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Stochastic Modeling Of Earthquakes And Option Pricing Using Bns-Gamma-Ou Model

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Dedication

to these amazing souls

Tau Nyarko Quashie, Mercy Quashie, Naomi Kaa,

Bright Asiedu and Emmanuel Asare-Asiedu,

with love

STOCHASTIC MODELING OF EARTHQUAKES AND STOCK PRICES USING BNS-GAMMA-OU MODEL

by

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Abstract

High frequency data are becoming increasingly popular these days. They are fundamental in basically every facet of people's lives. They are the determining factors in hedging in the field of finance. In geology, they help in the accurate prediction of earthquakes' magnitude which goes along way to help save lives and properties.

High frequency data are also used more and more frequently for speculations. For this reason, it is important not only for scientists to apply models allowing correct quantification of these data, but also to improve the efficiency of these models.

The Black-Scholes model, which is widely used because of its simplicity and comprehensiveness has so many drawbacks that a lot of literature has covered. Although Black-Scholes will undoubtedly continue to be useful for a very long time, it is clear that the model underlying its use is strongly at odds with the observed data.

The simulation of high frequency data has become essential in this fast paced world. The quest to predict the path of these kind of data in finance is essential to be able to curb loses or maximize profit in the field of seismic modeling. It is prudent to be able to find the magnitude of earthquake at any given time because of the numerous negative impact it has on properties and lives. These kinds of data are volatile which, makes them very difficult to simulate.

The Gamma-OU model and Black-Scholes model are known to be used to predict the path of earthquakes and stock-prices, respectively. The Gamma-OU works well with earthquake data, while the Black-Scholes model functions very well with option pricing.
This study is to present how to predict the path of high frequency data using the Barndorff-Nielsen and Shephard model with the Gamma-OU process and demonstrate that it is superior to the Gamma-OU model.

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Chapter 1

Introduction

1.1 Earthquakes

Earthquakes, simply referred to as quakes are seismic events that are characterized by motion within the Earth's crust caused by the release of stress piled up by eruption of the crust or by geological faults.

The first recorded earthquake occurred at Mount Tai in 1831, no known fatalities recorded. Ever since the first earthquake, the number of lives claimed by this phenomenon is more than hundreds of millions. In 2019 alone earthquakes with magnitude between $4.0 - 9.9$ occurred $12,985$ times and destroyed billions of dollars worth of properties, injured $8,989$ people and took 288 lives. Table 1.1 gives the break down. These figures even though staggering happens to be the lowest since the year 2000.

Japan is the country where most earthquakes are located. However Indonesia is the country with most earthquakes. Earthquakes cause more damages and fatalities in China and Iran than in any other country.

1.2 Option Pricing

Option pricing is the use variables such as stock price, exercise price, volatility, interest rate, time to expiration) to determine the value an option Stock prices cannot be ascertain deterministically, that is using a differential equation either by the ordinary or partial one. If they could be determined by a differential equation, then the stock exchange would be very different from what it is now, even somewhat boring.

Table 1.1: Break down of earthquakes in 2019

Month	Number of Injuries	Fatalities
January	113	5
February	143	4
March	188	8
April	321	20
May	38	3
June	283	13
July	415	31
August	92	9
September	2721	85
October	822	34
November	3624	63
December	229	13
Total	8989	288

In 1973, Black et al.[7] presented the use of Brownian motion to explain the pattern of volatile assets prices. The use of the Brownian motion to model the stock market prices has a drawback which is its independent increment property.This implies that the price at a current time must not affect the price at a future time, but the prices currently may have an effect in determining the future price. This drawback makes the Brownian motion process not meritorious to model stock price.

To model the price of a financial asset, we must always take into account disturbances of the market which are reflected in the volatility of asset price.

1.3 High-frequency Data

High-frequency data are a set of data that is collected at a very fast rate. [1].The fulcrum of these kinds of data is how they are collected in terms of time. High-frequency data generally refers to data that is collected at a very rapid rate. Data can be collected at different times intervals. This makes the rate of collection differ. The highest rate at which data can be collected every time that new information comes in is referred to as ultrafrequency data. The information that arrives can take different forms in different data sets. The most fundamental data must be prices and quantities, however, there might be more than one type of price and more than one type of quantity that can be reported.

Chapter 2

Related Literature

In this chapter, we provide a brief introduction to the theory of stochastic differential equation which is the bedrock of the BNS-Gamma-Ornstein-Unlenbeck process in Chapter 3. We will look at the notation and definitions necessary to accurately formulate the BNS-Gamma-Ornstein-Unlenbeck process.

2.1 Motivation

In a lot of games, a coin is tossed to determine who plays first. The result of such experiment is either head or tail. The complete description of the trajectory of this simple experiment is rather challenging. It is mostly assumed that both outcomes can occur when the probability is one-half. In this sense, we switch from deterministic domain to the of stochastic.

Table 2.1: Temperature of a rod in Deterministic and Stochastic sense

Type of Variable	Deterministic	Stochastic
Single Variable. (Temperature of a rod)	$\mathbb{R}$ $T = 46^o$	Random Variable $\mathbb{E}$, Variance, e.t.c
Dynamic Variable (Temperature of a rod for the first 3 seconds)	$\mathbb{R}_+ \to \mathbb{R}$ $T(1) = 38^o$ $T(2) = 47^o$ $T(2) = 56^o$	Stochastic Process

Table 2.1 illustrates how a quantity is moved from the Deterministic world to that of the Stochastic world.

The Theory of Stochastic Processes belongs to the field of mathematics known as Stochastic. Basically, there are three disciplines which are related to Stochastic. They are probability theory, mathematical statistics, and theory of the stochastic processes. However, there are other disciplines such as jump processes, stochastic calculus, stochastic differential equations, population dynamics, and many others. It is clear from the above table that, to talk of stochastic, we have to deal with random variables.

PROBABILITY SPACES

To define the "random", properly, it is essential to define what a probability space is. We commence with a set, we represent that set by Ω, which is made up of certain subsets that we will refer to as "events". A $\sigma-$algebra is a collection $\mathscr{U}$ of subsets of Ω with these properties:

i $\emptyset, \Omega \in \mathscr{U}$.

ii If $A \in \mathscr{U} \implies A^c \in \mathscr{U}$.

iii f $A_1, A_2, A_3, ... \in \mathscr{U}, \implies \bigcup_{k=1}^{\infty} A_k, \bigcap_{k=1}^{\infty} A_k \in \mathscr{U}$

Let $\mathscr{U}$ be $\sigma-$algebra of subsets of Ω. We denote $P : \mathscr{U} \to [0,1]$ a probability measure provided:

i $P(\emptyset) = 0$.

ii $P(\Omega) = 1$.

iii If $A_1, A_2, A_3, ... \in \mathscr{U}, \implies P\big(\bigcup_{k=1}^{\infty} A_k\big) \leq \sum_{k=1}^{\infty} P(A_k)$

iv If $A_1, A_2, A_3, ...$disjoint sets $\in \mathscr{U}, \implies P\big(\bigcup_{k=1}^{\infty} A_k\big) = \sum_{k=1}^{\infty} P(A_k)$

It follows that if A, B $\in \mathscr{U}$, **then**$A \subseteq B \implies P(A) \leq P(B)$.

A triple $(\Omega, \mathscr{U}, P)$ is called a probability space provided Ω is any set, $\mathscr{U}$ is a σ-algebra of subsets of Ω, and P is a probability measure on $\mathscr{U}$.

Remark

i A set $A \in \mathscr{U}$ is called an event; points $\omega \in \Omega$ are sample points.

ii $P(A)$is the *probability* of the event A.

iii A property which is true except for an event of probability zero is said to hold almost surely abbreviated "a.s".

Let $\Omega = \{\omega_1, \omega_2, \omega_3, ..., \omega_N\}$ be a finite set, and we assume the given values $0 \leq p_j \leq 1$ for $j = 1, 2, 3, ..., N$ which satisfies $\sum p_j = 1$. We let $\mathscr{U}$ to consist all subsets of Ω. For each set $A = \{\omega_1, \omega_2, \omega_3, ..., \omega_{j_m}\} \in \mathscr{U}$ with The smallest $\sigma-$ algebra containing all the open subsets of $\mathbb{R}^n$ is called the Borel $\sigma-$ algebra, denoted $\mathscr{B}$. Assuming that f is a nonnegative, integrable function, such that $\int_{\mathbb{R}^n} f dx = 1$. We define

$$P(B) := \int_B f(x)dx$$

for each $B \in \mathscr{B}$. Then $()\mathbb{R}^n, \mathscr{B}, P$ is a probability space. f is called the density probability measure P.

Every probability needs to be set in an appropriate $(\Omega, \mathscr{U}, P)$. The word "random," then have different interpretation to distinct models of $(\Omega, \mathscr{U}, P)$.

RANDOM VARIABLES

Let $(\Omega, \mathscr{U}, P)$ be a probability space. A mapping $\mathbf{X} : \Omega \to \mathbb{R}^n$ is called a $n-$ dimensional random variable if for each $B \in \mathscr{B}$, we have $\mathbf{X}^{-1}(B \in \mathscr{U})$. We thus say $\mathbf{X}$ is $\mathscr{U}-$ measurable.

Lemma Let $\mathbf{X} : \Omega \to \mathbb{R}^n$ be a random variable. Then

$$\mathscr{U}(\mathbf{X}) := \{\mathbf{X}^{-1}(B) | B \in \mathscr{B}\}$$

is a $\sigma-$algebra, called the $\sigma-$algebra generated by $\mathbf{X}$. Thus the smallest sub $-\sigma-$ algebra of $\mathscr{U}$ with respect to which $\mathbf{X}$. is measurable.

Proof

It is clear that $\{\mathbf{X}^{-1}(B)|B \in \mathscr{B}\}$ is a $\sigma-$algebra which is the smallest $\sigma-$algebra with respect to which $\mathbf{X}$, is measurable.

Indicator Function

Let $A \in \mathscr{U}$. The indicator function of A,

$$\chi_A(\omega) := \begin{cases} 1 & \text{if `}\omega \in A \\ 0 & \text{if } \omega \in A \end{cases}$$

is a random variable.

Simple Function

If $A_1, A-2, A-3, ..., A_m \in \mathscr{U}$ with $\Omega = \bigcup_{i=1}^{m} A_i$ and $a_1, a_2, a_3, ..., a_m$ are real numbers, then

$$\mathbf{X} = \sum_{i=1}^{m} a_i \chi_{A_i}$$

is a random variable, called a simple function.

Remark

The $\sigma-$algebra $\mathscr{U}(\mathbf{X})$ is seen to contain all relevant information about the random variable $\mathbf{X}$. If we have a random variable say $\mathbf{Y}$, that is if $\mathbf{Y} = \Phi(\mathbf{X})$, for a function Φ, then $\mathbf{Y}$ is $\mathscr{U}(\mathbf{X})-$ measurable. [13]

We now present the idea of stochastic rigorously by looking at this phenomenon .

If we fix a point $x_0 \in \mathbb{R}^n$ and take a look at the ordinary differential equation:

$$\begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{b}\big(\mathbf{x}(t)\big) \quad (t > 0) \\ \mathbf{X}(0) & = x_0 \end{cases} \tag{2.1}$$

where $\mathbf{x} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field and solution is the trajectory $\mathbf{x}(\cdot) : [0, \infty) \to \mathbb{R}^n$.

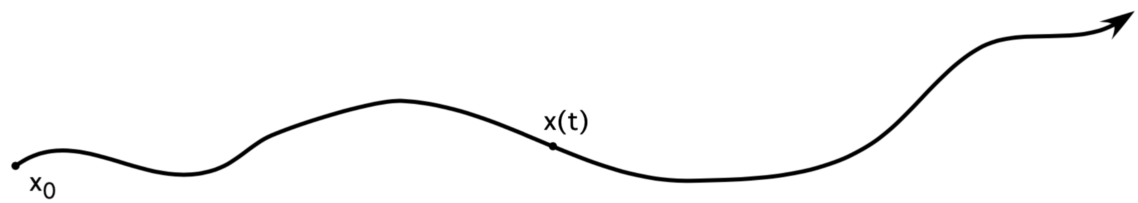

Figure 2.1: Path of a Differential Equation.

Fig 2.1 does not really depicts the experimental trajectories modeled by 2.1.

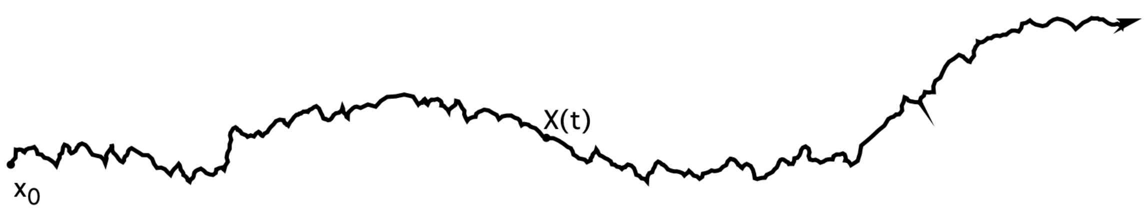

Figure 2.2: Path of a Stochastic Differential Equation.

This makes it essential to modify 2.1, to account for the possibility of randomness having an effect on the system. We thus rewrite the equation to account for the random effect that might disturb the system.

$$\begin{cases} \dot{\mathbf{X}}(t) & = \mathbf{b}\big(\mathbf{X}(t)\big) + \mathbf{B}\big(\mathbf{X}(t)\big)\xi(t) \quad (t > 0) \\ \mathbf{X}(0) & = x_0 \end{cases} \tag{2.2}$$

where $\mathbf{B} : \mathbb{R}^n \to \mathbb{M}^{n\times m}$ and $\xi(\cdot) := m$ - dimensional "white noise".

We will go ahead to define "white noise" $\xi(\cdot)$ rigorously and then go ahead to formally define$\mathbf{X}(\cdot)$, show that it has a unique solution, and discuss its asymptotic behavior dependence on $x_0, \mathbf{b}$, and $\mathbb{B}$.

2.2 Stochastic Processes.

Stochastic Processes are the bedrock of many disciplines which includes population dynamics, Insurance, Medicine,Qualitative Finance, Biology, Stochastic modeling,and many more.

2.2.1 Martingales

Filtration

Given $(\Omega, \mathscr{F}, P)$ and $T \geq 0$, we assume for every $0 \geq t \geq T$, there exists a $\sigma-$algebra $\mathscr{F}(t)$ such that $\mathscr{F}(t) \subset \mathscr{F}$ whenever $s \leq t$ then $\mathscr{F}(t) \subseteq \mathscr{F}(s)$.
Then $\{\mathscr{F}(t)\}_{0\geq t\geq T}$ is called a filtration associated with the space $(\Omega, \mathscr{F}, P)$. [10]

Given a stochastic process $\{X(t)\}_{0\geq t\geq T}$ is said to be adapted to the filtration $\{\mathscr{F}(t)\}_{0\geq t\geq T}$ if $X(t)$ is $\{\mathscr{F}(t)\}-$measurable, that is $X(t)^{-1}(\mathscr{B}) \in \mathscr{F}(t)$, for all Borel set $\mathscr{B} \in \mathbb{R}$

We look at the filtration which can be continuous or discrete, but our study will focus mainly on application where we have an end time hence discrete. Hence, given $\{\mathscr{F}_m\}_{m=0}^N$ is a given filtration and $\{X_n\}_{n=0}^N$ is a stochastic process adapted to the filtration $\{\mathscr{F}_n\}$, we will call $\{X_n\}$ to be a discrete martingale if

$$\mathbb{E}[X_{n+1}|\mathscr{F}_n] = X_n$$

and its continuous martingale is equivalent to:

$$\mathbb{E}[X_t|\mathscr{F}_s] = X_s \quad \text{a.s for} \;\; 0 \leq s \leq t \leq \infty.$$

Basically, martingales are special kinds adapted stochastic process. [19]

If

$$\mathbb{E}[X_t|\mathscr{F}_s] \leq X_s \quad \text{a.s for} \;\; 0 \leq s \leq t \leq \infty.$$

it is called a Supermartingale and when we have the inequality sign change to

$$\mathbb{E}[X_t|\mathscr{F}_s] \geq X_s \quad \text{a.s for} \;\; 0 \leq s \leq t \leq \infty.$$

it is called a Submartingale.

Theorem (Kolmogorov)

Let $X = \{X.\mathscr{F}, t \geq 0\}$ be an integrable process. Then, we define $\mathscr{F}_{t+} := \wedge_{\epsilon>0}\mathscr{F}_{t+}$ and also the partial augmentation of $\mathscr{F}$ and by $\tilde{\mathscr{F}}_t = \sigma(\mathscr{F}_{t+}, \mathscr{N})$. If $t \rightarrow \mathbb{E}(X_t)$ is continuous, there exists an $\tilde{\mathscr{F}}_t$ adapted stochastic process $\tilde{X} = \{\tilde{X}.\tilde{\mathscr{F}}, t \geq 0\}$ with sample paths which are right continuous, with left limits (CÀDLÀG) such that X and $\tilde{X}$ are modifications of each other.

Definition

A martingale $X = \{X.\mathscr{F}, t \geq 0\}$ is said to be an L^2-martingale or a square integrable martingale if $\mathbb{E}(X_2^t) < \infty$ for every $t \geq 0$.

Definition

A process $X = \{X.\mathscr{F}, t \geq 0\}$ is said to be L^p bounded if and only if $\sup_{t\geq 0} \mathbb{E}(|X_t|^p < \infty)$. The space of L^2 bounded martingales is denoted by M^2, and the subspace of continuous L^2 bounded martingales is denoted M_c^2.

Definition

A process $X = \{X.\mathscr{F}, t \geq 0\}$ is said to be uniformly integrable if and only if

$$\sup_{t\geq 0} \mathbb{E}(|X_t| 1_{|X_t|\geq N}) \to 0 \quad \text{as} \quad N \to \infty.$$

2.2.2 Stopping Times

Let $X = X_t, t \geq 0$ be a stochastic process. A stopping time with respect to X is a random time such that for each $n \geq 0$, he event $\{\tau = n\}$ is completely determined by (at most) the total information known up to time n,$\{X_0, ..., X_n\}$.In gambling, in where X_n denotes the total earnings after the n^{th} gamble, a stopping time $\tau$is thus a rule that tells the gambler at what time to stop gambling. The decision for a gambler to stop gambling depends on the information known at that time. [12]

2.2.3 Theorem (Doob's Martingale Inequalities).

Let M be a uniformly integrable martingale, and let $M^* := \sup_{t\geq 0} |M_t|$. Then:

i Maximal Inequality, for $\lambda > 0$

$$\lambda \mathbb{P}(M^* \geq \lambda) \leq \mathbb{E}[|M_\infty| 1_{M^* < \infty}]$$

ii $L^P maximal$ inequality, for $a < p < \infty$

$$||M^*||_p \leq \frac{p}{p-1} ||M_\infty||_p$$

The norm used in the Doob L^p is defined as

$$||M||_p = [\mathbb{E}(|M|^p)]^{\frac{1}{p}}$$

[11]

Why we need a Stochastic Integral

We now delve into the depths of why the ordinary integral cannot be used on a path at a time basis ((i.e separately for every $\omega \in \Omega$)). Assuming it is possible to do that, we set:

$$I_t(X) = \int_0^t X_s(\omega) dM_s(\omega)$$

for $M \in M_2^c$; but for a martingale which is not zeros a.s the total variation is not finite, even when it is evaluated on a closed interval let say $[0, T]$. This implies that the he Lebesgue-Stieltjes integral definition does not hold in such a situation. We rather use the quadratic variation that is appropriate in such situation.

Theorem

If T is a subordinator then its Lèvy symbol takes the form as shown in Bertoin,1999: [6]

$$\eta(u) = ibu + \int_0^\infty \big(\exp(iuy) - 1\big)\lambda(dy)$$

where $b \geq 0$ and the Lèvy measure λ satisfy the following conditions:

$$\lambda(-\infty, 0) = 0 \quad \text{and} \quad \int_0^\infty (y \wedge 1)\lambda(dy)$$

. A concise proof of this theorem can be found in Rogers and Williams [25].

2.3 Itô's Integral.

Earlier in this chapter, the quest was to develop a stochastic differential equation that is of the form:

$$\begin{cases} d\mathbf{X} = \mathbf{b}\big(\mathbf{X}, t\big)dt + \mathbf{B}\big(\mathbf{X}, \mathbf{t}\big)d\mathbf{W} \\ \mathbf{X}(0) = \mathbf{X}_0 \end{cases} \tag{2.3}$$

Definition 1 Let $\mathbf{W}(\cdot)$ be a one dimensional Brownian motion defined on the space $\big(\Omega, U, P\big)$. A family $\mathscr{F}$ of σ-algebras $\subseteq U$ is called a filtration with respect to $\mathbf{W}(\cdot)$ *if*

(a) $\mathscr{F}(t) \supseteq \mathscr{F}(s) \quad \forall\ t \geq s \geq 0$

(b) $\mathscr{F}(t) \supseteq \mathscr{W}(t) \quad \forall\ t \geq 0$

(c) $\mathscr{F}(t)$ *is independent of* $\quad \forall\ t \geq 0$

Basically $\mathscr{F}(t)$ contains all information needed at a particular time t.

Example $\mathscr{F}(t) := \mathscr{U}\Big(\mathscr{W}(s)(0 \leq s \leq t), \mathbf{X}_0\Big)$, such that $\mathbf{X}_0$ is a random variable that is independent of $\mathscr{W}^+(0)$.

Properties of Itô's Integral

For every constant $a, b \in \Re$ and for every $G, H \in \mathbb{L}^2(0, T)$ we have:

i

$$\int_0^T \big(aG + bH\big)\ d\mathscr{W} = a\int_0^T GdW + b\int_0^T HdW$$

Proof. This follows form the corresponding linearity property for step processes.

ii

$$E\bigg(\int_0^T G\ dW\bigg) = 0$$

Proof. This also follows form the corresponding linearity property for step processes.

iii

$$E\left(\left(\int_0^T G\ dW\right)^2\right) = E\left(\int_0^T G^2\ dt\right)$$

Proof. This also follows form the corresponding linearity property for step processes.

iv

$$E\left(\int_0^T G\ d\mathscr{W} \int_0^T H\ d\mathscr{W}\right) = E\left(\int_0^T GH\ dt\right)$$

Proof of (iv) results from (iii) and the identity $2ab = (a+b)^2 - a^2 - b^2$ [8]

Itô's Formula

Definition 2 By *absolute width* W of an interval $\mathbf{x} = [x^-, x^+]$, we mean twice the absolute half-width of $\mathbf{x}$, i.e.,

$$X(r) = X(s) + \int_s^r F\ dt + \int_s^R G\ dW$$

for some $F \in \mathbb{L}^1(0,T), G \in \mathbb{L}^2(0,T)$ *and* $0 \leq s \leq r \leq T$ *we say that* $X(\cdot)$ *has the* stochastic differential

$$dX = Fdt + GdW$$

for $0 \leq t \leq T$. *[14]*

Theorem 1 *Suppose that* $X(\cdot)$ *has a stochastic differential*

$$dX = Fdt + GdW$$

for $F \in \mathbb{L}^1(0,T), G \in \mathbb{L}^2(0,T)$. *Assuming* $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ *is continuous and that* $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ *exist and are continuous.*

$$Y(t) := u(X(t), t)$$

Then Y has the stochastic differential

$$dY = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2 dt$$
$$= \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}F + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2\right)dt + \frac{\partial u}{\partial x}GdW \tag{2.4}$$

we call 2.4 Itô's formula.

2.3.1 Lèvy process

The Lèvy process plays a significant role in this thesis which serves as building blocks for the BNS-Gamma-OU model. High frequency time series models studied are driven by Lèvy processes. To understand how they are used, some background material is needed.In this study, we use the Lèvy processes in mathematical finance and application to seismic data from a computational and applied view.

Definition 3 *Let* $X = (X(t), t \geq 0)$ *be a stochastic process defined on a probability space* $(\Omega, \mathscr{F}, P)$. *We say that it has independent increments if for each* $n \in N$ *and each* $0 \leq t_1 < t_2 < ... < t_{n+1} < \infty$*the random variables* $(X(t_{j+1})X(t_j),\ 1 \leq j \leq n)$ *are independent and that it has stationary increments if each* $X(t_{j+1})X(t_j) = X(t_{j+1}t_j)X(0)$. *We say that X is a Lèvy process if:*

1. $X(0) = 0$ *(a.s);*

2. *X has independent and stationary increments;*

3. *X is stochastically continuous, i.e. for all* $a > 0$ *and for all* $s \geq 0$

$$\lim_{t \to s} P(|X(t)X(s)| > a) = 0.$$

Note that in the presence of (1) and (2), (3) is equivalent to the condition

$$\lim_{t \to 0} P(|X(t)| > a) = 0$$

for all $a > 0$.

Brownian Motion

Definition (Brownian Motion). A one dimensional Brownian motion is an $\mathbb{R}-$ valued Lèvy process $B = (B_t), t \geq 0$ such that [18] :

i. $B_0 = 0 \quad \mathbb{P}-$ almost surely.

ii. B has independent increments, i.e for every $0 \leq t \leq s \leq u < v$, the random variable $B_v - B_u$ and $B - t - B_s$ are independent.

iii. B has stationary normally distributed increments with

$$B_t - B_s \sim \mathscr{N}(0, t - s),\ \forall t < s \geq 0$$

iv. B has $\mathbb{P}-$ a.s continuous paths.

The characteristic and the moment generating function of a Brownian motion are

$$\begin{aligned}\Phi_B(u) &= \mathbb{E}[\exp\{iuB_t\}] \\ &= \exp\{-\frac{1}{2}u^2 t\}\end{aligned}$$

and

$$\begin{aligned}m_t(u) &= \mathbb{E}[\exp\{uB_t\}] \\ &= \exp\{\frac{1}{2}u^2 t\}\end{aligned}$$

respectively since $B_t \sim \mathscr{N}(0, t), \forall t$ with $u \in \mathbb{R}$. A Brownian motion is an $\mathbb{R}^n$ valued process $(n \in \mathbb{N}^*)$

$$B = \big((B_t^{(1)}, B_t^{(2)}, B_t^{(3)}, ..., B_t^{(n)}), t \geq 0\big),$$

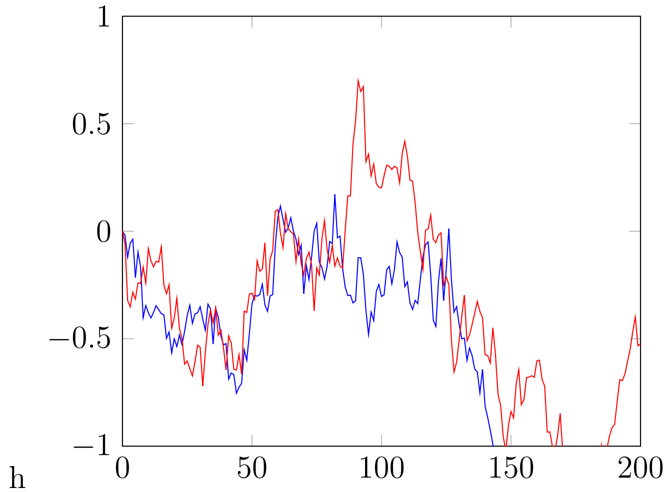

Figure 2.3: Two sample paths

where $\forall i = 1, 2, ..., n, n \in \mathbb{N}^*$, $B_t^{(i)}$ are n independent one-dimensional Brownian motion. Remark: The Brownian motion is a Markov process and has martingale properties.

The Poisson process

To define a Poisson process, we must first look at what is meant by a counting process to possess independent increments.

Definition 4 *A counting process is said to possess independent increments if the number of events that occur in disjoint time intervals are independent.*

A counting process is said to have stationary increments if the distribution of the number of events which occur in any interval depends only on the length of the interval. [17]

Definition 5 *A counting process* $(N(t), t \geq 0)$ *is said to be a Poisson process with mean* λt*, for* $\lambda > 0$ *if it satisfies the following conditions:*

- $N(0) = 0$
- $P(N(h) = 1) = \lambda h + \mathcal{O}(h)$
- *Process has stationary and independents increments.*

- $P(N(h) \geq 2) = \mathscr{O}(h)$

The compound Poisson process

Definition 6 *Let $(N(t), t \geq 0)$ be a Poisson process with rate λ, so that $\mathbb{E}[N(t)] = \lambda t$ for $t \geq 0$.Let $X_1, X_2,$ be iid random variables independent of N. Let $D(t)$ be the random sum*

$$D(t) \equiv \sum_{i=1}^{N(t)} X_i, \quad t \geq 0.$$

Then $D(t), t \geq 0)$ is a compound Poisson process. [26].

Definition 7 A subordinator is a one-dimensional Lèvy process that is non-decreasing (a.s.). These kind of processes can be considered as random models of time evolution since when ever $T = \big(T(t), t \geq 0\big)$ is a subordinator, we also have that $T(t) \geq 0$ a.s $\forall t > 0$ *and* $T(t_1) \leq T(t_2)$ *a.s whenever* $t_1 \leq t_2$ *[3].*

2.4 Ornstein-Uhlenbeck process

We now present the OrnsteinUhlenbeck process with Lèvy stochastic integral. Let us $(\mathbf{X}(t), t \geq 0)$ be a $\mathbb{R}^n-$ valued Lèvy and $f \in L^2(\mathbb{R}^+)$. We consider the Lèvy Integral $Y = (Y(t), t \geq 0)$ such that:

$$Y(t) = \int_o^t f(s) X ds \tag{2.5}$$

then Y according to Applebaum [3] has independent increment.

Let us take $f \in L^2(\mathbb{R})$ and a shift in $s \to s - t$ such that $f(s-t) \in L^2(\mathbb{R})$. We further assume that f is càdlàg. The moving-average process $Z = (Z(t), t \geq 0)$ is given by

$$Z(t) = \int_{-\infty}^t f(s-t) X ds \tag{2.6}$$

This moving-average process $Z = (Z(t), t \geq 0)$ is stationary. The Ornstein-Uhlenbeck process is basically a kind of moving average process. To obtain this kind let fix $\lambda > 0$,

$s \leq 0,\ t \geq 0$, taking $f(s) = e^{\lambda s}$ we have:

$$Z(t) = \int_{-\infty}^{t} f(s-t)dX(s) \tag{2.7}$$

$$= \int_{-\infty}^{t} e^{\lambda(s-t)}dX(s) \tag{2.8}$$

$$= \int_{-\infty}^{0} e^{\lambda(s-t)}dX(s) + \int_{0}^{t} e^{\lambda(s-t)}dX(s) \tag{2.9}$$

$$= e^{-\lambda t}Z(0) + \int_{0}^{t} e^{\lambda(s-t)}dX(s) \tag{2.10}$$

Example

A better model of Brownian motion given by the Ornstein-Uhlenbeck equation

$$\begin{cases} \ddot{Y} = -b\dot{Y} + \sigma\xi \\ Y(0) = Y_0, \dot{Y}(0) = Y_1. \end{cases} \tag{2.11}$$

where $Y(t)$ is the position of Brownian particle at the time t, Y_0, and Y_1 are given Gaussian variables. $\xi(\cdot)$ is the white noise, $b > 0$ and σ are the friction coefficient and diffusion coefficient respectively.

$$Y(t) = Y_0 + \int_0^t X ds \tag{2.12}$$

$$= E(Y_0) + \int_0^t E(X(s))ds \tag{2.13}$$

$$= E(Y_0) + \int_0^t e^{-bs}E(Y_1)ds \tag{2.14}$$

$$= E(Y_0) = +\left(\frac{1-e^{-bt}}{b}\right)E(Y_1) \tag{2.15}$$

$$V(Y(t)) = V(Y_0) + \frac{\sigma^2}{b^2}t + \frac{\sigma^2}{2b^3}(-3_4e^{-bt} - e^{-2bt}) \tag{2.16}$$

Ornstein-Uhlenbeck processes (OU processes) are used for the stochastic description of volatility in the Black-Scholes model [5].

Lèvy process with positive increments, $\sigma^2 = (\sigma_t^2)_{t\geq 0}$is an Ornstein-Uhlenbeck process driven by $L = (L_t)_{t\geq 0}$, $\mathbf{B} = (\mathbf{B}_t)_{t\geq 0}$ is a Brownian motion, $\lambda > 0, \rho \leq 0$. A Lèvy process $L = (L_t)_{t\geq 0}$ is called background driving Lèvy process (BDLP).

2.5 Stochastic Differential Equations

We now have all the tools to define what a Stochastic Differential Equation is. In this section we define Stochastic Differential Equations and look at the linear SDE's and examples finally look at some theorems and solutions.

Definition 8 An $\Re^n$-valued stochastic process $\mathbf{X}$ is a solution of the stochastic differential

$$\begin{cases} d\mathbf{X} = \mathbf{b}\big(\mathbf{X}, t\big)dt + \mathbf{B}\big(\mathbf{X}, \mathbf{t}\big)d\mathbf{W} \\ \mathbf{X}(0) = \mathbf{X}_0 \end{cases} \tag{2.17}$$

for $0 \leq t \leq T$.
where:

$\mathbf{W}(\cdot)$ *is an m-dimensional Brownian motion,*

$\mathbf{b} : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ *and*

$\mathbf{B} : \mathbb{R}^n \times [0, T] \to \mathbb{M}^{n\times m}$

A higher order SDE of the form

$$Y^{(n)} = f(t, Y, ..., Y^{(n-1)}) + g(tY, ...Y^{(n-1)})\xi \tag{2.18}$$

where ξ represents the "white noise".

Example Let $m = n = 1$ and we let g be a continuous function .Then the unique solution of

$$\begin{cases} d\mathbf{X} = g\mathbf{X}d\mathbf{W} \\ \mathbf{X}(0) = 1 \end{cases} \tag{2.19}$$

is

$$\mathbf{X}(t) = e^{-\frac{1}{2}\int_0^t g^2 ds + \int_0^t g d\mathbf{W}} \tag{2.20}$$

for $0 \leq t \leq T$. To verify the solution, we have that:

$$Y(t) := -\frac{1}{2}\int_0^t g^2 ds + \int_0^t g d\mathbf{W}$$

which satisfies

$$dY = -\frac{1}{2}g^2 dt + g d\mathbf{W}$$

Using the Itô's lemma for $u(x) = e^x$ gives

$$\begin{aligned} d\mathbf{X} &= \frac{\partial u}{\partial x}dY + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}g^2 dt \\ &= e^Y\big(-\frac{1}{2}g^2 dt + g d\mathbf{W} + \frac{1}{2}g^2 dt\big) \\ &= g\mathbf{X}d\mathbf{W} \end{aligned} \tag{2.21}$$

Example (Stock prices). Let $P(t)$ denote the price of a stock at time t. We can model the evolution of $P(t)$ in time by supposing that $\frac{dP}{P}$,the relative change of price, P evolves according to the SDE.

$$\frac{dP}{P} = \mu dt + \sigma d\mathbf{W}$$

for certain constants $\mu > 0$ and σ, called the drift and the volatility of the stock. Hence

$$dP = \mu P dt + \sigma dP\mathbf{W}; \tag{2.22}$$

and so

$$d\big(\log(P)\big) = \frac{dP}{P} - \frac{1}{2}\frac{\sigma^2 P^2 dt}{P^2}$$

By Itô's formula

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW$$

consequently

$$P(t) = p_0 e^{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right)t},$$

Since the prices is always positive, we assume $p_0 > 0$, hence we have

$$P(t) = p_0 + \int_0^t \mu P ds + \int_0^t \sigma P d\mathbf{W}$$

and $E\left(\int_0^t \sigma P d\mathbf{W}\right) = 0$ we see that

$$E(P(t)) = p_0 + \int_0^t \mu E(P(s)) ds$$

Then

$$E(P(t)) = p_0 e^{\mu t} \quad t \geq 0$$

2.6 Existence and Uniqueness of Stochastic Differential Equation

In this section, we will discuss the existence and uniqueness of solutions to a stochastic differential equation:

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{B}(t), \quad t \in [t_0, T] \tag{2.23}$$

with $x(t_0) = x_0$ as initial value, such that $0 \leq t_0 < \infty$, The essence of looking at this is to be able to answer the fundamental questions posed by such equations which are:

- Do solutions exist to such equations?
- If solutions exist, are they unique.?
- What are the behaviors of such solutions?
- How can such solutions be arrived at in practice?

In this discussion, we assume the following:

- $(\Omega, \mathscr{F}, P)$ be a probability space.
- $(\mathbf{B}_1(t), \mathbf{B}_1(t), \mathbf{B}_2(t), ..., \mathbf{B}_m(t))^T$ be a m-dimensional Brownian motion.
- x_0 be an $\mathscr{F}_{t_0}-$measurable, such that $0 \leq t_0 < T < \infty$ $\mathbb{R}^n-$ valued random variable where $\mathbb{E}|x_0|^2 < \infty$.
- $f : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^{n\times m}$ be Borel measurable.

Equation 2.23 is equivalent to the stochastic integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)d\mathbf{B}(s), \quad t_0 \leq t \leq T \tag{2.24}$$

Definition

A stochastic process $\{x(t)\}_{t_0 \leq t \leq T}$ is a solution to

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{B}(t),$$

with $x(t_0) = x_0$ as the initial value if these conditions are satisfied:

1. $\{x(t)\}$ is continuous and $\mathscr{F}_t-$ -adapted;

2. $\{f(x(t),t)\} \in \mathscr{L}^1([t_0,T];\mathbb{R}^n)$ and $\{g(x(t),t)\} \in \mathscr{L}^2([t_0,T];\mathbb{R}^{n\times m})$

3. The integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s),s)ds + \int_{t_0}^{t} g(x(s),s)d\mathbf{B}(s) \tag{2.25}$$

holds for every $t \in [t_0,T]$ with probability of 1.

Definition

A solution $\{x(t)\}$ to a stochastic differential equation is unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is, almost all their sample paths agree

$$P\{x(t) = \bar{x}(t) \ \text{ for all } t_0 \leq t \leq T\} = 1$$

Example with a unique solution

We consider the stochastic differential equation given by:

$$dN_t = rN_t dt + \alpha N_t d\mathbf{B}_t \tag{2.26}$$

Equation 2.26 can be equivalently written as

$$\frac{dN_t}{N_t} = rdt + \alpha d\mathbf{B}_t \tag{2.27}$$

This results in:

$$\int_0^t \frac{dN_s}{N_s} = rdt + \alpha d\mathbf{B}_t \quad (B_0 = 0) \tag{2.28}$$

In other to evaluate the left hand side, we apply the Itô's formula for the function

$$g(t,x) = \ln x, x > 0$$

to obtain

$$d(\ln N_t) = \frac{1}{N_t} \cdot dN_t + \frac{1}{2}\Big(- \frac{1}{N_t^2}\Big)(dN_t)^2 \tag{2.29}$$

Thus

$$d\frac{dN_t}{N_t} = d(\ln N_t) + \frac{1}{2}\alpha^2 dt \tag{2.30}$$

Therefore, from Equation 2.28 we conclude that:

$$\ln \frac{N_t}{N_0} = \big(r - \frac{1}{2}\alpha\big)t + \alpha \mathbf{B}_t \tag{2.31}$$

which is equivalent to:

$$N_t = N_0 \exp\Big\{ \big(r - \frac{1}{2}\alpha\big)t + \alpha \mathbf{B}_t) \Big\}$$

The solution N_t is a process of the form

$$X_t = X_0 \exp\{\mu t + \alpha B_t\}, \quad \mu, \alpha \ \text{ are constants} \tag{2.32}$$

Such processes are called geometric Brownian motion.

Remark

The following can be observed:

- If B_t is independent of N_0 we would have

$$\mathbb{E}[N_t] = \mathbb{E}[N_0] \exp\{rt\}$$

- To confirm that such is the situation we let

$$Y_t = \exp \alpha \mathbf{B}_t$$

- When Itô's formula we get:

$$dY_t = \alpha \exp\{\alpha B_t\} d\mathbf{B}_t + \frac{1}{2}\alpha^2 \exp\{\alpha \mathbf{B}_t\} dt$$

which is equivalent to:

$$Y_t = Y_0 + \alpha \int_0^t \exp\{\alpha B_s\} d\mathbf{B}_s + \frac{1}{2}\alpha^2 \int_0^t \exp\{\alpha \mathbf{B}_s\} ds$$

- Since

$$\mathbb{E}\left[\int_0^t \exp\{\alpha B_s\} d\mathbf{B}_s\right] = 0$$

we get

$$\mathbb{E}[Y_t] = \mathbb{E}[Y_0] + \frac{1}{2}\alpha^2 \int_0^t \mathbb{E}[Y_s] ds$$

Hence

$$\frac{d}{dt}\mathbb{E}[Y_t] = \frac{1}{2}\alpha^2 \mathbb{E}[Y_t], \quad \mathbb{E}[Y_0] = 1$$

- Therefore, we have:

$$\mathbb{E}[Y_t] = \exp\left\{\frac{1}{2}\alpha^2 t\right\}$$

Hence we conclude that :

$$\mathbb{E}[N_t] = \mathbb{E}[N_0] \exp\{rt\}$$

If $g(x, t) \equiv 0$, then the Stochastic Differential Equation 2.23 reduces to:

$$dx(t) = f(x(t), t)dt \quad t \in [t_0, T] \tag{2.33}$$

However the initial condition $x(t_0) = x_0$ can still be a random variable.

Example with infinitely many solutions

We consider the example

$$\dot{x} = 3x^{\frac{2}{3}} \quad t \in [t_0, T] \tag{2.34}$$

with initial condition $x(t_0) = 1_A$, where $A \in \mathscr{F}_{t_0}$. We observe that for each $0 < \alpha < T - t_0$, the stochastic process:

$$x(t) = x(t,\omega) = \begin{cases} (t - t_0 + 1)^3 & t_0 \leq t \leq T, & \omega \in A \\ 0 & t_0 \leq t \leq t_0 + \alpha, & \omega \notin A \\ (t - t_0 - \alpha)^3 & t_0 + \alpha < t \leq T, & \omega \notin A \end{cases}$$

is a solution of equation 2.35.

This initial problem has an infinitely many solutions.

Example with no solution

$$\dot{x} = x^2 \quad t \in [t_0, T] \tag{2.35}$$

with $x(t_0) = x_0$ as initial condition, a random variable which takes values larger than $\frac{1}{T - t_0}$. It is verified that initial value problem above has a unique solution

$$x(t) = \left(\frac{1}{x_0} - (t - t_0) \right)^{-1}$$

for $t_0 \leq t \leq t_0 + \frac{1}{x_0} < T$. However, there is no solution for this initial value problem which is defined for all $t \in [t_0, T]$.

2.6.1 Existence and uniqueness of solution

Theorem 2 *Let there exist two positive constants $\bar{K}$ and K such that these conditions hold:*

1. Lipschitz condition: $\forall x, y \in \mathbb{R}^n$ *and* $t \in [t_0, T]$

$$\max\{|f(x,t) - f(y,t)|^2, |g(x,t) - g(y,t)|^2\} \leq \bar{K}|x-y|^2$$

2. Linear growth condition: $\forall (x,t) \in \mathbb{R}^n \times [t_0, T]$

$$\max\{|f(x,t)|^2, |g(x,t)|2\} \leq K(1+|x|^2)$$

We assume x_0 to be a random variable which is independent of the $\sigma-$algebra $\mathscr{F}_\infty^{(m)}$ generated by $\mathbf{B}_s(\cdot), s \geq 0$ such that $\mathbb{E}|x_0|^2 < \infty$.
Then there exists a unique $t-$continuous solution $X_t(\omega)$ of the initial value problem in Equation 2.23. With the property $X_t(\omega)$ is adapted to the filtration $\mathscr{F}_\infty^{x_0}$ generated by x_0 and $\mathbf{B}_s(\cdot), s \leq t$. Furthermore, such solution belongs to $\mathscr{M}^2([t_0, T]; \mathbb{R}^n)$.
We will now discuss some lemmas that will aid us in the proof of the above theorem.

Theorem 3 *Let* $p \geq 2$ *and let* $g \in \mathscr{M}^2([t_0, T]; \mathbb{R}^{n \times m})$ *be such that*

$$\mathbb{E}\left[\int_0^T |g(s)|^p ds\right] < \infty$$

Then we have

$$\mathbb{E}\left[\int_0^T |g(s)|^p ds\right] \leq \left(\frac{(p(p-1))}{2}\right)^{\frac{p}{2}} T^{\frac{(p-2)}{2}} \mathbb{E}\left[\int_0^T |g(s)|^p ds\right]$$

In particular, the quality holds for $p = 2$.

Proof

For $0 \leq t \leq T$, set

$$x(t) = \int_0^t g(s) d\mathbf{B}(s)$$

using Itô's formula and Itô's integral properties, we have:

$$\begin{aligned}\mathbb{E}|x(t)|^p &= \frac{p}{2}\mathbb{E}\int_0^t \Big(|x(s)|^{p-2}|g(s)|^2 + (p-2)|x(s)|^{p-4}|x^T(s)g(s)|^2\Big)ds \\ &\leq \frac{p(p-1)}{2}\mathbb{E}\int_0^t |x(s)|^{p-2}|g(s)|^2 ds\end{aligned}$$

Using H$\acute{}$older's's inequality, for $1 \leq p, q \leq \infty$ such that $\frac{1}{p}+\frac{1}{q}=1$, $X \in L^p, Y \in L^q$ we have

$$\mathbb{E}|XY| \leq \big(\mathbb{E}|X|^p\big)^{\frac{1}{p}}\big(\mathbb{E}|Y|^q\big)^{\frac{1}{q}}$$

we have

$$\begin{aligned}\mathbb{E}|x(t)|^p &= \frac{p(p-1)}{2}\Big(\mathbb{E}\int_0^t |x(s)|^p ds\Big)^{\frac{p-2}{p}}\Big(\mathbb{E}\int_0^t |g(s)|^p ds\Big)^{\frac{2}{p}} \\ &\leq \frac{p(p-1)}{2}\Big(\int_0^t \mathbb{E}|x(s)|^p ds\Big)^{\frac{p-2}{p}}\Big(\mathbb{E}\int_0^t |g(s)|^p ds\Big)^{\frac{2}{p}}\end{aligned}$$

We note that $\mathbb{E}|x(t)|^p$ is non-decreasing in t, we obtain

$$\mathbb{E}|x(t)|^p = \frac{p(p-1)}{2}\Big[t\mathbb{E}|x(t)|^p\Big]^{\frac{p-2}{p}}\Big(\mathbb{E}\int_0^t |g(s)|^p ds\Big)^{\frac{2}{p}}$$

This the becomes:

$$\mathbb{E}|x(t)|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} t^{\frac{p-2}{p}}\mathbb{E}\int_0^t |g(s)|^p ds$$

This concludes the proof of this theorem.

Theorem 4 *Under the same assumptions in the previous theorem. This then becomes:*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t g(s)d\mathbf{B}(s)\Big|^p\Big] \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{p}}\mathbb{E}\int_0^T |g(s)|^p ds$$

Proof

We have that the stochastic integral

$$\int_0^t g(s)d\mathbf{B}(s)$$

is a martingale.

By the Doob martingale inequality, we have that this then becomes:

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t g(s)d\mathbf{B}(s)\right|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left|\int_0^T g(s)d\mathbf{B}(s)\right|^p$$

using the previous theorem, we obtain the desired inequality. theorem.

Theorem 5 (Gronwall's inequality) *Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded non-negative function on $[0, T]$, and let $v(\cdot)$ be a non-negative integrable function on $[0, T]$. [24]*
If

$$u(t) \leq c + \int_0^t v(s)u(s)ds, \qquad \forall\ 0 \leq t \leq T,$$

then

$$u(t) \leq c \exp\left\{\int_0^t v(s)ds\right\}, \qquad \forall\ 0 \leq t \leq T,$$

Proof

It is clear we have:

$$\log(z(t)) = \log(c) + \int_0^t \frac{v(s)u(s)}{z(s)}ds \leq \log(c) + \int_0^t v(s)ds$$

This implies that

$$z(t) \leq c \exp\left\{\int_0^t v(s)ds\right\}, \qquad \forall\ 0 \leq t \leq T$$

The required inequality follows since $u(t) \leq z(t)$.

Lemma

With the assumption that the linear growth condition holds,if $x(t)$ is a solution of equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{B}(t)$$

then

$$\mathbb{E}\big(\sup_{t_0 \le t \le T} |x(t)|^2\big) \le (1 + 3\mathbb{E}|x_0|^2) \exp 3k(T - t_0)(t - t_0 + 4)$$

In particular, $x(t)$ belongs to $\mathscr{M}^2([t_0, T; \mathbb{R}^n])$. [22]

Proof

For every integer $n \ge 1$, define the stopping time

$$\tau_n = \min\{T, \inf\{t \in [t_0, T] : |x(t)| \ge n\}\}$$

Set $x_n(t) = x(\min\{t, \tau_n\})\ \forall t \in [t_0, T]$.

X_n thus satisfy the equation

$$x_n(t) = x_0 \int_{t_0}^{t} f(x_n(s), s) I_{[[t_0, \tau_n]]}(s) ds + \int_{t_0}^{t} g(x_n(s), s) I_{[[t_0, \tau_n]]}(s) ds$$

We now use the elementary inequality

$$|a + b + c|^2 \le 3(|a|^2 + |b|^2 + |c|^2),$$

the linear growth condition and the Hölder's inequality we have that:

$$|x_n(t)|^2 \le 3|x_0|^2 + 3K(t - t_0) \int_{t_0}^{t} (1 + |x_n(s)|^2) ds + 3\left| \int_{t_0}^{t} g(x_n(s), s) I_{[[t_0, \tau_n]]}(s) ds \right|^2$$

We apply the linear growth condition and previous theorem, to obtain:

$$\mathbb{E}\big(\sup_{t_0\le s\le t}|x_n(s)|^2\big) \le 3\mathbb{E}|x_0|^2 + 3K(T-t_0)\int_{t_0}^{t}(1+\mathbb{E}|x_n(s)|^2 ds + 12\mathbb{E}\int_{t_0}^{t}|g(x_n(s),s)|^2 I_{[[t_0,\tau_n]]}(s)ds$$
$$\le 3\mathbb{E}|x_0|^2 + 3K(T-t_0+4)\int_{t_0}^{t}(1+\mathbb{E}|x_n(s)|^2 ds$$

Consequently

$$1+\mathbb{E}\big(\sup_{t_0\le s\le t}|x_n(s)|^2\big) \le 1+3\mathbb{E}|x_0|^2 + 3K(T-t_0+4)\int_{t_0}^{t}\big[1+\mathbb{E}(\sup_{t_0\le r\le s}|x_n(s)|^2)\big]ds$$

By Gronwall inequality we have:

$$1+\mathbb{E}\big(\sup_{t_0\le t\le T}|x_n(t)|^2\big) \le (1+3\mathbb{E}|x_0|^2)\exp\{3k(t-t_0)(T-t_0+4)\}$$

By Gronwall inequality we have:

$$\mathbb{E}\big(\sup_{t_0\le t\le \tau_n}|x_n(t)|^2\big) \le (1+3\mathbb{E}|x_0|^2)\exp\{3k(t-t_0)(T-t_0+4)\}$$

If we let $n\to\infty$ we get the desired inequality.

2.6.2 Existence and uniqueness of solutions

Uniqueness

If we let $x(t)$ and $\bar{x}$ be two solutions, by the previous lemma, both solutions belong to $\mathscr{M}^2([t_0,T];\mathbb{R}^n)$.
We have that

$$x(t)-\bar{x}(t) = \int_{t_0}^{t}\big(f(x(s),s)-f(\bar{x}(s),s)\big)ds + \int_{t_0}^{t}\big(g(x(s),s)-g(\bar{x}(s),s)\big)d\mathbf{B}(s), \quad (2.36)$$

Applying the H ölder inequality, the previous theorem and Lipschitz condition we have that:

$$\mathbb{E}\big(\sup_{t_0\le s\le t}|x(s)-\bar{x}(s)|^2\big)\le 2\bar{K}(T+4)\int_{t_0}^{t}\mathbb{E}\big(\sup_{t_0\le r\le s}|x(r)-\bar{x}(r)|^2\big)ds$$

Proof

$$\mathbb{E}\big(\sup_{t_0\le t\le T}|x(t)-\bar{x}(t)|^2\big)=0$$

Hence, $x(t)=\bar{x}(t)$ for all $t_0\le t\le T$ almost surely, concluding the proof of uniqueness of solutions.

2.6.3 Existence

We have that $x_0(t)\equiv x_0$ and, for $n=1,2,3,...$ define the Picard's iterations

$$x_n(t)=x_0+\int_{t_0}^{t}f(x_{n-1}(s),s)ds+\int_{t_0}^{t}g(x_{n-1}(s),s)d\mathbf{B}(s)$$

for $t\in[t_0,T]$, we have that $x(\cdot)\in\mathscr{M}^2([t_0,T];\mathbb{R}^n)$, by induction we have that $x_n(\cdot)\in\mathscr{M}^2([t_0,T];\mathbb{R}^n)$ since we have that

$$\mathbb{E}|x_n(t)|^2\le c_1+3K(T+1)\int_{t_0}^{t}\mathbb{E}|x_{n-1}(s)|^2ds$$

where $c_1=3\mathbb{E}|x_0|^2+3KT(T+1)$.

For any $k\ge 1$

$$\begin{aligned}\max_{1\le n\le k}\mathbb{E}|x_n(t)|^2&\le c_1+3K(T+1)\int_{t_0}^{t}\max_{1\le n\le n}\mathbb{E}|x_{n-1}(s)|^2ds\\&\le c_1+3K(T+1)\int_{t_0}^{t}\big(\mathbb{E}|x_0|^2+\max_{1\le n\le k}\mathbb{E}|x_n(s)|^2\big)ds\\&\le c_2+3K(T+1)\int_{t_0}^{t}\max_{1\le n\le k}\mathbb{E}|x_n(s)|^2ds\end{aligned}$$

where $c_2=c_1+3KT(T+1)\mathbb{E}|x_0|^2$.

Applying Gronwall's inequality we have that

$$\max_{1\le n\le k} \mathbb{E}|x_n(t)|^2 \le c_2 \exp\{3KT(T+1)\}$$

Since k is arbitrary, we have that

$$\mathbb{E}x_n(t)|^2 \le c_2 \exp\{3KT(T+1)\} \qquad \forall \;\; t_0 \le t \le T, n \ge 1.$$

We note that:

$$|x_1(t)-x_0(t)|^2 = |x_1(t)-x_0|^2 \le 2\bigg|\int_{t_0}^t f(x_0,s)ds\bigg|^2 + 2\big|\int_{t_0}^t g(x_0,s)d\mathbf{B}(s)\big|^2$$

Taking the expectation and using the linear growth condition we have:

$$\mathbb{E}|x_1(t)-x_0(t)|^2 \le 2K(t-t_0)^2(1+\mathbb{E}|x_0|^2) + 2K(t-t_0)(1+\mathbb{E}|x_0|^2) \le C$$

where $C = 2K(T-t_0+1)(T-t_0)(1+\mathbb{E}|x_0|^2)$

We claim that $n \ge 0$,

$$\mathbb{E}|x_{n+1}(t)-x_n(t)|^2 \le \frac{c[M(t-t_0)]^n}{n!}, \forall\; t_0 \le t \le T,$$

where $M = 2\bar{K}(T-t_0+1)$

By induction, we have that

$$\mathbb{E}|x_{n+1}(t)-x_n(t)|^2 \le \frac{c[M(t-t_0)]^n}{n!}$$

holds for $n+1$

We have that:

$$|x_{n+2}(t)-x_{n+1}(t)|^2 \le 2\big|\int_{t_0}^t [f(x_{n+1}(s),s)-f(x_n(s),s)]ds\big|^2 + 2\big|\int_{t_0}^t [g(x_{n+1}(s),s)-g(x_n(s),s)]\big|^2$$

We take the expectation and apply the Lipschitz condition we obtain:

$$
\begin{aligned}
|x_{n+2}(t) - x_{n+1}(t)|^2 &\leq 2\bar{K}(T - t_0 + 1)\mathbb{E}\int_{t_0}^{t} \left|x_{n+1}(s) - x_n(s)\right|^2 ds \\
&\leq M\int_{t_0}^{t} \mathbb{E}\left|x_{n+1}(s) - x_n(s)\right|^2 ds \\
&\leq M\int_{t_0}^{t} \frac{C[M(s-t_0)]^n}{n!} ds \\
&\leq M\int_{t_0}^{t} \frac{C[M(t-t_0)]^{n+1}}{(n+1)!} ds
\end{aligned}
$$

If we replace n with $n-1$ we have that:

$$
\sup_{t_0 \leq t \leq T} |x_{n+1}(t) - x_n(t)|^2 \leq 2\bar{K}(T-t_0)\int_{t_0}^{T} |x_n(s) - x_{n-1}(s)|^2 ds + 2\sup_{t_0 \leq t \leq T}\int_{t_0}^{T} |g(x_n(s), s) - g(x_{n-1}(s), s) d\mathbf{B}s|^2
$$

Taking the expectation and using the previous theorem, we have that

$$
\begin{aligned}
\left(\sup_{t_0 \leq t \leq T} |x_{n+1}(t) - x_n(t)|^2\right) &\leq 2\bar{K}(T - t_0 + 4)\int_{t_0}^{T} |x_n(s) - x_{n-1}(s)|^2 ds \\
&\leq 4M\int_{t_0}^{T} \frac{C[M(s-t_0)]^{n-1}}{(n-1)!} ds \\
&= \frac{4C[M(T-t_0)]^n}{n!} /
\end{aligned}
$$

Thus

$$
P\left\{\sup_{t_0 \leq t \leq T} |x_{n+1}(t) - x_n(t)| > \frac{1}{2^n}\right\} \leq \frac{4C[4M(T-t_0)]^n}{n!}
$$

Since

$$
\sum_{n=0}^{\infty} < \frac{4C[4M(T-t_0)]^n}{n!}
$$

, the Borel-Cantelli lemma yields for almost all $\omega \in \Omega$ there exists a positive integer

$n_0 = n_0(\omega)$ such that:

$$\sup_{t_0 \le t \le T} |x_{n+1}(t) - x_n(t)| \le \frac{1}{2^n}, \quad n \ge n_0.$$

This follows that with probability of 1, the partial sums:

$$x_0(t) + \sum_{i=0}^{n-1} [x_{i+1}(t) - x_i(t)] = x_n(t)$$

are convergent uniformly in $t \in [0, T]$.

We denote the limit by $x(t)$, we have that $x(t)$ is continuous and $\mathscr{F}_t-$ adapted. For every $t, \{x_n(t)_{n \ge 1}\}$ is a Cauchy sequence in L^2. Hence $x_n(t) \to x(t)$ in L^2.

If we let $n \to \infty$ in

$$\mathbb{E}|x_n(t)|^2 \le c_2 \exp\{3KT(T+1)\}$$

and we have

$$\mathbb{E}|x(t)|^2 \le c_2 \exp\{3KT(T+1)\}, \quad \forall \ t_0 \le t \le T.$$

Hence $x(\cdot) \in \mathscr{M}([t_0, T]; \mathbb{R}^n)$

To show that $x(t)$ satisfies the equation We obtain that

$$x(t) = \int_{t_0}^{t} f(x(s), s) ds + \int_{t_0}^{t} g(x(s), s) d\mathbf{B}(s) \tag{2.37}$$

We note that

$$\mathbb{E}\bigg| \int_{t_0}^{t} f(x_n(s), s) ds - \int_{t_0}^{t} f(x(s), s) ds \bigg|^2 + \mathbb{E}\bigg| \int_{t_0}^{t} g(x_n(s), s) d\mathbf{B}(s) - \int_{t_0}^{t} g(x(s), s) d\mathbf{B}(s) \bigg|^2$$
$$\le \bar{K}(T - t_0 + 1) \int_{t_0}^{T} \mathbb{E}|x_n(s) - x(s)|^2 ds \to 0$$

When we let $n \to \infty$ in

$$x_n(t) = x_0 + \int_{t_0}^{t} f_{n-1}(x(s), s)ds + \int_{t_0}^{t} g_{n-1}(x(s), s)d\mathbf{B}(s), \quad t_0 \le t \le T \tag{2.38}$$

we obtain

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)d\mathbf{B}(s), \quad t_0 \le t \le T \tag{2.39}$$

as desired.

Thus we have shown that the Picard's iterations $x_n(t)$ converges to the unique solution $x(t)$ of the equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)d\mathbf{B}(t)$$

2.7 Black-Scholes

Black-Scholes (BS) formula also known as Black-Scholes-Merton formula, was named after Fischer Black, Myron Scholes and Robert Merton. Myron Scholes and Robert Merton received the noble prize for Economics for laying the foundation for the most famous equation in the field of finance. [20]

Development of the Equation. To evaluate the price of a stock option we take into account the following parameters.

1. S = stock price.
2. E = exercise price.
3. t = time
4. ρ = risk-free interest rate.
5. T = time to expiration.

6. $\mu =$ average rate of growth of the stock.

7. $\sigma =$ standard deviation of log returns (volatility)

The option value can be written as

$$V(S, t, \sigma, \mu, E, T, r) \equiv V(S, t)$$

The equivalent relation holds because apart from S and t, which are the independent variables the rest of the variables are just parameters.

We denote a portfolio of one long option position by Π and a short position of some quantity, Δ of the underlying, S:

$$\Pi = V(S, t) - \Delta \cdot S. \tag{2.40}$$

We assume S follows a log-normal random walk hence we have that:

$$dS = \mu S dt + \sigma S dX$$

where $dX = N(0, 1)dt^{\frac{1}{2}}$ hence the average of $d\bar{x}^2 = dt$.

We now evaluate $d\Pi$ in equation 2.40.

$$d\Pi = dV = \Delta dS \tag{2.41}$$

The change in the value of the portfolio from t to $t+dt$ is: Using Taylor series and expanding to second order.

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 V}{\partial t^2} dt^2 + \frac{1}{2} \frac{\partial^2 V}{\partial S \partial t} dS dt \tag{2.42}$$

Using Itô's Lemma and substitution $dS^2 = \sigma^2 S^2 dt$

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \tag{2.43}$$

Hence we have:

$$d\Pi = \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 \S^2 \frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial S}dS - \Delta dS. \tag{2.44}$$

If we choose $\Delta = \frac{\partial V}{\partial S}$ the randomness (risk) is reduced to zero, then is referred to as Delta hedging.

Thus changes are completely risk-less, then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account. Thus

$$d\Pi = r\Pi dt$$

and that results in the Black-Scholes-Morten equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.45}$$

Remark

- There is no effect of the μ.
- There is no information about the kind of option being considered. Hence all the options follow the same equation. The information of different option would be found in the boundary condition and initial condition that would be included to completely solve the Black-Scholes-Morten equation.
- The equation is a parabolic equation.

Chapter 3

The BNS-Gamma-Ornstein-Uhlenbeck Process

Beginning with this chapter, we present the formulation of the BNS-Gamma-Ornstein-Uhlenbeck Process which will pave the way for its parameter estimation. We will first look at the Gamma distribution, then go ahead and look at the superposition of Gamma-OU process and finally present the BNS-Gamma-Ornstein-Uhlenbeck Process.

3.1 Definitions

Definition 9 Gamma function

$$\Gamma(a) = \int_0^\infty x^{a-1} \exp(-x)dx, \qquad a > 0. \tag{3.1}$$

It turns out that :

- $\Gamma(n) = (n-1)!$ for n a positive integer.
- $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

A pdf related to the Gamma function is gotten by normalizing. 3.5. This is achieved by dividing the equation by $\Gamma(a)$ and we get:

$$1 = \int_0^\infty \frac{1}{\Gamma(a)} x^{a-1} \exp(-x) dx, \tag{3.2}$$

This $\frac{1}{\Gamma(a)} x^{a-1} \exp(-x)$, is called the $\Gamma(a, 1)$ probability density function (pdf). A more general one is obtained by making change of variables and that results in the Gamma distribution.

Definition 10 A Gamma distribution random variable X with parameters α and β has a density function [21]

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), \quad x, \alpha, \beta > 0. \tag{3.3}$$

The Gamma distribution has a corresponding characteristic function defined as

$$\varphi(x) = \left(1 - \frac{ix}{\beta}\right)^{-\alpha} \tag{3.4}$$

$X \sim \Gamma(a, b)$ is infinitely divisible distribution. This enables us to conclude that it is a Gamma Lvy process.
A Self-decomposable distribution is a distribution that belongs to a subclass of infinitely divisible distribution.

Definition 11 *A random variable X is self-decomposable if, for all f , its characteristic function can be decomposed as,*$\varphi_X(z) = \varphi_{cX}(z) \times \varphi_c(z)$, *for some characteristic function* $\varphi_c(z), Z \in \mathbb{R}$.

3.2 Superposition of the Gamma-OU process

Definition 12 A Gamma distribution random variable X with parameters $\alpha > 0$ and $\beta > 0$ has a density function

$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp\big\{-\beta x\big\} \quad , x \geq 0 \tag{3.5}$$

It is self-decomposable, hence we have a stationary OU process$\{Y(t), t/geq0\}$ with gamma marginal distribution. It characteristic function is given as:

$$\varphi_X(Z) = \mathbb{E}[\exp\{izX\}] \tag{3.6}$$

$$= \big(1 - \frac{iz}{\beta}\big)^{-\alpha} \tag{3.7}$$

which has a convolution property on α according to [15] Grahovac et al. The stationary Gamma OU process $\{Y(t), t/geq0\}$ and a corresponding gamma marginal distribution has a cumulant generating function defined as :

$$\kappa_X(z) = \log \mathbb{E}\big[\exp\{izY(t)\}\big] \tag{3.8}$$

$$= -\alpha \log\big(1 - \frac{iz}{\beta}\big) \tag{3.9}$$

$$= \sum_{m=1}^{\infty} \frac{\alpha(iz)^m}{m\beta^m} \tag{3.10}$$

$$= \sum_{m=1}^{\infty} \frac{\alpha(m-1)!}{\beta^m} \frac{(iz)^m}{m!} \quad z < \beta \tag{3.11}$$

$$\tag{3.12}$$

The characteristic function $\kappa_X(Z)$ is analytic around the origin since at any point in the neighborhood of the origin, the Taylor series converges to the value of the function.

$$\kappa_X^{(m)} = \frac{\alpha(m-1)!}{\beta^m} \tag{3.13}$$

Hence for $m = 1, \kappa_X^{(1)} = \mathbb{E}(Y(t)) = \dfrac{\alpha}{\beta}$ and when $m = 2, \kappa_X^{(2)} = Var(Y(t)) = \dfrac{\alpha}{\beta^2}$. The corresponding covariance function is given as:

$$R_{Y(t)} = \mathrm{Var}(Y(t)) \exp\{-\lambda|t|\} = \frac{\alpha}{\beta^2} \exp\{-\lambda|t|\}. \tag{3.14}$$

Thus, we can now find the superposition of the stationary Gamma OU type process.We take $\{Y(t), t/geq0\}, k \geq 1$ are independent stationary OU Gamma processes with marginals $\Gamma(\alpha_k, \beta), k \geq 1$ such that $\alpha_k = \alpha k^{-(1+2(1-H))}, H \in (\frac{1}{2}, 1)$.

Let $\alpha(H) = \sum_{k=1}^{\infty} k^{-(1+2(1-H))}$ be Riemann zeta-function $\zeta(z)$ with $z = 1+2(1-H)$ and the same treatment for $IG-OU$ and the conditions been satisfied we have the superposition OU given by:

$$Y_{\infty}(t) = \sum_{k=1}^{\infty} Y^{(k)}(t), \quad t \geq 0, \tag{3.15}$$

has a marginal distribution $\Gamma\bigg(\sum_{k=1}^{\infty} \alpha_k, \beta \bigg)$ and the covariance function is given as:

$$R_{Y_{\infty}}(t) = \sum_{k=1}^{\infty} \mathrm{Var}\big[Y^{(k)}(t)\big] \exp\{-\lambda(k)t\} \tag{3.16}$$

$$= \frac{1}{\beta^2} \sum_{k=1}^{\infty} \alpha_k \exp\{-\lambda(k)t\}. \tag{3.17}$$

A particular choice of $\lambda^{(k)} =$, we obtain a long range dependent stationary Gamma-OU process.

Simulation Algorithm for Gamma-OU process

The Gamma-OU process simulation through the Background Driving Lvy Process(BDLP) is given by the following algorithm:

1. Simulation of the Poisson process $\big(N(t), t \geq 0\big))$ with intensity parameter $a\lambda t$ in the time points $n\Delta t, \quad n = 1, 2, 3, ...$

(a) Simulation of the uniform independent random numbers $u_n \sim Uniform(0,1)$

(b) Simulation of the independent exponential random numbers $x_n \sim Exp(b)$

$$x_n = \log(u_n)/b$$

.

(c) Let:

$$s_0 = 0,\ s_n = s_{n-1} + x_n$$

(d) Sample the path of the Poisson process $N = (N_t)_{t\geq 0}$ in the time points $n\Delta t$. $N_0 = 0,\ N_{n\Delta t} = \sup(k : s_n \leq n\Delta t),\ n > 1.$

2. Sample the path of the Gamma-Ou process $X = \big(X_t\big))_{t\geq 0}$ in the time points $n\Delta t$.

$$X_{n\Delta t} = \exp\left(-\lambda\Delta t\right)X_{(n-1)\Delta t} + \sum_{n=N_{(n-1)\Delta t}}^{N_{n\Delta t}} X_n, \quad X_0 > 0$$

3.3 BNS-Gamma-OU Model

We now develop the Barndorff-Nielsen and Shephard (BNS) model that would be used to predict the value of the high-frequency data at a time t and then use the model to predict the path of the Stochastic Differential Equation.

The BNS model is reformulated from the Black-Scholes(BS) [16]. Stochastic Differential Equation discussed in Chapter 2. For the Black-Scholes the log of the option price S at time t in 2.45 satisfy the equation .

$$d\log(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma d\bar{W}_t, \quad \log(S_0). \tag{3.18}$$

We let $\log(S_t) = Z_t$ and $\log(S_0) = Z_0$ 3.18 becomes:

$$dZ_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma d\bar{W}_t, \quad Z_0 \tag{3.19}$$

Since the volatility is dynamic and random we let σ be stochastic.The volatility follows a Gamma distribution.The price of an asset will jump decrease when there a jump in volatility takes place. When there the stock price is stable asset price moves continuously, along with the continuously decreasing volatility.

Barndorff-Nielsen and Shephard proposed a model where σ^2 is defined by an Ornstein Uhlenbeck process [4]. [23]

Thus σ^2 satisfy the Stochastic Differential Equation

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + d\bar{z}_{\lambda t} \tag{3.20}$$

such that $x = \{\bar{x}_t, t \geq 0\}$ is a Lvy process with a subordinator. Assuming that $\bar{x}$ has density for it Lvy measure and it has no drift. We thus have:

$$dZ_t = \left(\mu - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t d\bar{W}_t + \rho d\bar{Z}_{\lambda t}, \quad \log(S_0) = Z_o = x_0. \tag{3.21}$$

where ρ serves as the positive leverage, while S_t is Wiener process. The sample path for a stock price is given by:

$$Z_{n\Delta t} = Z_{(n-1)\Delta t} + \big(\mu - \frac{\sigma_{n\Delta t}^2}{2}\big)\Delta t + \sigma_{n\Delta t}\big(\Delta t\big)^{\frac{1}{2}}\nu_n + \rho\big(Z_{\lambda n\Delta t} - Z_{\lambda(n-1)\Delta t}\big) \tag{3.22}$$

Simulation Algorithm for BNS-Gamma-OU process

BNS Model with Gamma-OU process is simulate by taking the following steps:

1. Generate Poisson process by the method above with the parameter of $a\lambda$.

2. Calculate the number of jumps in each interval.

$$Z_{n\Delta t} = Z_{(n-1)\Delta t} + \big(\mu - \frac{\sigma^2_{n\Delta t}}{2}\big)\Delta t + \sigma_{n\Delta t}\big(\Delta t\big)^{\frac{1}{2}}\nu_n + \rho\big(Z_{\lambda n\Delta t} - Z_{\lambda(n-1)\Delta t}\big) \tag{3.23}$$

3. Obtain S_t, stock price, by $S_t = \exp(Z_t)$;

4. Obtain option price by $\exp(-\mu t)\max(S_t - X_t, 0)$.

Chapter 4

Numerical Results and Application To Real Live Data

4.1 Introduction

This chapter is devoted to documentation of numerical results that demonstrate the performances of Gamma-OU and BNS-Gamma-OU models and compare them by applying them to high frequency data arising from finance and geology, that is stock and earthquake respectively. MATLAB version R2020a was used to perform the simulations.

4.2 Application to Data from Earthquakes

The algorithm for superposition of Gamma-OU process was implemented. The data used for the simulation was earthquake data from California for the year 1973. This data was extracted from the data set provided by the United States Geological Survey. The area associated with the data is carefully selected to avoid disturbance due to noise from unrelated activities. The most crucial factor was that the area should not be too small else fitting of the data would be close to impossible and should not be too big to prevent undue interference from activities that are not related to the seismic under study. The dimensions of the area taken which was a quadrilateral selected as $\pm(0.1^o - 0.2^o)$ and $\pm(0.2^o - 0.4^o)$ in latitude and longitude respectively. The recorded magnitude of the earthquake assumed that the magnitude has no dimension and it ranges between $1-12$. The moment magnitude is what is recorded.

Ten random paths were generated and their mean was computed and plotted as shown with the legend in black. 4.1 illustrates the results.

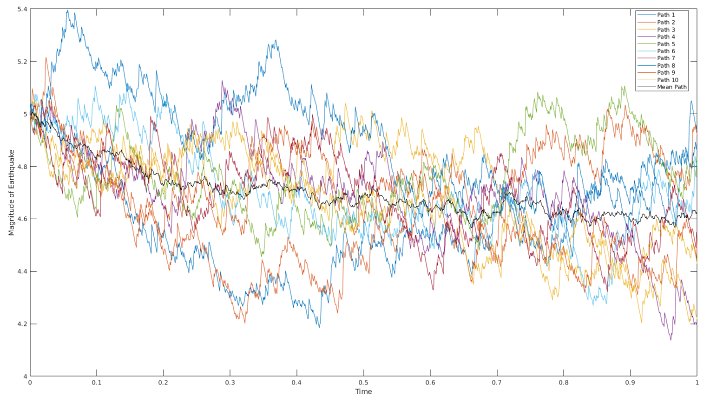

Figure 4.1: Sample paths of Gamma-OU process for Earthquake

With the same data set, we implemented the BNS-Gamma-OU model using the algorithm discussed under section 3.3.

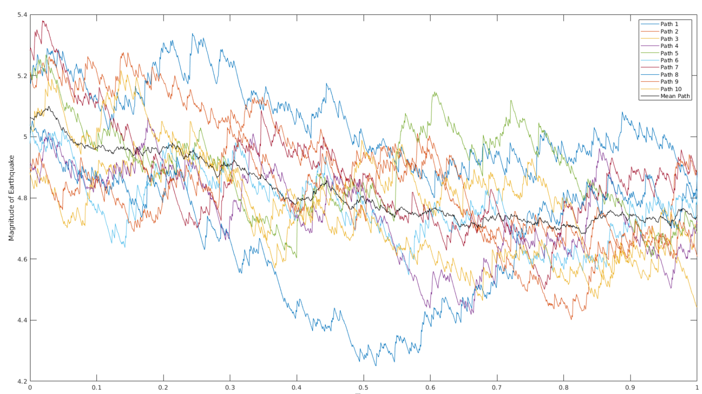

Figure 4.2: Sample paths of BNS-Gamma-OU process for Earthquake

4.3 Application to Data from Stock Market

The data set used for the simulation is the Standard and Poors 500 index frequently referred to as S & P 500. This data is a high frequency data. The log returns of S & P 500 are what was used. S & P 500 index is one of the most widely traded and closely watched stock index. It is regarded as the best single goal of the United States of America equities' market. [9]

The S & P 500 index includes a representative sample of 500 leading companies in the US economy. Although the S & P 500 focuses on the large cap segment of the market with over 75 percent coverage of the US equities. It is viewed as the proxy of the market.[2]

The algorithm for of Gamma-OU process was implemented for the S & P 500 price data available with one-minute frequency from August 1st,1997 through to September 28th, 2005.

We once again simulated 10 results and found the mean and plotted it as well, as shown in figure 4.3.

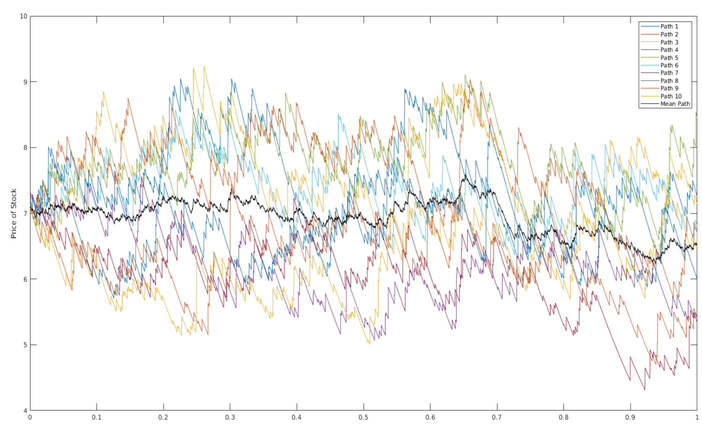

Figure 4.3: Sample paths of Gamma-OU process for Stock price

The data set of S & P 500 was used to simulate the BNS-Gamma-OU model, the same routine was done, 10 different paths were generated and the mean of these paths was found and plotted as well. The plot is shown in figure 4.4.

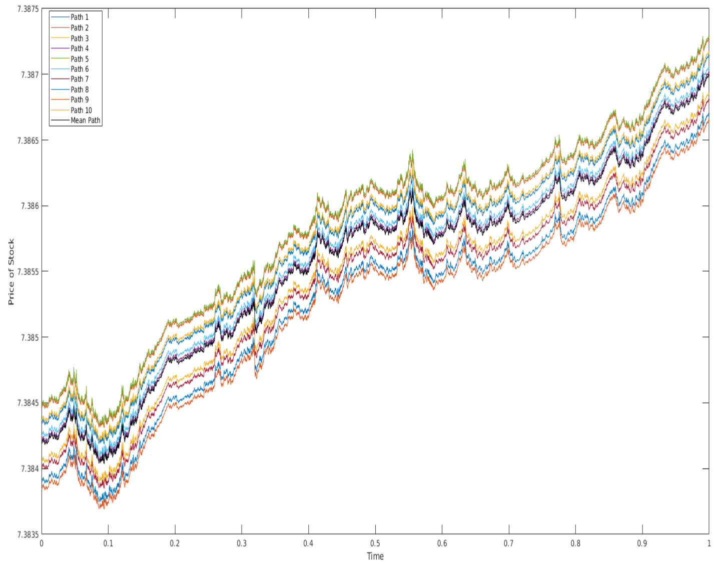

Figure 4.4: Sample paths of BNS-Gamma-OU process for Stock price

It is significant to note that the path taken by the BNS-Gamma-OU model was almost that of the original. We noted that no matter the random generator used to path was almost the same. When they were all normalized and plotted, all the paths almost lied on each other.

4.4 Discussion of Results

To access the performance of the two models, we used the method of calibration and also calculating the residuals. For the earthquake, 100 of the mean simulated points from Gamma-OU were selected at random and plotted against the corresponding time on the BNS-Gamma-OU and the actual values. It was seen from figure 4.5 that the BNS-Gamma-OU model points were significantly close to that of the actual magnitude of the earthquake than that of the Gamma-OU.

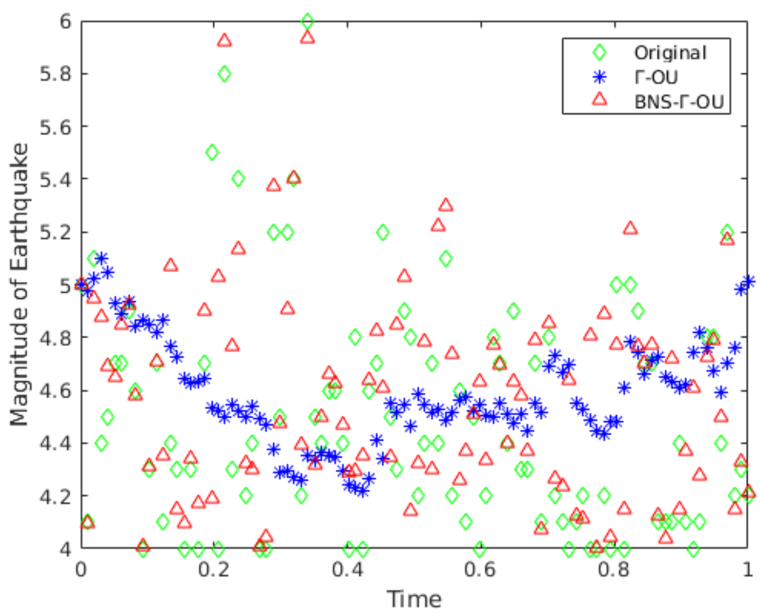

Figure 4.5: BNS-Gamma-OU and Gamma-OU calibrations for the Magnitude of Earthquake

The residuals were calculated using the Root Mean Square Error (RMSE) we had for Gamma-OU 0.1767 and that of BNS-Gamma-OU we had 0.1023. shown in 4.1

Similarly we performed the calibration using 100 data points that were selected randomly from the actual data set of S & P 500 and plotted against the corresponding mean simulated results from Gamma-OU and BNS-Gamma-OU. We found that the simulated results from

Gamma-OU were close to the actual prices but the results from BNS-Gamma-OU were very close.in most cases than that of the Gamma. This is shown figure in 4.6. This was also

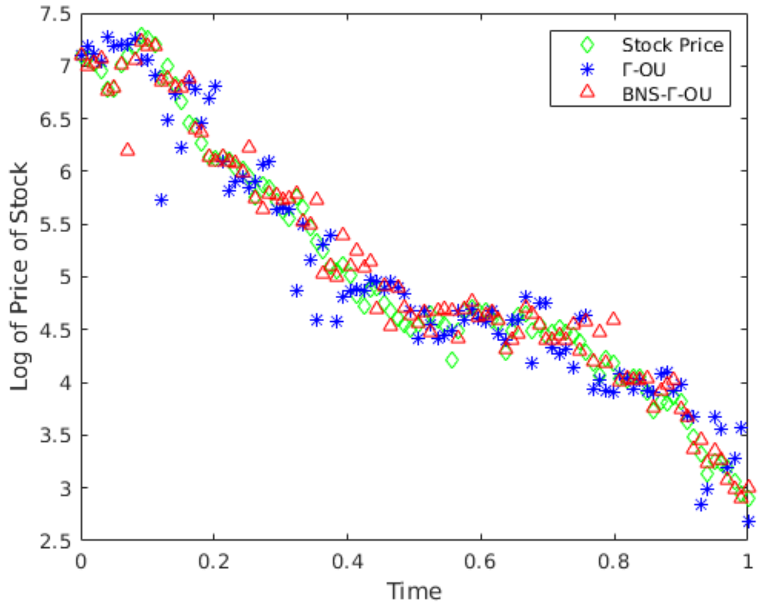

Figure 4.6: BNS-Gamma-OU and Gamma-OU calibrations for the stock price.

Table 4.1: Root Mean Square Error Comparison

RMSE	**Gamma-OU**	**BNS-Gamma-OU**
Earthquake	0.1767	0.1023
Option Pricing	0.2173	0.1006

further collaborated when the residuals were calculated using the Root Mean Square Error (RMSE). For Gamma-OU the RMSE was found to be 0.2173 and that of BNS-Gamma-OU was less than half of that of the Gamma-OU which was 0.1006 as shown in 4.1.

Chapter 5

Conclusion and Future Work

5.1 Introduction

In this chapter we make concluding remarks by highlighting the main results of this work and look at other areas where further work can be done to augment the results we have arrived at.

5.2 Conclusion

We developed a more superior model to Gamma-OU model to model high frequency data. We illustrated how the BNS Gamma-OU has a better Root Mean Square Error as compared to the Gamma-OU model, also when both were calibrated against the original data once again the BNS Gamma-OU was very close to the real data. It is worthy to note that since both the Gamma-OU and BNS Gamma-OU models are mean-reverting, hence they tend to drift towards their mean function. The BNS-Gamma-OU model will converge faster to the mean function as compared to the Gamma-OU model. BNS-Gamma-OU model proved to be better than the Gamma-Ou model and Black-Scholes models for predicting option price. The Black-Scholes model for predicting the earthquake data did not produce any significant results but the BNS-Gamma-OU and Gamma-OU did. We thus conclude that the BNS-Gamma-OU model is excellent to predict high frequency data such as option price and magnitude of earthquake.

5.3 Future Work

A natural extension of this present work is to look at the BNS-Gamma-OU Model. Further studies can be done on BNS models that are driven by other processes instead of Gamma-OU, such as Meixner-OU, IG-OU and NIG-OU processes.

Secondly, further studies could be done in applying Lvy models to high frequency data that the time can be stochastic.

An issue that remains to be addressed is how is the parameter ρ in the BNS-Gamma-OU model. What range would it give an accurate solution further works could be done to study this parameter and how it relates to specific type of high frequency data.

Finally, we can look at the convergence rate of both the Gamma-OU and BNS-Gamma-OU model and the factors that affect the order of convergence.

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Curriculum Vitae

Mandela Bright Quashie was born on December 25, 1992 to Emmanuel Asare Asiedu and Mercy Quashie. He entered Kwame Nkrumah University of Science and Technology (KNUST) after High school in 2011. He graduated in the top 10 in 2016 and was privileged to serve as a research and teaching Assistant. While pursuing his degree at KNUST he served at various leadership position including Lane Representative of the University Hall, the Editor of University Hall, The Deputy Judicial Chair of AMS-KNUST, a member of the Student Parliament and the JCR President of the University Hall (Katanga).

After his National Service in Ghana, he was part of the select few among thousands of applicants scholarship to purse a masters' degree at the African Institute for Mathematical Sciences in 2017, there he was elected as the MasterCard Scholars. He graduated in 2018 having met Field Medalist and Noble Laureates.

In pursuit of becoming a research scientist and academician in the field of applied mathematics, he entered the Graduate School at The University of Texas at EL Paso in the fall of 2018 for another masters degree in Mathematical Sciences to lay a foundation for a PhD in Applied Mathematics. He was a regular contributor and participant in a number of workshops and seminars under Mathematics, Computational Science, Data Science and Statistics. He won a research grant under MasterCard Foundation Research Fund. While pursuing a master's degree in Mathematics he worked as a Teaching and Research Assistant. He was also earned a Graduate Certificate in Applied and Computational Mathematics whiles pursuing his studies at UTEP

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