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Laplacian Spectra Of Kneser-Like Bipartite Graphs

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to my

MOTHER

with love

LAPLACIAN SPECTRA OF KNESER-LIKE BIPARTITE GRAPHS

by

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THESIS

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Abstract

Given $a, b \in \mathbb{N}$ such that $a > b$ we define a Kneser-like bipartite graph $G(a, b)$, whose two bipartite sets of vertices represent the a-subsets and b-subsets of $S = \{1, \ldots, a + b + 1\}$, and whose edges are pairs of vertices X and Y such that $X \cap Y = \emptyset$. We prove that the eigenvalues of the Laplacian matrix of graphs $G(a, 1)$ are all nonnegative integers. In fact, we describe these eigenvalues, and their respective multiplicities.

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Chapter 1

Introduction

1.1 Historical Background

In mathematics it is often convenient to represent a graph as a matrix. Adjacency and Laplacian matrices are often common ways of representing graphs. Throughout recent decades mathematicians have shown how Laplacian matrices show important properties of graphs.

The use of Laplacian matrices (which we define shortly) were first motivated by the famous Matrix Tree Theorem which tells us the number of spanning trees of a graph G on n vertices is equal to the absolute value of the determinant of any $(n-1) \times (n-1)$ submatrix of its Laplacian. [3]

The spectrum of these Laplacian matrices have also been studied profoundly in recent decades by mathematicians. A special, interesting case are Laplacian integral graphs, which are simply graphs whose Laplacian spectrum consists entirely of integers. Dr. Russell Merris, for example, has shown that if graph G is connected, r-regular, and is Laplacian integral graph on n vertices, then the spectrum of G is Laplacian integral if and only if $G = K_n$. [2]

Another important property of a graph that we get from its Laplacian matrix is its algebraic connectivity, which is simply the second smallest eigenvalue of the matrix. The eigenvectors corresponding to this eigenvalue are now widely known as Fiedler Vectors. These Fiedler vectors have been found to be useful in algorithms for distributed memory parallel processors.

In this thesis we analyze Laplacian matrices of certain bipartite graphs that Dr. Art

Duval from the University of Texas at El Paso and Dr. Jeremy Martin from the University of Kansas discovered. We explain why these matrices are Laplacian integral. In fact, we will see that Laplacian spectra of these graphs have strictly nonnegative integers, a rare phenomenon.

1.2 Preliminaries

As it is common in algebraic literature, we will denote J as the matrix of all ones and $\mathbf{j}$ as the vector of all ones.

Also, as we will be dealing with binomial coefficients quite often, it is important to make an important distinction. $\binom{n}{k}$ are all combinations made from choosing $k \in \mathbb{N}$ elements of $n \in \mathbb{N}$, whereas $\binom{N}{k}$ represents all the combinations of k-subsets ($k \in \mathbb{N}$) of set N. We will clearly state what set N will be in each case in order to avoid confusion.

Definition 1.1. A *graph* G is a pair of sets (V, E), where V is a finite non-empty set of elements called *vertices*, and E is a set of unordered pairs of distinct vertices called *edges*.

Definition 1.2. If the vertices of a graph G can be partitioned into two non-empty sets so that no edge joins two vertices in the same set, then G is called *bipartite*. The two sets are called *partite* sets.

Definition 1.3. The *degree* of a vertex v, denoted $\deg(v)$, is the number of vertices adjacent to v.

Definition 1.4. Let G be a graph on n vertices labelled $1, \ldots, n$. The *adjacency matrix* of G on n vertices is the $n \times n$ matrix $A = |a_{ij}|$ where

$$a_{i,j} = \begin{cases} 1, & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{if } i = j, \text{ or } i \neq j \text{ and } i \text{ is not adjacent to } j \end{cases}$$

Definition 1.5. The *degree matrix* $D(G) = \operatorname{diag}(\deg v : v \in V)$ is the diagonal matrix indexed by V with the vertex-degrees on the diagonal.

Definition 1.6. The difference

$$L(G) = D(G) - A(G)$$

is called the *Laplace matrix* (or *Laplacian*) of G.

In other words, a Laplace matrix will have (-1)s where the adjacency matrix had 1s, the degree of each vertex $v \in V$ down the diagonal, and 0s everywhere else. With that said we can use an equivalent definition of the Laplace matrix as follows:
Let G be a graph on n vertices labelled $1, \ldots, n$. The *Laplacian matrix* of G is the $n \times n$ matrix $L = [\ell_{i,j}]$ where

$$\ell_{i,j} = \begin{cases} -1, & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ d_i, & \text{if } i = j \end{cases}$$

Definition 1.7. An $m \times m$ symmetric real matrix M is said to be *positive semidefinite* or non-negative definite if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n$. Formally,

$$M \text{ positive semi-definite} \Longleftrightarrow x^T M x \geq 0 \text{ for all } x \in \mathbb{R}^n$$

Definition 1.8. The set of all eigenvalues of a Laplace (or Laplacian) matrix L is known as the *Laplacian spectrum* of L.

Recall that the nullity of a matrix is the dimension of the null space of a matrix and rank is the dimension of the row space of a matrix. We now introduce a very well-known theorem involving the two terms.

The Fundamental Theorem of Linear Algebra. *For any $k \times l$ matrix M, $Rank(M) + Nullity(M) = l$.*

This next lemma is a well known theorem which we now state and show the proof, as it is instructive and straightforward.

Lemma 1.9. *The eigenvalues of the matrix* J_m *are* 0 *and* m *with multiplicities* $m-1$ *and* 1*, respectively.*

Proof. Let the square matrix of all ones, J_m, be given. First, we can easily see that m is an eigenvector of J_m with the corresponding eigenvector $\mathbf{j}$ as,

$$J_m\mathbf{j} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} m \\ m \\ \vdots \\ m \end{bmatrix} = m\mathbf{j}.$$

The solutions $\mathbf{x}$ of $J_m\mathbf{x} = \mathbf{0}$ satisfy

$$x_1 = -x_2 - x_3 - \cdots - x_m.$$

Therefore, every vector in the null space is of the form

$$\mathbf{x} = \begin{bmatrix} -x_2 - x_3 - \cdots - x_m \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_m \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

for some scalars $x_2, x_3, \dots, x_m$. It follows that,

$$\left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{m-1} \right\}$$

is a basis of the null space of J_m. □

Chapter 2

Vertices and Edges

In this chapter we concern ourselves with the construction of $G(a, b)$ in anticipation of analyzing their Laplacian matrices. We look over some useful properties of the graph such as the number of vertices, the degree of vertices and connectedness. A concrete example is shown to clarify the nature of $G(a, b)$.

2.1 Construction of graphs $G(a, b)$

Given numbers a and b, where $a, b \in \mathbb{N}$, and $a > b$, we can construct the bipartite graph $G(a, b)$ in the following way:

We begin with the set $S = \{1, 2, \ldots, a+b+1\}$. The vertices of the first partite set will be all the sets of cardinality a that can be chosen from set S. The set of all such vertices will be denoted as $\mathcal{A}$.

Similarly, the vertices of the second partite set will be all the sets of cardinality b that can be chosen from set S. The set of all such vertices will be denoted as $\mathcal{B}$. An edge is formed between a vertex $X \in \mathcal{A}$ and a vertex $Y \in \mathcal{B}$ if and only if $X \cap Y = \emptyset$.

Every vertex $X \subset \mathcal{A}$ has cardinality a and every vertex $Y \subset \mathcal{B}$ has cardinality b. We denote these graphs by $G(a, b)$, furthermore, we denote the Laplacian matrix of the graph $G(a, b)$ by $L(G(a, b))$ from this point on. Also, throughout this thesis we let $n = a + b + 1$.

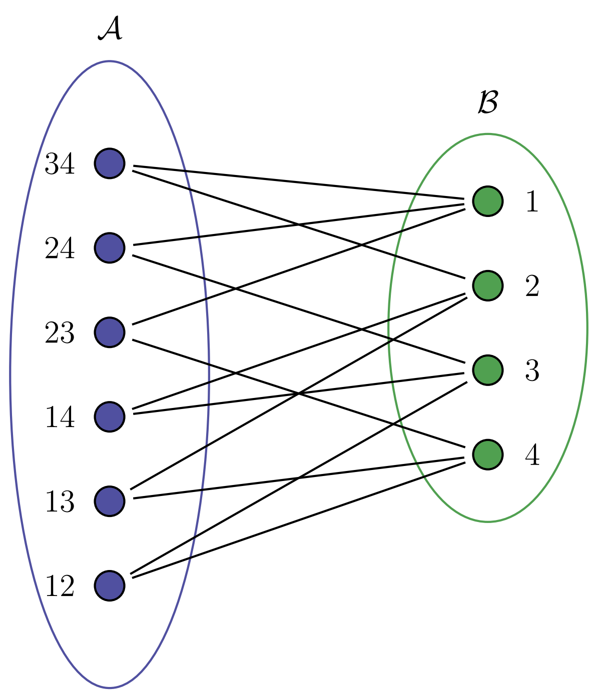
We will also refer to the cardinality of sets $\mathcal{B}$ and $\mathcal{A}$ as β and α, respectively, from this point forward. Finally, as commonly seen in combinatorial matrix books, $\sigma(G)$ represents the spectrum of $L(G)$.

2.2 $G(2,1)$ Example

Given $a = 2$ and $b = 1$ we can construct $G(2,1)$ in the manner explained above in Section 2.1.

Starting with the set $S = \{1,2,3,4\}$ we can begin generating vertices. The vertices of the first partite set will be all the sets of cardinality $a = 2$ that can be chosen from set S. We thus get $\binom{4}{2} = 6$ vertices. These 2-subsets of S are,
$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\} = \mathcal{A}$.

Similarly, the vertices of the second partite set will be all the sets of cardinality $b = 1$ that can be chosen from set S. We thus get $\binom{4}{1} = 4$ vertices. These 1-subsets are $\{\{1\},\{2\},\{3\},\{4\}\} = \mathcal{B}$. Now, an edge is formed between a vertex $X \subset \mathcal{A}$ and a vertex $Y \subset \mathcal{B}$ if and only if $X \cap Y = \emptyset$. This finally gives us $G(2,1)$ as shown below.



2.3 Number of Vertices of $G(a, b)$

Theorem 2.1. *Given $a, b \in \mathbb{N}$ where $a > b$, the number of vertices of $G(a, b)$ is*

$$\binom{a+b+2}{b+1} = \binom{a+b+2}{a+1}.$$

Proof. Let $a, b \in \mathbb{N}$ where $a > b$ be given. From here we get that set $S = \{1, 2, \ldots, a+b+1\}$ from which our vertices will form. By using the symmetry and Pascal's identity of binomial coefficients we get our desired results through a series of equalities. We note that the number of vertices is,

$$\binom{n}{a} + \binom{n}{b} = \binom{n}{n-a} + \binom{n}{b} = \binom{n}{b+1} + \binom{n}{b} = \binom{n+1}{b+1} = \binom{a+b+2}{b+1}$$

or

$$\binom{n}{a} + \binom{n}{b} = \binom{n}{a} + \binom{n}{n-b} = \binom{n}{a} + \binom{n}{a+1} = \binom{n+1}{a+1} = \binom{a+b+2}{a+1}.$$

□

2.4 Degree of Each Vertex of Graph $G(a, b)$

Theorem 2.2. *The degree of every vertex $Y \in \mathcal{B}$ is $a + 1$.*

Proof. Let set $S = \{1, 2, \ldots, a + b + 1\}$ and let Y (a b-subset of S) be an arbitrary vertex of $\mathcal{B}$. Because vertex $Y \subset \mathcal{B}$ cannot share an edge with any vertex $X \subset \mathcal{A}$ that has an element of S in common, we remove the elements in the set (or vertex) Y from the set S, which are a total of b elements removed. Then, we choose combinations of length a from the remaining elements in S, as these vertices will share an edge with Y. The degree of vertex $Y \subset \mathcal{B}$ will then be,

$$\binom{n-b}{a} = \binom{a+b+1-b}{a} = \binom{a+1}{a} = a + 1.$$

□

Theorem 2.3. *The degree of every vertex $X \in \mathcal{A}$ is $b+1$.*

Proof. In this proof, similar to the previous, we let set $S = \{1, 2, \ldots, a+b+1\}$ and let X (an a-subset of S) be an arbitrary vertex of $\mathcal{A}$. Because vertex $X \subset \mathcal{A}$ cannot share an edge with any vertex $Y \subset \mathcal{B}$ that has an element of S in common, we remove the elements in the set (or vertex) X from the set S, which are a total of a elements removed. Then, we choose combinations of length b from the remaining elements in S as these vertices will share an edge with X. The degree of vertex $X \subset \mathcal{A}$ will then be

$$\binom{n-a}{b} = \binom{a+b+1-a}{b} = \binom{b+1}{b} = b+1.$$

□

2.5 Connectedness of $G(a, b)$

Definition 2.4. A graph is *connected* if there exists a path between every pair of distinct vertices.

Definition 2.5. The *distance* between two vertices X and Y is the length of the shortest path between X and Y.

Theorem 2.6. *$G(a, b)$ is connected and thus, has one component.*

The idea is to show connectedness between two arbitrary vertices by following a path that jumps from vertices in $\mathcal{A}$ to vertices in $\mathcal{B}$ (and vice versa) in a progressive manner until the path is completed.

Proof. Let A and A' be arbitrary vertices in $\mathcal{A}$. Our goal is to find a connected path from A to A'. To be able to so, we will denote $p \leq a$ as the number of elements of A that are not in A'. Given A_i we define $A_{i+1} = (A \setminus \{v\}) \cup \{a'\}$ such that $a' \in A'$, $a' \notin A$ and $v \notin A'$. Therefore, the number of elements of A_{i+1} that are not in A' is $(p-1)$. We define $B_i = \overline{A_i \cup A_{i+1}}$ for $i \in \{1, 2, ..., \binom{n}{b}\}$. This is not difficult to see as $A_i = A$ and A_{i+1} have

$a+1$ distinct elements of S, and $n-(a+1)=b$, so there must be a vertex B_i that is adjacent to both vertices. We now find a path from A to A'.

If $p=1$, then $A=A_1$ and A' are adjacent to B_1 as $A'=A_{1+1}$.

If $p>1$, then we follow the connected path $A_1, A_{1+1}, \ldots, A_{1+(p-2)}, A_{1+(p-1)}, A_{1+p}=A_k$ that is guaranteed to be connected by vertices B_i.

From Theorem 2.2 we saw that every vertex $Y \in \mathcal{B}$ is connected to a vertex in $\mathcal{A}$. Therefore, by showing that there is a path between any two arbitrary vertices in $\mathcal{A}$, we have shown that $G(a,b)$ is connected. □

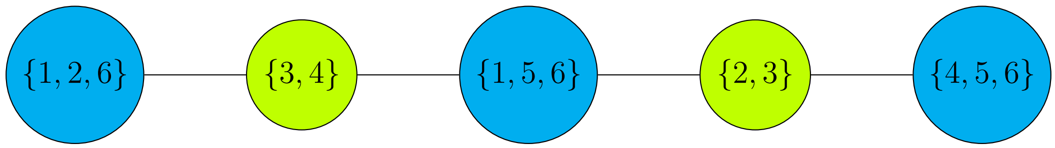
Remark. $G(a,b)$ can have a maximum diameter of $2p$, and p is bounded above by a. Therefore, $\operatorname{diam}(G) \leq 2a$.

2.6 Example showing $G(3,2)$ is Connected

As stated in Theorem 2.6, it suffices to show that there is a path between any two vertices in a partite to show the whole graph is connected. For this example we look at $G(3,2)$. We will show that vertices $A=\{1,2,6\}, A'=\{4,5,6\} \in \mathcal{A}$ are indeed connected.

Here, A is of length $a=3$ and has one element of S in common with A', meaning that $p=2$ in this example. Now, following the explanation of the proof above, we first find a vertex $A_{1+1} \in \mathcal{A}$ that is connected to $A=A_1$ and agrees with A_1 in every element but one, call it a'. That one element of A is an element of vertex A' that is not in A, so $a'=\{5\}$. Now, $A_{1+1}=\{1,5,6\}$. Note how by comparing our new vertex A_{1+1} with A' we now see that these vertices disagree in only $1=p-1$ element now.

To show $A=A_1$ is connected to A_{1+1}, we find a vertex $B_1 \in \mathcal{B}$ that shares an edge with both vertices. Again, from the proof of Theorem 6, we know vertex $B_1=\overline{A_1 \cup A_{1+1}}$, so $B_1=\{3,4\}$. We repeat this process and see that vertex $A_1=\{1,2,6\}$ is connected to vertex $B_1=\{3,4\}$ which is connected to vertex $A_{1+1}=\{1,5,6\}$. From here on we follow the connected path $B_{1+1}=\{2,3\}, A_{1+2}=A'=\{4,5,6\}$, showing that vertex A is connected to vertex A'.


$\{1,2,6\}$
$\{3,4\}$
$\{1,5,6\}$
$\{2,3\}$
$\{4,5,6\}$

Chapter 3

Laplace Eigenvalues of $G(a, 1)$

In this chapter we review properties already known about Laplacian matrices and go into the investigation as to why the Laplacian matrix of $G(a, 1)$ always has integer eigenvalues. In fact, we prove that these Laplacian eigenvalues are 0, $a + b + 2$, $b + 1$, 1, and $a + 2$ with multiplicities $1, 1, \binom{n}{a} - \binom{n}{b}, a + 1$, and $a + 1$, respectively.

3.1 $L(G(2, 1))$ Example

Recall from Section 2.2 the list of all vertices from the two partite sets
$\mathcal{A} \cup \mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.
The Laplacian matrix of $G(2, 1)$ will then be,

$$L(G(2,1)) = \left(\begin{array}{cccc|cccccc}
3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 3 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\
0 & 0 & 3 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 3 & -1 & -1 & 0 & -1 & 0 & 0 \\
\hline
0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right)$$

As we can observe, the Laplacian matrix of $G(2, 1)$ can be split into four smaller matrices to form a nice block-form matrix,

$$\begin{pmatrix} 3I & C \\ C^T & 2I \end{pmatrix}.$$

3.2 Laplacian Matrix Properties

In Section 1.2, we saw that by definition, all Laplacian matrices are symmetric, meaning that if we call the top right matrix of the block form C, then the lower left matrix of the block form must be C^T. Also, because the degree of every vertex of $\mathcal{B}$ is $(a+1)$ and the degree of every vertex of $\mathcal{A}$ is $(b+1)$ (as shown in Theorems 2.2 and 2.3), we can generalize the block form of $L(G(a,b))$ to

$$\begin{pmatrix} (a+1)I & C \\ C^T & (b+1)I \end{pmatrix}.$$

This block form will be of great use for proofs to come.

We now move on to state the dimensions of these submatrices. The dimensions of matrices $(a+1)I$, C, C^T, and $(b+1)I$ are $\binom{n}{b}\times\binom{n}{b}$, $\binom{n}{b}\times\binom{n}{a}$, $\binom{n}{a}\times\binom{n}{b}$, $\binom{n}{a}\times\binom{n}{a}$, respectively.

Since the sum of all eigenvalues equals the trace of the matrix, we have

$$\sum_{i=1}^{n} \lambda_i(G) = \sum_{v\in V} \deg(v) = 2|E(G)|$$

Moreover, since the sum of the absolute values of the off-diagonal entries in each row of a Laplacian matrix is equal to the diagonal entry of the row, it follows by the Gershgorin Disc Theorem that all eigenvalues of a Laplacian matrix are nonnegative real numbers [3]. Therefore, Laplacian matrices are postive semidefinite.

3.3 Eigenvalue 0

Without further ado, we now begin looking at our first eigenvalue of $L(G(a,1))$.

Theorem 3.1. *The multiplicity of* 0 *as an eigenvalue of the Laplacian matrix of* G *is equal to the number of connected components of* G. *[1]*

This result implies that $\lambda_1(G) = 0$ is a simple eigenvalue of $L(G)$ if and only if the graph G is connected. Because all $G(a,b)$ are connected, as shown from the proof of Theorem 2.6, 0 is a simple eigenvalue for all $L(G(a,b))$, which we prove next.

Theorem 3.2. $\lambda_1(G) = 0$ *is a simple eigenvalue of* $L(G(a,b))$ *and its corresponding eigenvector is* $(1,\ldots,1)^T = \mathbf{j}$.

Proof. Let $L(G(a,b))$ be given. The block form of $L(G(a,b))$ is, as we know,

$$\begin{pmatrix} (a+1)I & C \\ C^T & (b+1)I \end{pmatrix} = M.$$

By definition of Laplace matrix, every row of the matrix sums to 0. Therefore, multiplying matrix M by the vector $(1,\ldots,1)^T = \mathbf{j}$ should give us the 0 vector, i.e.

$$M\mathbf{j} = 0\mathbf{j}.$$

Finally, we know the multiplicity of this eigenvalue is 1 (simple) from Theorem 3.1. □

3.4 Eigenvalue $a+b+2$

Theorem 3.3. $a+b+2$ *is an eigenvalue of* $L(G(a,b))$ *with corresponding eigenvector* $((a+1)\mathbf{j},(-b-1)\mathbf{j})^T$ *where the first* $\binom{n}{b}$ *components of the eigenvector are* $(a+1)$ *and the last* $\binom{n}{a}$ *components are* $(-b-1)$.

Proof. Given $G(a,b)$, we have our known Laplace matrix in block form. We take the product of this matrix and the eigenvector $((a+1)\mathbf{j},(-b-1)\mathbf{j})^T$. So, we have,

$$\begin{pmatrix} (a+1)I & C \\ C^T & (b+1)I \end{pmatrix} \begin{pmatrix} (a+1)\mathbf{j} \\ (-b-1)\mathbf{j} \end{pmatrix}$$

which clearly results in a vector of length $\binom{n}{b} + \binom{n}{a}$. First, we focus on the first $\binom{n}{b}$ components of said dot product. We first get the obvious product $(a+1)^2$. Next, we know that by definition of Laplace matrix that there are exactly $(a+1)$ (-1)'s in every row of matrix C. This means our next product is $(-1)(a+1)(-b-1)$. Adding both results we get that the first $\binom{n}{b}$ components of the dot product is $(a+1)^2+(-1)(a+1)(-b-1)$. Similar reasoning is used for the other $\binom{n}{a}$ components of the dot product. We know by definition of Laplace matrix that matrix C^T has exactly $(b+1)$ (-1)'s in every row. This, again, gives us our product $(-1)(b+1)(a+1)$. Lastly, our final product will be $(b+1)(-b-1)$. Adding both of these results we get that the last $\binom{n}{a}$ components of the product is $(-1)(b+1)(a+1)+(b+1)(-b-1)$. Therefore, the product of the block form matrix and the eigenvector is,

$$\begin{pmatrix} ((a+1)^2+(-1)(a+1)(-b-1))\mathbf{j} \\ ((-1)(b+1)(a+1)+(b+1)(-b-1))\mathbf{j} \end{pmatrix} = (a+b+2)\begin{pmatrix} (a+1)\mathbf{j} \\ (-b-1)\mathbf{j} \end{pmatrix}$$

□

3.5 Eigenvalue $b+1$

To show $b+1$ is an eigenvalue of $L(G(a,1))$ we will be focusing on the linear independence of the row space of matrix C^T. For $L(G(a,1))$, since $b=1$, our eigenvalue is $b+1=1+1=2$.

Lemma 3.4. *The column vectors of matrix C^T corresponding to matrix $L(G(a,b))$ are linearly independent.*

Proof. In Section 3.2 we saw that C^T has dimensions $\binom{n}{a}\times\binom{n}{b}$. Now, we solve for vector v in the equation $C^Tv=0$ as we do to find the nullity of matrices. The rows in C^T are made of all the $(b+1)$-subsets of $\mathcal{B}$. This means that we will have $\binom{n}{a}$ linear equations (with $\binom{n}{b}$ unknowns) that make up the system for finding the nullity of C^T. These equations will be all the combinations of the form

$$-\sum_{z\in\binom{[n]}{b+1}} x_z = 0.$$

where $[n]$ represents the sets made from all $\binom{n}{b}$ combinations.

Given two subsets of $\binom{[n]}{b+1}$, our linear equations can be of the form

$$-x_1 - x_2 - \cdots - x_{p-1} - x_p = 0$$
$$-x_1 - x_2 - \cdots - x_{p-1} - x_q = 0.$$

This implies that $x_p = x_q$. By choosing equations pairwise in the manner we did to show that $x_p = x_q$, we can show that $x_1 = x_2 = \cdots = x_{b+1}$. Now considering an arbitrary equation, say,

$$-x_1 - x_2 - \cdots - x_p = 0$$

we can get the following equalities,

$$-x_1 - x_2 - \cdots - x_p = -x_1 - x_1 - \cdots - x_1 = -(b+1)x_1 = 0$$

which finally implies that $x_1 = x_2 = \cdots = x_{\binom{n}{a}} = 0$. □

Theorem 3.5. *$b+1$ is an eigenvalue of $L(G(a,b))$ with multiplicity $\binom{n}{a} - \binom{n}{b}$.*

Proof. Let $L(G(a,b))$ be given. The block form is this matrix is then,

$$\begin{pmatrix} (a+1)I & C \\ C^T & (b+1)I \end{pmatrix}.$$

From Lemma 3.4, we know that all the columns in matrix C^T are linearly independent. Therefore, the rows of C are linearly independent. By definition, $\text{Rank}(C) = \binom{n}{b}$. Now, by The Fundamental Theorem of Linear Algebra, there are $\binom{n}{a} - \binom{n}{b}$ vectors which form a basis of the null space of C, we will call these vectors v. These vectors are of length $\binom{n}{a}$, but we can extend them to be of length $\binom{n}{a} + \binom{n}{b}$ (as it should be) by adding $\binom{n}{b}$ 0s to the beginning of the vector. So the new vectors would then be of the form $(0\mathbf{j}, v)^T$. These $\binom{n}{a} - \binom{n}{b}$ vectors are all eigenvectors of the Laplace matrix of $G(a,b)$ with eigenvalue $(b+1)$ as we would have,

$$\begin{pmatrix} (a+1)I & C \\ C^T & (b+1)I \end{pmatrix} \begin{pmatrix} 0\mathbf{j} \\ v \end{pmatrix} = \begin{pmatrix} 0\mathbf{j} \\ (b+1)v \end{pmatrix} = (b+1) \begin{pmatrix} 0\mathbf{j} \\ v \end{pmatrix}.$$

□

3.6 Eigenvalues 1 and $a+2$

As we previously saw, we had two eigenvalues with the same multiplicity, namely 0 and $a+b+2$. Here, we take a look at another pair of eigenvalues with the same mulitplicity. Before going on to state and prove the next theorem, it is important to realize something about matrix C, which we label as Lemma 3.6.

Lemma 3.6. *Any two rows of matrix C in $L(G(a,1))$ have a (-1) in the same component exactly once.*

Proof. We have two cases to prove here in order to show that it is indeed exactly one common component in which two rows have a (-1). For the first, suppose two arbitrary rows, c_1 and c_2, of matrix C of the block form of $L(G(a,1))$ have a (-1) in the same component more than once. Because in the block form of $L(G(a,1))$, matrix C has exactly two (-1)'s in every column, if c_1 and c_2 have a (-1) in the same component more than once, then the vertex defined by exactly those two rows (having a (-1) in that row component) appears more than once, contradicting our combination construction. For the second case, suppose two arbitrary rows, c_1 and c_2, of matrix C of the block form of $L(G(a,1))$ do not have a (-1) in any same component. Then, we would be missing the vertex of $\mathcal{A}$ that is defined by exactly those two rows. □

Theorem 3.7. *Both* 1 *and* $a+b+1 = a+2$ *are eigenvalues of* $L(G(a,1))$ *with multiplicity* $a+b = a+1$.

Proof. Let $L(G(a,1))$ be given. The block form of the matrix is then,

$$\begin{pmatrix} (a+1)I & C \\ C^T & 2I \end{pmatrix}$$

The eigenvector corresponding to eigenvalue $a+2$, will be named $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$. As it turns out, the eigenvector corresponding to eigenvalue 1 will be $\left(\begin{smallmatrix} v \\ -aw \end{smallmatrix}\right)$. We can see this with a system of equations where both products result in the same equation. So, we multiply what we claim to be the eigenvector by the block form of $L(G(a,1))$, and we get the following:

$$\begin{pmatrix} (a+1)I & C \\ C^T & 2I \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = (a+2) \begin{pmatrix} v \\ w \end{pmatrix}$$

Leading us to,

$$\begin{aligned} (a+1)v + Cw &= (a+2)v \\ C^T v + 2w &= (a+2)w. \end{aligned}$$

This now reduces to,

$$\begin{aligned} Cw &= v \\ C^T v &= aw. \end{aligned}$$

We finally get our equation,

$$CC^T v = av \tag{3.1}$$

As previously stated, we get equation (3.1) with eigenvalue 1 and its corresponding eigenvector shown next.

$$\begin{pmatrix} (a+1)I & C \\ C^T & 2I \end{pmatrix} \begin{pmatrix} v \\ -aw \end{pmatrix} = \begin{pmatrix} v \\ -aw \end{pmatrix}$$

This leads us to,

$$\begin{aligned} av + v - aCw &= v \\ C^T v - 2aw &= -aw. \end{aligned}$$

Next,

$$av = aCw$$
$$C^T v = aw.$$

After, we solve for v and w giving us,

$$v = Cw$$
$$\frac{C^T v}{a} = w.$$

This, again, gives us our equation,

$$CC^T v = av. \tag{3.2}$$

We see that equations (3.1) and (3.2), are in fact, the same. Now, recall that the dimension of C for $L(G(a,1))$ is $n \times \binom{n}{a}$, and the dimension of C^T is $\binom{n}{a} \times n$. $CC^T = M$ is therefore a matrix with dimensions $n \times n$. Furthermore, M has entries $(a+1)$ on the diagonal and, by Lemma 3.6, 1's everywhere else. Now consider, matrix $M - aI = J$. The eigenvalues of this matrix are n (with multiplicity 1) and 0 (with multiplicity $n-1$), as seem in Lemma 1.9. The eigenvectors of M corresponding to eigenvalue 0 are then all possibilities for our vector v. Vector v is of length $\binom{n}{b} = \binom{n}{1} = n$. So, to get w, the rest of the $\binom{n}{a}$ components of our eigenvectors, we just recall

$$\frac{C^T v}{a} = w$$

where we simply plug in every vector v which is multiplied by matrix C^T and then divided by a to give us vector w. □

The reason we do not consider the eigenvector corresponding to eigenvalue n from our $n \times n$ J matrix is because that eigenvector is the vector $\mathbf{j}$ (as we saw in Lemma 1.9). Plugging in vector $\mathbf{j}$ into the equation (3.1) yields,

$$CC^T v = av \implies CC^T \mathbf{j} = a\mathbf{j} \implies ((n-1) + (a+1))\mathbf{j} = a\mathbf{j}$$

which is clearly false, as $(n-1) + (a+1) > a$.

3.7 Conclusion

In the case of $L(G(a,1)$, we have $\binom{n}{a} = \frac{(a+2)(a+1)}{2}$ vertices of length a and $\binom{n}{b} = \binom{a+2}{1} = a+2$ vertices of length b. This means $L(G(a,1))$ has $\frac{(a+2)(a+1)}{2} + a + 2 = \frac{(a+3)(a+2)}{2}$ eigenvalues.

Now, recall the multiplicity of eigenvalue $b+1 = 2$ which was $\binom{n}{a} - \binom{n}{b}$. In this case since $b = 1$, we have that $\binom{n}{a} - \binom{n}{b} = \frac{(a+2)(a+1)}{2} - (a+2) = \frac{(a-1)(a+2)}{2}$. Next, recall that both eigenvalue 0 and eigenvalue $a+b+1$ had multiplicity 1. Also, eigenvalues 1 and $a+2$ had multiplicity $(a+1)$. If we add these multiplicities together we get,

$$\frac{(a-1)(a+2)}{2} + 1 + 1 + (a+1) + (a+1) = \frac{(a-1)(a+2)}{2} + 2a + 4 = \frac{(a+3)(a+2)}{2}$$

which is precisely the total amount of eigenvalues $L(G(a,1))$ should have. Therefore, we can conclude that $L(G(a,1))$ has strictly nonnegative integer eigenvalues!

Chapter 4

Laplace Eigenvalues of $G(a, 2)$

4.1 Eigenvalue 0

In Section 3.3 we saw from Theorem 3.1 that 0 is a simple eigenvalue of $L(G)$ if and only if the graph G is connected. From Theorem 2.6 we saw that every graph $G(a, b)$ is connected. Therefore, 0 is also a simple eigenvalue of $L(G(a, 2))$.

4.2 Eigenvalue $a + b + 2$

From Theorem 3.3 of Section 3.4 we recall that $a + b + 2$ is an eigenvalue for $L(G(a, b))$. Therefore, $a + 4$ is also an eigenvalue of $L(G(a, 2))$.

4.3 Eigenvalue $b + 1$

From Theorem 3.5 in Section 3.5 we also proved that $b + 1$ was an eigenvalue of of matrix $L(G(a, b))$. This means $3 = b + 1$ is therefore an eigenvalue of $L(G(a, 2))$ with multiplicity $\binom{n}{a} - \binom{n}{b} = \binom{n}{a} - \binom{n}{2}$.

4.4 Conjectured Eigenvalues of $L(G(a, 2))$

In Section 3.6 we showed that 1 and $a + b + 1$ were both eigenvalues of $L(G(a, 1))$ with the same multiplicity by realizing that the product of matrices CC^T gave us matrix J. From there we were then able to figure the eigenvectors corresponding to the eigenvalues 1 and

$a + 2$.

In the case of matrices $L(G(a,2))$, the product of matrices CC^T would not give us matrix J because Lemma 3.6 does not hold for matrices $L(G(a,2))$. However, through computer computations we can not only see that 1 and $a + b + 1$ tend to be eigenvalues, but we also see a pattern when it comes to the eigenvectors. We then have the following conjecture:

Conjecture 4.1. *Both* 1 *and* $a+b+1 = a+3$ *are eigenvalues of* $L(G(a,2))$ *with eigenvectors* $\begin{pmatrix} v \\ \frac{-3w}{2} \end{pmatrix}$ *and* $\begin{pmatrix} v \\ w \end{pmatrix}$*, respectively, with multiplicity* $a + b = a + 2$*.*

With graphs $G(a,2)$ we noticed that the Laplacian matrices had a new pair of eigenvalues. Through computer computations we got the following conjecture,

Conjecture 4.2. *Both* 2 *and* $a + b = a + 2$ *are eigenvalues of* $L(G(a,2))$ *with the same multiplicity, namely,* $\frac{a^2+3a}{2}$*.*

Chapter 5

Further Work

5.1 Eigenvalues of $L(G(a,b))$

Through numerous computer computations, many observations of the eigenvalues of matrices $L(G(a,b))$ were made. In fact, the biggest conjecture to be proven here (first noticed by Dr. Art Duval and Dr. Jeremy Martin) would be,

Conjecture 5.1. *The eigenvalues of the Laplacian matrix of graphs $G(a,b)$ are strictly nonnegative integers.*

5.2 Related Conjectures

Several patterns were seen when executing these numerous computer computations through the use of Sage [4]. This led us to our last 2 conjectures. First,

Conjecture 5.2. *$\sigma(G(a,b))$ is symmetrical.*

This means that in the spectrum of $L(G(a,b))$ always has pairs of eigenvalues with the same multiplicities, except for eigenvalue $b+1$ which is our midpoint reference for the symmetry.

For example, the eigenvalues of $L(G(3,1))$ are $6,0,2,1,5$ with multiplicities $1,1,5,4,4$, respectively. The list of all the eigenvalues can be arranged in the following way: $0,1,1,1,1,2,2,2,2,2,5,5,5,5,6$. This list is symmetrical with respect to our midpoint $b+1=2$ as 0 and 6 are a pair of eigenvalues that have the same multiplicity and 1 and 5 are pairs.

And second, a very intriguing observation was made giving us the conjecture:

Conjecture 5.3. *Given* $a, b, c, d \in \mathbb{N}$, *if* $a+b = c+d$ *and* $c > a > b > d$, *then* $\sigma(G(c,d)) \subset \sigma(G(a,b))$.

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Curriculum Vitae

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