-loop</td>
<td>Nondominated sorting genetic algorithm II</td>
</tr>
<tr>
<td>Chen, et al, 2019</td>
<td>In this paper, the CLSC models consider the case that the manufacturer can improve the product quality.</td>
<td>Forward/Reverse</td>
<td>Game theory</td>
</tr>
<tr>
<td>Taleizadeh, et al, 2018</td>
<td>This paper explores a dual-channel closed-loop supply chain (CLSC) system consisting of a manufacturer and a retailer.</td>
<td>Closed-loop</td>
<td>Game theory</td>
</tr>
<tr>
<td>Zhang, et al, 2017</td>
<td>An uncertain CLSC system with hybrid recycling channels, applies fuzzy control to restrain the bullwhip effect.</td>
<td>Closed-loop</td>
<td>Fuzzy control</td>
</tr>
<tr>
<td>Zhang and Zao, 2015</td>
<td>This model consists of a material recovery subsystem with a third-party reverse logistics provider, and a manufacturer remanufacturing recovery subsystem, as well as a cost-based switching signal vector</td>
<td>Closed-loop</td>
<td>Fuzzy robust control</td>
</tr>
<tr>
<td>Hosoda and Disney, 2018</td>
<td>A dynamic CLSC with first-order auto-regressive demand and return processes is analyzed, with the assumption that two processes are cross-correlated.</td>
<td>Forward/Reverse</td>
<td>Time series</td>
</tr>
<tr>
<td>De Giovanni, 2018</td>
<td>In the battery sector, retailers can offer a joint maximization incentive to manufacturers to push</td>
<td>Closed-loop</td>
<td>Optimal control</td>
</tr>
</tbody>
</table>
up green activity program efforts and use the return rate as a marketing lever.

Kumar, et al, 2017  
This paper presents green SC and of reverse logistics programs in manufacturing organizations.  
Forward/ Reverse  
Evolutionary algorithms

Banakis, et al, 2017  
This paper presents a CLSC in agri-food where the medium used mushrooms are recovered, and used as raw material for the production of new growing medium.  
Closed loop  
Multi-objective optimization

Jindal and Sangwan, 2014  
In this paper, a multi-product, multi-facility capacitated CLSC framework is proposed in an uncertain environment.  
Closed loop  
Mixed integer linear programming

Ozceylan and Paksoy, 2013  
In this paper, a new mixed integer mathematical model for a CLSC network, which includes both forward and reverse flows with multi-periods and multi-parts is proposed.  
Closed loop  
Mixed integer linear programming

Huang, et al, 2009  
This paper focus on modeling with uncertainties, in the CLSC for an effective, and robust control the dynamic system.  
Closed loop  
Robust control

Hassanzadeh Amin and Zhang, 2013  
A general CLSC network is configured which consists of multiple customers, parts, products, suppliers, remanufacturing subcontractors, and refurbishing sites.  
Closed loop  
Mixed-integer nonlinear programming

Compartmental analysis has been applied in several engineering and science areas, Table 4.2, summarizes a review of papers within areas of application.

<table>
<thead>
<tr>
<th>Article</th>
<th>Paper Scope</th>
<th>Area of application</th>
</tr>
</thead>
<tbody>
<tr>
<td>Martínez-Luaces, 2018</td>
<td>In this paper, mixing problems lead to linear ordinary differential equation (ODE) systems, and the corresponding qualitative analysis about the ODE solutions and their stability or asymptotical stability are included.</td>
<td>Chemical engineering</td>
</tr>
<tr>
<td>Bassingthwaighte, et al, 2012</td>
<td>This paper presents, in pharmacokinetics, compartmental models for describing the curves of concentration-time of a drug following administration, to guide a defining dosage.</td>
<td>Biology systems</td>
</tr>
<tr>
<td>Henley, et al, 2015</td>
<td>This paper analyzes the relation between a compartmental circuit and a data-based, input-output model of dynamic cerebral autoregulation, and dynamic CO2-vasomotor reactivity.</td>
<td>Biomedical engineering</td>
</tr>
</tbody>
</table>
Rahimian, et al, 2018  A modified Pinsky–Rinzel pyramidal model is proposed by replacing its complex nonlinear equations with piecewise linear approximation, via compartmental modeling.  Biomedical engineering

Griffiths, et al, 2000  This paper examines epidemic models for HTV/AIDS, and show that the equations with a proper linearization, achieves analytic solutions.  Medicine

Berglind, et al, 2012  In this paper, a compartmental model is examined with a stochastic differential equation to model concentrations of free-leucine in the plasma.  Medicine

4.2 MATHEMATICAL MODELING

4.2.1 Problem description and notation: Compartmental analysis

Compartmental analysis incorporates the rate change in concentration for a solute, which follows the following first-order differential equation:

\[
\frac{ds}{dt} = \varphi_{\text{in}} - \varphi_{\text{out}}
\]  (1)

Where:

- \(S(t)\) denotes the amount of solute in the tank at time \(t\).
- \(\varphi_{\text{in}}\) input rate of solute
- \(\varphi_{\text{out}}\) output rate of solute

![Figure 4.1: Tank level representation for the compartmental analysis problem](image)

In general, equation 1 can expressed from Figure 4.1, as:
\[ \frac{dM}{dt} = \phi_i - \phi_o \frac{M}{V} \]  

(2)

In terms of inventory management, to approximate a proper analogy for the mixture problem system, Figure 4.2 analyzes a single node of a supply chain echelon such as:

![System dynamics representation for the capacitated control problem](image)

Figure 4.2: System dynamics representation for the capacitated control problem

A first order differential equation describes the dynamics of Figure 4.2:

\[ \frac{dI}{dt} = D_r - \phi_o \frac{I}{C} \]  

(3)

Where: Dr corresponds to the demand rate; \( \phi_o \) (throughput); I (inventory level); C( Capacity).

Figure 4.3 presents two-node interaction, which is described by the first-order differential equation, in equation 4 such as:

\[ \frac{dl_j}{dt} = \phi_i \frac{l_i}{c_j} - \phi_j \frac{l_j}{c_j} \]  

(4)

![Two echelon representation for the capacitated control problem](image)

Figure 4.3: Two echelon representation for the capacitated control problem

Considering that a differential equation approximates the continuum, the main interest of this project is to represent in a state-space formulation the dynamics for the capacity control for a CLSC. Recalling, a state-space formulation is the minimal set of states, which describes the behavior of a dynamical system.
4.2.2 CLSC mathematical modeling: System 1

A proposed CLSC network for non-perishable products with the general structure:

Figure 4.4: General structure for the CLSC network

The system dynamics for the general CLSC network, from Figure 4, has the following state-space equations:

Supplier tier:

\[ \frac{dI_s}{dt} = \frac{D_r}{c_s} - \frac{\varphi_{sm}}{c_s} I_s \]  \hspace{1cm} (5)

Manufacturer tier:

\[ \frac{dQ_m}{dt} = \frac{\varphi_{sm}}{c_m} I_s + \frac{\varphi_{rm}}{c_m} I_r - \frac{\varphi_{md}}{c_m} Q_m \]  \hspace{1cm} (6)

Distributor tier:

\[ \frac{dI_d}{dt} = \frac{\varphi_{md}}{c_d} Q_m - \frac{\varphi_{dc}}{c_d} I_d \]  \hspace{1cm} (7)

Consumer tier:
\[
\frac{dI_c}{dt} = \frac{\phi_{dc}}{c_c} I_d - \frac{\phi_{cc}}{c_c} I_c - \frac{y}{c_c} I_c
\]  
(8)

Collector tier:

\[
\frac{dI_{co}}{dt} = \frac{\phi_{cc}}{c_{co}} I_c - \frac{\phi_{co}}{c_{co}} I_{co}
\]  
(9)

Recycler tier:

\[
\frac{dI_R}{dt} = \frac{\phi_{coR}}{c_R} I_{co} - \frac{\phi_{rm}}{c_R} I_R
\]  
(10)

4.2.3 Four-tier CLSC mathematical modeling: System 2

Considering the dynamics of the forward CLSC, which consists of four echelons such as Suppliers, manufacturers, distributors, and consumers. Based on Figure 4.5, the set of ordinary differential equations, and applying equations (3) and (4), the dynamics that describes the forward CSLC are:

For echelon 1:

Supplier 1

\[
\frac{dI_1}{dt} = d_a - (\phi_{11} + \phi_{12}) \frac{I_1}{c_1}
\]  
(11)

Supplier 2

\[
\frac{dI_2}{dt} = d_b - (\phi_{13} + \phi_{14}) \frac{I_2}{c_2}
\]  
(12)

For echelon 2:

Manufacturer 1

\[
\frac{dQ_1}{dt} = \phi_{11} \frac{Q_1}{C_3} - (\phi_{21} + \phi_{22}) \frac{Q_1}{C_3}
\]  
(13)

Manufacturer 2
\[
\frac{dQ_2}{dt} = (\phi_{12}\frac{I_3}{C_4} + \phi_{13}\frac{I_2}{C_4}) - (\phi_{23} + \phi_{24})\frac{Q_2}{C_4} \quad (14)
\]

Manufacturer 3

\[
\frac{dQ_3}{dt} = \phi_{14}\frac{I_2}{C_5} - (\phi_{25} + \phi_{26})\frac{Q_3}{C_5} \quad (15)
\]

For echelon 3:

Distributor 1

\[
\frac{dI_3}{dt} = \phi_{31}\frac{Q_1}{C_6} - \phi_{34}\frac{I_3}{C_6} \quad (16)
\]

Distributor 2

\[
\frac{dI_4}{dt} = \phi_{22}\frac{Q_1}{C_7} + \phi_{23}\frac{Q_2}{C_7} - \phi_{35}\frac{I_4}{C_7} \quad (17)
\]

Distributor 3

\[
\frac{dI_5}{dt} = \phi_{24}\frac{Q_2}{C_8} + \phi_{25}\frac{Q_3}{C_8} - \phi_{36}\frac{I_5}{C_8} \quad (18)
\]

Distributor 4

\[
\frac{dI_6}{dt} = \phi_{26}\frac{Q_3}{C_9} - \phi_{37}\frac{I_6}{C_9} \quad (19)
\]

For echelon 4:

Consumer 1

\[
\frac{dI_7}{dt} = \phi_{34}\frac{I_3}{C_6} + \phi_{35}\frac{I_4}{C_7} - y_1I_7 \quad (20)
\]

Consumer 2

\[
\frac{dI_8}{dt} = \phi_{36}\frac{I_5}{C_8} + \phi_{37}\frac{I_6}{C_9} - y_2I_8 \quad (21)
\]
Figure 4.5: Forward CLSC dynamics
The reverse CLSC is present in Figure 4.6, with the following dynamics:

Reverse dynamics echelon 1:

Consumer 1

\[
\frac{dI_9}{dt} = -\frac{(\phi_{R1} + \phi_{R2})}{cc_1} I_9 - y_1 I_9
\]  

(22)
Consumer 2

\[
\frac{dI_{10}}{dt} = -\frac{(\varphi_{R3} + \varphi_{R4})}{C_{C2}} I_{10} - y_2 I_{10}
\]  

(23)

Reverse dynamics echelon 2:

Collector 1

\[
\frac{dI_{11}}{dt} = \frac{\varphi_{R1}}{C_{C01}} I_9 - \frac{\varphi_{R5}}{C_{C01}} I_{11}
\]  

(24)

Collector 2

\[
\frac{dI_{12}}{dt} = \frac{\varphi_{R2}}{C_{C02}} I_9 + \frac{\varphi_{R6}}{C_{C02}} I_{10} - \frac{(\varphi_{R6} + \varphi_{R7})}{C_{C02}} I_{11}
\]  

(25)

Collector 3

\[
\frac{dI_{13}}{dt} = \frac{\varphi_{R4}}{C_{C03}} I_{10} - \frac{\varphi_{R8}}{C_{C03}} I_{13}
\]  

(26)

Reverse dynamics echelon 3:

Recycler 1

\[
\frac{dI_{14}}{dt} = \frac{\varphi_{R5}}{C_{R1}} I_{11} + \frac{\varphi_{R6}}{C_{R1}} I_{12} - \frac{(\varphi_{R9} + \varphi_{R10})}{C_{R1}} I_{14}
\]  

(27)

Recycler 2

\[
\frac{dI_{15}}{dt} = \frac{\varphi_{R7}}{C_{R2}} I_{12} + \frac{\varphi_{R8}}{C_{R2}} I_{13} - \frac{(\varphi_{R11} + \varphi_{R12})}{C_{R2}} I_{15}
\]  

(28)

Reverse dynamics echelon 4:

Reverse Manufacturer 1

\[
\frac{dQ_4}{dt} = \frac{\varphi_{R1}}{C_{M1}} I_1 + \frac{\varphi_{R9}}{C_{M1}} I_{14} - \frac{\varphi_{TR1}}{C_{M1}} Q_4
\]  

(29)

Reverse Manufacturer 2
\[
\frac{dQ_5}{dt} = \frac{\varphi_{12}}{CM_2} I_1 + \frac{\varphi_{13}}{CM_2} I_2 + \frac{\varphi_{R16}}{CM_2} I_{14} + \frac{\varphi_{R11}}{CM_2} I_{15} - \frac{\varphi_{TR2}}{CM_2} Q_5 \quad (30)
\]

Reverse Manufacturer 3

\[
\frac{dQ_6}{dt} = \frac{\varphi_{14}}{CM_3} I_2 + \frac{\varphi_{R12}}{CM_3} I_{15} - \frac{\varphi_{TR3}}{CM_3} Q_6 \quad (31)
\]

### 4.3 STABILITY ANALYSIS

#### 4.3.1 System 1: Stability analysis

**Theorem 1.** For the dynamical system in equations (5)-(10), an equilibrium point \(x^*\) is stable if all the eigenvalues of \(J^*\), the Jacobian evaluated at \(x^*\), have negative real parts such as:

\[
\lambda_1 < -\frac{\varphi_{sm}}{c_s}, \quad \lambda_2 < -\frac{\varphi_{md}}{c_m}, \quad \lambda_3 < -\frac{\varphi_{dc}}{c_d}, \quad \lambda_4 < -\frac{\varphi_{CC}}{c_c}, \quad \lambda_5 < -\frac{\varphi_{CoR}}{CC_o}, \quad \lambda_6 < -\frac{\varphi_{rm}}{c_R}
\]

**Proof.** Developing the Jacobian for the dynamical system is:

\[
J = \begin{bmatrix}
-\frac{\varphi_{sm}}{c_s} & 0 & 0 & 0 & 0 & 0 \\
\frac{\varphi_{sm}}{c_s} & -\frac{\varphi_{md}}{c_m} & 0 & 0 & 0 & 0 \\
0 & \frac{\varphi_{md}}{c_m} & -\frac{\varphi_{dc}}{c_d} & 0 & 0 & 0 \\
0 & 0 & \frac{\varphi_{CC}}{c_c} & \frac{-\varphi_{CCo+\varphi}}{c_c} & 0 & 0 \\
0 & 0 & 0 & \frac{\varphi_{CoR}}{CC_o} & -\frac{\varphi_{CoR}}{CC_o} & 0 \\
0 & 0 & 0 & 0 & \frac{\varphi_{CoR}}{c_R} & -\frac{\varphi_{CoR}}{c_R}
\end{bmatrix}
\]

Calculating the eigenvalues for: \(|\lambda I - J| = 0\), we have:

\[
\lambda_1 < -\frac{\varphi_{sm}}{c_s}, \quad \lambda_2 < -\frac{\varphi_{md}}{c_m}, \quad \lambda_3 < -\frac{\varphi_{dc}}{c_d}, \quad \lambda_4 < -\frac{\varphi_{CC}}{c_c}, \quad \lambda_5 < -\frac{\varphi_{CoR}}{CC_o}, \quad \lambda_6 < -\frac{\varphi_{rm}}{c_R}
\]

We can conclude that all the eigenvalues of the Jacobian have negative real parts, by considering that production rates and capacities are positive numbers.
4.3.2 System 2: Stability analysis

Theorem 2. If the forward CLSC dynamical system with equations (11)-(21) is stable, and the reverse CLSC dynamical system with equations (22)-(31) is stable, therefore the overall CLSC system is stable.

Proof. By calculating the eigenvalues for:

\[ |\lambda_1 - J| = 0 \]

for the forward CLSC dynamical system such as:

\[
\begin{align*}
\lambda_{f1} &< -\frac{(\phi_{11}+\phi_{12})}{c_1}, & \lambda_{f2} &< -\frac{(\phi_{13}+\phi_{14})}{c_2}, & \lambda_{f3} &< -\frac{(\phi_{21}+\phi_{22})}{c_3}, & \lambda_{f4} &< -\frac{(\phi_{23}+\phi_{24})}{c_4}, & \lambda_{f5} &< -\frac{(\phi_{25}+\phi_{26})}{c_5}, \\
\lambda_{f6} &< -\frac{\phi_{34}}{c_6}, & \lambda_{f7} &< -\frac{\phi_{35}}{c_7}, & \lambda_{f8} &< -\frac{\phi_{36}}{c_8}, & \lambda_{f9} &< -\frac{\phi_{37}}{c_9}, & \lambda_{f10} &< -y_1, & \lambda_{f11} &< -y_2
\end{align*}
\]

Also, calculating the eigenvalues for the reverse CLSC dynamical system we have:

\[
\begin{align*}
\lambda_{r1} &< -\frac{(\phi_{R1}+\phi_{R2})}{cc_1}, & \lambda_{r2} &< -\frac{(\phi_{R3}+\phi_{R4})}{cc_2}, & \lambda_{r3} &< -\frac{\phi_{R5}}{cc_3}, & \lambda_{r4} &< -\frac{\phi_{R6}+\phi_{R7}}{cc_4}, & \lambda_{r5} &< -\frac{\phi_{R8}}{cc_5}, \\
\lambda_{r6} &< -\frac{(\phi_{R9}+\phi_{R10})}{cr_1}, & \lambda_{r7} &< -\frac{(\phi_{R11}+\phi_{R12})}{cr_2}, & \lambda_{r8} &< -\frac{\phi_{TR1}}{cm_1}, & \lambda_{r9} &< -\frac{\phi_{TR2}}{cm_2}, & \lambda_{r10} &< -\frac{\phi_{TR3}}{cm_3}
\end{align*}
\]

By considering that production rates and capacities are positive numbers, we can conclude that all the eigenvalues of the Jacobians have negative real parts, therefore the overall CLSC is stable.
Chapter 5: Present value Hamiltonian-Optimal control for a production-inventory system

Enabling industrial production systems requires competitive and adaptive solutions to ensure profitability in markets. In PI systems, decision-makers require to achieve and approximate, online decisions. Proper manner, resilient, and sustainable PI systems present solutions to decrease operations management costs such as production, logistics, and quality and enhances real-time or near real-time decision making. Based on this, dynamic approaches for production-inventory systems are valuable to maximize profits and minimize costs over the smart manufacturing system. This chapter emphasizes the characteristics that make a production system resilient and sustainable from the systems modeling and control theory perspectives. Optimal control presents a tradeoff between optimality and stability conditions of the proposed dynamical system.

This chapter presents an optimal control approach for a class of resilient and sustainable production-inventory system (PI). Modeling, analysis, and control for a PI system are present. For systems modeling purposes, a resilient mixed capacity/third order PI is analyzed, considering an emergy sustainability index. Resiliency factor and sustainability index are features enhance proper PI characteristics for the system. Optimality, robustness, and stability are the top-level characteristics which encompass a suitable control. A proper Lyapunov stability analysis is present over a third-order nonlinear dynamical system. Optimal control formulation is conducted via Pontryagin maximum principle, with a present value-Hamiltonian approach (PVH).
5.1 INTRODUCTION

Nowadays, enabling industrial production systems requires competitive and adaptive solutions to ensure profitability in markets. Inventory management which involves the planning and control of inventory, typically, yields significant cost savings (Chen and Chen, 2019). Competitive companies require to produce high-quality products and respond to fast changes in demands, within a low-cost production profile (Boukas, 2006). In PI systems, decision-makers require to achieve and approximate, online decisions. Proper manner, PI integrates solutions to decrease operations management costs such as production, logistics, and quality and enhances real-time or near real-time decision making. Also, the production processes assume, all the products are perfect and the demand rate is constant (Li, et al, 2017).

Based on this, dynamic approaches for production-inventory systems are valuable to maximize profits and minimize costs over the manufacturing system. The modeling and analysis of dynamic production-inventory control systems in manufacturing enterprises is essential, for operations and planning control (Bijulal, et al, 2011). This paper presents PI from the systems modeling and control theory perspectives. Control Theory is the area of engineering, which analyze the time response of physical systems, by applying differential equations (Ortega and Lin, 2004). In order to solve the optimal production-inventory control problem, optimal control theory is commonly applied to production-inventory systems (Wang and Chan, 2015). Optimal control presents a tradeoff between optimality and stability conditions of the proposed dynamical system. Optimal control applications have been proved in economics, management science, and industry (Khmelnitsky and Gerchak, 2002 From control theory approaches, stochastic and deterministic models have been proposed to analyze the production-inventory systems. In (Tan, 2002), (Zhao and Melamed, 2007), (Ioannidis, et al, 2008), (Kogan, et al, 2010), (Kogan, et al, 2017), (Wang,
(et al, 2018), (Li, et al, 2018), presents PI systems from a stochastic control perspective, while in (Nandola and Rivera, 2013), (Ignaciuk, 2015), (Doganis, et al, 2008) which present deterministic control perspective. For mathematical modeling purposes, production-inventory systems are deterministic (Tadj, et al, 2006) and stochastic (Khoury, 2016), with linear and nonlinear dynamic performance. Capacity level, inventory level, production rate, production level, etc. are the main parameters to be analyzed in production-inventory systems. To present a mathematical model, for a class of third-order nonlinear dynamical system, which approximates a PI system behavior applying fluid and classical mechanics analogies.

This research chapter:
1) Develops a mathematical model exploring the dynamic nature of resilient and sustainable production-inventory systems.
2) Presents the relation between optimality, robustness, and stability for PI systems.
3) Establishes the main characteristics of production-inventory system from a systems and control theory perspective.
4) Presents the design process of a third-order nonlinear dynamical system which captures a resilience factor and an emergy sustainability index.
5) Demonstrates stability analysis via Lyapunov direct method for the third-order nonlinear system.
6) Develops a proper linearization of the dynamical system to proceed with a feedback control system design.
7) Introduces the PVH-optimal control approach for energy-based formulation via the Pontryagin maximum principle.

The rest of chapter is developed as follows. First, systems theory and then, control features for PI systems are reviewed and mathematical modeling on PI is discussed in Section 5.2. Stability
analysis is presented in Section 5.3. Optimal control formulation is provided in Section 5.4 while case study results are developed and present in Section 5.5.

5.2 MATHEMATICAL MODELING

5.2.1 Optimality, robustness, and stability

In systems and control theory, optimality, robustness, and stability are the top-level characteristics which encompass a suitable control system. Control systems for a proper operation in a process, require a certain level of optimality, robustness, and stability conditions. PI systems require real-time control approaches to make real-time or near real-time online decisions. The role of mathematical modeling is to theoretically approximate the operation of factories, production-inventory systems, and supply chains. A good approach is to justify proper analogies with mixed variables from fluid mechanics and electromechanical systems domain. Therefore, in this paper, to define those characteristics that are mandatory for automation science practitioners to achieve a PI system:

Definition 1. Principle of Optimality

An optimal policy has the property that independently of initial state and decisions, the subsequent decisions must constitute the optimal policy regard to the state resulting from the first decision (Bellman, 1957).

Definition 2. Robustness

In control systems, robustness is defined as the ability to maintain the performance characteristics under the presence of system variations or perturbations (Stengel, and Ray, 1991).

Theorem 1. Lyapunov stability

The equilibrium state xe=0 of the system \( \dot{x}(t) = f(x, t) \), is asymptotically stable if a Lyapunov function \( V(x, t) \) can be found such that \( \dot{V}(x, t) < 0 \) for all \( x \neq 0 \) and \( t \geq t_0 \).
5.2.2 Systems theory and control for PI

Production-inventory systems present a complex dynamical, to stability and dynamic performance (Al-Khazraji, et al, 2018). For modeling, analysis, and control of PI systems, several approaches have been conducted: classical control theory (transfer function) and modern control (state-space). In Illustration 5.1, our interest is to graphically, demonstrate the relation of a PI system as a subset of SM and show that these terms are related to optimality, robustness, and stability properties from a systems and control theory perspective. Based on definition 1, which is related to the principle of optimality and definition 2, which is related to robustness, and Theorem 1 (Lyapunov stability).

![Illustration 5.1: Optimality, robustness, and stability for PI systems.](image)

As Illustration 5.1 shows, PI systems are at the intersection of these three concepts mentioned above. In this work, a PI system is mathematically modeled by a third-order nonlinear system of ordinary equations. This method is also known as a model-based systems theory and control theory-oriented approach in the context of optimal control.

This chapter defines nine characteristics that from a systems and control theory perspectives are necessary for PI systems such as trackability, observability, controllability, identifiability, dissipativity, passivity, entropy, sensitivity, and profitability (refer to Table 5.1).
These features are required in different levels of application within the context of automation science and control systems, while the optimality, robustness, and stability dimensions for PI systems are presented. In Table 5.1, O refers to Optimality, S to Stability and R to Robustness.

**Table 5.1: Systems theory and control feature for PI systems**

<table>
<thead>
<tr>
<th>Control feature</th>
<th>Definition</th>
<th>O</th>
<th>S</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trackability</td>
<td>Trackability deals with the dynamical plant behavior from the initial to the final moment. Refers to the plant property, it is independent of the controller and control. (Gruyitch, 2018)</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Observability</td>
<td>imply that internal state variables of the system can be externally measured.</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Controllability</td>
<td>Ability of a controller to arbitrarily alter the functionality of the system plant.</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Identifiability</td>
<td>Refers that for a class of models, for the identification method of the system, a unique solution is achieved.</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Dissipativity</td>
<td>A system is dissipative if the increase in its energy, during a time interval, is less than that supply rate energy. (Bao, and Lee, 2007)</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passivity</td>
<td>In passivity, if the difference between stored energy and supplied energy is positive the feedback system will be stable in an input-output sense (Loria, and H. Nijmeijer, 2004)</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entropy</td>
<td>Refer to the measure of the cost of performing different performance criteria, to calculate the maximum work that can be extracted from the repeated operation of feedback-controlled systems. (Cao, and M. Feito, 2009)</td>
<td>X</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>Relates to a measure of the system’s characteristics dependence of a particular element. (Shinners, 1998)</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Profitability</td>
<td>Provide economical solutions during a time horizon, which captures tradeoffs between revenue and costs.</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>
5.2.3 Systems modeling and notation: Equations of motion

Considering the dynamic nature of PI systems, it becomes of special interest to approximate online solutions for decision-makers to generate tradeoffs between the maximization of profits and the minimization of cost the supply chain network. The main contribution of this paper is to present a mathematical model captures the features PI systems via optimal control theory applying the Pontryagin maximum principle.

Based on this, a mathematical model for the mixed capacity-cyber physical production-inventory system involves a third-order state space formulation, in which a vertical motion for inventory level is considered, coupled by a resiliency coefficient (spring) and an emergy sustainability index (damper), and horizontal production level flow is presented. Both, the resiliency coefficient and the emergy sustainability index are dimensionless and real positive parameters. The resiliency coefficient is related to the ability of the system to deal with changes in demand, inventory, and production levels. While the emergy sustainability index measures the contribution of the process to the economy of environmental loading; this index dissipates the energy of the system through the damper.

Figure 5.1 presents a PI system adapted from mathematical models previously published in (Davizon, et al, 2014), (Davizon, et al, 2015) for the spring-mass-damper dynamical system, and from papers which analyze production-inventory systems via fluid analogies (Schwartz, et al, 2006) with applications in Semiconductor Manufacturing in (Wang, et al, 2007), (Schwartz, et al, 2009) and demand modeling oriented approaches for inventory management presented in (Schwartz, and Rivera, 2010), (Schwartz, and Rivera, 2014).
The third order mixed PI systems refers to two subsystems: (subsystem 1) a tank-level system and (subsystem 2) a mass-spring-damper, which coupled represents a novel PI systems realization. This PI system approximate a manufacturing system, in continuous time.

![Third-order mixed production inventory system](image)

**Figure 5.1: Third-order mixed production inventory system**

The tank inventory level is mathematically considered as a mass variable-mechanical oscillator, subject to an external force, and such as it is presented in its canonical form in Equation (1).

\[
m(t)\ddot{h} + \beta \dot{h} + kh = d(t)
\]  

(1)

Systems modeling parameters are presented in Table 5.2. To propose the tank-spring-damper dynamical system, as an analogy for a mixed capacity/production inventory system.
Table 5.2: System modeling parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
<th>Type of parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d(t)$</td>
<td>Demand</td>
<td>Manipulated variable (input)</td>
</tr>
<tr>
<td>$h(t)$</td>
<td>Inventory level</td>
<td>Controlled variable (output)</td>
</tr>
<tr>
<td>$q_0(t)$</td>
<td>Production level</td>
<td>Controlled variable (output)</td>
</tr>
<tr>
<td>$C$</td>
<td>Capacity</td>
<td>Fixed parameter</td>
</tr>
<tr>
<td>$P$</td>
<td>Production system fluency (density flow)</td>
<td>Fixed parameter</td>
</tr>
<tr>
<td>$K$</td>
<td>Resiliency factor</td>
<td>Fixed parameter</td>
</tr>
<tr>
<td>$B$</td>
<td>Emergy sustainability index</td>
<td>Fixed parameter</td>
</tr>
<tr>
<td>$V$</td>
<td>Volume</td>
<td>Fixed parameter</td>
</tr>
</tbody>
</table>

The horizontal production level flow is described by the following first-order equation:

$$
\dot{q}_0 = \frac{1}{ac} (d(t) - q_0)
$$

(2)

Assuming that laminar flow is present, and by the relation in which: $\rho = m(t)/V$, where $\rho$ is production-inventory system fluency (fluid density); $V$ is the volume defined by $V = C \cdot h(t)$, and $C$ is the capacity of the system. This produces:

$$
m(t) = \rho C h(t)
$$

(3)

Substituting equation (3) in (1):

$$
\rho C \ddot{h} + \beta \dot{h} + k h = u(t)
$$

(4)

After mathematical manipulation, solving for $\ddot{h}$:

$$
\ddot{h} + \left( \frac{\beta}{\rho c} \right) \dot{h} + \left( \frac{k}{\rho c} \right) = \left( \frac{1}{\rho c} \right) \left( \frac{u}{h} \right)
$$

(5)

Based on equation (5), a state-space formulation for the system, considering:

$x_1 = h$ and $x_2 = \dot{h}$, is:
\[ \dot{x}_1 = x_2 \]  
\[ \dot{x}_2 = -\left(\frac{\beta}{\rho c}\right) \frac{x_2}{x_1} - \left(\frac{k}{\rho c}\right) + \frac{1}{\rho c} \left(\frac{u(t)}{x_1}\right) \]  
\[ (6) \]
\[ \dot{x}_2 = -\left(\frac{\beta}{\rho c}\right) x_2 - \left(\frac{k}{\rho c}\right) x_1 \frac{k}{\rho c} + \frac{1}{\rho c} \left(\frac{u(t)}{x_1}\right) \]  
\[ (7) \]

To calculate the equilibrium points, we proceed with equations (6) and (7), from which:

\[ h_e(t) = e^{-\frac{k}{\rho c} t} \] and \[ u_e = K h_e. \] (See Appendix).

The full state space description for the system requires to represent equation (2), with equations (6) and (7), as the following:

\[ \dot{x}_3 = -\left(\frac{1}{\rho c}\right) (x_3 - u(t)) \]  
\[ (8) \]

### 5.2.4 Linearization

To proceed, a proper linearization of equations (6) and (7) around the equilibrium point needs to be obtained. In the form: \( \dot{x} = Ax + Bu \), we have:

\[ A = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} \]  
\[ (9) \]

where:

\[ F_1 = \dot{x}_1 = x_2 \]  
\[ F_2 = \dot{x}_2 = -\frac{\beta}{\rho c} x_2 - \frac{k}{\rho c} x_1 = \frac{u}{\rho c x_1} \]  
\[ (10) \]
\[ (11) \]

Applying derivatives in equations (10) and (11), and evaluating for the equilibrium point:

\[ \frac{\partial F_1}{\partial x_1} = 0, \quad \frac{\partial F_1}{\partial x_2} = 1, \quad \frac{\partial F_2}{\partial x_1} = -\frac{\beta}{\rho c} \left(1 + \frac{k}{\beta}\right) \quad \text{and} \quad \frac{\partial F_2}{\partial x_2} = -\frac{\beta}{\rho c} \]

For matrix B:

\[ B = \begin{bmatrix} \frac{\partial F_1}{\partial u} \\ \frac{\partial F_2}{\partial u} \end{bmatrix} u_e \]  
\[ (12) \]
\[
\frac{\partial F_1}{\partial u} = 0 \quad \text{and} \quad \frac{\partial F_2}{\partial u} = \frac{1}{\rho c}
\]

Finally, the linear state-space formulation for the dynamic system is:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-\frac{1}{\rho c} (\beta + k) & -\frac{\beta}{\rho c} & 0 \\
0 & 0 & -\frac{1}{ac}
\end{bmatrix}\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
\frac{1}{ac}
\end{bmatrix} u(t) \quad (13)
\]

5.3 STABILITY ANALYSIS

Lyapunov stability analysis for the third-order nonlinear mixed PI systems is conducted via the direct method. The candidate Lyapunov function is:

\[
V = \frac{q}{2} x_1^2 + \frac{1}{2} x_2^2 + \frac{1}{2} x_3^2 
\]

(14)

Applying the derivative in time to equation (14):

\[
\dot{V} = qx_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3 
\]

(15)

Substituting equations (6-8) into equation (15):

\[
\dot{V} = qx_1 x_2 + x_2 \left(-a \frac{x_2}{x_1} - b\right) + x_3 \left(-\frac{1}{ac} x_3\right) 
\]

(16)

Where \(a = \frac{\beta}{\rho c}\) and \(b = \frac{K}{\rho c}\). After some algebraic manipulations, equation (16) reduces to:

\[
\dot{V} = qx_1 x_2 - a \frac{x_2^2}{x_1} - bx_2 - \left(\frac{1}{ac}\right) x_3^2 
\]

(17)

Considering that: \(|x_1||x_2| = w^2\), equation (17) is:

\[
\dot{V} = qw^2 - (a + b)|w| - \left(\frac{1}{ac}\right) x_3^2 
\]

(18)

Applying the following conditions: \(q=0\) and \(K= -\beta\), asymptotic stability is achieved and
\[-\left(\frac{1}{ac}\right)x_3^2 \leq 0, \text{ which implies } \dot{V} \leq 0.\]

### 5.4 OPTIMAL CONTROL

Applications of optimal control theory in engineering and sciences are broad and diverse and can be found in aerospace, chemical, electrical, mechanical, and industrial engineering. Economists use the term dynamic optimization to address optimal control theory problems.

Applications of optimal control in management sciences and operations research are (1) pricing with dynamic demand and production costs; (2) scheduling and production planning problems; (3) distribution and transportation in logistics networks; (4) optimal transfer of technology, and (5) optimal remanufacturing and recycling in closed-loop supply chains, among others. Decision-makers in management sciences and operations research use optimal control theory to map optimal control methods towards optimization methods based on the nature of the complexity of the manufacturing system.

The main contribution of this paper is to present a model-based systems theory and optimal control theory approach based on a mathematical model captures the features of PI systems via optimal control theory applying the Pontryagin maximum principle.

#### 5.4.1 Optimal control for PI system

An optimal control (OC) is defined as an admissible control, minimizes a functional objective (Naidu, 2003).

Definition 3. Given a dynamical system with initial condition \(x_0\), which evolves in time according to the state-space equation \(\dot{x} = f(x, u, t)\), to find an admissible control and make the functional objective to achieve its maximum.

A general mathematical form for an optimal control problem is:
\[
\min_u J(u) = \frac{1}{2} \int_{t_0}^{t_f} F(x, u, t) \, dt + S[x(t_f)]
\]

s.t.
\[
\begin{align*}
\dot{x} &= f(x, u, t) \\
x(t_0) &= x_0, x \in X, u \in U
\end{align*}
\]

### 5.4.2 Present value Hamiltonian-Optimal control problem

The proposed third-order nonlinear dynamical system via linearization for this research follows the general PVH-optimal control definition:

\[
\min_u J(u) = \frac{1}{2} \int_{t_0}^{t_f} e^{-\delta t} W(x, u, t) \, dt
\]

s.t.
\[
\begin{align*}
\dot{x} &= f(x, u, t) \\
x(t_0) &= x_0, x \in X, u \in U
\end{align*}
\]

Where \( \delta \) is a general discount factor. By applying, the Pontryagin maximum principle, for the PVH-optimal control problem in Equation (20) as is developed and presented in (Cerda-Tena, 2012).

\[
H(x, u, \rho, t) = W(x, u, t) e^{-\delta t} + \rho f(x, u, t)
\]

With the following conditions:

i) \( \dot{\rho} = -\frac{\partial H}{\partial x} = -\frac{\partial W}{\partial x} e^{-\delta t} - \rho \frac{\partial f}{\partial x} \) with \( \rho(t_f) = 0 \)

ii) \( \max H \) for \( u \in \Omega \)

iii) \( \dot{x} = f(x, u, t) \) for \( x(t_0) = x_0 \)

Defining the PVH such as \( \Pi = He^{\delta t} \), this gives:

\[
\Pi = W(x, u, t) + \rho e^{\delta t} f(x, u, t)
\]

Let \( q(t) = \rho(t)e^{\delta t} \), solving for \( \rho(t) \):
\[ \rho(t) = e^{-\delta t} q(t) \]  

By the derivative of Equation (23) with respect to time:

\[ \dot{\rho}(t) = -\delta e^{-\delta t} q(t) + e^{-\delta t} \dot{q}(t) \]  

From: \( \Pi(x, u, q, t) = W(x, u, t) + q(t)f(x, u, t) \), we know that:

\[ \dot{\rho} = -\frac{\partial W}{\partial x} e^{-\delta t} - \rho \frac{\partial f}{\partial x} \]  

Equating equations (24) and (25), after some algebra:

\[ \dot{q} = -\frac{\partial H}{\partial x} + \delta q \]  

with \( q(t_f) = 0 \)

5.5 RESULTS

5.5.1 Case study

Applying the PVH for energy-based optimal control problem a general formulation such as in equation (20) is presented. The following problem is addressed as:

\[ \min_u J(u) = \int_{t_0}^{t_f} e^{-\delta t} u^2 \, dt \]  

s.t.

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\left(\frac{\beta}{\rho C}\right) x_1 - \left(\frac{k}{\rho C}\right) + \frac{1}{\rho C} \left(\frac{u(t)}{x_1}\right) \\
\dot{x}_3 &= -\left(\frac{1}{ac}\right) (x_3 - u(t))
\end{align*} \]  

Defining the Hamiltonian similar to equation (22), and applying equation (26), for the system described in (27):

\[ \Pi(x, u, t) = u^2 + q_1 x_2 + q_2 \left(-Ax_1 - Bx_2 + C_0 u(t)\right) + q_3 D(-x_3 + u(t)) \]  

(28)
\[
\dot{q}_1 = Aq_2 + \delta q_1 \\
\dot{q}_2 = (Bq_2 - q_1) + \delta q_2 \\
\dot{q}_3 = Dq_3 + \delta q_3
\]  
(29) \hspace{1cm} (30) \hspace{1cm} (31)

Finding an optimal control such as \( \frac{\partial \Pi}{\partial u} = 0 \), this yields:

\[
u^* = -\frac{1}{2} (C_0q_2 + Dq_3)
\]  
(32)

Where: \( A = C_0(\beta + K) \); \( B = C_0\beta \); \( C_0 = \frac{1}{\rho c} \); \( D = \frac{1}{\alpha c} \)

5.5.2 Simulations

Simulations were performed in MATLAB, for the PVH based optimal control system described in Equation 27. For mathematical modeling purposes, we worked with a linearization of the system model which was conducted within the equilibrium points of the third-order nonlinear PI systems.

The system modeling quantities are summarized in Table 5.3:

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>C</th>
<th>( \rho )</th>
<th>( K )</th>
<th>( \beta )</th>
<th>( \delta )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>QUANTITY</td>
<td>10</td>
<td>1000</td>
<td>500</td>
<td>100</td>
<td>0.7</td>
<td>1</td>
</tr>
</tbody>
</table>

In Figure 5.2, the inventory level for PI systems achieves a maximum overshoot considering the dynamic nature of the system and stabilizes in a time scale of nine. Once the maximum inventory is achieved, around the four-time scale, half of the maximum inventory level is reached. Considering that the inventory level and inventory rates were developed from a second-order differential equation with a time-varying approach as a function of the mass, the simulations conducted led towards a dynamic response in which the steady-state is reached near to the time scale of ten.
The vertical process description is achieved via the analysis in inventory level, as the first output description of the mixed-PI systems dynamical system.

To quantify the inventory rate as a change measure in the inventory over time, Figure 5.3 presents a minimum overshoot and stabilizes a time scale of eight. The inventory rate approximates the time derivative of the inventory level which has been presented previously in Figure 5.2.

Figure 5.2: Inventory level for PI system

Figure 5.3: Inventory rate for PI system
The horizontal flow description for the third-order mixed-PI systems achieves a second output analysis for the production level which is shown in Figure 5.5. The maximum production level is achieved at the time scale of nine. This result is related to the time when steady-state conditions are reached (as shown previously in Figures 5.2 and 5.3).

A tradeoff between inventory level, inventory rate, and production levels reflects that once the steady-state is reached, the proper inertia of the dynamical system decreases the production goals in a specific time horizon.

Vertical and horizontal flow descriptions for the third-order PI systems are based on a system modeling linearization around the equilibrium points.

![Production level for Mixed Production-Inventory (PVI)](image)

**Figure 5.4: Production level for PI system**

Figure 5.5 shows the demand level as the single input variable; it presents a maximum level at initial conditions. Once PI systems reach steady-state conditions, an inflection level in the demand profile is achieved by the time scale of nine. At the end of the time horizon, the demand reaches a minimum amount, which implies that low inventory level and inventory rates are present. A tradeoff between production level and demand level is related, considering the horizontal flow description of the system.
This chapter developed a PVH-based optimal control for PI system mathematical modeling system. Systems and control theory perspective features for PI systems were defined such as trackability, observability, controllability, identifiability, dissipativity, passivity, entropy, sensitivity, and profitability. A third-order nonlinear mixed-PI systems mathematical model was developed following a model-based control system engineering and optimal control approach. A mathematical linearization for the third-order mixed-PI systems was presented which takes into account a vertical and horizontal flow description. In future work, our interest is to develop a four echelon cascade system analysis for the mixed-PI systems in which more scenarios will be addressed. Also, we expect to work with a nonlinear model predictive control approach for the nonlinear PI system descriptions of the dynamical system. The interaction of automation science and control systems with model-based systems engineering plays an important role in the context of mixed-PI systems. System identification techniques will be of interest to explore, for data-driven control-oriented approaches. Finally, fault-tolerant control analysis is important to be addressed to consider reliability approaches over each of the parameters and elements in the mixed-PI systems.
Chapter 6: Conclusions and future work

This research work, based on previous chapters, presents the following conclusions and future work, to continue improving the state of the art in mathematical modeling, optimal control, and stability analysis for supply chains.

6.1 CONCLUSIONS

In chapter 3, a serial supply chain was present considering the application of compartmental analysis as a mathematical model for a class of linear supply chain system, with a proper sensitivity and stability analysis for the dynamic supply chain. An energy-based optimal control problem for the serial supply chain was conducted, with results-oriented to inventory level and demand rate with stable and negative decreasing profiles.

In chapter 4, closed-loop supply chains were developed via compartmental analysis for a single-input single-output system, and two input-two output systems, in which inventory levels and demand rate, present a stable behavior consider the linear supply chain system nature.

In chapter 5, a production-inventory system was developed with a third-order dynamical behavior, applying Pontryagin Maximum principle via present-value Hamiltonian optimal control. Proper Lyapunov stability analysis for the dynamical system was conducted, taking into account the simulations for the production-inventory system.

6.2 FUTURE WORK

To extend the results achieved in this research work, to develop the following:

Mathematical modeling.

- To extend the application of compartmental analysis for closed-loop supply chains and the circular economy taking into account the sustainability impacts in manufacturing supply chains.
- To develop stochastic mathematical models to incorporate uncertainty levels within the supply chains.
- To incorporate the time-delayed impact in the dynamic supply chains, via the use of lead times.

Control oriented approaches.
- To develop novel control structures incorporating: Model Predictive Control, and Economic Model Predictive Control, for the previous mathematical models developed.
- To extend robust control strategies for supply chains which incorporates uncertainty in inventory levels and demand rates.
- To apply control based computational intelligence approaches such as chaos control, Fuzzy control, Neural networks, etc. for novel supply chains.
- To extend results for data-driven control in supply chains.

Stability analysis.
- To incorporate novel stability analysis for control structures with extended mathematical modeling for dynamic linear supply chains.
- To develop Lyapunov stability analysis for nonlinear supply chains, with time-delayed state-space formulations.
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Glossary

Mathematical model: A mathematical model is a mathematical description of the behavior of some real-life system or phenomenon. Such as physical, sociological, economics, etc.

System: In systems science, by definition, a system is a group of components which interacts towards keeping a collection of relationships with the sum of the systems themselves to other entities.

Supply chain management: Refers to the cooperation process management of materials and information flows between supply chains.

Closed-loop supply chains: Closed-loop supply chains integrates forward and reverse logistics, towards the main interest to process finished goods in a re-manufacturing analysis, to achieve sustainable solutions for enterprises.

Optimal control: An optimal control is defined as an admissible control, which minimizes a functional objective.
Appendix

Equations (6) and (7) are used to calculate the equilibrium points such as: \( \dot{x}_1 = \dot{x}_2 = 0 \). Therefore, Equation 6 is reduced to:

\[
- \left( \frac{k}{\rho C} \right) + \frac{1}{\rho C} \left( \frac{u(t)}{x_1} \right) = 0 \tag{A1}
\]

Recalling \( x_1 = h \), this provide the equilibrium point:

\[
u_e = kh_e \tag{A2}\]

With the assumption that \( u = 0 \), in equation (5) this leads to:

\[
- \frac{\dot{h}}{h} - \left( \frac{k}{\beta} \right) = 0 \tag{A3}
\]

Integrating over time, finally, this leads to:

\[
h_e(t) = e^{-\frac{k}{\beta t}} \tag{A4}\]
Vita

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