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Applications Of Ornstein-Uhlenbeck Type Stochastic Differential Equations

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APPLICATIONS OF ORNSTEIN-UHLENBECK TYPE STOCHASTIC DIFFERENTIAL EQUATIONS

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This Dissertation is dedicated to my parents

John Tweneboah

and

Christiana Tweneboah

with love

APPLICATIONS OF ORNSTEIN-UHLENBECK TYPE STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

In this dissertation, we show with plausible arguments that the Stochastic Differential Equations (SDEs) arising on the superposition and coupling system of independent Ornstein-Uhlenbeck process is a new method available in modern literature that takes the properties and behavior of the data into consideration when performing the statistical analysis of the time series.

The time series to be analyzed is thought of as a source of fluctuations, and thus we need a model that takes this behavior into consideration when performing such analysis. Most of the standard methods fail to take into account the physical behavior of the time series, and some of the models are not completely stochastic. Thus in an attempt to overcome the modeling problems associated with the memory-less property models used in the traditional methods, we propose a continuous-time stationary and non-negative stochastic differential equation that is useful for describing a unique type of dependence in a sequence of events.

The Ornstein-Uhlenbeck type SDE offers plenty of analytic flexibility which is not available to more standard models such as the geometric Gaussian Ornstein-Uhlenbeck processes. Moreover, the SDE provides a class of continuous time processes capable of exhibiting long memory behavior. The presence of long memory suggests that current information is highly correlated with past information at different levels. This facilitates prediction.

The proposed SDE is applied to two different sets of real data; financial and geophysical time series. In the analysis of the time series, we show that the SDE makes new properties and estimate parameters that are useful for making inferences and predicting these types of events.

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Chapter 1

Introduction

Most areas of science rely on Big Data and correctly modeling these data will help answer some of the key research questions in data science, statistics and related fields. Big data is a term applied to ways to analyze, extract information from, or otherwise deal with data sets that are too large or complex to be dealt with by classical data-processing application software. Big data has one or more of the following characteristics: high volume, high velocity, high variety and high veracity. That is the data sets are characterized by huge amounts (volume) of frequently updated data (velocity) in various types, such as numeric, textual, audio, images and videos (variety) with high quality (veracity).

The objective of this dissertation is to introduce a new method to enhance the understanding of "extreme events" in data science, statistics and related fields. The main interest is verifying using theoretical and practical framework that our proposed new method describes accurately the behavior of financial indices and earthquake series. We show with plausible arguments that the stochastic differential equations arising on the superposition and coupling system of independent Ornstein-Uhlenbeck process is a new method available in modern literature that takes the physical behavior of the data into consideration when modeling and performing the statistical analysis of the time series. In the paragraphs that follow, we will review some literature that have been dedicated to the modeling of complex data sets in finance and geophysics.

In recent years, due to the huge amount of data available in the financial market, there has been a constant interest by researchers and practitioners to develop models to describe these data sets. Modeling and analyzing these financial sampled data helps investors, practitioners and researchers make useful inference and predictions. The financial data mentioned are data

sets that are related to stock market crashes. A stock market crash is a sudden decline of stock prices across a significant cross-section of a stock market, resulting in a significant loss of paper wealth. Market crashes are often influenced by panic as much as by underlying economic factors [58]. There has been a growing literature in financial economics analyzing the behavior of major stock indices. Most of these literature are based on deterministic and probabilistic models to depict various aspects of the mathematical and statistical modeling of major stock indices. In deterministic models, the output of the model is fully determined by the initial conditions and parameter values. On the other hand, stochastic models possess some intrinsic randomness, that is, the same set of initial conditions and parameter values will lead to a group of different outputs.

One of the first models developed for describing the evolution of stock prices is the Brownian motion. This model assumes that the increment in the logarithm of the prices follows a diffusive process with Gaussian distribution [59]. However, the empirical study of some financial indices shows that in the short time intervals the associated probability density function has greater kurtosis than a Gaussian distribution [30], and that the Brownian motion does not describe correctly the evolution of financial indices near a market crash. The authors in Refs. [30], [35] and [33] tried to overcome this issue by using a stable non-Gaussian Lévy process that takes into account the long correlation scales. To be specific, the authors in Ref. [30] proved that the scaling of the probability distribution of a Standard & Poor's 500 stock index can be described by a non-Gaussian process with dynamics that, for the central part of the distribution, correspond to a Lévy stable process. Also in the work by Ref. [35], the authors studied the statistical properties of financial indices from developed and emergent markets. They performed the analysis of different financial indices near a crash for both developed and emergent markets by using a normalized truncated Lévy walk model. Later the authors in Ref. [33], studied the correlations, memory effects and other statistical properties of several stocks. The authors verified that the behaviors of the stock returns were compatible with that of continuous time Lévy processes. Furthermore the authors concluded that stochastic volatility models, jump diffusion models and general Lévy processes are useful for the modeling of

financial time series. Other researchers also described and modeled the behavior of a financial market before a crash by analyzing high frequency financial sampled data (see [31], [6] and [10]). For example, by using the Ising type model the authors in Ref. [31] studied high frequency market data leading to the Bear Stearns crash [18] which occurred in mid March 2008. They predicted the time when stock prices experience phase transition. A phase transition is a change in state from one phase to another. In this context, the two states are buy and sell. Thus if all traders decision changes from a buy and aligned to a sell phenomenon this leads to a market crash.

Next, we review some literature that have been used to describe geophysical data specifically earthquake time series. As the knowledge of the geophysical mechanisms that drive seismic events have increased, so have the corresponding mathematical and statistical model representations. In fact a good estimation of the seismic hazard in a region requires the prediction of time, location and magnitude of future seismic events [37].

As in the modeling of stock indices, in geophysics several deterministic and probabilistic models have been studied to describe the temporal evolution of earthquake sequences. Probabilistic models such as the long term correlations have been applied to the occurrences of seismic events (see [29]). In Ref. [29] the authors showed that the long term correlations can explain both the fluctuations of magnitudes and their interoccurrence times in the seismic records. In a different study, the authors in Ref. [44] used a stochastic finite fault source model to estimate ground motion in northeastern India for intermediate depth events originating in the Indo-Burmese tectonic domain. Accelerograms from eight events with magnitudes ranging from Mw $4.8 - 6.4$ were used to estimate the input source and site parameters of the stochastic finite fault source model. Stochastic modeling of the ground acceleration due to an earthquake using an existing deterministic formulation was presented by Ref. [15]. The author constructed a non-stationary stochastic model by making use of the well-known ω square model for source time function of the earthquake. The author argued that, the result of applying this procedure is a model whose main parameter has a physical interpretation, and therefore a validation based on criteria other than statistical goodness of fit is also possible.

The authors in Ref. [60] proposed a stochastic differential equation (SDE) to simulate Group Delay Time (GDT) of earthquake ground motion. They expressed the random characteristic of the GDT using the SDE whose mean and variance processes where defined by ordinary differential equations and solved the SDE of GDT using the Milstein approximation scheme. In their work, the efficiency of the developed model was demonstrated by comparing the simulated results with the original one. The most interesting model in recent time has been the scale invariant functions and Lévy models which have been used to estimate parameters related to some major events [34]. In this study the authors, by looking at the preceding data collected before a major earthquake estimated the parameters leading to these critical events. The modeling approach used was similar to [31], where they described the behavior of the market before a financial crash.

Most of the models reviewed above have in common the fact that they are based upon the Gaussian assumption, that is, we can describe the behavior of the time series by studying the variance of the generated diffusion process. This fact is not completely accurate because empirical study [30] of some financial stock indices suggests that the Gaussian assumption is typically inappropriate because asset returns often exhibit excess kurtosis and asymmetries [4]. Furthermore, most of the models described in previous literature fails to take into account the physical behavior of the financial and earthquake data, and some of the models are not completely stochastic. As in reality many phenomena are influenced by random noise, behavior of the noise should be reflected in the model [25]. Therefore there is the need to understand the general principles of the mathematical and statistical modeling of datasets which describes the actual realizations.

In an attempt to overcome the modeling problems associated with the memory-less property models described in previous literature and also incorporate the physical behavior of the time series, we propose a continuous-time stationary and non-negative stochastic differential equation that is useful in describing a unique type of dependence in a sequence of events. In finance and econometrics, this stochastic model accounts for the stochastic nature of both the process of price fluctuations and the process of trade durations [23]. Continuous-time stochas-

tic volatility models are now popular ways to describe many "critical phenomena" because of their flexibility in accommodating most stylized facts of time series data such as moderate and high frequency data. The authors in Ref. [9] proposed a class of models where the volatility behaved according to an Ornstein-Uhlenbeck process driven by a positive Lévy process with a non-Gaussian component. This model type has many applications in many fields of science and other disciplines [51]. There are also known applications within the context of finance and econometric [9]. The Ornstein-Uhlenbeck process is a mean reverting process which is widely used for modeling interest rates and commodities among many others.

In this dissertation, we implement very flexible classes of processes that incorporate long-range dependence, that is, they have a slowly polynomially decaying autocovariance function and self-similarity like properties that are capable of describing some of the key distributional features of typical financial and geophysical time series . In order to capture realistic dependence structures, we combine and couple system of independent Ornstein-Uhlenbeck processes driven by a $\Gamma(a, b)$ process which is a Lévy process. This selection is supported by the fact that generalized Lévy models are suitable for describing these type of time series, see [34]. The advantage of the superposition and coupling system of independent $\Gamma(a, b)$ Ornstein-Uhlenbeck processes is that it offers plenty of analytic flexibility which is not available for more standard models such as the geometric Gaussian Ornstein-Uhlenbeck processes. Moreover, superposition of Ornstein-Uhlenbeck processes provide a class of continuous time processes capable of exhibiting long memory behavior. The presence of long memory suggests that current information is highly correlated with past information at different levels. This facilitates prediction. The methodology used in this work can be applied to other disciplines such as biology, bioinformatics, medicine and in social sciences.

The dissertation is organized in two parts followed by a conclusion. The first part addresses the theory of Stochastic Processes and Stochastic Differential Equations and the numerical simulations that verify the theoretical results. The second part shows applications of the model to real data. Each application addresses a particular way to work with the Stochastic Differential

Equations for describing the behavior of financial and geophysical time series.

Chapter 2 reviews the necessary definitions from probability theory, stochastic and Lévy processes. We will also discuss the $\Gamma(a, b)$ distribution and provide a detail description and definitions of a stochastic differential equation and its solution methods.

In Chapter 3 we discuss the non Gaussian Ornstein-Uhlenbeck processes in detail, derive some important results and pave the way for the proposed superposed and coupled $\Gamma(a, b)$ Ornstein-Uhlenbeck model. The main characteristic of the proposed stochastic model will be discussed and shown. Simulation methods to generate realizations of the model are also presented.

The results illustrated from Chapters 4 to the end are original results of this dissertation and have produced several papers, some already published ([37, 36, 32]) and more publications are expected to results from this dissertation.

The second part of the dissertation begins with Chapter 4 in which the stochastic differential equations is applied to the study of well developed and emergent market indices. For the time series arising on financial indices near a crash for both well developed and emergent markets, we estimate the daily closing values. Chapter 5 is dedicated to analyzing using a stochastic differential equation the second-by-second and minute-by-minute sampled financial data from the Bear Stearns companies and estimating parameters that are useful for making inferences and predicting these types of events. This results may help an investor or practitioner who lacks insider information but has at their disposal all the information contained in the equity prices discover that a crash is imminent and take the necessary precautions.

Chapter 6 is dedicated to the modeling of earthquake series. An important example and potential application of this work is for analyzing the effect of events that occurred very far in the past, for example decades ago, might have on the occurrence of present and future events. This type of analysis might help to better understand how tectonic stress decays and accumulates during long period of time. Chapter 7 is dedicated to describing the effects of geophysical time series arising between different regions in the same geographical area

by using coupled systems of stochastic differential equations. The objective is to model the correlation and effects of earthquake series occurring at different regions.

The dissertation ends with a short conclusion and with a research project concerning further applications of this methodology.

Chapter 2

Stochastic Processes and Stochastic Differential Equations

This chapter is divided into two parts. The first part describes the theory of stochastic processes and the second is dedicated to the theory of stochastic differential equations. In the first part of this chapter we give definitions, properties and examples of stochastic processes that will be useful throughout this dissertation. In the second part, we will discuss in detail stochastic differential equations and their solution methods.

Stochastic processes and stochastic differential equations play a fundamental role in Mathematical Finance, as well as other fields of science, such as Physics (turbulence), Engineering (telecommunications, dams), Actuarial Science (insurance risk) and several others. General reference works on stochastic processes and stochastic differential equations are given by [43],[48],[49],[50],[47] and [2].

2.1 Necessary definitions from probability theory

We begin the first part of this chapter with necessary definitions from probability theory. The following definitions can be found in many literature for example [13], [40], [43] and references therein.

We will use the term experiment in a very general way to refer to some process that produces a random outcome.

Definition 2.1.1 (Sample Space)**.** *The set Ω, of all possible outcomes of a particular experiment is called the sample space for the experiment.*

Definition 2.1.2 (σ-algebra). *A σ-algebra is a collection of sets $\mathcal{F}$ of Ω satisfying the following condition:*

1. $\emptyset \in \mathcal{F}$.
2. *If $F \in \mathcal{F}$ then its complement $F^c \in \mathcal{F}$.*
3. *If $F_1, F_2, \ldots$ is a countable collection of sets in $\mathcal{F}$ then their union $\cup_{n=1}^{\infty} F_n \in \mathcal{F}$*

Definition 2.1.3 (Probability measure). *Let $\mathcal{F}$ be a $\sigma-$algebra on Ω. A probability measure on $\mathcal{F}$ is a real-valued function P on $\mathcal{F}$ with the following properties.*

1. $P(A) \geq 0$ *for* $A \in \mathcal{F}$.
2. $P(\Omega) = 1, P(\emptyset) = 0$.
3. *If $A_n \in \mathcal{F}$ is a disjoint sequence of events, i.e. $A_i \cap A_j = \emptyset$, for $i \neq j$, then*

$$P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

Definition 2.1.4 (Probability Space). *A probability space is a triplet $(\Omega, \mathcal{F}, P)$ where Ω is a sample space, $\mathcal{F}$ is a $\sigma-$algebra on Ω and P is a probability measure $P : \mathcal{F} \to [0, 1]$.*

Definition 2.1.5 (Measurable functions). *A real-valued function f defined on Ω is called measurable with respect to a sigma algebra $\mathcal{F}$ in that space if the inverse image of the set B, defined as $f^{-1}(B) \equiv \{\omega \in E : f(\omega) \in B\}$ is a set in σ-algebra $\mathcal{F}$, for all Borel sets B of $\mathbb{R}$.*

Definition 2.1.6 (Random variable). *A random variable $\mathbf{X}$ is any measurable function defined on the probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^n$.*

Suppose we have a random variable $\mathbf{X}$ defined on a space $(\Omega, \mathcal{F}, P)$. The σ algebra generated by $\mathbf{X}$ is the smallest σ algebra in $(\Omega, \mathcal{F}, P)$ that contains all the pre-images of sets in $\mathbb{R}$ through $\mathbf{X}$. That is:

$$\sigma(\mathbf{X}) = \sigma\left(\{\mathbf{X}^{-1}(B) \mid \text{ for all } B \text{ Borel sets in } \mathbb{R}\}\right)$$

Table 2.1: Examples of random variables

Experiment	Random variable
Toss two dice	$\mathbf{X}$ = sum of the numbers
Flip a coin 5 times	$\mathbf{X}$ = sum of heads in 5 flips

This concept is necessary to make sure that we may calculate any probability related to the random variable $\mathbf{X}$.

For every random variable $\mathbf{X}$, we can associate a function called the cumulative distribution function of $\mathbf{X}$ which is defined as follows:

Definition 2.1.7 (Cumulative distribution function)**.** *Given a random vector* $\mathbf{X}$ *with components* $\mathbf{X} = (X_1, \ldots, X_n)$*, its cumulative distribution function (cdf) is defined as:*

$$F_{\mathbf{X}}(x) = P(\mathbf{X} \leq x) = P(X_1 \leq x_1, \ldots X_n \leq x_n) \text{ for all } x.$$

Definition 2.1.8 (Continuous and Discrete function)**.** *A random variable* $\mathbf{X}$ *is continuous if* $F_{\mathbf{X}}(x)$ *is continuous function of* x*. A random variable* $\mathbf{X}$ *is discrete if* $F_{\mathbf{X}}(x)$ *is step function of* x*.*

Associated with a random variable $\mathbf{X}$ and its cumulative distribution function $F_{\mathbf{X}}$ is another function, called the probability density function (pdf) or probability mass function (pmf). The terms pdf and pmf refer to the continuous and discrete cases of random variables respectively.

Definition 2.1.9 (Probability mass function)**.** *The probability mass function (pmf) of a discrete random variable* $\mathbf{X}$ *is given by*

$$f_{\mathbf{X}}(x) = P(\mathbf{X} = x) \text{ for all } x$$

Definition 2.1.10 (Probability density function)**.** *The probability mass function (pdf),* $f_{\mathbf{X}}(x)$ *of a continuous random variable* $\mathbf{X}$ *is the function that satisfies*

$$F(\mathbf{x}) = F(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{\mathbf{X}}(t_1, \ldots, t_n) dt_n \ldots dt_1.$$

2.2 Stochastic Processes

Definition 2.2.1 (Stochastic Process)**.** *A Stochastic process is a parametrized collection of random variables* $\{X(t) : t \in \mathcal{I}\}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *and assuming values in* $\mathbb{R}^n$*, where* $\mathcal{I}$ *is an index set.*

The notations X_t and $X(t)$ are used interchangeably to denote the value of the stochastic process at index value t.

2.2.1 The Index Set $\mathcal{I}$

The set $\mathcal{I}$, that indexes the stochastic process determines the type of stochastic process. Below we give some examples.

- If the index set is defined as $\mathcal{I} = \{0, 1, 2 \ldots\}$ we obtain the discrete-time stochastic processes. We shall denote the process as $\{X_n\}_{n\in\mathbb{N}}$ in this case.
- If the index set is defined as $\mathcal{I} = [0, \infty)$, we obtain the continuous-time stochastic processes. We shall also denote the process as $\{X_t\}_{t\geq 0}$. In most instances, t represents time.
- The index set can be multidimensional. For example, if $\mathcal{I} = [0, 1] \times [0, 1]$ we may be describing the structure of some surface where for instance $X(x, y)$ could be the value of some electrical field intensity at position (x, y).

2.2.2 The State Space $\mathcal{S}$

The state space is the domain space of all the random variables X_t. Since we are discussing about random variables and random vectors, then necessarily $\mathcal{S} \subseteq \mathbb{R}$ or $\mathbb{R}^n$. This domain space can be defined using integers, real lines, n-dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces. We present some examples as follows:

- If $\mathcal{S} \subseteq \mathbb{Z}$, then the process is integer valued or a process with discrete state space.

- If $\mathcal{S} = \mathbb{R}$, then X_t is a real-valued process or a process with a continuous state space.
- If $\mathcal{S} = \mathbb{R}^k$, then X_t is a k-dimensional vector process.

The state space $\mathcal{S}$ can be more general (for example, an abstract Lie algebra), in which case the definitions work very similarly except that for each t we have X_t measurable functions.

2.2.3 Stationary and Independent Components

Definition 2.2.2 (Independent Components)**.** *For any collection $\{t_1, t_2, \ldots, t_n\}$ of elements in $\mathcal{I}$ if the corresponding random variables $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are independent then, the joint distribution $F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}$ is the product of the marginal distributions $F_{X_{t_i}}$, where $i = 1, \ldots, n$.*

Definition 2.2.3 (Strictly Stationary)**.** *A stochastic process X_t, is said to be strictly stationary if the joint distribution function of the vectors:*

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \quad \textit{and} \quad (X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h})$$

are the same for all $h > 0$ and all arbitrary selection of index points $\{t_1, t_2, \ldots, t_n\}$ in $\mathcal{I}$.

Definition 2.2.4 (Weak stationary)**.** *A stochastic process X_t, is said to be weak stationary if X_t has finite second moments for any t and if the covariance function $Cov(X_t, X_{t+h})$ depends only on h for all $t \in \mathcal{I}$.*

Remark 2.2.1. *A strictly stationary process with finite second moments (so that covariance exists) is going to be automatically weak stationary. The reverse is not true.*

The concept of weak stationarity was developed because of the practical way in which we observe stochastic processes. While strict stationarity is a very desirable concept it is not possible to test it with real data. To show strict stationarity means we need to test all joint distributions. However in real life the samples we gather are finite so this is not possible. Instead, we can test the stationarity of the covariance matrix which only involves bivariate distributions.

Many phenomena can be described by stationary processes. In addition, many classes of processes eventually become stationary if observed for a long time. The white noise process is an example of a strictly stationary process. However, some of the most common processes encountered in practice – the Poisson process and the Brownian motion – are not stationary. However, they have stationary and independent increments. We define this concept next.

2.2.4 Stationary and Independent Increments

In order to discuss the increments for stochastic processes, we assume that the index set $\mathcal{I}$ has a total order, that is for any two elements a and b in $\mathcal{I}$ either $a \leq b$ or $b \leq a$. We note that a two dimensional index set for example $\mathcal{I} = [0,1] \times [0,1]$ does not have this property.

Definition 2.2.5 (Independent increments)**.** *A stochastic process X_t is said to have independent increments if the random variables:*

$$X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2},\ \ldots,\ X_{t_n} - X_{t_{n-1}}$$

are independent for any n and any choice of the sequence $\{t_1, t_2, \ldots, t_n\}$ in $\mathcal{I}$ with $t_1 < t_2 < \cdots < t_n$.

Definition 2.2.6 (Stationary increments)**.** *A stochastic process X_t is said to have stationary increments if for $s, t \in T$ with $s \leq t$, the increment $X_t - X_s$ has the same distribution as X_{t-s}.*

Notice that this is not the same as stationarity of the process itself. In fact, with the exception of the constant process there exists no process with stationary and independent increments which is also stationary.

Definition 2.2.7 (Quadratic Variation for stochastic processes)**.** *Let X_t be a stochastic process on the probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t\in\mathcal{I}}$. Let $\pi_n = (0 = t_0 < t_1 < \ldots t_n = t)$ be a partition of the interval $[0, t]$. We define the quadratic variation process*

$$[X, X]_t = \lim_{\|\pi_n\|\to 0} \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}|^2,$$

where the limit of the sum is defined in probability.

The quadratic variation process is a stochastic process. The quadratic variation may be calculated explicitly only for some classes of stochastic processes. In fact the stochastic processes used in finance have finite second order variation. The third and higher order variations are all zero while the first order is infinite. This is the fundamental reason why the quadratic variation has such a big role for stochastic processes used in finance.

2.2.5 Filtration and Standard Filtration

In the case where index set $\mathcal{I}$ possesses a total order relationship, we can discuss about the information contained in the process $X(t)$ at some moment $t \in \mathcal{I}$. To quantify this information we generalize the notion of sigma algebras by introducing a sequence of sigma algebras: the filtration.

Definition 2.2.8 (Filtration)**.** *A probability space $(\Omega, \mathcal{F}, P)$ is a filtered probability space if and only if there exists a sequence of sigma algebras $\{\mathcal{F}_t\}_{t\in\mathcal{I}}$ included in $\mathcal{F}$ such that $\mathcal{F}$ is an increasing collection i.e.:*

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \quad \forall s \leq t, \quad s, t \in \mathcal{I}.$$

A filtration is called *complete* if its first element contains all the null sets of $\mathcal{F}$.

Definition 2.2.9 (Right and Left Continuous Filtrations)**.** *A filtration $\{\mathcal{F}_t\}_{t\in\mathcal{I}}$ is right continuous if and only if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all t, and the filtration is left continuous if and only if $\mathcal{F}_t = \mathscr{F}_{t-}$ for all t.*

Throughout this dissertation, we shall assume that any filtration is right continuous.

Definition 2.2.10 (Adapted stochastic process)**.** *A stochastic process $\{X_t\}_{t\in I}$ defined on a filtered probability space $(\Omega, \mathscr{F}, P, \{\mathcal{F}_t\}_{t\in\mathcal{I}})$ is called adapted if and only if X_t is $\mathcal{F}_t$-measurable for any $t \in \mathcal{I}$.*

This is an important concept since in general, $\mathcal{F}_t$ quantifies the flow of information available at any moment t. By requiring that the process be adapted, we ensure that we can calculate probabilities related to X_t based solely on the information available at time t. In addition, since the filtration by definition is increasing, this also means that we can calculate the probabilities at any later moment in time.

In some cases, we are only given a standard probability space (i.e. without a separate filtration defined on the space). This corresponds to the case where we assume that all the information available at time t comes from the stochastic process X_t itself. In this instance, we will be using the standard filtration generated by the process $\{X_t\}_{t\in\mathcal{I}}$ itself. Let

$$\mathcal{F}_t = \sigma(\{X_s : s \leq t, s \in \mathcal{I}\}),$$

denote the sigma algebra generated by the random variables up to time t. The collection of sigma algebras $\{\mathcal{F}_t\}_t$ is increasing and the process $\{X_t\}_t$ is adapted with respect to it.

A stochastic process $\{Y_t\}$ is called a modification of a stochastic process $\{X_t\}$, if

$$\mathbb{P}[X_t = Y_t] = 1 \quad \text{for} \quad t \in [0, \infty). \tag{2.1}$$

Two stochastic processes $\{X_t\}$ and $\{Y_t\}$ are identical in law, written as

$$\{X_t\} \stackrel{\mathrm{d}}{=} \{Y_t\}, \tag{2.2}$$

if the systems of their finite-dimensional distributions are identical. We discuss the concept of finite-dimensional distributions as follows.

Let $\{X_t\}_{t\in\mathcal{I}}$ be a stochastic process. For any $n \geq 1$ and for any subset $\{t_1, t_2, \ldots, t_n\}$ of $\mathcal{I}$ we denote with $F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}$ the joint distribution function of the variables $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$. The statistical properties of the process X_t are completely described by the family of distribution functions $F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}$ indexed by the n and the t_i's. If we can describe these finite-dimensional joint distributions for all n and t's we completely characterize the stochastic process.

2.2.6 Time series

Definition 2.2.11 (Time series)**.** *If a random variable X is indexed to time, usually denoted by t, the observations $\{X_t, t \in \mathbb{T}\}$ is called a time series, where $\mathbb{T}$ is a time index set (for example, $\mathbb{T} = \mathbb{Z}$, the integer set).*

Definition 2.2.12 (Continuous time series)**.** *A time series $\{X_t\}$ is said to be discrete when observations are taken only at specific times, usually equally spaced.*

An example is the realization of a binary process. The binary process is a special type of time series which arises when observations can take one of only two values, usually denoted by 0 and 1. They occur in many fields including communication theory.

Definition 2.2.13 (Discrete time series)**.** *A time series* $\{X_t\}$ *is said to be continuous when observations are made continuously through time.*

The term "discrete" is used for series of this type even when the measured observation is a continuous variable. In this dissertation, we will be discussing discrete time series, where the observations are taken at equal time intervals.

Examples of time series includes the Dow Jones Industrial Averages, historical data on sales, inventory, customer counts, interest rates, costs, etc. Time series are usually plotted via line charts and are often used in statistics, signal processing, pattern recognition, mathematical finance, weather forecasting, earthquake prediction, and largely in several domain of applied sciences and engineering which involves temporal measurements. Methods of analyzing time series constitute an important area of research in several fields. As mentioned earlier in Chapter 1 of this dissertation, the goal of this work is to develop new method for the statistical analysis of time series arising in finance and geophysics. Figure 2.1 is an example of earthquake time series corresponding to a set of magnitude 3.0-3.3 aftershocks of a recent magnitude 5.2 intraplate earthquake which occurred in Clifton, Arizona on June 26, 2014 and Fig. 2.2 is an example of a high-frequency financial returns time series (per minute) corresponding to the Bank of America Corporation (BAC) stock index.

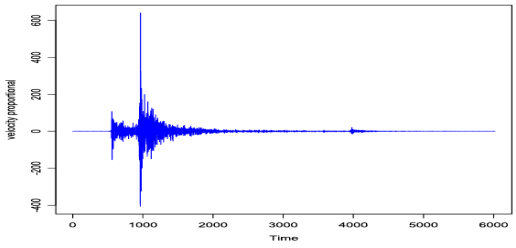

Figure 2.1: The time series data of arrival phased from earthquake occurring on 7/26/14.

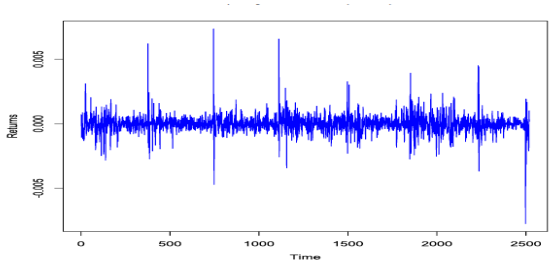

Figure 2.2: Time series data of financial returns from Bank of America (BAC) stock index.

When modeling finite number of random variables, a covariance matrix is usually computed to summarize the dependence between these variables. For a time series $\{X_t\}_{t=-\infty}^{\infty}$ we need to model the dependence over infinite number of random variables. The concepts of autocovariance and autocorrelation functions provide us a tool for this purpose.

Definition 2.2.14 (Autocovariance function)**.** *The autocovariance function of a time series* $\{X_t\}$ *with* $Var(X_t) < \infty$ *is defined by*

$$\gamma_X(s,t) = Cov(X_s, X_t) = E[(X_s - E[X_s])(X_t - E[X_t])]$$

With autocovariance functions, we can define the covariance stationarity, or weak stationarity

Definition 2.2.15 (Stationarity)**.** *The time series* $\{X_t, t \in \mathbb{Z}\}$ *(where* $\mathbb{Z}$ *is the set on integers) is stationary if*

1. $E[X_t^2] < \infty$ *for all* $t \in \mathbb{Z}$.
2. $E[X_t] = \mu$ *for all* $t \in \mathbb{Z}$.
3. $\gamma_X(s,t) = \gamma_X(s+h, t+h)$ *for all* $s, t, h \in \mathbb{Z}$

Based on Definition 2.2.15, we can rewrite the autocovariance function of a stationary process as

$$\gamma_X(h) = Cov(X_t, X_{t+h}) \text{ for } t, h \in \mathbb{Z}$$

Definition 2.2.16 (Autocorrelation function)**.** *The autocorrelation function of a stationary time series* $\{X_t\}$ *is defined by*

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)},$$

where $\gamma_X(h) = Cov(X_t, X_{t+h})$ *and* $\gamma_X(0) = Cov(X_t, X_t)$.

Remark 2.2.2. *When the time series* X_t *is stationary, we must have*

$$\rho_X(h) = \rho_X(-h)$$

Definition 2.2.17 (Strict Stationary)**.** *The time series* $\{X_t, t \in \mathbb{Z}\}$ *is said to be strict stationary if the joint distribution of* $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$ *is the same as* $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h})$.

Most statistical forecasting methods are based on the assumption that the time series can be rendered approximately stationary through the use of mathematical transformations. This is because stationary data series is relatively easy to predict that is, one can simply forecast that its statistical properties will be the same in the future as they have been in the past. The predictions for the stationarized series can then be "untransformed," by reversing whatever mathematical transformations were previously used, to obtain predictions for the original series. Non-stationary data on the other hand are unpredictable and cannot be modeled or forecasted. The results obtained by using non-stationary time series may be false in that they may indicate a relationship between two variables where actually one does not exist. In order to receive consistent, reliable results, the non-stationary data needs to be transformed into stationary data. Examples of non-stationary data includes the population of United States, income, price changes, and several others.

2.3 Examples of stochastic processes

A Markov process is a simple type of stochastic process in which the time order in a sequence of events plays a significant role i.e. the present state can influence the probability of what happens next. We present a formal definition as follows:

2.3.1 Markov processes

Definition 2.3.1 (Markov process)**.** *The stochastic process X_t is a Markov process if the following property are satisfied: For every $s \in T$ and $t \in T$ with $s < t$, and for every $H \in \mathcal{F}_s$ and $x \in S$, the conditional distribution of X_t given H and $X_s = x$ is the same as the conditional distribution of X_t just given $X_s = x$:*

$$P(X_t \in A | H, X_s = x) = P(X_t \in A | X_s = x)$$

for all $A \subset S$.

The complexity of Markov processes depends greatly on whether the time space or the state space are discrete or continuous. We will assume that both are discrete, that is we assume that time space $T = N$ and the state space S is countable. The Brownian motion process and the Poisson process are both examples of Markov processes [45] in continuous time, whereas simple random walks on the integers are examples of Markov processes in discrete time [20]. We will discuss the Brownian motion, Poisson process and random walks later in this chapter.

2.3.2 Martingales

Definition 2.3.2 (Martingales)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A martingale sequence of length n is a set of variables $X_1, X_2, \ldots, X_n$ and corresponding σ-algebras $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ that satisfy the following relations:*

1. *Each X_i is an integrable random variable adapted to the corresponding σ-algebra $\mathcal{F}_i$.*
2. *The $\mathcal{F}_i$'s form a filtration.*
3. *For every $i \in [1, 2, \ldots, n-1]$, we have:*

$$X_i = \mathbf{E}\left[X_{i+1} | \mathcal{F}_i\right].$$

This process has the property that the expected value of the future given the information we have today is going to be equal to the known value of the process today. In French (a

martingale means a winning strategy. This is because for gamblers, a martingale is a betting strategy where the stake doubled each time the player loses. Players follow this strategy because, since they will eventually win, they argue they are guaranteed to make money. Examples of martingales are given below:

1. Let $X_{t+1} = X_t \pm b_t$ where $+b_t$ and $-b_t$ occur with equal probability b_t is measurable $\mathcal{F}_t$, and the outcome $\pm b_t$ is measurable $\mathcal{F}_{t+1}$ (i.e. my "bet" b_t can only depend on what has happened so far and not on the future, but my knowledge$\mathcal{F}_t$ includes the outcome of all past bets). Then $\{X_t|\mathcal{F}_t\}$ is a martingale.

2. A random ± 1 walk is a martingale.

2.3.3 Simple random walk

A random walk is a stochastic sequence $\{X_n\}$, defined by

$$X_n = \sum_{t=1}^{n} X_t$$

where X_t are independent and identically distributed random variables (i.i.d.).

The random walk is simple if $X_t = \pm 1$, with $P(X_t = 1) = p$ and $P(X_t = -1) = 1 - p$. A simple random walk is symmetric if the particle has the same probability for each of the neighbors. We recall that the simple random walk is both a martingale that is $E(X_{t+s}|X_t) = X_t$ and a stationary Markov process that is the distribution of $X_{t+s}|X_t = k_t, \ldots, X_1 = k_1$ depends only on the value k_t.

2.3.4 The Brownian Motion (Wiener process)

The Brownian motion also called the Wiener process is a continuous-time stochastic process.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A Brownian motion is a stochastic process B_t with the following properties:

1. $B_0 = 0$.

2. With probability 1, the function $t \to B_t$ is continuous in t.

3. The process B_t has stationary and independent increments.

4. The increment $B_{t+s} - B_s$ has a $N(0,t)$ distribution, where $N(0,t)$ denotes the normal distribution with mean 0 and variance t.

2.4 Lévy Processes

Stochastic processes are mathematical models of time evolution of random phenomena. Therefore the index t is usually taken for time. The most basic process modeled for continuous random motions is the Brownian motion or Wiener process and that for jumping random motions is the Poisson process. The Brownian motion was described in the previous section. The Poisson process will be described in this section.

Before we start our discussion of Lévy processes, we present the following definitions.

Definition 2.4.1 (Stochastic Continuity). *A stochastic process $\{X_t\}$ on $\mathbb{R}^n$ is stochastically continuous or continuous in probability if, for every $t \geq 0$ and $\epsilon > 0$,*

$$\lim_{s \to t} \mathbb{P}[|X_s - X_t| > \epsilon] = 0 \tag{2.3}$$

Definition 2.4.2 (Characteristic Function). *The Characteristic Function ϕ of a random variable X is the Fourier Stieltjes transform of the distribution function $F(x) = \mathbb{P}(X \leq x)$:*

$$\phi_X(u) = \mathbb{E}[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} dF(x), \tag{2.4}$$

where i is the imaginary number.

One important property of the characteristic function is the fact that for any random variable X, it always exists, it is continuous, and it determines X univocally. If X and Y are independent random variables then:

$$\phi_{X+Y}(u) = \phi_X(u)\phi_Y(u). \tag{2.5}$$

The following are some of the the functions, related to the characteristic function which we will use in this dissertation:

- The cumulant function: $k_X(u) = \log \mathbb{E}[e^{-uX}] = \log \phi(iu)$.
- The cumulant characteristic function or characteristic exponent:

$$\psi_X(u) = \log \mathbb{E}[e^{iuX}] = \log \phi(u),$$

or equivalently

$$\phi_X(u) = e^{\psi(u)}. \tag{2.6}$$

Definition 2.4.3 (Infinitely Divisible Distribution, [50]). *Suppose $\phi(u)$ is the characteristic function of a random variable X. If for every positive integer n, $\phi(u)$ is also the nth power of a characteristic function, we say that the distribution is infinitely divisible. Equivalently, in terms of X for any n:*

$$X = Y_1^{(n)} + \ldots + Y_n^{(n)}$$

where $Y_i^{(n)}, i = 1, \ldots n$, are independently and identically distributed random variables, all following a law with characteristic function $\phi(z)^{\frac{1}{n}}$.

We begin this section with the definition of a Lévy Process.

Definition 2.4.4 (Lévy Process). *A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^n$ is a Lévy process if the following conditions are satisfied.*

1. *For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent. That is, the process has independent increments.*
2. *$X_0 = 0$.*
3. *The distribution of $X_{s+t} - X_s$ does not depend on s. That is, the process has stationary increments.*
4. *It is stochastically continuous.*

5. *There is* $\Omega_0 \in \mathcal{F}$ *with* $\mathbb{P}[\Omega_0] = 1$ *such that, for every* $\omega \in \Omega_0$, $X_t(\omega)$ *is right-continuous in* $t \geq 0$ *and has left limits in* $t > 0$.

A Lévy process on $\mathbb{R}^n$ is called an n-dimensional Lévy process. The law at time t of a Lévy process is completely determined by the law of X_1. The only degree of freedom we have in specifying a Lévy process is to define its distribution at a single time. The following theorem describes the one-to-one relationship between Lévy processes and infinitely divisible distributions.

Theorem 2.4.1 (Infinite Divisibility of Lévy Processes). *Let* $X = \{X_t, t \geq 0\}$ *be a Lévy process. Then* $X = \{X_t, t \geq 0\}$ *has infinitely divisible distributions F for every t. Conversely if F is an infinitely divisible distribution there exists a Lévy Process* $X = \{X_t, t \geq 0\}$ *such that the distribution of* X_1 *is given by F.*

We can further write

$$\phi_{X_t}(u) = \mathbb{E}[e^{-iuX_t}] = e^{t\psi(u)}$$

where $\psi_X(u) = \log(\phi(u))$ is the characteristic exponent as in (3.25). The characteristic exponent $\psi(u)$ of a Lévy Process satisfies the following *Lévy- Khintchine formula* ([47]):

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbb{I}_{\{|x|<1\}})\nu(dx), \tag{2.7}$$

where $\gamma \in \mathbb{R}, \sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with

$$\int_{-\infty}^{\infty} \inf\{1, x^2\}\nu(dx) = \int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty. \tag{2.8}$$

From (2.7), we observe that, generally a Lévy process consist of 3 independent parts namely: a linear deterministic part, a Brownian part, and a pure jump part. We say that the corresponding infinitely divisible distribution has a Lévy triplet $[\gamma, \sigma^2, \nu(dx)]$. The measure ν is called the Lévy measure of X.

Definition 2.4.5 (Lévy measure). *Let* $\{X_t : t \geq 0\}$ *be a Lévy Process on* $\mathbb{R}^n$. *The measure* ν *on* $\mathbb{R}^n$ *defined by;*

$$\nu(A) = \frac{1}{t}\mathbb{E}\left(\sum_{0<s\leq t} \mathbb{I}_{\{\Delta X_s \in A\}}\right), \quad A \in \mathcal{B}(\mathbb{R}) \tag{2.9}$$

is called a Lévy measure. The measure $\nu(A)$ dictates how jumps occur. In particular jumps of sizes in the set A occur according to a Poisson process with parameter $\nu(A) = \int_A \nu(dx)$. In other words, $\nu(A)$ is the expected number of jumps per unit time, whose size belongs to A.

A Lévy measure has no mass at the origin, but singularities that is infinitely many jumps can occur near the origin (small jumps). Special attention has to be considered on small jumps. The sum of all jumps smaller than some $\epsilon > 0$ may not converge. For instance, consider the example where the Lévy measure $\nu(dx) = \frac{dx}{x^2}$, as we move closer to the origin there is an increasingly large number of small jumps and $\int_{-1}^{1} |x|\nu(dx) = +\infty$. But the integral $\int_{-1}^{1} x^2\nu(dx) = \int_{-1}^{1} x^2\frac{dx}{x^2} = \int_{-1}^{1} dx$ is still finite. However, as we move away from the origin, $\nu([-1,1]^c)$ is finite and we do not experience any difficulties with the integral in (2.8) being finite. Brownian motion has continuous sample paths with no jumps and as such $\Delta X_t = 0$. On the hand, Poisson process with rate parameter a and jump sizes equal to 1 is a pure jump process with $\Delta X_t = 1$ and Lévy measure $\nu(A) = \begin{cases} a & \text{if } \{1\} \in A \\ 0 & \text{if } \{1\} \notin A \end{cases}$

If the Lévy measure is of the form $\nu(dx) = u(x)dx$, then $u(x)$ is known as the Lévy density. The Lévy density has properties similar to a probability density, however, it need not be integrable and must have zero mass at the origin.

2.4.1 Properties of Lévy Processes

If $\sigma^2 = 0$ and $\int_{-1}^{+1} |x|\nu(dx) < \infty$, it follows from standard Lévy process theory that the process is of finite variation see [48], [47]. Moreover, there is a finite number of jumps in any finite interval and the process is said to be of finite activity.

The Brownian motion is of infinite variation, therefore a Lévy process with a Brownian component is of infinite variation. A pure jump Lévy process is of infinite variation if and only if $\int_{-1}^{+1} |x|\nu(dx) = \infty$. In this instance, special attention has to paid to the small jumps. Basically, the sum of all jumps smaller than $\epsilon > 0$ does not converge. However, the sum of the

jumps compensated by their mean does not converge. This peculiarity leads to the necessity of the compensator term $iux\mathbb{I}_{\{|x|<1\}}$ in (2.7).

2.5 Examples of Lévy Processes

In this section we give examples of some popular Lévy processes and describe the main properties, which we will use in this dissertation systematically. We will start with subordinators. Next, we will present some examples of Lévy processes that live on the real line. Much attention will be paid to their density function, their characteristic function, their Lévy triplets and some important properties. We compute moments, variance, skewness and kurtosis, if possible. For more examples of Lévy processes, see [49], [50] and [2].

2.5.1 Poisson Process

Definition 2.5.1 (Poisson Process). *A stochastic process $N = \{N_t, t \geq 0\}$ with intensity parameter $\lambda > 0$ is a Poisson process if it fulfills the following conditions:*

1. $N_0 = 0$.
2. *The process has independent increments.*
3. *The process has stationary increments.*
4. *For $s < t$ the random variable $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t-s)$:*

$$\mathbb{P}[N_t - N_s = n] = \frac{\lambda^n (t-s)^n}{n!} \mathrm{e}^{-\lambda(t-s)}$$

.

The Poisson process is the simplest of all the Lévy processes. It is based on the Poisson distribution, which depends on the parameter $\lambda > 0$ and has the the following characteristic function:

$$\phi_{Poisson}(u;\lambda) = \exp(\lambda(\exp(iu) - 1)).$$

The Poisson distribution lives on the non-negative integers $k = \{0, 1, 2, \dots\}$ and the probability mass function at point k is given by:

$$f(k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

Since the Poisson distribution is infinitely divisible, we can define a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter $\lambda > 0$ as the process which starts at zero, has independent and stationary increments and where the increments over a time interval of length $s > 0$ follows the Poisson(λs) distribution. The Poisson process is an increasing pure jump process, with jump sizes equal to 1. The time between two consecutive jumps follows an exponential distribution with mean λ^{-1}, that is a $\Gamma(1, \lambda)$ law. The moments of the Poisson distribution are given in Table 2.2.

Table 2.2: Moments of the Poisson distribution with intensity λ

	Poisson(λ)
Mean	λ
Variance	λ
Skewness	$\frac{1}{\sqrt{\lambda}}$
Kurtosis	$3 + \lambda^{-1}$

2.5.2 Compound Poisson Process

Definition 2.5.2 (Compound Poisson Process)**.** *A compound Poisson process with intensity parameter λ and a jumps size distribution L is a stochastic process $Z = \{Z_t, t \geq 0\}$ defined as:*

$$Z_t = \sum_{k=1}^{N_t} \chi_k \tag{2.10}$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter λ and $(\chi_k, k = 1, 2, \dots)$ is an independently and identically distributed sequence.

The sample paths of $Z = \{Z_t, t \geq 0\}$ are piecewise constant and the value of the process at time t, Z_t, is the sum of N_t random numbers with law L. The jump times have the same law as those of the Poisson process $N = \{N_t, t \geq 0\}$. The ordinary Poisson process corresponds to the case where $\chi_k = 1, k = 1, 2, \ldots$. The characteristic function of Z_t is given by

$$\mathbb{E}[\exp(iuZ_t)] = \exp\left(t \int_{-\infty}^{\infty} (\exp(iux) - 1)\nu(dx)\right) \quad \forall u \in \mathbb{R}, \tag{2.11}$$

where ν is called the *Lévy measure* of process $Z = \{Z_t, t \geq 0\}$. ν is a positive measure on $\mathbb{R}$ but not a probability measure since $\int \nu(dx) = \lambda \neq 1$.

2.5.3 The Gamma Process

Definition 2.5.3 (Gamma Process). *A stochastic process $X = \{X_t, t \geq 0\}$ with parameters a and b is a Gamma process if it fulfills the following conditions:*

1. $X_0 = 0$.
2. *The process has independent increments.*
3. *The process has stationary increments.*
4. *For $s < t$ the random variable $X_t - X_s$ has a Gamma$(a(t-s), b)$ distribution.*

A random variable X has a Gamma distribution $\Gamma(a, b)$ with rate and shape parameters $a > 0$ and $b > 0$ respectively, if its density function is given by:

$$f_X(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \mathrm{e}^{-bx}, \quad \forall x > 0. \tag{2.12}$$

The moments of the $\Gamma(a, b)$ distribution are given in Table 2.3.

.

The Gamma process is a non-decreasing Lévy process and its characteristic function is given by:

$$\phi(u; a, b) = (1 - \frac{iu}{b})^{-a}. \tag{2.13}$$

Table 2.3: Moments of the $\Gamma(a,b)$ distribution

	$\Gamma(a,b)$
Mean	$\frac{a}{b}$
Variance	$\frac{a}{b^2}$
Skewness	$2a^{\frac{1}{2}}$
Kurtosis	$3(1+2a^{-1})$

2.5.4 Inverse Gaussian Process

Let $T^{(a,b)}$ be the first time a standard Brownian motion with drift $b > 0$, that is $\{W_s + bs, s \geq 0\}$, reaches a positive level $a > 0$. The random time follows the inverse Gaussian, $IG(a,b)$, law and has a characteristic function:

$$\phi(u;a,b) = \exp(-a(\sqrt{-2iu+b^2} - b)).$$

The inverse Gaussian distribution is infinitely divisible and we define the inverse Gaussian process $X = \{X_t, t \geq 0\}$, with parameters $a, b > 0$, as the process which starts at zero and has independent and stationary increments such that

$$\begin{aligned} \mathbb{E}[\exp(iuX_t)] &= \phi(u;at,b) \\ &= \exp(-at(\sqrt{-2iu+b^2} - b)). \end{aligned} \tag{2.14}$$

2.6 Subordination of Lévy processes

Subordination is a transformation of a stochastic process to a new stochastic process through a random time change by increasing Lévy process (subordinator) independent of the original process. The new process is called a subordinate to the original one. The idea of subordination was introduced by Bochner [12] in 1949.

Definition 2.6.1 (Subordinator)**.** *A subordinator is a one-dimensional Lévy process that is non-decreasing almost surely. Such processes can be thought of as a random model of time evolution,*

since if $T = (T(t), t \geq 0)$ *is a subordinator we have,*

$$T(t) \geq 0 \quad a.s \quad \textit{for each} \quad t > 0,$$

and

$$T(t_1) \leq T(t_2) \quad a.s \quad \textit{whenever} \quad t_1 \leq t_2.$$

Theorem 2.6.1. *If* T *is a subordinator, then its Lévy symbol takes the form*

$$\eta(u) = ibu + \int_0^\infty (e^{iuy} - 1)\lambda(dy), \tag{2.15}$$

where $b \geq 0$ *and the Lévy measure* λ *satisfies the additional requirements*

$$\lambda(-\infty, 0) = 0 \quad \textit{and} \quad \int_0^\infty (y \wedge 1)\lambda(dy) < \infty.$$

Conversely, any mapping from $\mathbb{R}^d \to \mathbb{C}$ *of the form* (2.15) *is the Lévy symbol of a subordinator.*

The proof of Theorem 2.6.1 can be found in [47]. The pair (b, λ) is the characteristic of the subordinator T.

Some classical examples of a subordinators are the Poisson process, compound Poisson process (if and only if the $(\chi_k, k = 1, 2, \dots)$ in (2.10) are all $\mathbb{R}^+ -$ valued), Gamma process and many others. For more examples of subordinators, see Ref. [2].

We begin the second part of this chapter which is dedicated to the discussion of stochastic differential equation. We start our discussion with the concept of deterministic differential equations.

2.7 Deterministic Differential Equations

The theory of differential equations is the provenience of classical calculus and it motivated the creation of differential and integral calculus. A differential equation is an equation involving

an unknown function and its derivative. The idea underlying a differential is simple. Given a functional relationship

$$f(t, x(t), x'(t), x''(t), \dots) = 0, \quad 0 \le t \le T \tag{2.16}$$

involving the time t, an unknown function $x(t)$ and its derivative. The solution of the differential equation (2.16) is to find a function $x(t)$ which satisfies (2.16).

The simplest differential equations are those of order 1. They involve only $t, x(t)$ and the first derivative $x'(t)$. The standard form for the first-order differential equation in the unknown function $x(t)$ is

$$x'(t) = \frac{dx(t)}{dt} = a(t, x(t)), \quad x(0) = x_0, \tag{2.17}$$

where $a(x, t)$ is a known function and the derivative $x'(t)$ appears only on the left side of (2.17).

Example 2.7.1 (Exponential growth model)**.** *Consider the simple population growth model*

$$\frac{dN}{dt} = k(t)N(t), \quad N(0) = N_0 \tag{2.18}$$

where $N(t)$ is the size of the population at time t, and $k(t)$ is the relative rate of growth at time t. Integration on both sides yields the solution

$$\int \frac{dN}{N(t)} = \int k(t)dt \implies N(t) = N_0 e^{\int k(t)dt}$$

Example 2.7.2 (Separation of variables)**.** *Suppose the right-hand side of* (2.17) *can be separated into a product of two functions:*

$$x'(t) = \frac{dx}{dt} = a_1(t)a_2(x(t)). \tag{2.19}$$

Equation (2.19) *can be rewritten as*

$$\frac{dx}{a_2(x(t))} = a_1(t). \tag{2.20}$$

Integration on both sides yields the solution

$$\int_{x(0)}^{x(t)} \frac{dx}{a_2(t)} = \int_0^t a(s)ds. \tag{2.21}$$

On the left-hand side we obtain a function $x(t)$, on the right-hand side a function of t. Thus, we have obtain an explicit form of the function $x(t)$.

Remark 2.7.1. *Integrating both sides of* (2.17) *, one obtains an equivalent* integral equation:

$$x(t) = x(0) + \int_0^t a(s, x(s))ds. \tag{2.22}$$

The transformed equation is generally not used to find the solution of (2.19)*. It however gives an idea of how we could define a stochastic integral equation.*

In the exponential growth model in Example 2.7.1, it might happen that $k(t)$ is not completely known, but subject to some random environmental effects i.e.

$$k(t) = b(t) + \text{ “noise”}$$

where we do not know the exact behavior of the noise term, only its probability distribution. The function $b(t)$ is assumed to be nonrandom.

2.8 Itô Integrals

We begin this section by finding a mathematical interpretation of the “noise” term in the equation of Example 2.7.1. We recall that,

$$\frac{dN}{dt} = (b(t) + \text{ “noise” })N(t) \tag{2.23}$$

More generally, we can rewrite the above in the form

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot \text{ “noise”} \tag{2.24}$$

where $b(t, X_t)$ and $\sigma(t, X_t)$ are some given deterministic functions. If we consider the case where the noise is 1-dimensional, we can describe the noise term by some stochastic process W_t, so that

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \cdot W_t \tag{2.25}$$

The stochastic process W_t has the following properties:

(i) For $t_1 \neq t_2$ implies that the stochastic processes W_{t_1} and W_{t_2} are independent.

(ii) The stochastic process $\{W_t\}$ is stationary, i.e. the joint distribution of $\{W_{t_1+t}, \ldots, W_{t_k+t}\}$ does not depend on t.

(iii) $E[W_t] = 0$ for all t.

It turns out that there does not exist any suitable stochastic process satisfying properties (i) and (ii) i.e. such a W_t cannot have a continuous paths. However we can represent W_t as a generalized stochastic process called the *white noise process*. Here generalized means that the process can be constructed as a probability measure on the space of tempered distributions on $[0, \infty)$, and not as a probability measure on the much smaller space $\mathbb{R}^{[0,\infty)}$.

If we let $0 = t_0 < t_1 < \ldots < t_m = t$ we can discritize (2.25) as follows:

$$X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)W_k\Delta t_k \tag{2.26}$$

where $X_j = X(t_j)$, $W_k = W_{t_k}$ and $\Delta t_k = t_{k+1} - t_k$

Replacing $W_k\Delta t_k$ by $\Delta V_k = V_{t_k+1} - V_{t_k}$ in (2.26) where $\{V_t\}_{t\geq 0}$ is a suitable stochastic process. The properties (i), (ii) and (iii) on W_t suggest that V_t should be stationary. The only process with continuous paths is the Brownian motion Bt. Therefore we substitute V_t with B_t in eqrefexample-population-ito2 to obtain:

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta B_j \tag{2.27}$$

Assuming that the limit of the right hand side of (2.27) exist when $\Delta t_j \to 0$, then applying the usual integration notation we obtain

$$X_t = X_0 + \int b(s, X_s)ds + \int \sigma(s, X_s)dB_s \tag{2.28}$$

where the first integral on the right-hand side is a Riemann integral, and the second one is an Itô stochastic integral. We would adopt as a convention that (2.27) really means that $X_t = X_t(\omega)$ is a stochastic process satisfying (2.28).

We will proceed to prove the existence of

$$\int_0^t f(s,\omega)dB_s(\omega), \tag{2.29}$$

where $B_t(\omega)$ is a 1- dimensional Brownian motion starting at the origin, for a wide class of functions $f : [0,\infty] \times \Omega \to \mathbb{R}$.

Definition 2.8.1 (The Itô integral)**.** *Let $f \in V(S,T)$. Then the Itô integral of f is defined by*

$$\int_s^T f(t,\omega)dB_t(\omega) = \lim_{x\to\infty} \int_s^T \phi_n(t,\omega)dB_t(\omega), \tag{2.30}$$

where ϕ_n is a sequence of elementary functions such that

$$\mathbb{E}\left[\int_s^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0 \quad as \quad n \to \infty. \tag{2.31}$$

From Definition 2.8.1, we get the following,

Corollary 2.8.1 (The Itô Isometry)**.**

$$\mathbb{E}\left[\left(\int_s^T f(t,\omega)dB_t\right)^2\right] = \mathbb{E}\left[\left(\int_s^T f^2(t,\omega)dt\right)\right] \quad \forall \quad f \in V(S,T). \tag{2.32}$$

Corollary 2.8.2 (The Itô Isometry)**.** *If $f(t,\omega) \in V(S,T)$ and $f_n(t,\omega) \in V(S,T)$ for $n = 1,2,\ldots$ and $\mathbb{E}\left[\int_s^T f_n(t,\omega) - f(t,\omega)dt\right] \to 0$ as $n \to \infty$, then*

$$\int_s^T f_n(t,\omega)dB_t(w) \to \int_s^T f(t,\omega)dB_t(w) \quad in \quad L^2(P) \quad as \quad n \to \infty. \tag{2.33}$$

Theorem 2.8.1 (Integration by parts)**.** *Suppose $f(s,\omega) = f(s)$ only depends on s and that f is continuous and of bounded variation in $[0,t]$. Then*

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s. \tag{2.34}$$

2.8.1 Properties of the Itô integral

Theorem 2.8.2. *Let $f, g \in V(S,T)$ and let $0 \leq S < U < T$. Then*

1. $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$.
2. $\int_S^T (cf + g) dB_t = c \int_S^T f dB_t + \int_S^T g dB_t$, *for* $c \in \mathbb{R}$.
3. $\mathbb{E}\left[\left(\int_S^T f dB_t\right)\right] = 0$.
4. $\int_S^T f dB_t$ *is* $\mathcal{F}_T-$ *measurable.*

Another important property of the Itô integral is the fact that it is a martingale.

Definition 2.8.2. *A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t \geq 0}$ of σ algebras $\mathcal{M}_t \subset \mathcal{F}$ such that*

$$0 \leq s < t \implies \mathcal{M}_s \subset \mathcal{M}_t.$$

An $n-$ dimensional stochastic process $\{\mathcal{M}_t\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $\{\mathcal{M}_t\}_{t \geq 0}$ if

(i) $\mathcal{M}_t$ *is* $\mathcal{M}_t-$ *measurable for all* t.

(ii) $\mathbb{E}\left[|\mathcal{M}_t| < \infty\right]$ *for all* t.

(iii) $\mathbb{E}\left[\mathcal{M}_s | \mathcal{M}_t\right] = \mathcal{M}_t$ *for all* $s \geq t$.

The expectation in (ii) and the conditional expectation in (iii) is taken with respect to $\mathbb{P} = \mathbb{P}^0$.

Example 2.8.1. *Brownian motion B_t in $\mathbb{R}^n$ is a martingale with respect to the $\sigma-$ algebras $\mathcal{F}_t$ generated by $\{B_s; s \leq t\}$, because*

$$\begin{aligned} \mathbb{E}[|B_t|]^2 \leq \mathbb{E}[|B_t|^2] &= |B_0|^2 + nt \quad \text{and if} \quad s \geq t \quad \text{then} \\ \mathbb{E}[B_s | \mathcal{F}_t] &= \mathbb{E}[B_s - B_t + B_t | \mathcal{F}_t] \\ &= \mathbb{E}[B_s - B_t | \mathcal{F}_t] + \mathbb{E}[B_t | \mathcal{F}_t] \\ &= 0 + B_t. \end{aligned}$$

From Example 2.8.1, Itô integrals are martingales. Thus Itô integral gives an important computational advantage, even though it does not behave so nicely under transformations.

2.9 Itô Lemma

In the last section we discussed the Itô stochastic integral. The integral $\int_s^T f(t,\omega)dB_t$ is now known to be defined, however we do not know the tools to calculate Itô stochastic integrals. The objective of this section is to provide such a too, the Itô lemma. The Itô lemma is a very useful tool for evaluating Itô integrals. For example

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t \quad \text{or} \quad \frac{1}{2}B_t^2 = \frac{1}{2}t + \int_0^t B_s dB_s \tag{2.35}$$

The image of the Itô integral $B_t = \int_0^t dB_s$ by the map $g(x) = \frac{1}{2}x^2$ is not again an Itô integral of the form

$$\int_0^t f(s,\omega)dB_s(\omega)$$

but a combination of a dB_s- and a $ds-$ integral:

$$\frac{1}{2}B_t^2 = \frac{1}{2}t + \int_0^t B_s dB_s. \tag{2.36}$$

It turns out that if we introduce Itô stochastic integrals as sums of dB_s- and $ds-$ a integral then this family of integrals is stable under smooth maps.

Definition 2.9.1 ($1-$ dimensional Itô Stochastic Integral). *Let B_t be a $1-$ dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$. A $1-$ Itô stochastic integral is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ which is of the form of*

$$X_t = X_0 + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dB_s, \quad 0 \le t \le T, \tag{2.37}$$

so that

$$\mathbb{P}\left[\int_0^t b(s,\omega)^2 ds < \infty \quad \forall \quad t \ge 0\right] = 1 \tag{2.38}$$

and

$$\mathbb{P}\left[\int_0^t |a(s,\omega)| ds < \infty \quad \forall \quad t \ge 0\right] = 1 \tag{2.39}$$

If X_t is an Itô stochastic integral of the form of =(2.37) then (2.37) in its differential form is

$$dX_t = adt + bdB_t, \quad X_0(\omega) = Y(\omega). \tag{2.40}$$

$B = (B_t, t \geq 0)$ denotes Brownian motion, and a and b are deterministic functions.

Theorem 2.9.1 (The $1-$ dimensional Itô lemma)**.** *Let X_t be an Itô stochastic differential equation given by*

$$dX_t = adt + bdB_t, \quad X_0(\omega) = Y(\omega). \tag{2.41}$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$, that is g is twice continuously differentiable on $[0,\infty) \times \mathbb{R}$. Then

$$Y_t = g(t, X_t)$$

is also an Itô stochastic differential equation, and

$$dY_t = \frac{\partial g(t,X_t)}{\partial t}dt + \frac{\partial g(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 g(t,X_t)}{\partial x^2} \cdot (dX_t)^2, \tag{2.42}$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt. \tag{2.43}$$

Proof. Assume that g, $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x^2}$ are bounded and assume that $a(t,\omega)$ and $b(t,\omega)$ are elementary functions. Using Taylor's theorem we get

$$\begin{aligned} g(t,X_t) &= g(0,X_0) + \sum_j \Delta g(t_i, X_j) = g(0,X_0) + \sum_j \frac{\partial g}{\partial t}\Delta t_j + \sum_j \frac{\partial g}{\partial x}\Delta X_j \\ &+ \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta X_j) + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 + \sum_j R_j \end{aligned} \tag{2.44}$$

where $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$ etc. are evaluated at the points (t_j, X_{t_j}), $\Delta t_j = t_{j+1} - t_j$, $\Delta X_j = X_{t_{j+1}} - X_{t_j}$, $\Delta g(t_j, X_j) = g(t_{j+1}, X_{t_{j+1}}) - g(t_j, X_j)$ and $R_j = O(|\Delta t_j|^2 + |\Delta X_j|^2)\ \forall \quad j$.
If $\Delta t_j \to 0$ then

$$\sum_j \frac{\partial g}{\partial t}\Delta t_j = \sum_j \frac{\partial g}{\partial t}(t_j, X_j)\Delta t_j \to \int_0^t \frac{\partial g}{\partial t}(s, X_s)ds \tag{2.45}$$

$$\sum_j \frac{\partial g}{\partial x}\Delta t_j = \sum_j \frac{\partial g}{\partial t}(t_j, X_j)\Delta X_j \to \int_0^t \frac{\partial g}{\partial t}(s, X_s)dX_s. \tag{2.46}$$

Moreover, since a and b are elementary functions, we get

$$\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 = \sum_j \frac{\partial^2 g}{\partial x^2} a_j^2 (\Delta t_j)^2 + \sum_j \frac{\partial^2 g}{\partial x^2} a_j b_j (\Delta t_j)(\Delta B_j) + \sum_j \frac{\partial^2 g}{\partial x^2} b_j (\Delta B_j)^2, \tag{2.47}$$

where $a_j = a(t_j, \omega)$ and $b_j = b(t_j, \omega)$.

The first two terms in (2.47) tend to 0 as $\Delta t_j \to 0$. For example

$$\mathbb{E}\left[\left(\sum_j \frac{\partial^2 g}{\partial x^2} a_j b_j (\Delta t_j)(\Delta B_j)\right)^2\right] = \sum_j \mathbb{E}\left[\left(\sum_j \frac{\partial^2 g}{\partial x^2} a_j b_j\right)^2\right](\Delta t_j)^3 \to 0 \quad \text{as} \quad \Delta t_j \to 0.$$

Claim:

The last term in (2.47) tends to

$$\int_0^t \frac{\partial^2 g}{\partial x^2} b^2 ds \quad \text{in} \quad L^2(\mathbb{P}), \quad \text{as} \quad \Delta t_j \to 0.$$

To prove this claim, put $u(t) = \frac{\partial^2 g}{\partial x^2}(t, X_t) b^2(t, \omega)$, $u_j = u(t_j)$ and consider

$$\mathbb{E}\left[\left(\sum_j u_j (\Delta B_j)^2 - \sum_j u_j \Delta t_j\right)^2\right] = \sum_{i,j} \mathbb{E}[u_i u_j ((\Delta B_i)^2 - \Delta t_i)((\Delta B_j)^2 - \Delta t_j)].$$

If $i < j$ then $u_i u_j ((\Delta B_i)^2 - \Delta t_i)$ and $(\Delta B_j)^2 - \Delta t_j$ are independent so the terms vanish in this case, and similarly if $i > j$. So we are left with

$$\begin{aligned}
\sum_j \mathbb{E}[u_j^2 ((\Delta B_j)^2 - \Delta t_j)^2] &= \sum_j \mathbb{E}[u_j^2] \cdot \mathbb{E}[(\Delta B_j)^4 - (\Delta B_j)^2 \Delta t_j + (\Delta t_j)^2] \\
&= \sum_j \mathbb{E}[u_j^2] \cdot (3(\Delta t_j)^2 - 2(\Delta t_j)^2 + (\Delta t_j)^2) \\
&= 2 \sum_j \mathbb{E}[u_j^2] \cdot (\Delta t_j)^2 \to 0 \quad \text{as} \quad \Delta t_j \to 0.
\end{aligned}$$

Thus we have established that

$$\sum_j (\Delta B_j)^2 \to \int_0^t u(s) ds \quad \text{in} \quad L^2(\mathbb{P}) \quad \text{as} \quad \Delta t_j \to 0$$

and this is often expressed by the formula

$$(dB_t)^2 = dt. \tag{2.48}$$

The argument above also proves that $\sum_j R_j \to 0 \quad \text{as} \quad \Delta t_j \to 0$. This completes the proof of the Itô lemma. □

Remark 2.9.1. *It is enough that $g(t,x)$ is C^2 on $[0,\infty) \times U$, if $U \subset \mathbb{R}$ is an open set such that $X_t(\omega) \in U$ for all $t \geq 0, \omega \in \Omega$. Moreover, it is sufficient that $g(t,x)$ is C^1 with respect to t and C^2 with respect to x.*

Next we present the general Itô formula.

Theorem 2.9.2 (The general Itô formula)**.** *Suppose $\mathbf{B}_t = (B_1(t), \ldots, B_d(t))$ is a d-dimensional Brownian motion. Recall that each component is a standard Brownian motion.*

Let $\mathbf{X} = (X_1(t), X_2(t), \ldots, X_d(t))$, be an n-dimensional Itô process, that is,

$$dX_i(t) = \mu_i dt + \sigma_{i1} dB_1(t) + \sigma_{i2} dB_2(t) + \cdots + \sigma_{id} dB_d(t) \tag{2.49}$$

for all i from 1 to n. Equation (2.49) can be represented in matrix form as

$$d\mathbf{X}_t = U dt + \Sigma d\mathbf{B}_t,$$

where

$$U = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}, \text{ and } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1d} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nd} \end{pmatrix}$$

Suppose that $f(t,x) = (f_1(t,\mathbf{x}), f_2(t,\mathbf{x}), \ldots, f_m(t,\mathbf{x}))$ is a function defined on $[0,\infty) \times \mathbb{R}^n$ with values in $\mathbb{R}^m$ with $f \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R}^n)$.

Then, the process $\mathbf{Y}_t = f(t, \mathbf{X}_t)$ is also an Itô process and its component k is given by

$$dY_k(t) = \frac{\partial f_k}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i} dX_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j} (dX_i(t))(dX_j(t)),$$

for all k from 1 *to m. The last term is calculated using the following rules:*

$$dtdt = dB_i(t)dt = dtdB_i(t) = dB_i(t)dB_j(t) = 0, \quad \forall i \neq j$$
$$dB_i(t)dB_i(t) = dt.$$

The proof of the above theorem follows from the proof of Theorem 2.9.1.

2.10 One-dimensional Stochastic Differential Equation

A one-dimensional stochastic differential equation can be understood as a deterministic differential equation which is perturbed by random noise.

We recall the stochastic differential equation (2.25) defined in Section 2.8 of this chapter:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \tag{2.50}$$

where $b(t, x)$ and $\sigma(t, x)$ are both in $\mathbb{R}$ and W_t is a 1-dimensional "white noise". As mentioned earlier in Section 2.8, the Itô interpretation of (2.50) is that X_t satisfies the stochastic integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{2.51}$$

or in differential form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{2.52}$$

We obtained dB_t in (2.52) by replacing the white noise in (2.50) with $\frac{dB_t}{dt}$ and then multiply by dt.

Given a stochastic differential equation, two question that normally arises are

1. Can one obtain existence and uniqueness theorems for such equations? What are the properties of the solutions?

2. How can one solve such equations?

We now discuss the existence and uniqueness question 1 above.

2.10.1 An Existence and Uniqueness Result

Theorem 2.10.1 (Existence and Uniqueness theorem for stochastic differential equations)**.** *Let $T > 0$ and $a(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, b(\cdot,\cdot) : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|); \quad x \in \mathbb{R}^n, t \in [0,T] \tag{2.53}$$

for some constant C, (where $|\sigma|^2 = \sum |\sigma_{ij}|^2$) and such that

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x-y|; \quad x,y \in \mathbb{R}^n, t \in [0,T] \tag{2.54}$$

for some constant D. Let Z be a random variable which is independent of the $\sigma-$ algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(\cdot), s \geq 0$ and such that

$$\mathbb{E}[|Z|^2] < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad 0 \leq t \leq T, X_0 = Z \tag{2.55}$$

has a unique $t-$ continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and

$$B_s(\cdot); s \leq t \quad \textit{and} \tag{2.56}$$

$$\mathbb{E}\left[\int_0^T |X_t|^2 dt\right] < \infty. \tag{2.57}$$

Conditions (2.53) and (2.54) are natural in view of the following two simple examples from deterministic equations i.e. $\sigma = 0$.

Example 2.10.1. *1. The equation*

$$\frac{dX_t}{dt} = X_t^2, \quad X_0 = 1 \tag{2.58}$$

corresponding to $b(x) = x^2$ (which does not satisfy condition (2.53) *has the (unique) solution*

$$X_t = \frac{1}{1-t}; \quad 0 \leq t < 1.$$

Thus it is impossible to find a global solution (defined for all t) in this case. More generally, condition (2.53) *ensures that the solution $X_t(\omega)$ of* (2.55) *does not explode, that is, $|X_t(\omega)|$ does not tend to ∞ in a finite time.*

2. *The equation*

$$\frac{dX_t}{dt} = 3X_t^{2/3}; \quad X_0 = 0 \tag{2.59}$$

has more than one solution. In fact, for any $u > 0$ the function

$$X_t = \begin{cases} 0 & \text{for} \quad t \leq a \\ (t-u)^3 & \text{for} \quad t > a \end{cases}$$

solves (2.59). *In this case $b(x) = 3x^{2/3}$ does not satisfy the Lipschitz condition* (2.54) *at $x = 0$. Thus condition* (2.54) *guarantees that* (2.55) *has a unique solution. Here uniqueness means that if $X_1(t,\omega)$ and $X_2(t,\omega)$ are two $t-$ continuous processes satisfying equations* (2.55), (2.56) *and* (2.57) *then*

$$X_1(t,\omega) = X_2(t,\omega) \quad \text{for} \quad \text{all} \quad t \leq T, a.s. \tag{2.60}$$

For the second question, the Itô lemma is the used as a solution method to many stochastic differential equations.

2.10.2 Weak and Strong Solutions

Definition 2.10.1 (Strong Solution)**.** *A strong solution to the Itô stochastic differential equation* (2.37) *is a stochastic process $X_t = (X_t, t \in [0,T])$ which satisfies the following conditions:*

- *X_t is adapted to the Brownian motion, that is, at time t it is a function of $B_s, s \leq t$.*
- *The integrals occurring in* (2.37) *are well defined as Riemann and Itô stochastic integrals respectively.*

- *X_t is a function of the underlying Brownian sample path and of the coefficient functions $b(t,x)$ and $\sigma(t,x)$. Thus a strong solution to (2.37) is based on the path of the underlying Brownian motion.*

Equivalently, a solution X_t to a stochastic differential Equation is called a strong solution if the version of B_t of Brownian motion is given in advance and the solution X_t constructed from it is $\mathcal{F}_t^Z-$ adapted.

Definition 2.10.2 (Weak Solution)**.** *A solution X_t to a stochastic differential Equation is called a weak solution if we are only given the functions $b(t,x)$ and $\sigma(t,x)$ and ask for a pair of processes $((\tilde{X}_t, \tilde{B}_t), \tilde{\mathcal{H}}_t)$ on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ such that condition (2.54) holds. $\mathcal{H}_t$ is an increasing family of $\sigma-$algebras such that $\tilde{X}_t$ is $\mathcal{H}_t-$ adapted and $\tilde{B}_t$ is an $\mathcal{H}_t-$ Brownian motion, that is $\tilde{B}_t$ is a Brownian motion, and $\tilde{B}_t$ is a a martingale with respect to $\mathcal{H}_t$.*
For these solutions, the path behavior is is not essential, we are only interested in the distribution X_t. Weak solutions X_t are sufficient in order to determine the distributional characteristics of X_t such as the expectation, variance and covariance functions of the process.

Remark 2.10.1. *A strong solution is of course also a weak solution, but the converse is not true in general.*

2.11 Multi-dimensional Stochastic Differential Equations

If a finite number of SDEs are given in a model, then the multi-dimensional case should be considered. This situation can arise for the modeling of several phenomena, e.g., the price evolution of multiple stocks, interest rates, volatilities and several others see [43], [28].

For the case of random fluctuation in higher dimensions (i.e. $m \geq 2$), let $\mathbf{B}_t = (B_t^1, B_t^2, \ldots, B_t^m)^T$ denote $m-$ dimensional Brownian motion at time t. In this case, the deterministic (drift) part $b : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$, namely, $\mathbf{b}(X_t, t)$ is a measurable vector process the diffusion part $\sigma : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^{d\times m}$, namely, $\boldsymbol{\sigma}(X_t, t)$ is a measurable matrix-valued process.

For the underlying probability space $(\Omega, \mathbf{F}, P)$, adapted to the filtration $(\mathbf{F})_{t\geq 0}$, a $d-$ dimensional stochastic process $\mathbf{X} = (\mathbf{X}_t : t \in [0,\infty))$ is represented by d differential equations,

$d-$ dimensional initial vector and suitable conditions.

If we write a coupled system of SDEs with d processes of states and $m-$ dimensional Brownian motions, we obtain:

$$d\mathbf{X}_t = b(\mathbf{X}_t, t)dt + \sigma(\mathbf{X}_t, t)d\mathbf{B}_t \tag{2.61}$$

where

$$\mathbf{X}_t = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_d(t) \end{bmatrix}, b(\mathbf{X}_t, t) = \begin{bmatrix} b_1(\mathbf{X}_t, t) \\ b_2(\mathbf{X}_t, t) \\ \vdots \\ b_d(\mathbf{X}_t, t) \end{bmatrix}, d\mathbf{B}_t = \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ \vdots \\ dB_m(t) \end{bmatrix}$$

$$\sigma(\mathbf{X}_t, t) = \begin{bmatrix} \sigma_{11}(\mathbf{X}_t, t) & \sigma_{12}(\mathbf{X}_t, t) & \dots & \sigma_{1m}(\mathbf{X}_t, t) \\ \sigma_{21}(\mathbf{X}_t, t) & \sigma_{22}(\mathbf{X}_t, t) & \dots & \sigma_{2m}(\mathbf{X}_t, t) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{d1}(\mathbf{X}_t, t) & \sigma_{d2}(\mathbf{X}_t, t) & \dots & \sigma_{dm}(\mathbf{X}_t, t) \end{bmatrix},$$

For comprehensive study of the stochastic differential equation and its solution methods, see [40], [43], [28] and references therein.

2.12 Simulation of stochastic differential equations

In this section we will discuss the simulation of SDEs focusing mainly on the Euler–Maruyama method [38] and Euler–Milstein scheme [41] for approximating the sample path of SDE's. Please refer to [22] for other techniques used to generate sample paths of SDEs. We begin our discussion with the Euler–Maruyama method which is also known as the Euler method.

2.12.1 Euler–Maruyama scheme for approximating stochastic differential equations

The Euler–Maruyama method is a technique used to approximate the numerical solution of a stochastic differential equation. It is a generalization of the Euler method for approximating ordinary differential equations to stochastic differential equations. Consider the following SDE:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{2.62}$$

with initial condition $X_0 = x_0$ and B_t a 1-dimensional standard Brownian motion. The solution of (2.62) is given as:

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s. \tag{2.63}$$

Therefore, approximating the path of X_t is equivalent to approximating the integral. There are several ways to approximate the first integral, the second however has to be approximated using the Euler scheme. The Euler method uses a simple rectangular rule. First assume that the interval $[0, t]$ is divided into n equal subintervals. This implies that the increment is $\Delta t = t/n$ and that the points are $t_0 = 0, t_1 = \Delta t, \ldots, t_i = i\Delta t, \ldots, t_n = n\Delta t = t$. Thus using X_i to denote X_{t_i} we have:

$$\begin{cases} X_0 &= x_0 \\ X_i &= X_{i-1} + b(t_{i-1}, X_{i-1})\Delta t + \sigma(t_{i-1}, X_{i-1})\Delta B_i, \quad \forall i \in \{1, 2, \ldots, n\}, \end{cases}$$

where ΔB_i is the increment of a standard Brownian motion over the interval $[t_{i-1}, t_i]$. We recall that the Brownian motion has independent and stationary increments, so it follows that each of such increment is independent of all others and is distributed as a normal (Gaussian) random variable with mean 0 and variance the length of the time sub-interval (i.e. Δt). Therefore, the standard deviation of the increment is $\sqrt{\Delta t}$.

The Euler–Maruyama algorithm for generating the sample paths of example 2.63 using a fixed number of paths, n, and discretization interval, Δt are presented in Algorithm 1.

Algorithm 1 Generating a sample path using Euler–Maruyama's method to estimate $\theta = E[f(X_t)]$.

---

for $j = 1$ to n **do**

  $t = 0$; $\hat{X} = X_0$

  for $k = 1$ to $[T/\Delta t] =: m$ **do**

    generate $Z \sim N(0,1)$

    set $\hat{X} = \hat{X} + b(t, \hat{X})\Delta t + \sigma(t, \hat{X})\sqrt{\Delta t}Z$

    set $t = t + \Delta t$

  end for

  set $f_j = f(\hat{X})$

end for

set $\hat{\theta}_n = (f_1 + \ldots + f_n)/n$

set $\hat{\sigma}_n^2 = \sum_{j=1}^{n}(f_j - \hat{\theta}_n)^2/(n-1)$

set approximately $100(1-\alpha)\%$ $\quad$ CI $= \hat{\theta}_n \pm z_{1-\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}}$

This algorithm can be extended to general b and σ functions by creating separate functions. The scheme can also be generalized to approximate multidimensional SDEs. In the multidimensional case, $X_t \in \mathbb{R}^d$, $B_t \in \mathbb{R}^p$ and $b(t, X_t) \in \mathbb{R}$ are vectors and $\sigma(t, X_t) \in \mathbb{R}^{d\times p}$ is a matrix. The Euler–Maruyama method gives a first-order approximation for the stochastic integral.

The Euler–Milstein method discussed in the next subsection provides an improvement by including second-order terms.

2.12.2 Euler–Milstein scheme for approximating stochastic differential equations

The Euler–Milstein scheme is a technique for approximating the numerical solution of a stochastic differential equation. The idea in this scheme is to consider expansions on the coefficients b and σ. This method is applied when the coefficients of the process are functions of only the main process i.e. do not depend on time. The scheme is designed to work with SDEs of the type

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

with initial conditions $X_0 = x_0$. We consider expansions on the coefficients $b(X_t)$ and $\sigma(X_t)$ using Itô's lemma. We then obtain:

$$db(X_t) = b'(X_t)dX_t + \frac{1}{2}b''(X_t)(dX_t)^2$$

and

$$d\sigma(X_t) = \left(b'(X_t)b(X_t) + \frac{1}{2}b''(X_t)\sigma^2(X_t)\right)dt + b'(X_t)\sigma(X_t)dB_t.$$

Writing out the integral form from t to u for any $u \in (t, t + \Delta t]$, we obtain

$$b_u = b_t + \int_t^u \left(b'_s b_s + \frac{1}{2}b''_s\sigma_s^2\right)ds + \int_t^u b'_s\sigma_s dB_s$$
$$\sigma_u = \sigma_t + \int_t^u \left(\sigma'_s b_s + \frac{1}{2}\sigma''_s\sigma_s^2\right)ds + \int_t^u \sigma'_s\sigma_s dB_s,$$

where we used the notation $b_u = b(X_u)$. Substituting these expressions in the original SDE, we obtain

$$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} \left(b_t + \int_t^u \left(b_s' b_s + \frac{1}{2} b_s'' \sigma_s^2 \right) ds + \int_t^u b_s' \sigma_s dB_s \right) du$$
$$+ \int_t^{t+\Delta t} \left(\sigma_t + \int_t^u \left(\sigma_s' b_s + \frac{1}{2} \sigma_s'' \sigma_s^2 \right) ds + \int_t^u \sigma_s' \sigma_s dB_s \right) dB_u$$

In this expression, we eliminate all terms which will produce higher orders than Δt after integration. That means eliminating terms of the type $dsdu = O(\Delta_t^2)$ and $dudB_s = O(\Delta t^{\frac{3}{2}})$. The only terms remaining other than simply du and ds are the ones involving $dB_u dB_s$ since they are of the right order. Thus, after eliminating the terms, we obtain:

$$X_{t+\Delta t} = X_t + b_t \int_t^{t+\Delta t} du + \sigma_t \int_t^{t+\Delta t} dB_u + \int_t^{t+\Delta t} \int_t^u \sigma_s' \sigma_s dB_s dB_u \tag{2.64}$$

For the last term, we apply Euler discretization in the inner integral as follows:

$$\begin{aligned} \int_t^{t+\Delta t} \left(\int_t^u \sigma_s' \sigma_s dB_s \right) dB_u &\approx \int_t^{t+\Delta t} \sigma_t' \sigma_t (B_u - B_t) dB_u \\ &= \sigma_t' \sigma_t \left(\int_t^{t+\Delta t} B_u dB_u - B_t \int_t^{t+\Delta t} dB_u \right) \\ &= \sigma_t' \sigma_t \left(\int_t^{t+\Delta t} B_u dB_u - B_t B_{t+\Delta t} + B_t^2 \right). \end{aligned} \tag{2.65}$$

For the integral term inside, recall that

$$\int_0^t B_u dB_u = \frac{1}{2}(B_t^2 - t).$$

Therefore, applying for t and $t + \Delta t$ and taking the difference, we obtain

$$\int_t^{t+\Delta t} B_u dB_u = \frac{1}{2}(B_{t+\Delta t}^2 - t - \Delta t) - \frac{1}{2}(B_t^2 - t)$$

Therefore, substituting back into (2.65), we have

$$\begin{aligned} \int_t^{t+\Delta t} \left(\int_t^u \sigma_s' \sigma_s dB_s \right) dB_u &\approx \sigma_t' \sigma_t \left(\frac{1}{2}(B_{t+\Delta t}^2 - B_t^2 - \Delta t) - B_t B_{t+\Delta t} + B_t^2 \right) \\ &= \sigma_t' \sigma_t \left(\frac{1}{2} B_{t+\Delta t}^2 + \frac{1}{2} B_t^2 - B_t B_{t+\Delta t} - \Delta t \right) \\ &= \sigma_t' \sigma_t \left(\frac{1}{2}(B_{t+\Delta t} - B_t)^2 - \Delta t \right). \end{aligned}$$

We recall that $B_{t+\Delta t} - B_t$ is the increment of the Brownian motion which we know is $N(0, \Delta t)$ or $\sqrt{\Delta t}Z$, where $Z \sim N(0,1)$. In summary, for the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0,$$

the Euler–Milstein scheme starts with $X_0 = x_0$ and for each successive point, we first generate $Z \sim N(0,1)$ and then calculate the next point as

$$X_{t+\Delta t} = X_t + b(X_t)\Delta t + \sigma(X_t)\sqrt{\Delta t}Z + \frac{1}{2}\sigma'(X_t)\sigma(X_t)\Delta t(Z^2 - 1) \tag{2.66}$$

where σ' denotes the derivative of $\sigma(x)$ with respect to x.

Remark 2.12.1. *When $\sigma'(X_t) = 0$, i.e. the diffusion term does not depend on X_t, the Euler–Milstein scheme is equivalent to the Euler–Maruyama method.*

The Euler–Milstein method can also be generalized to approximate multidimensional SDEs.

Chapter 3

Construction of Ornstein-Uhlenbeck type Stochastic Differential Equation

Many phenomena in natural, health, finance and biological processes can be better described and analyzed by the help of mathematical and statistical modeling. A wide range of literature on mathematical modeling of dynamical processes can be found for deterministic differential equations, e.g., ordinary differential equations (ODEs) [1] [11], partial differential equations (PDEs) [53], integro-differential equations (IDEs) [3] and partial integro-differential equations (PIDEs) [54], whereby the element of noise is not considered. However, many phenomena are influenced by random fluctuation, the behavior of noise in differential equations should be explained. For that reason, Stochastic Differential Equations (SDEs) are very useful. There are several areas of applications where SDEs are given, for example, in public health, finance, economics, geophysics, population dynamics, engineering and social sciences.

In this chapter we will discuss a mean reverting process; the Ornstein-Uhlenbeck processes which was introduced by [9] as a model to describe volatility in finance. We will derive some important results of the Ornstein-Uhlenbeck processes and pave the way for our model: One-dimensional and Multi-dimensional Ornstein-Uhlenbeck type Stochastic Differential Equation. Simulation methods used to generate the realizations of the model will be given.

3.1 Introduction

Mean reversion models can be defined as the property to always revert to a certain constant or time varying level with limited variance around it. For example, if we pluck the guitar string,

the string will revert to its place of equilibrium.

The first description of a mean reversion process was given by Uhlenbeck and Ornstein, [56]. Uhlenbeck and Ornstein argued that the total force on a particle falling through a fluid should arise from a combination of random bombardments by the molecules of the fluid and also a macroscopic frictional force, which acts to dampen the motion. According to Newton's laws of motion this net force equals the rate of change of momentum and so we have the differential equation:

$$m\frac{dX}{dt} = -\lambda m X + m\frac{dB}{dt}, \tag{3.1}$$

where X is the velocity, λ is a positive constant and the derivative "$\frac{dB}{dt}$" describes the random velocity changes due to molecular bombardment. Equation (3.1) acquires a meaning as soon as we interpret it as a stochastic differential equation. We therefore obtain the Langevin equation which was named after physicist Paul Langevin,

$$dX(t) = -\lambda X(t)dt + dB(t). \tag{3.2}$$

In order to generalize (3.2), we replace B by a Lévy process $Z = \{Z_t, t \geq 0\}$, to obtain

$$dX(t) = -\lambda X(t)dt + dZ(t), \tag{3.3}$$

which we call the Ornstein-Uhlenbeck process. These Lévy driven processes are known as non-Gaussian Ornstein-Uhlenbeck processes. They were introduced by Barndorff-Nielsen and Shephard, [9] to describe volatility in finance. The negative sign appearing in (3.3) makes the process mean-reverting. Mean-reverting means that the process eventually revert to their long-term mean or average level.

In the next section, we provide a brief introduction to Ornstein-Uhlenbeck process and develop the theory and properties of this process. In particular, we study the definitions, properties and solutions of (3.3).

3.2 One-dimensional Ornstein-Uhlenbeck Processes

A continuous time stationary and non-negative stochastic process $X = \{X(t)\}$ is said to be a process of the Ornstein-Uhlenbeck type if it is c*á*dl*á*g (i.e. it is right continuous and has a left limit at every point) and satisfies the stochastic differential equation:

$$dX(t) = -\lambda X(t)dt + dZ(t), X_0 > 0, \quad \lambda \in \mathbb{R}^+. \tag{3.4}$$

where $Z = \{Z_t, t \geq 0\}$ is a Lévy process and the rate parameter λ is a positive number. The process $Z = \{Z_t, t \geq 0\}$ is termed the background driving Lévy process or subordinator corresponding to the process $X(t)$. The subordinator Z is a pure jump Lévy process therefore (3.4) is a jump process. The Lévy process $Z = \{Z_t, t \geq 0\}$ is of bounded variation on finite intervals and from (2.30) integrals such as $\int_0^t f(s)dZ(t)$ exist and are well defined when f is a continuous function.

3.2.1 Solution of the One-dimensional Ornstein-Uhlenbeck Processes

For $\lambda > 0$ and $t > 0$, we can find an explicit solution to (3.4). Define

$$g(t, X_t) = \mathrm{e}^{\lambda t} X_t \tag{3.5}$$

Recall the $1-$ dimensional Itô lemma described in Chapter 2 of this dissertation i.e.

$$d\left(g(t, X_t)\right) = \left[\frac{\partial g}{\partial t}(t, X_t) + a(t, X_t)\frac{\partial g}{\partial X_t}(t, X_t) + \frac{1}{2}(b(t, X_t))^2\frac{\partial^2 g}{\partial X_t^2}(t, X_t)\right]dt + b(t, X_t)\frac{\partial g}{\partial X_t}(t, X_t)dZ_t$$

Applying the Itô lemma to (3.5),

$$\begin{aligned} d(\mathrm{e}^{\lambda t}X_t) &= \left[\lambda X_t \mathrm{e}^{\lambda t} - \lambda X_t \mathrm{e}^{\lambda t}\right]dt + e^{\lambda t}dZ_t \\ &= \mathrm{e}^{\lambda t}dZ_t \end{aligned} \tag{3.6}$$

Integrating both sides of (3.6) and dividing through by $\mathrm{e}^{\lambda t}$ we obtain:

$$X_t = \mathrm{e}^{-\lambda t}X_0 + \int_0^t \mathrm{e}^{-\lambda(t-s)}dZ_s \tag{3.7}$$

Since (3.7) is a function of $Z_s, s \leq t$ it implies that the solution is a strong solution by Definition 2.10.1.

Barndorff-Nielsen and Shephard [9], proposed a continuous time stationary and strictly increasing process for the squared volatility process $X = \{X(t)\}_{t\geq 0}$ of a financial asset. The proposed model for the squared volatility process is an Ornstein-Uhlenbeck model which is of the form,

$$dX(t) = -\lambda X(t)dt + dZ(\lambda t), X_0 > 0, \quad \lambda \in \mathbb{R}^+. \tag{3.8}$$

The timing in the background driving Lévy process $dZ(\lambda t)$ of (3.8) is selected in order that regardless of the value of λ the marginal distribution of $X = \{X(t)\}_{t\geq 0}$ remains unchanged. That is the marginal distribution of $X = \{X(t)\}_{t\geq 0}$ does not depend on λ. Therefore taking into account the unusual timing in the background driving Lévy process, we can rewrite (3.7) as

$$X_t = \mathrm{e}^{-\lambda t}X_0 + \int_0^t \mathrm{e}^{-\lambda(t-s)}dZ(\lambda s) \tag{3.9}$$

which is the solution to (3.8).

Remark 3.2.1. *The background driving Lévy process of* (3.9)*, $Z = \{Z(\lambda t), t \geq 0\}$ is an increasing process and the initial condition $X_0 > 0$, we have that the process X_t is strictly positive and is bounded from below by the function $X_0 \exp(-\lambda t)$. The form of* (3.9) *implies the autocorrelation function of X_t will decay exponentially with the rate λ.*

3.2.2 Superposition of the Ornstein-Uhlenbeck Processes

In this section we present the main result upon which this dissertation is centered. To model the component of (3.9) so as to offer plenty of analytic flexibility and ensure correlation structures for the process X_t, we consider the sum of m independent Ornstein-Uhlenbeck processes, that is

$$X(t) = \sum_{i=1}^{m} w_i X_i \mathrm{e}^{-\lambda_i t} + \int_0^t \sum_{i=1}^{m} w_i \mathrm{e}^{-\lambda_i(t-s)}dZ(\lambda_i s) \quad t \geq 0, \tag{3.10}$$

where $\sum_{i=1}^{m} w_i = 1$.

A two component model of (3.10) is given by

$$X(t) = w_1 X_1(t) + w_2 X_2(t) \quad \text{and} \quad w_1 + w_2 = 1,$$

where each component process is an independent Ornstein-Uhlenbeck process with rate parameters λ_1 and λ_2. Therefore we have,

$$\begin{aligned} X(t) = w_1 X_1 \mathrm{e}^{-\lambda_1 t} + \int_0^t w_1 \mathrm{e}^{-\lambda_1 (t-s)} dZ(\lambda_1 s) \\ + w_2 X_2 \mathrm{e}^{-\lambda_2 t} + \int_0^t w_2 \mathrm{e}^{-\lambda_2 (t-s)} dZ(\lambda_2 s), \quad t \geq 0. \end{aligned} \tag{3.11}$$

which, by a change of variable can also be written as

$$\begin{aligned} X(t) = w_1 X_1 \mathrm{e}^{-\lambda_1 t} + w_1 \mathrm{e}^{-\lambda_1 t} \int_0^{\lambda_1 t} \mathrm{e}^s dZ(s) \\ + w_2 X_2 \mathrm{e}^{-\lambda_2 t} + w_2 \mathrm{e}^{-\lambda_2 t} \int_0^{\lambda_2 t} \mathrm{e}^s dZ(s), \quad X_1, X_2 > 0, \lambda_1, \lambda_2 > 0 \quad \text{and} \quad t \geq 0. \end{aligned} \tag{3.12}$$

3.3 Multi-dimensional Ornstein-Uhlenbeck Processes

The system of SDEs understudy is the formulation of a 2-dimensional stochastic process $\mathbf{X}(t)$ driven in terms of two differential equation. From these two SDEs, the first one $\{X_1(t)\}$ describes physical process which is affected by location of an event, while the second SDE $\{X_2(t)\}$ describes location of an event, which is affected by the physical process.

Then, our coupled system satisfies the stochastic differential equation:

$$\begin{cases} dX_1(t) = -\lambda_1 X_1(t) dt + \sigma_{11} dZ_1(t) + \sigma_{12} dZ_2(t), & \lambda_1 \in \mathbb{R}^+ \\ dX_2(t) = -\lambda_2 X_2(t) dt + \sigma_{21} dZ_1(t) + \sigma_{22} dZ_2(t), & \lambda_2 \in \mathbb{R}^+ \end{cases} \tag{3.13}$$

with $\mathbf{X}(0) = (X_0^1, X_0^2)^T$, where $X_0^1 > 0$ and $X_0^1 > 0$ denotes the initial condition for $X_1(t)$ and $X_2(t)$ respectively, $Z_1(t) = Z(\lambda_1 t)_{t \geq 0}, Z_2(t) = Z_1(\lambda_2 t)_{t \geq 0}$ are Lévy processes and λ_1, λ_2 are the intensity parameters. The parameters σ_{11}, σ_{22} determines the volatility of the system and σ_{12}, σ_{21} describes the correlation of the system. The processes Z_1 and Z_2 are termed the

background driving Lévy processes (BDLP). The intensity parameters describes the velocity at which the time series reverts towards its mean value. The volatility parameters are used to tune the effect of $\mathbf{Z}_t$ on $\mathbf{X}_t$. A higher value implies more randomness or haphazardness in the system. Choosing $\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_{21} = 0$ reduces (3.13) to a deterministic differential equation. The processes $X_1(t)$ and $X_2(t)$ are correlated if σ_{12}=$\sigma_{21} \neq 0$.

In matrix notation, we can rewrite (3.13) as:

$$d\mathbf{X}(t) = A\mathbf{X}(t)dt + \sum_{i=1}^{2} B_i(t)d\mathbf{Z}(\lambda t) \tag{3.14}$$

where

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, A = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, B_1(t) = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{21} \end{pmatrix}, B_2(t) = \begin{pmatrix} \sigma_{12} & 0 \\ 0 & \sigma_{22} \end{pmatrix} \text{ and}$$

$$\mathbf{Z}(\lambda t) = \begin{pmatrix} Z_1(\lambda t) \\ Z_2(\lambda t) \end{pmatrix}$$

Equation (3.13) is an example of a multi-dimensional SDE discussed in Chapter 2. Differently from the general structure already defined in Chapter 2, in our proposed model, we decrease the model complexity in order to increase the stability of the model (that is, to decrease its sensitivity) against various forms of noise and perturbation and also the noise term is a Lévy process instead of the classical Brownian motion.

Thus we have a 2- dimensional stochastic differential equation. This particular system can be solved easily by finding the solution for X_1 and X_2. However, we are looking for a more general theory.

3.3.1 Solution of the Multi-dimensional Ornstein-Uhlenbeck Processes

From (3.14) we have the following system of SDEs:

$$d\mathbf{X}(t) = A\mathbf{X}(t)dt + B_1 d\mathbf{Z}(\lambda t) + B_2 d\mathbf{Z}(\lambda t) \tag{3.15}$$

for an $n-$ dimensional process $\mathbf{X}(t)$, where A, B_1 and B_2 are matrices.

We rewrite (3.15) as

$$e^{-At}d\mathbf{X}(t) - e^{-At}A\mathbf{X}(t)dt = e^{-At}B_1 d\mathbf{Z}(\lambda t) + B_2 d\mathbf{Z}(\lambda t) \tag{3.16}$$

where for any general $n \times n$ matrix A, we define e^A to be an exponential matrix of the form:

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \tag{3.17}$$

where $A^0 = I$ is the identity matrix.

From (3.16), we observe that the left hand side is related to

$$d(e^{-At}dX(t)).$$

To achieve this, we apply the 2-dimensional version of the Itô formula (Theorem 2.9.2) to the two coordinate functions f_1, f_2 of

$$f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2 \text{ given by } f(t, x_1, x_2) = e^{-At} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

we obtain that

$$d(e^{-At}d\mathbf{X}(t)) = e^{-At}d\mathbf{X}(t) - e^{-At}A\mathbf{X}(t)dt. \tag{3.18}$$

Substituting (3.18) into (3.16) and taking into account the unusual timing in the BDLP, we obtain the solution:

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-s)}B_1 d\mathbf{Z}(\lambda s) + \int_0^t e^{A(t-s)}B_2 d\mathbf{Z}(\lambda s) \tag{3.19}$$

by integration by parts.

Remark 3.3.1. The issue when calculating the solution is to calculate the exponential e^{At}, when A is a $n \times n$ matrix. The idea is to write the matrix A as $A = PDP^{-1}$, where P is invertible matrix and D is a diagonal matrix. We can find P and D by diagonalizing the matrix.

Definition 3.3.1 (Diagonalizable). *A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, i.e. if $A = PDP^{-1}$ where P is invertible and D is a diagonal matrix.*

We state the following theorem without proof.

Theorem 3.3.1 (Diagonalizable theorem). *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, $A = PDP^{-1}$ with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. The diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.*

From Theorem 3.3.1, if we can find n linearly independent eigenvectors for $n \times n$ matrix A, then it is diagonalizable. In addition, we can use the eigenvectors and their corresponding eigenvalues to find an invertible matrix P and diagonal matrix D required to show that A is diagonalizable.

In order to proceed further with our proposed Ornstein-Uhlenbeck type model, we need the following definition of self-decomposability.

Definition 3.3.2. (Self-decomposability) A probability distribution P on $\mathbb{R}$ is said to be self-decomposable or belong to the Lévy class L if, for all $\lambda > 0$, there exists a probability distribution P_λ on $\mathbb{R}$ such that

$$\phi(\nu) = \phi_\lambda(\nu)\phi(e^{-\lambda}\nu); \quad \nu \in \mathbb{R}$$

where ϕ and ϕ_λ refers to the characteristic functions of P and P_λ, respectively. A random variable X with law in the Lévy class L is self-decomposable.

The following statement gives the relationship between self-decomposability and infinite divisibility. If the probability distribution P on $\mathbb{R}$ is self-decomposable then the distribution P_λ is infinitely divisible. The idea of self-decomposability is linked to that of stationary autoregressive processes of order 1. In essence, processes of these nature are those for which the stationary distribution is self-decomposable [9].

An important characterization of the Lévy class L as a subclass of the set of all infinitely divisible distributions in terms of the Lévy measure is given by the following theorem:

Theorem 3.3.2. *Let $v(dx)$ denote the Lévy measure of an infinitely divisible measure P on $\mathbb{R}$. Then the following statements are equivalent:*

(i) *P is self-decomposable.*

(ii) *The functions on the positive half-line given by $v((-\infty, -e^s])$ and $v([e^s, \infty))$ are both convex.*

(iii) *$v(dx)$ is of the form $v(dx) = u(x)dx$ with $|x|u(x)$ increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.*

If $u(x)$ is differentiable, then the necessary and sufficient condition (ii) may be expressed as,

$$u(x) + x'u(x) \leq 0, \quad \forall \quad x \neq 0. \tag{3.20}$$

If the Lévy density, $u(x)$, of a distribution is known then (3.20) is useful for determining whether it is self-decomposable. The proof of the equivalence $(i), (ii)$ and (iii) of Theorem 3.3.2 can be found in [5]. We proceed to show how (3.20) is obtained from condition (ii). We begin by defining the function $k(s)$ by:

$$k(s) = v([e^s, \infty)) = \int_{e^s}^{\infty} u(x)dx.$$

We know that $k''(s) \geq 0$ for all s since $k(s)$ is convex. Thus,

$$k'(s) = -e^s u(e^s) \tag{3.21}$$

and

$$k''(s) = -e^s u(e^s) - e^{2s}u'(e^s). \tag{3.22}$$

Therefore,

$$\begin{aligned} & -e^s u(e^s) - e^{2s}u'(e^s) \geq 0 \\ \iff & -u(e^s) - e^s u'(e^s) \geq 0 \\ \iff & u(e^s) + e^s u'(e^s) \leq 0 \\ \iff & u(x) + xu'(x) \leq 0 \quad \forall \quad x > 0. \end{aligned}$$

The following theorem provides a link between self-decomposability and Lévy processes.

Theorem 3.3.3. *If X is self-decomposable, then there exists a stationary stochastic process $X(t)$ and a Lévy process $Z(t)$ such that $X(t) \overset{\mathrm{d}}{=} X$ and*

$$\begin{aligned} X(t) &= w_1 X_1 \mathrm{e}^{-\lambda_1 t} + w_1 \mathrm{e}^{-\lambda_1 t} \int_0^{\lambda_1 t} \mathrm{e}^s dZ(s) \\ &\quad + w_2 X_2 \mathrm{e}^{-\lambda_2 t} + w_2 \mathrm{e}^{-\lambda_2 t} \int_0^{\lambda_2 t} \mathrm{e}^s dZ(s), \forall \quad \lambda_1, \lambda_2 > 0 \quad \textit{for} \quad t \geq 0. \end{aligned} \tag{3.23}$$

Conversely, if $X(t)$ is a stationary stochastic process and $Z(t)$ is a Lévy process so that $X(t)$ and $Z(t)$ satisfy (3.23) *for all $\lambda_1, \lambda_2 > 0$ then X is self-decomposable, where $\overset{\mathrm{d}}{=}$ means equality in distribution.*

The following theorem is the analogous for the multi-dimensional case.

Theorem 3.3.4. *If X is self-decomposable, then there exists a stationary stochastic process $X(t)$ and a Lévy process $Z(t)$ such that $X(t) \overset{\mathrm{d}}{=} X$ and*

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-s)} B_1 d\mathbf{Z}(\lambda s) + \int_0^t e^{A(t-s)} B_2 d\mathbf{Z}(\lambda s) \textit{ for } \quad t \geq 0. \tag{3.24}$$

Conversely, if $X(t)$ is a stationary stochastic process and $Z(t)$ is a Lévy process such that $X(t)$ and $Z(t)$ satisfy (3.24) *for all λ then X is self-decomposable, where $\overset{\mathrm{d}}{=}$ means equality in distribution.*

In this work, our approach to developing our model is to specify a parametric form for the marginal distribution of (3.12) and (3.13) and then work out the corresponding distribution of the background driving Lévy process. We will do this by specifying a distribution for the Lévy process and construct the model via the background driving Lévy process.

In order to proceed with our model, we recall the definition of the notations for the cumulant function and the cumulant characteristic function of a random variable X:

- The cumulant function: $k_X(u) = \log \mathbb{E}[\mathrm{e}^{-uX}] = \log \phi(iu)$.
- The cumulant characteristic function or characteristic exponent:

$$\psi_X(u) = \log \mathbb{E}[\mathrm{e}^{iuX}] = \log \phi(u),$$

or equivalently

$$\phi_X(u) = \mathrm{e}^{\psi(u)}. \tag{3.25}$$

It turns out that the background driving Lévy process, Z, and the stationary Ornstein-Uhlenbeck process, $X(t)$, are linked through the formula,

$$\psi_{X(t)}(u) = \int_0^\infty \psi_{Z(1)}(\mathrm{e}^{-s}u)ds \tag{3.26}$$

and

$$\psi_{Z(1)}(u) = u(\psi_{X(t)}(u))'. \tag{3.27}$$

While

$$k_{X(t)}(u) = \int_0^\infty k_{Z(1)}(\mathrm{e}^{-s}u)ds \tag{3.28}$$

and

$$k_{Z(1)}(u) = u(k_{X(t)}(u))'. \tag{3.29}$$

3.4 Lévy density and the tail mass function

Suppose we select a probability distribution $\mathcal{P}$ on $\mathbb{R}_+$ which is self-decomposable. Then by Theorem 3.3.3 there exist a stationary Ornstein-Uhlenbeck process such that $X(t) \overset{\mathrm{d}}{=} \mathcal{P} \ \forall \ t$ and is driven by the background driving Lévy process. We denote the Lévy measure of $Z(1)$ by $W(dx)$ and its corresponding density by w. The log-Laplace transform of $Z(1)$, which is denoted $k_{Z(1)}(u)$, may be expressed as

$$\begin{aligned} k_{Z(1)}(u) &= \log \mathbb{E}[\mathrm{e}^{-uZ(1)}] \\ &= \log \phi_{Z(1)}(iu) \\ &= -\int_0^\infty (1-\mathrm{e}^{-ux})W(dx) \\ &= -\int_0^\infty (1-\mathrm{e}^{-ux})w(x)dx, \end{aligned} \tag{3.30}$$

Equation (3.30) follows from (2.11). Similarly, for the $X(t)$ process which has Lévy density denoted by $u(x)$,

$$\begin{aligned} k_{X(t)}(u) &= \log \mathbb{E}[e^{-uX(t)}] \\ &= \log \phi_{X(t)}(iu) \\ &= -\int_0^\infty (1 - e^{-ux})u(x)dx. \end{aligned} \tag{3.31}$$

From Lévy-Khintchine representations of $Z(1)$ and $X(t)$ in equations (3.30) and (3.31), we obtain the relation,

$$w(x) = -u(x) - x\frac{du}{dx}, \tag{3.32}$$

this is as a result of the fact that if the Lévy density u of $X(t)$ of the self-decomposable distribution $\mathcal{P}$ is differentiable, then it is related to the Lévy density w of $Z(1)$. Below is the proof of the Lévy density relation.

Proof. From (3.31),

$$k_{X(t)}(u) = \int_0^\infty (1 - e^{-ux})u(x)dx.$$

Differentiating both sides with respect to u yields

$$\begin{aligned} \frac{k_{X(t)}}{du} &= \frac{d}{du}\int_0^\infty (e^{-ux} - 1)u(x)dx \\ &= \int_0^\infty \frac{d}{du}(e^{-ux} - 1)u(x)dx \\ &= \int_0^\infty -xe^{-ux}u(x)dx \\ &= \int_0^\infty e^{-ux}(-xu(x))dx. \end{aligned} \tag{3.33}$$

From (3.29), we have that

$$k_{Z(1)}(u) = u\frac{k_{X(t)}}{du},$$

this implies that,

$$\begin{aligned} k_{Z(1)}(u) &= u\int_0^\infty e^{-ux}(-xu(x))dx \\ &= -\int_0^\infty (xu(x))d(e^{-ux} - 1) \\ &= \int_0^\infty (e^{-ux} - 1)d(-xu(x)), \end{aligned} \tag{3.34}$$

where the last inequality follows from integration by part.

From (3.30), we have

$$k_{Z(1)}(u) = \int_0^\infty (\mathrm{e}^{-ux} - 1)w(x)dx. \tag{3.35}$$

Comparing (3.34) and (3.35) we obtain the desired relation

$$w(x) = -u(x) - xu'(x).$$

□

The next theorem that follows provides a link between self-decomposability and Lévy processes.

Theorem 3.4.1. (Jurek and Vervaat, [24]) *A random variable X is self-decomposable if and only if X has a representation of the form*

$$X = \int_0^\infty \mathrm{e}^{-t} dZ(t), \tag{3.36}$$

where $Z(t)$ is a Lévy process. The Lévy measures U and W of X and $Z(1)$ respectively are related by

$$U(dx) = \int_0^\infty W(\mathrm{e}^t dx)dt. \tag{3.37}$$

There exist a relationship between the Lévy process Z and the stationary stochastic process $X(t)$. The tail mass functions of $W(dx)$ is defined by

$$\begin{aligned} W^+(x) &= \int_x^\infty w(dt)dt \\ &= W([x, \infty)) \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} W^-(x) &= \int_x^\infty w(dt)dt \\ &= W((-\infty, x]). \end{aligned} \tag{3.39}$$

Then the Lévy measure W of $Z(1)$ is related to the Lévy density u of the self-decomposable stationary distribution $\mathcal{P}$ of $X(t)$ through the expression

$$W^+ = xu(x) \quad \text{for} \quad x > 0 \quad \text{and} \quad W^- = |x|u(x) \quad \text{for} \quad x < 0. \tag{3.40}$$

Equation (3.40) can be expressed in terms of the Lévy density as follows,

$$u(x) = \begin{cases} \frac{1}{x}W^+ & \text{for } x > 0 \\ \frac{1}{|x|}W^- & \text{for } x < 0 \end{cases} \tag{3.41}$$

The inverse function of W^+ is denoted by

$$W^{-1}(x) = \inf\{y > 0 : W^+ \leq x\}. \tag{3.42}$$

We now consider instances where the relations discussed above is applied. In particular we look at the Gamma Ornstein-Uhlenbeck process. The Gamma ($\Gamma(a, b)$) process defined in (2.12) has a Lévy density given by:

$$u(x) = ax^{-1}\mathrm{e}^{-bx}, x \neq 0.$$

Using relation (3.32), the Lévy density corresponding to the background driving Lévy process is given as:

$$\begin{aligned} w(x) &= -u(x) - x\frac{du}{dx} \\ &= -ax^{-1}\mathrm{e}^{-bx} - x(-ab\mathrm{e}^{-bx}x^{-1} - ax^{-2}\mathrm{e}^{-bx}) \\ &= ab\mathrm{e}^{-bx}, x \neq 0. \end{aligned} \tag{3.43}$$

The corresponding upper tail integral is

$$\begin{aligned} W^+(x) &= xu(x) \\ &= x \cdot x^{-1}\mathrm{e}^{-bx} \\ &= a\mathrm{e}^{-bx}, x \neq 0, \end{aligned} \tag{3.44}$$

and its corresponding inverse function is

$$\begin{aligned} W^{-1}(x) &= \inf\{y > 0 : W^+ \leq x\} \\ &= \inf\{y > 0 : a\mathrm{e}^{-bx} \leq x\} \\ &= \inf\{y > 0 : y \geq -\frac{1}{b}\log(\frac{x}{a})\} \\ &= \max\{0, -\frac{1}{b}\log(\frac{x}{a})\} \end{aligned} \tag{3.45}$$

3.5 Parameter Estimation of the one-dimensional $\Gamma(a,b)$ Ornstein-Uhlenbeck type Model

The stationary law of our proposed model (3.12) is given by a $\Gamma(a,b)$ distribution of (2.12), which immediately explains the name superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck Model. Since the background driving Lévy process of the superposed Ornstein-Uhlenbeck model is driven by a $\Gamma(a,b)$ distribution which is a compound Poisson, thus, the superposed $\Gamma(a,b)$-Ornstein-Uhlenbeck process jumps a finite number of times in a compact time interval.

We aim at estimating the model parameters a, b, and λ_1 of (3.12) using a sample of equally spaced observations. Before we proceed with the parameter estimations, we state an assumption of a Lévy measure.

If F_Z denotes the Lévy measure of $Z(1)$, we will assume that there exists a constant $M > 0$ such that

$$\int_{|x|>1} \mathrm{e}^{vx} F_Z(dx) < \infty, \forall \quad |v| \leq M. \tag{3.46}$$

3.5.1 Estimation of the shape parameter a and rate parameter b of the $\Gamma(a,b)$ Ornstein-Uhlenbeck Model

Proposition 3.5.1 relates the theoretical moments of $Z(1)$ with the theoretical moments of the stationary distribution of $\{X(t)\}_{t\geq 0}$.

Proposition 3.5.1. Suppose that $\{Z_t\}_{t\geq 0}$ is a Lévy process such that $E(Z(1)) = \mu < \infty$ and $Var(Z(1)) = \sigma^2 < \infty$. Let M be the largest constant satisfying (3.47) and assume that $\lambda_1, \lambda_2 > 0$. Then the following are true.

1. $E(X_0) = \mu$.
2. $Var(X_0) = \frac{\sigma^2}{2}$.

From Proposition 3.5.1 the parameters mean μ and variance σ^2 relates to rate a and shape

b parameters as follows:

$$a = \frac{2\mu^2}{\sigma^2} \quad \text{and} \quad b = \frac{2\mu}{\sigma^2}. \tag{3.47}$$

For details of the proof, see [52].

3.5.2 Estimation of the intensity parameter λ_1 of the $\Gamma(a, b)$ Ornstein-Uhlenbeck Model

We estimate the intensity parameter λ_1 by using the following proposition.

Proposition 3.5.2. *Suppose that $\{X_t\}_{t\geq 0}$ is a stochastic process defined in* (3.12) *with $\lambda_2 = \lambda_1 + \epsilon$ where $\lambda_1 > 0, \lambda_2 > 0$ and $\epsilon > 0$. For $k \geq 0$, define the autocorrelation function for the process given in* (3.12) *as*

$$\rho(k) = w_1 e^{-\lambda_1 |k|} + w_2 e^{-\lambda_2 |k|}, \tag{3.48}$$

where $w_1 + w_2 = 1$. Then our estimate for λ_1 is

$$\hat{\lambda}_1 = -log(\hat{\rho}(1)) \tag{3.49}$$

Proof. Since X_t is a stochastic process, we can define an autocorrelation function so that:

$$\rho(k) = w_1 e^{-\lambda_1 |k|} + w_2 e^{-\lambda_2 |k|},$$

If we assume $\lambda_2 = \lambda_1$ for $\epsilon \approx 0$ then we have

$$\rho(k) = w_1 e^{-\lambda_1 |k|} + w_2 e^{-\lambda_1 |k|}$$

$$\rho(k) = e^{-\lambda_1 |k|} (w_1 + w_2)$$

Taking log on both sides of the last equation and solving of λ_1 we obtain:

$$\lambda_1 = -\frac{log(\rho(k) - log(w_1 + w_2))}{k}$$

Without loss of generality, we take $k = 1$. Therefore our estimate is

$$\hat{\lambda}_1 = -log(\hat{\rho}(1))$$

where $\hat{\rho}(1)$ denotes the empirical autocorrelation function of lag 1 based on the data $x_0, x_1, \ldots, x_n$.

□

Remark 3.5.1. *Once λ_1 is estimated, we adjust λ_1 to obtain λ_2 in order to fit the one-dimensional superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model.*

3.6 Parameter Estimation of the 2-dimensional $\Gamma(a,b)$ Ornstein-Uhlenbeck type Model

In the 2-dimensional scenario (3.13), the goal is to find an estimate of the parameters a, b $\lambda_1, \lambda_2, \sigma_{11}$, σ_{12}, σ_{21} and σ_{22} given $N+1$ data points. The estimates for a, b, λ_1 and λ_2 follows from the one-dimensional case discussed in the previous section.

3.7 Simulation techniques

In this section we discuss the simulation techniques for the for the 1-dimensional and 2-dimensional Lévy driven Ornstein-Uhlenbeck Model. We will base the simulation of the 1-dimensional stochastic differential equation on,

$$\begin{aligned} X(t) &= w_1 X_1 \mathrm{e}^{-\lambda_1 t} + w_1 \mathrm{e}^{-\lambda_1 t} \int_0^{\lambda_1 t} \mathrm{e}^s dZ(s) \\ &\quad + w_2 X_2 \mathrm{e}^{-\lambda_2 t} + w_2 \mathrm{e}^{-\lambda_2 t} \int_0^{\lambda_2 t} \mathrm{e}^s dZ(s), \quad X_1, X_2 > 0, \lambda_1, \lambda_2 > 0 \quad \text{and} \quad t \geq 0. \end{aligned}$$

and that of the 2-dimensional stochastic differential equation will be based on,

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-s)} B_1 d\mathbf{Z}(\lambda s) + \int_0^t e^{A(t-s)} B_2 d\mathbf{Z}(\lambda s)$$

The solutions of the stochastic differential equation will be simulated via the background driving Lévy process and then by approximating the corresponding integrals. We recall in Chapter 2 that the background driving Lévy process for the $\Gamma(a,b)$ Ornstein-Uhlenbeck model is a compound Poisson process.

3.7.1 Simulation of the 1-dimensional stochastic model via the background driving Lévy process

The superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck Model simulation algorithm is discussed below.

To simulate the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model, $X = \{X(t)\}_{t\geq 0}$ in the time points $t = n\Delta t, n = 0, 1, 2, \ldots$ via its background driving Lévy process is as follows:

(i) Simulate a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter $a\lambda$.

(ii) Calculate the number of jumps in each compact interval.

(iii) Set the sample path of the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model,

$$\begin{aligned} X_{n\Delta t} = w_1 X_{(n-1)\Delta t} e^{-\lambda_1 \Delta t} + w_1 \sum_{N_{(n-1)\Delta t}+1}^{N_{n\Delta t}} \chi_n \exp(-U_n \lambda \Delta t) \\ + w_2 X_{(n-1)\Delta t} e^{-\lambda_2 \Delta t} + w_2 \sum_{N_{(n-1)\Delta t}+1}^{N_{n\Delta t}} \chi_n \exp(-U_n \lambda \Delta t) \end{aligned} \tag{3.50}$$

3.7.2 Simulation of the 2-dimensional stochastic model via the background driving Lévy process

To simulate the 2-dimensional $\Gamma(a, b)$ Ornstein-Uhlenbeck model, $(X_1, X_2) = (\{X_1(t)\}_{t\geq 0}, \{X_2(t)\}_{t\geq 0})$ in the time points $t = n\Delta t, n = 0, 1, 2, \ldots$ via its background driving Lévy process is as follows:

1. Simulate a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter $a\lambda_1$ and $a\lambda_2$.

2. Calculate the number of jumps in each compact interval.

3. Set the sample path of the 2-dimensional $\Gamma(a, b)$ Ornstein-Uhlenbeck model,

$$\begin{aligned} X^1_{n\Delta t} &= X^1_{(n-1)\Delta t} e^{-\lambda_1 \Delta t} + \sigma_{11} \sum_{N_{(n-1)\Delta t}+1}^{N_{n\Delta t}} \chi_n \exp(-U_n \lambda_1 \Delta t) + \sigma_{12} \sum_{N_{(n-1)\Delta t}+1}^{N_{n\Delta t}} \chi_n \exp(-U_n \lambda_2 \Delta t) \\ X^2_{n\Delta t} &= X^2_{(n-1)\Delta t} e^{-\lambda_2 \Delta t} + \sigma_{21} \sum_{N_{(n-1)\Delta t}+1}^{N_{n\Delta t}} \chi_n \exp(-U_n \lambda_1 \Delta t) + \sigma_{22} \sum_{N_{(n-1)\Delta t}+1}^{N_{n\Delta t}} \chi_n \exp(-U_n \lambda_2 \Delta t) \end{aligned} \tag{3.51}$$

Remark 3.7.1. The exponential term and the independent uniform random numbers U_n in the sum allow jumps to happen in each time interval.

A Matlab module was developed to simulate the process described above. We simulated independent paths of our model using different time steps.

Chapter 4

Stochastic differential equations applied to the study of financial time series

4.1 Introduction

In this chapter we apply the superposed Ornstein-Uhlenbeck to the statistical analysis of well developed and emergent financial market indices. The statistical properties of the temporal series analyzing the evolution of financial markets have been of great importance in the study of financial indices [35]. For example the scale invariance in the behavior of financial indices near a crash has been studied in Ref. [19].

As the knowledge of the mechanisms that drive financial markets has increased, so have the corresponding mathematical model representations. One of the first models for describing the evolution of prices is the Brownian motion. This model assumes that the increment in the logarithm of the prices follows a diffusive process with Gaussian distribution [59]. However, the empirical study of some financial indices shows that in short time intervals the associated probability density function has greater kurtosis than a Gaussian distribution [30], and that the Brownian motion does not describe correctly the evolution of financial indices near a market crash.

The authors in Ref. [35] tried to over this issue by using a stable non-Gaussian Lévy process that takes into account the long correlation scales. The authors showed that the Lévy distribution describes the evolution of the financial indices near a crash for financial markets. However, the model used was not completely stochastic and also the price evolution of financial indices in a market may be stochastically dependent, that is, there is a correlation

between number of events. Therefore in an attempt to overcome the modeling problems associated with the memory-less property models described in previous literature, we propose a continuous-time stationary and non-negative stochastic process that is useful in describing a unique type of dependence in a sequence of events.

Continuous-time stochastic volatility models are now popular ways to describe many "critical phenomena" because of their flexibility in accommodating most stylized facts of time series data such as moderate and high frequency data. In the work of Barndorff-Nielsen and Shephard [9], they proposed a class of models where the volatility behaved according to an Ornstein-Uhlenbeck process driven by a positive Lévy process with a non-Gaussian component. This model has many applications in many fields of science and other disciplines. There are known applications of the non-Gaussian Ornstein-Uhlenbeck process within the context of finance and econometrics [9]. It is a mean reverting process which is widely used for modeling interest rates and commodities among many others.

In this chapter, we implement very flexible classes of processes that incorporate long-range dependence, i.e. they have a slowly polynomially decaying autocovariance function and self-similarity like properties that are capable of describing some of the key distributional features of typical financial indices corresponding to developed and emergent markets. In order to capture realistic dependence structures, we combine independent Ornstein-Uhlenbeck processes driven by a Lévy process. This selection is also supported by the fact that generalized Lévy models are suitable for describing these type of time series, see [34].

The advantage of the superposition of the independent Ornstein-Uhlenbeck processes is the fact that it offers plenty of analytic flexibility which is not available for more standard models such as the geometric Gaussian Ornstein-Uhlenbeck processes. Moreover, superposition of Ornstein-Uhlenbeck processes provide a class of continuous time processes capable of exhibiting long memory behavior. The presence of long memory suggests that current information is highly correlated with past information at different levels. This facilitates prediction. Our main focus is to verify that the Superposed Ornstein-Uhlenbeck model describes accurately the behavior of financial indices for both developed and emergent markets. The

methodology used in this work can be applied to data sets from other disciplines such as health, biology, bioinformatics, medicine and in social sciences.

In the next section we describe the source of our data sets and also present the numerical simulation results when our model is applied to the data sets.

4.2 Background of Financial Time Series

Most of the studies mentioned before have been done with financial indices of developed markets that have a great volume of transactions. In this work we analyze financial indices corresponding to developed and emergent markets.

We studied emergent market indices corresponding to three countries: Brazil (BOVESPA), from 04-27-1993 to 10-22-2001; Argentina (MERVAL), from 10-8-1996 to 10-22-2001 and Hong Kong (HSI), from 01-2-1991 to 10-25-2001. The number of data points for BOVESPA , MERVAL and HSI is 2100, 1250 and 2675 respectively. We also analyzed the Standard and Poor's 500 (S & P 500), a major index of the New York Stock Exchange. In the latter case, the data corresponds to a period from 01-3-1950 to 06-14-2005 with 13,951 data points. The daily close values were used in our analyses.

The stochastic behavior of two stock prices are showed in Figures 4.1, and 5.2. The figures provide a good perspective on the trending direction of the price. This behavior illustrates the time evolution of the S&P 500 and BVSP stock exchanges with its volatility. From the figures, we observe that the variability of the data is not constant but changes over time. This is an indication of possible non-stationarity in volatility, and the periods of high volatility tends to be correlated. The sharp drop in prices leads to a high value of the volatility, which decreases to the normal level after several units of time. Due to the randomness of the financial time series, it is appropriate to use a stochastic model to fit the financial data in order to capture this physical behavior.

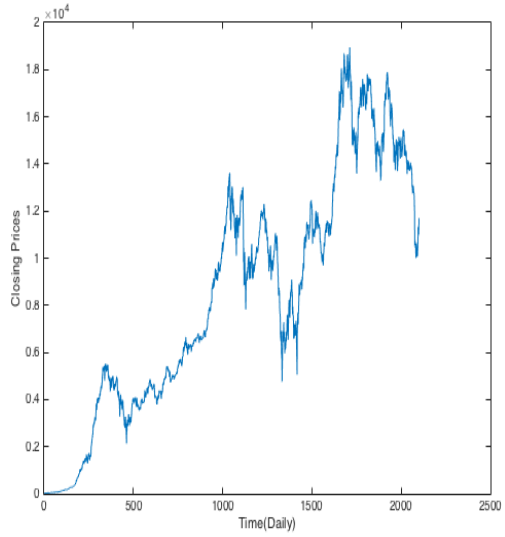

Figure 4.1: The closing prices of daily trading observations from the BVSP stock exchange.

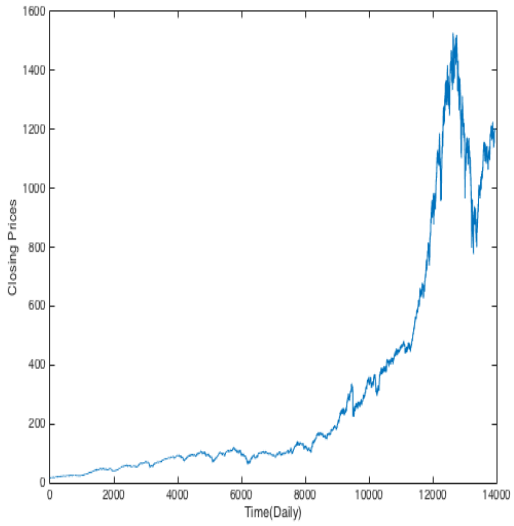

Figure 4.2: The closing prices of daily trading observations from the S&P 500 stock exchange.

4.3 Numerical Simulation and Results

In the numerical study of the time series arising in finance, we use data collected from four different financial indices to estimate the daily closing values near a crash for both well developed and emergent markets. In the time series data points, the local variance of the series was larger when the level of the series was higher. We therefore normalized the data sets, by taking logarithm of the time series data points. By performing this change of scale, it is likely that a stationary or integrated model can be fitted after the transformation see [21] for details.

The superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model is applied to the time series arising in finance. First of all, we estimate the model parameters a, b and λ_1 using the relations given in Equations 3.47 and 3.49 respectively. Next, we simulate the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model using the steps described in Chapter 3 of this dissertation. We simulated independent paths of our model using different time steps.

We compared our model that is, the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model to the ordinary $\Gamma(a,b)$ Ornstein-Uhlenbeck model to check which of them best fits the data. In order to investigate our model fit, we computed the root mean square error for each region. The root mean square error indicates how well fitted is our model with respect to the given data set.

4.3.1 Real Data Analysis of the Financial Indices

Table 4.1 summarizes the results of the estimation of parameters for the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model.

Tables 4.2 and 4.3 summarizes our numerical results for each index when the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model and the ordinary $\Gamma(a,b)$ Ornstein-Uhlenbeck model were applied to real financial data series respectively. λ_2 was not estimated however, we obtained it by adjusting λ_1 in order to fit the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model.

From the numerical results obtained in Tables 4.2 and 4.3 , we conclude that in all instances the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model performed considerably well compared to

Table 4.1: Estimation of Parameters: λ_1, a, b.

Financial Indices	Number of observations	λ_1	a	b
BOVESPA	2100	0.0017	5.4289	6.4362e-04
MERVAL	1250	0.2285	26.2184	0.0477
HSI	2674	0.0021	14.7389	0.0015
S&P 500	13,951	3.2448e-04	1.2718	0.0042

Table 4.2: Numerical results for the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model.

Financial Indices	λ_1	λ_2	w_1	w_2	RMSE
BOVESPA	0.0017	10	0.2	0.8	0.9098
MERVAL	0.2285	16	0.01	.99	0.2535
HSI	0.0021	6	0.1	0.9	0.2621
S&P 500	3.2448e-04	25	0.4	0.6	0.9228

Table 4.3: Numerical results for the $\Gamma(a, b)$ Ornstein-Uhlenbeck model.

Financial Indices	λ_1	RMSE
BOVESPA	0.0017	5.5643
MERVAL	0.2285	0.7668
HSI	0.0021	1.1970
S&P 500	3.2448e-04	2.5031

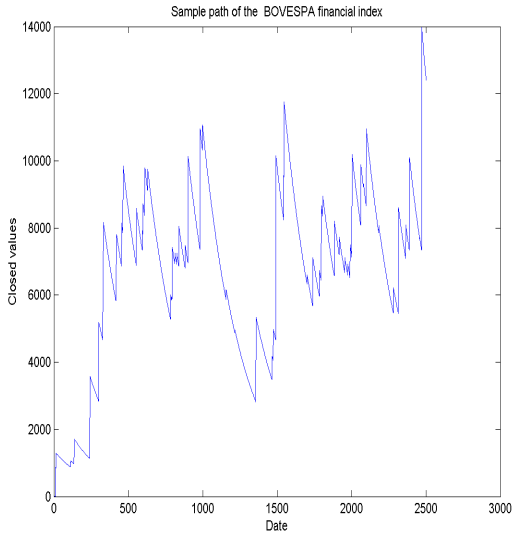

Figure 4.3: Sample Path of the BOVESPA financial index.

the results for the ordinary $\Gamma(a, b)$ Ornstein-Uhlenbeck model. This is due to the fact that, the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model is a weighted sum of solutions. Moreover, because the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model need not be identically distributed, it offers a lot of flexibility in the model.

Figures 4.3, 4.4, 4.5 and 4.6 show the sample path for the simulated data sets corresponding to BOVESPA, MERVAL, HSI and S&P 500 respectively.

4.4 Concluding remarks

In this work, we implemented flexible classes of processes that incorporate long-range dependence, i.e. they have a slowly polynomially decaying autocovariance function and self-similarity like properties and that are capable of describing some of the key distributional features of a financial time series.

In order to capture more realistic dependence structures, we combined two independent $\Gamma(a, b)$ Ornstein-Uhlenbeck processes. We simulated data from our proposed model; the super-

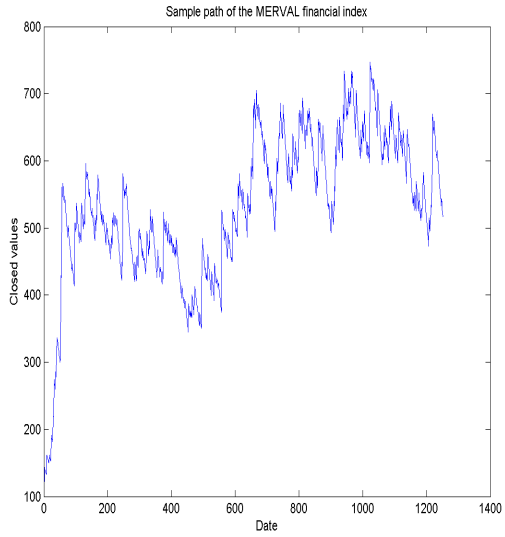

Figure 4.4: Sample Path of the MERVAL financial index.

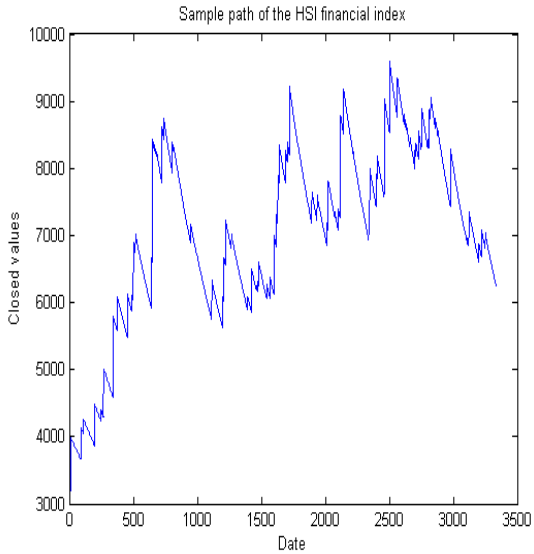

Figure 4.5: Sample Path of the HSI financial index.

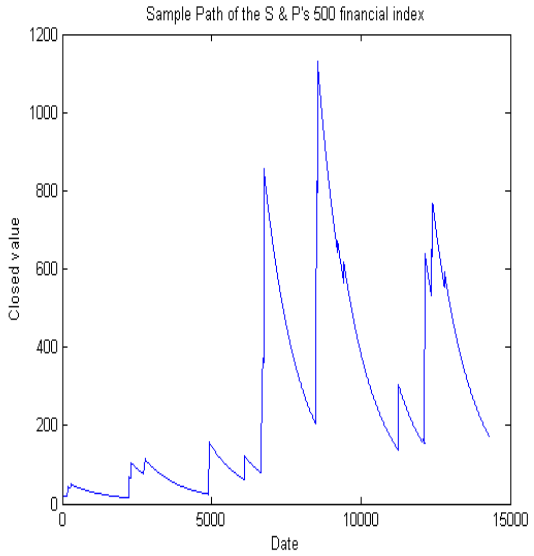

Figure 4.6: Sample Path of the S&P 500 financial index.

posed $\Gamma(a, b)$ Ornstein-Uhlenbeck process to estimate the daily closing values of the financial indices near a crash for both well developed and emergent markets. We compared our numerical results to the ordinary $\Gamma(a, b)$ Ornstein-Uhlenbeck process. Looking at the computed model fit; the root mean square error, the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck process fitted better than the $\Gamma(a, b)$ Ornstein-Uhlenbeck process.

In previous studies by [35] and [7], the authors concluded that the generalized Lévy models where very suitable to describe critical events including financial crashes and earthquakes. The solution to our stochastic differential equation is a Lévy model, so the very good fitting obtained reinforces the previous conclusions.

Superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck processes provide a class of continuous time processes which exhibits long memory behavior. The presence of long memory suggests that current information is highly correlated with past information at different levels, what may facilitate prediction.

Chapter 5

Analysis of high frequency financial time series by using a stochastic differential equation

5.1 Introduction

This chapter is devoted to the analysis of the Bear Stearns collapse. The Bear Stearns collapse is viewed as the first sign of the risk management meltdown of investment bank industry in September 2008. We analyze using a stochastic differential equation the second-by-second and minute-by-minute sampled financial data from the Bear Stearns companies and subsequently estimate parameters that are useful for making inferences and predicting these types of events.

Due to the advanced technology associated with Big Data and data availability, the effective modeling and analyzes of different kinds of high frequency financial time has been a major concern to reseachers and practitioners. For example, various modeling techiques have been used to study market crashes by analyzing financial time series (see Refs. [33, 17, 27]). Other researchers also described and modeled the behavior of a financial market before a crash by analyzing high frequency financial sampled data (see Refs. [31, 6, 10]). As a result of the huge amount of data available on the financial market, analyzing these type of financial sampled data helps investors, practitioners and researchers make useful inference and predictions.

In this chapter, we use a stochastic differential equation arising on the sum of two Ornstein-Uhlenbeck processes driven by a Lévy process to analyze high frequency financial sampled data. The stochastic model captures more realistic dependence structures since we combined

two independent Gamma Ornstein-Uhlenbeck processes. The model incorporates long-range dependence and self-similarity like properties that describes some of the key distributional features of a typical high frequency financial time series. This selection is supported by the fact that generalized Lévy models are suitable for describing these type of time series, see Refs. [34, 36]. Additionally, due to the randomness of high frequency financial time series, it is appropriate to use a stochastic model to fit the financial data in order to capture this physical behavior. Moreover, because the model need not be identically distributed, it offers a lot of flexibility in the model. This stochastic model has many applications in geophysics [55, 37], finance and econometrics [36, 9]. Using the model, we analyze how soon an investor or practitioner who was lacking insider information but had at their disposal all the information contained in the equity prices could have discovered that a crash is imminent and take the necessary precautions.

5.2 Background of the High Frequency Financial Time Series

On Friday, March 14, 2008 at about 9:14am, JP Morgan Chase together with the Federal Reserve Bank of New York announced an emergency loan to Bear Stearns (about 29 billion, terms undisclosed) to prevent the firm from becoming insolvent. This bailout was declared to prevent the very likely crash of the market as a result of the fall of one of the biggest investment banks at the time. This measure proved to be insufficient for keeping the firm alive. This collapse is viewed as the first sign of the risk management meltdown of investment bank industry in September 2008 and the subsequent global financial crisis and recession.

The high frequency data samples consist of the return values within minute-by-minute and second-by-second data for 3 Bear Stearns companies over a seven day period which we know is relevant for market crash behavior in the US market; March 10- March 18, 2008. The companies analyzed includes: Bank of America Corporation (BAC), Disney (DIS) and JPMorgan Chase (JPM).

The stochastic behavior of the return values for the minute-by-minute and second-by-

second sampled data are showed in figures 5.1-5.6. The figures provide some behavior on the trending direction of the return values. This behavior illustrates the time evolution of the BAC, DIS and JPM stock exchanges with its volatility.

Due to the randomness of the high frequency financial time series, it is appropriate to use a stochastic model to fit the financial data in order to capture this physical behavior.

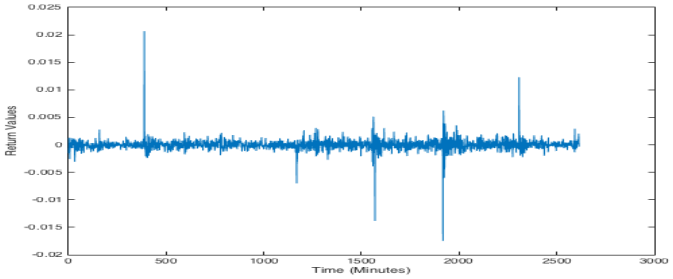

Figure 5.1: The return values within minute-by-minute for the BAC stock exchange.

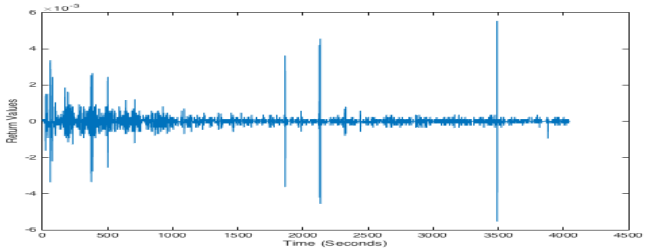

Figure 5.2: The return values within second-by-second for the BAC stock exchange.

5.3 Analysis of the High Frequency Financial Time Series

The stochastic model is applied to the high frequency financial time series as follows. Firstly, we estimate the model parameters a, b and λ_1 using the relations given in (3.47) and (3.49) respectively. Since we are considering a two component independent Ornstein-Uhlenbeck processes, we need to estimate two intensity parameters (λ_1 and λ_2). As mentioned earlier,

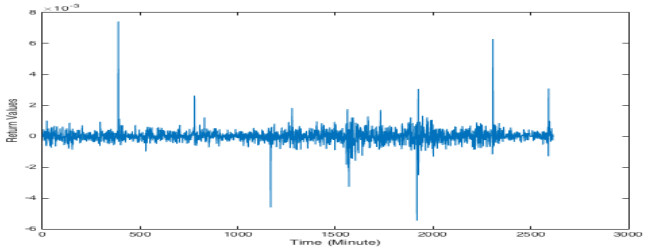

Figure 5.3: The return values within minute-by-minute for the DIS stock exchange.

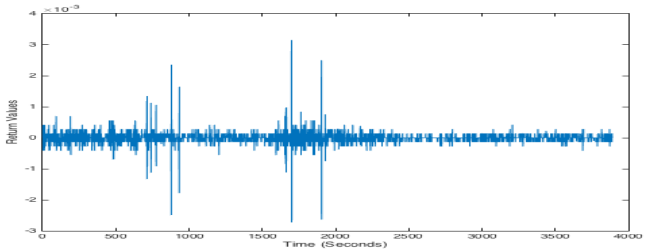

Figure 5.4: The return values within second-by-second for the DIS stock exchange.

Equation (3.49) is the relation used to estimate λ_1. The second parameter λ_2 is obtained by adjusting λ_1 (i.e. $\lambda_2 = \lambda_1 \pm \epsilon$) where epsilon is a positive real number. We keep adjusting λ_1 till we fit our model.

Next, we simulate data from the model following the discussions in Chapter 3 Section 3.5. We simulated independent path of our model using different time steps. Finally, we compared the random time series generated from the model to the original high frequency financial time series. We investigated our model fit by computing the RMSE for each company. The RMSE estimates measure how well our model fits the given data sets. In this work, we used the RMSE estimates to measure the fit of the random time series generated from our stochastic model and the original high frequency financial time series observed for the minute and second sampled data.

Tables 5.1 and 5.2 summarizes the results of estimation of parameters for the stochastic model applied to the minute-by-minute and second-by-second sampled data respectively.

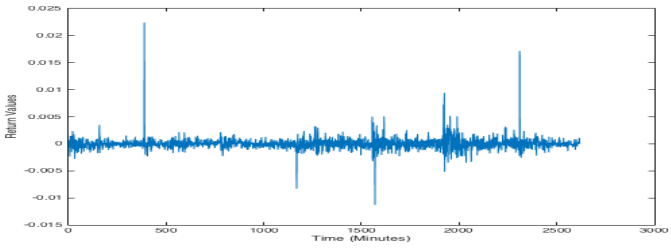

Figure 5.5: The return values within minute-by-minute for the JPM stock exchange.

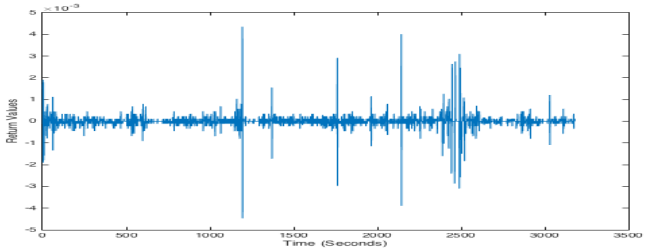

Figure 5.6: The return values within second-by-second for the JPM stock exchange.

Tables 5.3 and 5.4 summarizes the numerical results of the stochastic model for minute-by-minute and second-by-second sample data respectively.

Table 5.5 compares the numerical results of the minute and second data.

Table 5.1: Estimation of Parameters for the minute-by-minute sampled data: λ_1, a, b

Minute-by-Minute Data	λ_1	a	b
BAC	2.4069	1.5208e-04	17.2135
DIS	2.5153	2.8859e-04	55.8833
JPM	1.7785	6.5772e-04	33.9127

Table 5.2: Estimation of Parameters for the second-by-second sampled data: λ_1, a, b.

Second-by-Second Data	λ_1	a	b
BAC	0.8632	2.0246e-04	63.9561
DIS	0.9019	2.0141e-04	112.1105
JPM	1.0788	1.8693e-04	66.8734

Table 5.3: Numerical results of the stochastic model for minute-by-minute sample data.

Financial Indices	λ_1	λ_2	v_1	v_2	RMSE
BAC	2.4069	2.4070	0.5	0.5	0.0012
DIS	2.5153	2.5150	0.5	0.5	4.316e-04
JPM	1.7785	1.7780	0.5	0.5	0.0011

Table 5.4: Numerical results of the stochastic model for second-by-second data.

Financial Indices	λ_1	λ_2	v_1	v_2	RMSE
BAC	0.8632	0.8630	0.5	0.5	3.687e-04
DIS	0.9019	0.9015	0.5	0.5	2.154e-04
JPM	1.0788	1.0785	0.5	0.5	9.911e-04

Table 5.5: Comparison of the numerical results of the minute and second data.

Financial Indices	Minute (RMSE)	Second (RMSE)
BAC	0.0012	3.687e-04
DIS	4.316e-04	2.154e-04
JPM	0.0011	9.911e-04

5.4 Significance of the results obtained

The solution of our stochastic model is known to be everywhere continous and purely non-negative process [8]. This concept is captured in Figures 6.2 and 6.3 for both the minute and second sampled data. Also, per the RMSE estimates obtained (see Table 6.3), we conclude that our stochastic model fits the second-by-second sampled data well compared to the minute-by-minute sampled data. Our model will help analyze the effect that previous market crashes will have on present and future crashes since it provides a class of continuous time processes that exhibits long memory behavior. The presence of long memory behavior suggests that present data is correlated with past data which may facilitate prediction [36].

Additionally, the model parameters are very useful for making inferences and predictions. This is because the intensity parameters λ_1 and λ_2 reports how our model reacts to perturbation and also describes the mean reversion speed. From the analysis, we observe that the intensity parameters for the second by second high frequency times series are smaller compared to the minute by minute high frequency time series. This results suggests that in the long run, the closing prices for the stock exchanges of the minute by minute data will approach its mean closing prices faster than the second by second data.

In summary the studies performed in this dissertation will help an investor or practitioner who lacks insider information but has at their disposal all the information contained in the equity prices discover that a crash is imminent and take the necessary precautions. Other possible applications of this work is in physics– describe critical phenomena, biology, medicine and several other disciplines.

Chapter 6

Stochastic Differential Equation of Earthquakes Series

6.1 Introduction

This chapter is devoted to modeling earthquake time series. The stochastic differential equation is applied to the study of earthquakes by fitting the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model to earthquake sequences in South America containing very large events (Mw$\geq$8). We are modeling the magnitude of the earthquakes in seismic sequences that include very large earthquakes. We selected the regions where the most recent (since 2000) Mw $\geq$ 8 earthquakes have occurred in the western hemisphere, the South American segment of the Pacific ring of fire. This is one of the most seismically active zones in the planet, where earthquakes are mainly related to the tectonic boundary where the Nazca tectonic plate subducts underneath the South American tectonic plate [26]. This is the tectonic boundary where the largest earthquakes ever recorded occurred, the 1960 magnitude 9.5 Chile earthquake [14]. Other significant event is the 2010 Mw 8.8 Maule Chile earthquake (see Refs. [42, 16, 57]), which generated a large Tsunami and caused a great number of fatalities. This event ranks as the fifth largest earthquake ever recorded by a seismograph. More recently, in September 2015 a Mw 8.3 event struck in central Chile near the city of Illapel [39]. This is the largest moment magnitude earthquake that has occurred in more than two years worldwide. The selected study regions are very active and present very high seismic hazards.

6.2 Earthquake Time Series

The earthquakes data was obtained from the IRIS Data Management Center http://www.iris.edu. We obtained the magnitude, location and time origin of any M $\geq$ 4.0 earthquake that occurred within the regions surrounding the most recent (since 2000) very large earthquakes (Mw$\geq$8) in the western hemisphere. These give us 4 data sets for very active regions. The regions were defined as rectangles limited by the aftershock activity that occurred after the Mw $\geq$ 8 earthquakes (see Table 6.1). We consider the aftershocks-limited areas as areas where tectonic stress can be transmitted efficiently, thus there might exist a degree of dependency between the earthquakes within each of these regions. All of these earthquakes occurred in South America along the tectonic boundary where the Nazca tectonic plate subducts underneath the South American tectonic plate [26]. This is the tectonic boundary where the largest earthquakes ever recorded occurred, the 1960 magnitude 9.5 Chile earthquake [14]. Figure 6.1 shows the magnitude and time origin of the earthquakes for the 4 study regions.

Table 6.1: Coordinates of the four study regions.

Region	**Area**		**Largest Earthquake**		
	Longitude	**Latitude**	**Time**	**Magnitude**	**Epicenter**
1	-15.0° to -13.0°	-78.0° to -75.0 °	08/15/07	8.0	-13.38° , -77.56°
2	-18.5° to -15.5°	-75.5° to -69.5°	06/23/01	8.3	-16.30°, -73.56°
3	-22.0° to -19.0°	-72.0° to -69.0°	04/01/14	8.0	-19.63°, -70.86°
4	-40.0° to -30.0°	-77.0° to -70.0°	02/27/10	8.8	-36.15°, -72.93°

In [46] the authors showed that the unique characterization and properties of some active seismic regions with reference to the time series of data points satisfies the linear Ornstein-Uhlenbeck process. Therefore in this study, allowing for diffusion and the nonlinear terms in our stochastic process may be good for some geographical regions but will fail significantly for most of the other regions.

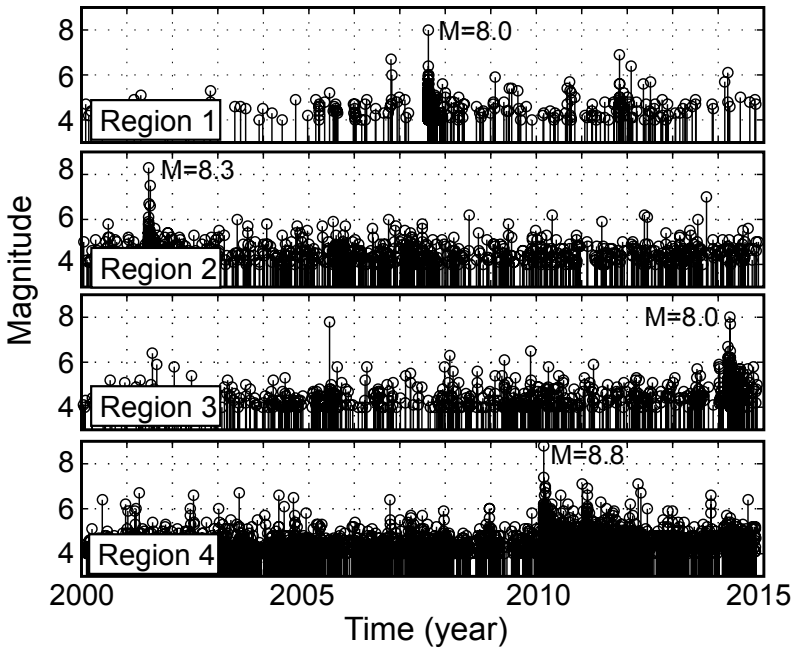

Figure 6.1: Magnitude as a function of time origin of earthquakes for the 4 study regions defined in Table 6.1.

6.3 Numerical Simulation and Results

In the numerical study of the time series arising in seismology, we used data collected from a location at a given time to estimate the magnitude of the earthquakes at a given location, where the real magnitude is known. The magnitudes recorded in the data set was used.

In the time series data points, the local variance of the series was larger when the level of the series was higher. We therefore normalized the data sets by taking logarithm of the time series data points. By performing this change of scale, it is likely that a stationary or integrated model can be fitted after the transformation (see [21] for details).

The superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model is applied to the geophysical time series as follows. We first estimate the model parameters a, b and λ_1 using the relations given in (3.47) and (3.49) respectively. Next, we simulate data from the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model following the discussions in Chapter 3 Section 3.5. We simulated independent path of our model using different time steps. Finally, we compared the random time series generated from the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model to the original earthquake time series. In order to investigate our model fit, we computed the root mean square error (RMSE) for each region. The RMSE estimates measure how well our model fits the given data sets. In this work, we used the RMSE estimates to measure the fit of the random time series generated from our proposed model and the original earthquake time series observed.

Physical interpretations of the parameters used in our model are as follows. The rate parameter, a controls the rate of jump arrivals of the earthquake time series. If a is large, then the time series will be more concentrated; if a is small then it will be more spread out. The shape or scale parameter, b describes the jump size of the earthquake time series rather than simply shifting it. λ_1 and λ_2 are the rate parameters of our superposed $\Gamma(a, b)$- Uhlenbeck model that measures how the model react to perturbations.

Table 6.2: Estimation of Parameters: λ_1, a, b.

Regions	Number of observations	λ_1	a	b
Region 1	389	2.1776	158.9808	34.4993
Region 2	1000	3.0955	196.4128	43.2430
Region 3	892	1.6348	151.7649	33.0730
Region 4	5000	1.6344	205.7332	46.0565

Table 6.3: Data Results for superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model.

Regions	λ_1	λ_2	w_1	w_2	RMSE
Region 1	2.1776	2.9776	0.50	0.50	0.1066
Region 2	3.0955	4.6165	0.60	0.40	0.0948
Region 3	1.6348	2.3348	0.45	0.55	0.1017
Region 4	1.6344	2.6344	0.60	0.40	0.0941

6.3.1 Results from analysis of earthquake time series

Results of the earthquake time series for the four (4) regions discussed are presented. Table 6.2 summarizes the results of the estimation of parameters for the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model.

Table 6.3 and 6.4 summarizes our numerical results for the earthquake time series when the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model and the ordinary $\Gamma(a, b)$ Ornstein-Uhlenbeck model were applied to real earthquake data series respectively. We obtained λ_2 by adjusting λ_1 in order to fit the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model.

The RMSE estimates for the random time series generated from the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck model and the original earthquake time series are also presented in Table 6.3. Similarly, the RMSE estimates for the random time series generated from the ordinary $\Gamma(a, b)$ Ornstein-Uhlenbeck model and the original earthquake time series are also presented

Table 6.4: Data Results for $\Gamma(a,b)$ Ornstein-Uhlenbeck model.

Regions	λ_1	RMSE
Region1	2.1776	0.1207
Region2	3.0955	0.1025
Region3	1.6348	0.1030
Region4	1.6344	0.0991

in Table 6.4.

Based on the RMSE estimates obtained in Tables 6.3 and 6.4, it is evident that the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model fit well than the ordinary $\Gamma(a,b)$ Ornstein-Uhlenbeck model. This is due to the fact that, the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model is a weighted sum of solutions. Moreover, because the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck model need not be identically distributed, it offers a lot of flexibility in the model.

Figures 6.2, 6.3, 6.4 and 6.5 reports the sample path for the simulated data sets corresponding to Regions 1, 2, 3 and 4. The sample path refers to the path of the simulated superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck process. The horizontal axis is time, t and the vertical is the superposed $\Gamma(a,b)$ Ornstein-Uhlenbeck process, X_t. The path is dominated by jumps and also illustrates the fact that the process is everywhere continuous and non-negative.

The very good fit of the observed magnitudes of the earthquakes with the stochastic differential equations, supports the use of this methodology for the study of earthquakes sequence

6.4 Significance of the results obtained

In this chapter, we implemented flexible classes of processes that incorporate long-range dependence, i.e. they have slow polynomially decaying autocovariance function and self-similarity like properties and that are capable of describing some of the key distribution features of typical geophysical time series. In order to capture more realistic dependence

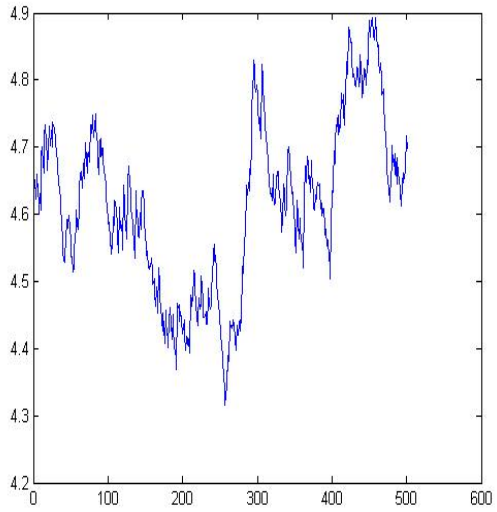

Figure 6.2: A Sample Path of the superposed $\Gamma(a, b)$-OU process with $\lambda_1 = 2.1776$, $\lambda_2 = 2.9776$, $a = 158.9808$ and $b = 34.4993$ for region 1.

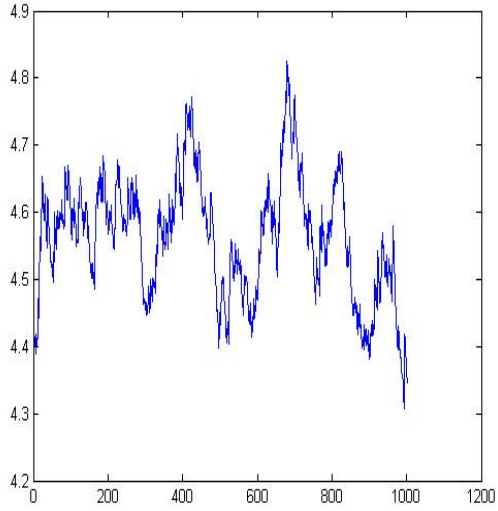

Figure 6.3: A Sample Path of the superposed $\Gamma(a, b)$-OU process with $\lambda_1 = 3.0955$, $\lambda_2 = 4.6165$, $a = 196.4128$ and $b = 43.2430$ for region 2.

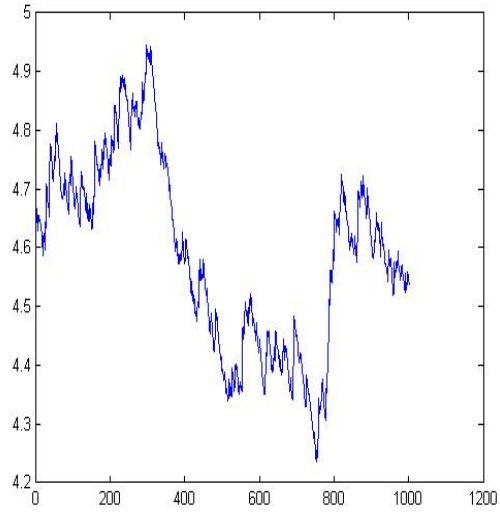

Figure 6.4: A Sample Path of the superposed $\Gamma(a, b)$-OU process with $\lambda_1 = 1.6348$, $\lambda_2 = 2.3348$, $a = 151.7649$ and $b = 33.0730$ for region 3.

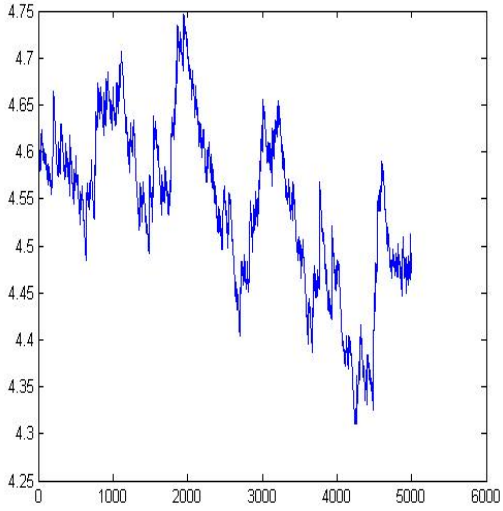

Figure 6.5: A Sample Path of the superposed $\Gamma(a, b)$-OU process with $\lambda_1 = 1.6344$, $\lambda_2 = 2.6344$, $a = 205.7332$ and $b = 46.0565$ for region 4.

structures, we summed two independent $\Gamma(a, b)$ Ornstein-Uhlenbeck processes.

We generated random time series from our proposed model; the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck process to estimate the magnitude of the earthquake for the four regions. Looking at the computed model fit; the root mean square error, the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck process gives a good fit. We also compared the random time series generated from the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck process to the random time series generated from the ordinary $\Gamma(a, b)$ Ornstein-Uhlenbeck process. In this instance, the superposed $\Gamma(a, b)$ Ornstein-Uhlenbeck process gave a better fit than the $\Gamma(a, b)$ Ornstein-Uhlenbeck process. This is evident in their RMSE estimates.

In previous works [35] and [7], the authors concluded that the generalized Lévy models where very suitable to describe critical events including earthquakes. The solution to our stochastic differential equation is a Lévy model, so the very good fit obtained reinforces the previous conclusions. We obtained very good fits of the observed magnitudes of the earthquakes with the stochastic differential equations (see Table 6.3), which supports the use of this modeling methodology for the study of earthquakes sequences.

An important example and potential application of this work is for analyzing the effect of events that occurred very far in the past, for example decades ago, might have on the occurrence of present and future events. This type of analysis might help to better understand how tectonic stress decays and accumulates during long period of time.

Chapter 7

Modeling earthquake series dependencies by using coupled system stochastic differential equations

In the Chapters 4- 6 of this dissertation, we described the modeling of phenomena by using a stochastic differential equation in a way that combines the aspects on how present occurrences will affect future occurrences and also allows for the fact that there will be small random effects shifting a natural phenomena state in future periods. In reality phenomena such as the ones described in this dissertation are influenced by random noise thus it is appropriate for the behavior of the noise to be reflected in the differential equations. Stochastic differential equations play an important role in applications of different kinds of fields; however in this study we focus on geophysics.

In this chapter, we model the influence of any natural or physical phenomena and the effect on each other, by using a coupled system of two stochastic differential equations. We test our approach and present a numerical example by using real world earthquake series. In this application, we describe the correlation of earthquake series from four regions in the country Chile by using coupled system of $\Gamma(a, b)$ Ornstein-Uhlenbeck stochastic differential equations. The objective is to model the dependency and effects of earthquake series occurring at four different regions within the same geographical area.

7.1 Background of Earthquake Time Series

The data used in this chapter is the same as the data sets used in Chapter 6 of this book. However, the novelty of this application is different from that of Chapter 6. In this chapter, we compare the four regions from the country Chile and simulate data to investigate the correlation of earthquake series between the regions. In that, we discuss if the occurrence of earthquake in one region affects the occurrence of another. For completeness of this chapter, we briefly present the data background.

The earthquakes series used in this study was obtained from the IRIS Data Management Center http://www.iris.edu. We obtained the magnitude, location and time origin of any M $\geq$ 4.0 earthquake that occurred within the regions surrounding the most recent (since 2000) very large earthquakes (Mw$\geq$8) in the western hemisphere. These comprised of 4 earthquake series for four regions shown is Figure 7.1. We consider the aftershocks-limited areas as areas where tectonic stress can be transmitted efficiently, thus there might exist a degree of dependency between the earthquakes within each of these regions. All of these earthquakes occurred in South America along the tectonic boundary where the Nazca tectonic plate subducts underneath the South American tectonic plate [26]. This is the tectonic boundary where the largest earthquakes ever recorded occurred, the 1960 magnitude 9.5 Chile earthquake [14].

Before we proceed to simulate data using our proposed model, we compute the correlation matrix between the four regions to verify if there exist any form of dependency between the four regions. The correlation matrix for all the four regions are given below:

	Region 1	Region 2	Region 3	Region 4
Region 1	1.00	-0.0046	-0.0089	-0.0386
Region 2	-0.0046	1.00	-0.0464	0.0220
Region 3	-0.0089	-0.0464	1.00	-0.0313
Region 4	-0.0386	0.0220	-0.0313	1.00

Each cell in the table shows the correlation between two regions. The line of 1.00's going from the top left to the bottom right is the main diagonal, which shows that each region

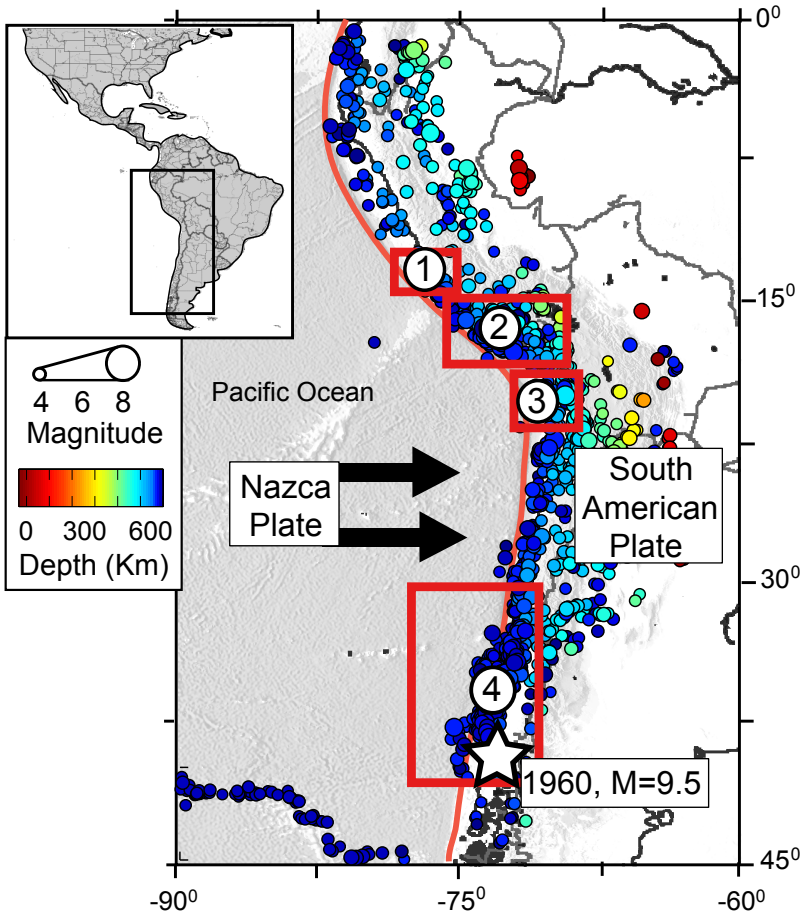

Figure 7.1: Map shows the spatial distribution of earthquakes (colored circles) along the tectonic boundary (red line) between the Nazca and the South American tectonic plates. Red rectangles represent the four study regions. Inset shows the area in larger map. Color represents earthquake's depths. White circles mark the location of the very large earthquakes (Mw$\geq$8). For reference, white star marks the location of the largest earthquake ever recorded.

always perfectly correlates with itself. For example the correlation between regions 1 and 2 is -0.0046 and that between regions 2 and 3 is -0.0464. This matrix is symmetrical with the same correlation shown above the main diagonal being a mirror image of those below the main diagonal. From the above matrix, we observe that there exist some correlation between the four regions therefore it is appropriate to model the earthquake series by using our proposed coupled system.

7.2 Application

In this section, we present the numerical results of the geophysical time series arising in seismology. The earthquake series from the four regions constituted data collected from a location at a given time to estimate the magnitude of the earthquakes at a given location, where the real magnitude is known. The earthquake magnitudes recorded in the data set was used in our analysis.

One charecteristics of the earthquake series data points is that, the local variance of the series was larger when the level of the series was higher. We therefore transformed the data into a different scale by taking logarithm of the time series data points. By performing this change of scale, it is likely that a stationary model can be fitted after the transformation (see [21] for details).

The 2-dimensional $\Gamma(a, b)$ Ornstein-Uhlenbeck model is applied to the earthquake series as follows. We first estimate the model parameters a, b and λ corresponding to each region by using the relations given in (3.47) and (3.49) respectively.

Next, we simulate data from the our model following the discussions in Chapter 3 Section 3.5. We simulated independent path of our model using different time steps. Finally, we compared the simulated time series generated from the 2-dimensional $\Gamma(a, b)$ Ornstein-Uhlenbeck model to the original earthquake time series. In order to investigate our model fit, we computed the root mean square error (RMSE) for all the four regions in Chile. The RMSE estimates measure how well our proposed model fits the original earthquake time series.

Table 7.1: Data and parameter estimates

Regions	Number of observations	X_0	λ	a	b
Region 1	389	4.70	2.1776	158.9808	34.4993
Region 2	1000	5.00	3.0955	196.4128	43.2430
Region 3	892	5.00	1.6348	151.7649	33.0730
Region 4	5000	4.70	1.6344	205.7332	46.0565

7.3 Results

The numerical results of the earthquake time series for the four (4) regions discussed are presented. Table 7.1 summarizes the numerical results of the estimation of parameters for four regions understudy. In the table, the second and third column represents the number of data points represents and initial points X_0 for the earthquake time series. The fourth, fifth and sixth columns in Table 7.1 represents the estimated parameters from real data.

Table 7.2 summarizes the numerical results for the earthquake time series when the 2-dimensional $\Gamma(a, b)$ Ornstein-Uhlenbeck model was applied to real earthquake data series. In the table the parameters σ_{11} and σ_{22} corresponds to the volatility of our earthquake time series and σ_{12} and σ_{21} describes the correlation between the regions understudy. For example, the results displayed in the first row of Table 7.2, represents coupled system of processes between Regions 1 and 2. In this scenario, we modeled the two regions as a coupled system where the occurrence of earthquake at one region has an effect on the occurrence of earthquake at another. From the table, we observed that for all the coupled regions, the values for $\sigma_{12} = \sigma_{21}$ and all these values were different from zero. We recall that if $\sigma_{12} = \sigma_{21} = 0$, then there exist no correlation between the regions understudy.

The error estimates for the random time series generated our 2-dimensional stochastic model and the original earthquake time series are also presented in Table Table 7.2. We observe that two error estimates are estimated in Table 7.2. The first estimate is when the simulated

Table 7.2: Data Results for 2-dimensional $\Gamma(a, b)$ Ornstein-Uhlenbeck model.

Regions	σ_{11}	σ_{12}	σ_{21}	σ_{22}	RMSE 1	RMSE 2
Region 1 and Region 2	0.5168	-0.0046	-0.0046	0.4583	0.3736	0.3563
Region 1 and Region 3	0.5168	-0.0089	-0.0089	0.5514	0.1576	0.20630
Region 1 and Region 4	0.5168	-0.0386	-0.0386	0.4411	0.1189	0.2733
Region 2 and Region 3	0.4389	-0.0464	-0.0464	0.5514	0.3774	0.5997
Region 2 and Region 4	0.4389	0.0220	0.0220	0.4411	0.31251	0.22890
Region 3 and Region 4	0.5514	-0.0313	-0.0313	0.4411	0.2499	0.3374

results arising from the first region is compared with the actual time series and the second estimate is when the the simulated results arising from the second region is compared with the actual time series from that same region. One interesting property with this results is the fact that, the results where generated from a coupled system. The results obtained indicate that the stochastic differential equation model provides a reasonable fit to the earthquake data.

Figures 7.2–7.7, reports the sample path for the simulated earthquake time series for the coupling regions. In the figures, the black colored path represents the sample path originating from the first stochastic differential equation $\{X_1(t)\}$ and the red colored path represents the sample path originating from the second stochastic differential equation $\{X_2(t)\}$. The path is dominated by jumps and also illustrates the fact that the process is everywhere continuous and non-negative.

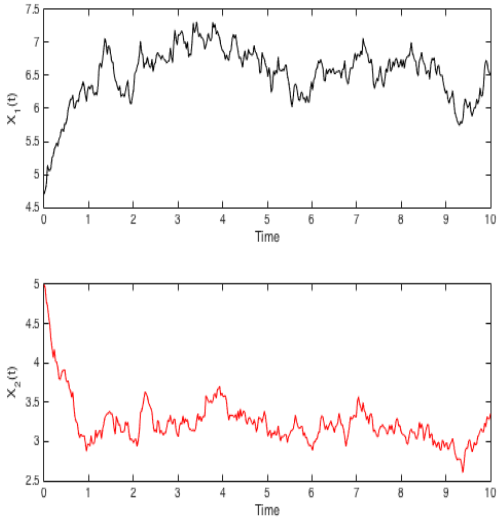

Figure 7.2: Sample Path of the coupled stochastic model for regions 1 and 2.

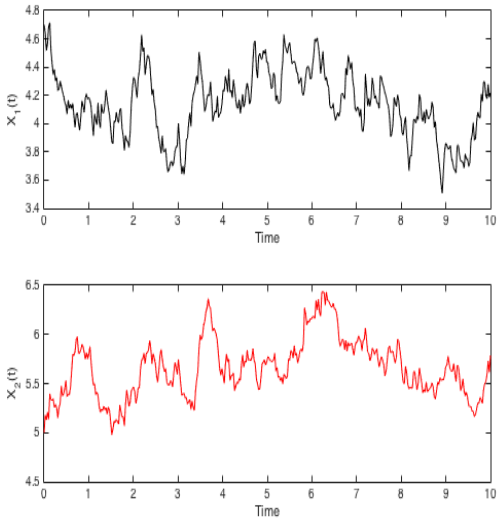

Figure 7.3: Sample Path of the coupled stochastic model for regions 1 and 3.

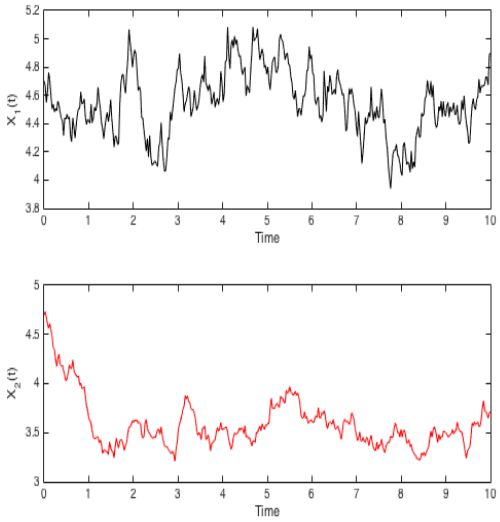

Figure 7.4: Sample Path of the coupled stochastic model for regions 1 and 4.

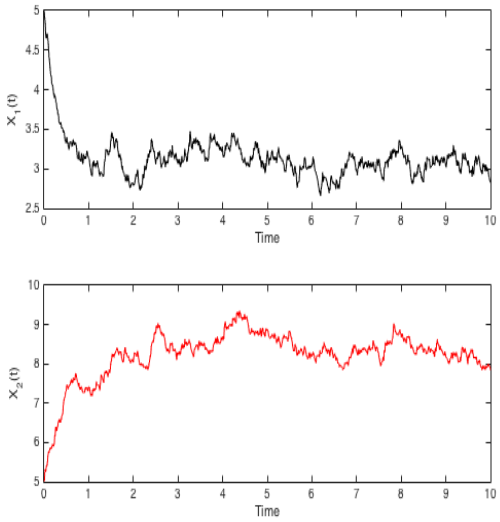

Figure 7.5: Sample Path of the coupled stochastic model for regions 2 and 3.

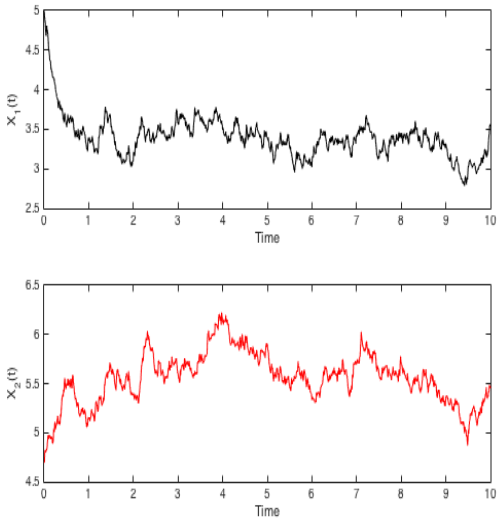

Figure 7.6: Sample Path of the coupled stochastic model for regions 2 and 4.

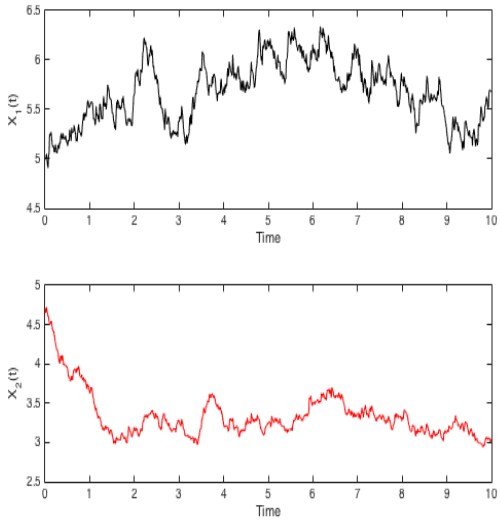

Figure 7.7: Sample Path of the coupled stochastic model for regions 3 and 4.

7.4 Concluding remarks

In the chapter, we modeled the correlation of earthquake series from four regions in the country Chile by using coupled system of $\Gamma(a, b)$ Ornstein-Uhlenbeck Stochastic Differential Equations. The goal is to model the dependency and effects of earthquake series occurring at all the regions and determine if the occurrence of earthquake at one region has the effect an effect on the another region. The modeling approached used here is very similar to the work by the authors in Ref. [37] except the fact that in this study we considered coupled system of stochastic differential equations. Based on the results obtained the proposed model provides a reasonable fit to the earthquake data. The selection of this model is justifiable since from the correlation matrix obtained from all the four earthquake series in the country Chile, there exist some dependency between the occurrences of earthquakes.

Chapter 8

Conclusion

In this dissertation we introduced a new method of the modeling and statistical analysis of complex time series to enhance the understanding of extreme events. By using the superposition and coupling system of $\Gamma(a, b)$ Ornstein-Uhlenbeck stochastic differential equation, we showed that proposed method accurately describes the behavior of financial indices and earthquake series due to two essential property: firstly, the model is completely stochastic and secondly, the method takes into account the physical behavior of the time series when modeling the data. This was proved and justified using the $\Gamma(a, b)$ Ornstein-Uhlenbeck process. We compared the results from our proposed model with those derived from different methods currently adopted to model financial and geophysical time series. Most of the traditional models have in common the fact that they are based upon the Gaussian assumption. However, the empirical study of some financial indices shows that in short time intervals the associated probability density function has greater kurtosis than a Gaussian distribution [4, 30]. In addition the models described in previous literature fails to take into account the behavior of the financial and earthquake data, and some of the models are not completely stochastic.

In this dissertation, we implemented very flexible classes of processes that incorporate long-range dependence, i.e. they have a slowly polynomially decaying autocovariance function and self-similarity like properties that are capable of describing some of the key distributional features of typical financial and geophysical time series. In order to capture realistic dependence structures, we combined independent Ornstein-Uhlenbeck processes driven by a Lévy process. In addition to model the correlation between different regions in lying in the same geographical area where an earthquake series was recorded we used coupled system of independent Ornstein-Uhlenbeck processes driven by a Lévy process. These selection process

is supported by the fact that generalized Lévy models are suitable for describing these type of time series [34]. Furthermore, the benefits of using the superposed and coupled system of independent Ornstein-Uhlenbeck processes is that the model offers plenty of analytic flexibility which is not available for more standard models such as the geometric Gaussian Ornstein-Uhlenbeck processes. In addition, the superposition of Ornstein-Uhlenbeck processes provide a class of continuous time processes capable of exhibiting long memory behavior. The presence of long memory suggests that current information is highly correlated with past information at different levels.

This dissertation focuses on theory and on application, as a way to check the efficiency of the new method of analysis. The efficiency of the superposed and coupled system of $\Gamma(a, b)$ Ornstein-Uhlenbeck processes were illustrated by showing this analysis in action on real complex data, that is, financial and geophysical time series. The financial time series used comprised of two frequency; moderate frequency (daily) and high frequency (minute and second). The geophysical time series also consisted of high frequency earthquake time series. Based on the computed model fit; the root mean square error, the proposed $\Gamma(a, b)$ Ornstein-Uhlenbeck stochastic differential equation gives a good fit. This is because the simulated results is very similar to the actual realizations. Additionally, the model parameters obtained from our statistical analysis are very useful for making inferences and predictions. For example the intensity parameters λ_1 and λ_2 describes the mean reversion speed of the specific time series and explains how our model reacts to perturbation. From the analysis in Chapter 5, we observe that the intensity parameters for the second by second high frequency times series are smaller compared to the minute by minute high frequency time series. This results suggests that in the long run, the closing prices for the stock exchanges of the minute by minute data will approach its mean closing prices faster than the second by second data. The results obtained will give an idea of the statistics of the future stock paths. This will help investors or practitioners who lacks insider information but has at their disposal all the information contained in the equity prices discover that a crash is imminent and take the necessary precautions. From the geophysical point of view, the results obtained will describe how the

effect of events that occurred very far in the past, for example decades ago, might have on the occurrence of present and future events. This type of analysis might help to better understand how tectonic stress decays and accumulates during long period of time. Based on the results of this dissertation, we conclude that the superposed and system of $\Gamma(a, b)$ Ornstein-Uhlenbeck stochastic differential equation accurately detect the statistical and temporal properties of the financial and geophysical time series. Several application in data science, statistics and related disciplines will be discussed in future work. Another future work will be to investigate the time an awareness should be raised for a high magnitude earthquake or market crash.

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Curriculum Vitae

Osei Kofi Tweneboah is a PhD Candidate in the Computational Science Program at the University of Texas at El Paso (UTEP). In August 2015, he earned a Master of Science degree in Mathematics from UTEP under the guidance of Prof. Maria C. Mariani. Prior to his enrollment in UTEP, He obtained a Bachelor of Science degree in Mathematics from Kwame Nkrumah University of Science and Technology, Kumasi, Ghana.

While pursuing his PhD in Computational Science he worked as a Teaching and Research Assistant at the Computational Science Program and Mathematical Sciences department at UTEP and also as an adjunct faculty at El Paso Community College. Osei has coauthored sixteen (16) papers in Peer-Reviewed Journals and has co-presented 28 research papers and posters. Based on his outstanding academic performance, the College of Science at the University of Texas at El Paso awarded him the Academic and Research Excellence Graduate Student award in Mathematics in May 2015. He has also won many research and travel grants from the Computational Sciences Program, Department of Mathematical Sciences, UTEP College of Science, and the UTEP Graduate School.

Osei was the President and Co-founder of the African Students Organization and Computational Science at the University of Texas at El Paso during the 2018-2019 and 2016-2018 academic years respectively. After graduation, Osei will pursue a career in a research university as a professor and or a career in the industry.

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