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Classification Of The Subalgebras Of The Algebra Of All 2 By 2 Matrices

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CLASSIFICATION OF THE SUBALGEBRAS OF THE ALGEBRA OF ALL 2 BY 2 MATRICES

by

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Abstract

Classification of the subalgebras of the familiar algebra of all $n \times n$ real matrices over the real numbers can get quite unwieldy as all subalgebras are of dimension ranging from 1 to n^2. Classification of the subalgebras of the algebra of all 2×2 real matrices over the real numbers is an interesting first start.

Since $\mathbf{M}_2(\mathbb{R})$ is of dimension 4 then its possible subalgebras are of dimension $1, 2, 3,$ or 4. The one-dimensional subalgebra and four-dimensional subalgebra need little to no attention. The two-dimensional and three-dimensional subalgebras however turn out to be of significance.

It turns out there is only one one-dimensional subalgebra and one four-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$. The former being fairly simple and the latter being trivial. The investigation of the two-dimensional and three-dimensional subalgebras is not as brief. Therefore, the goal of this thesis is to answer the following question:

> *Up to an isomorphism, how many distinct two-dimensional and three-dimensional subalgebras of* $\mathbf{M}_2(\mathbb{R})$ *are there?*

We show here that up to an isomorphism there are three distinct two-dimensional subalgebras and one distinct three-dimensional subalgebra.

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Chapter 1

Introduction

This chapter is intended to ease us into the real-algebra of $n \times n$ real matrices. We begin with some familiar definitions that are immediately called upon in the definition of an algebra over a ring, or later simply called an algebra. Then to better understand isomorphisms between algebras we go over needed homomorphisms. Finally an algebra is defined and then follows that, for a given n, the set of all $n \times n$ real matrices is easily equipped to be a real-algebra.

1.1 Ring Definition

First we adopt a familiar definition of a ring in anticipation of the algebra definition.

Definition 1 *A ring is a set* $\mathbf{R}$ *together with two binary operations*

$$+ \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

and

$$\cdot \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

satisfying the following properties (as is customary we write $a + b$ *in place of* $+(a, b)$ *and write* ab *in place of* $\cdot(a, b)$ *for all* $a, b \in \mathbf{R}$*):*

- $(\mathbf{R}, +)$ *is an abelian group*
- $(\mathbf{R}, \cdot)$ *is a monoid*
- $a(b + c) = ab + ac$ *and* $(b + c)a = ba + ca$ *for all* $a, b, c \in \mathbf{R}$.

If in addition $(\mathbf{R}\backslash\{\textit{identity for }+\},\cdot)$ *is an abelian group then* $\mathbf{R}$ *is called a field.*
Naturally a subset $\mathbf{S} \subset \mathbf{R}$ *is said to be a subring of* $\mathbf{R}$ *if* $(\mathbf{S},+)$ *is a subgroup of the group* $(\mathbf{R},+)$ *and also* $(\mathbf{S},\cdot)$ *is a submonoid of the monoid* $(\mathbf{R},\cdot)$.

It is of particular importance that $(\mathbf{R},\cdot)$ is a monoid so that a ring indeed contains unity, call it $1 \in \mathbf{R}$.

1.2 Vector Space Definition

Once more in anticipation of the algebra definition, we now define a vector space.

Definition 2 *Let* $\mathbf{F}$ *be a field.*
A vector space over $\mathbf{F}$ *(or vector space if* $\mathbf{F}$ *is fixed for a particular discussion) is an abelian group* $\mathbf{G}$ *(with binary operation* $+$*) together with a scalar multiplication map*

$$\cdot\colon \mathbf{F} \times \mathbf{G} \to \mathbf{G}$$

that satisfy the following properties for all $a, b \in \mathbf{F}$ *and* $m, n \in \mathbf{G}$ *(as is customary we write* am *in place of* $\cdot(a, m)$ *for all* $a \in \mathbf{F}$ *and* $m \in \mathbf{G}$*):*

- $a(m + n) = am + an$
- $(a + b)m = am + bm$
- $(ab)m = a(bm)$
- $1m = m$.

A subset $\mathbf{H} \subset \mathbf{G}$ *is said to be a vector subspace of* $\mathbf{G}$ *(or subspace of* $\mathbf{G}$*, or a subspace if* $\mathbf{G}$ *is fixed for a particular discussion) if* $\mathbf{H}$ *is a subgroup of the group* $\mathbf{G}$ *that is also closed under the scalar multiplication on* $\mathbf{G}$.

1.3 Homomorphisms

In anticipation of algebra isomorphisms we now define ring and vector space homomorphisms.

Definition 3 *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be rings.*
A function $f\colon \mathbf{R} \to \mathbf{S}$ *is a ring homomorphism if for all* $a, b \in \mathbf{R}$

- $f(a + b) = f(a) + f(b)$,
- $f(ab) = f(a)f(b)$, *and*
- $f(1_{\mathbf{R}}) = 1_{\mathbf{S}}$ *where* $1_{\mathbf{R}}$ *is the unity in* $\mathbf{R}$ *and* $1_{\mathbf{S}}$ *is the unity in* $\mathbf{S}$.

Definition 4 *Let* $\mathbf{G}$ *and* $\mathbf{H}$ *be vector spaces over a field* $\mathbf{F}$.
A function $f\colon \mathbf{G} \to \mathbf{H}$ *is a vector space homomorphism if*

- $f(m_1 + m_2) = f(m_1) + f(m_2)$ *for all* $m_1, m_2 \in \mathbf{G}$, *and*
- $f(am) = af(m)$ *for all* $a \in \mathbf{R}$ *and for all* $m \in \mathbf{G}$.

1.4 Algebra Definition

We now define an algebra over a field. Briefly, an algebra is both a vector space and a ring such that the scalar multiplication and ring multiplication interact in a pleasing way.

Definition 5 *Let* $\mathbf{F}$ *be a field.*
An abelian group $\mathbf{G}$ *together with a scalar multiplication map*

$$\cdot\colon \mathbf{F} \times \mathbf{G} \to \mathbf{G}$$

and a binary operation

$$*\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$$

is a $\mathbf{F}$*-algebra (or algebra if* $\mathbf{F}$ *is fixed for a particular discussion) if the following hold:*

- $(\mathbf{G}, \cdot)$ *is a vector space over* $\mathbf{F}$

- $(\mathbf{G}, *)$ *is a ring, with the ring addition being the same as the vector space addition*

- *For all* $r \in \mathbf{F}$ *and* $m, n \in \mathbf{G}$ *the scalar multiplication and ring multiplication satisfy the following identity:*
$$r(mn) = (rm)n = m(rn).$$

A subset $\mathbf{H} \subset \mathbf{G}$ *is said to be a subalgebra of* $\mathbf{G}$ *if* $\mathbf{H}$ *is a subspace of* $\mathbf{G}$ *and also a subring of* $\mathbf{G}$.

1.5 Isomorphic Algebras

As we are prone to sort together like algebras, we now explicitly interpret said effort. In short, two algebras are the same if they only differ by name.

Definition 6 *Two algebras* $\mathbb{A}_1$ *over* $\mathbf{F}$ *and* $\mathbb{A}_2$ *over* $\mathbf{F}$ *are isomorphic, or* $\mathbb{A}_1 \cong \mathbb{A}_2$*, if there exists a bijective function* $f\colon \mathbb{A}_1 \to \mathbb{A}_2$ *that is both a vector space homomorphism and ring homomorphism. That is,* f *satisfies:*

- $f(am_1 + bm_2) = af(m_1) + bf(m_2)$ *for all* $a, b \in \mathbf{F}$ *and for all* $m_1, m_2 \in \mathbb{A}$

- $f(m_1 m_2) = f(m_1)f(m_2)$ *for all* $m_1, m_2 \in \mathbb{A}$

- $f(1_{\mathbb{A}_1}) = 1_{\mathbb{A}_2}$ *where* $1_{\mathbb{A}_1}$ *is the unity in* $\mathbb{A}_1$ *and* $1_{\mathbb{A}_2}$ *is the unity in* $\mathbb{A}_2$.

1.6 $\mathbf{M}_n(\mathbb{R})$

We now briefly introduce some needed notation for real matrices.

Definition 7 $\mathbb{R}$ *denotes the field of real numbers.*

Definition 8 *Fix a positive integer n. Denote the $n \times n$ matrix with a single one in the (i, j) entry and zeroes elsewhere by E_{ij}.*
Examples for $n = 2$ and $n = 3$ respectively:

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then for an arbitrary $n \times n$ matrix M where $m_{ij} \in \mathbb{R}$ is the (i, j) entry we have

$$M = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} E_{ij}.$$

This representation of M as a linear combination of the E_{ij}'s is unique and later serves a particular importance. For readability we have the following.

Definition 9 $\mathbf{M}_n(\mathbb{R})$ *denotes the set of all $n \times n$ real matrices. That is, an $n \times n$ matrix M is an element of $\mathbf{M}_n(\mathbb{R})$ if and only if $m_{ij} \in \mathbb{R}$ for all $1 \leq i, j \leq n$. I denotes the $n \times n$ identity matrix whose (i, j) entry is 1 whenever $i = j$ and 0 whenever $i \neq j$.*

Definition 10 *Let $M \in \mathbf{M}_n(\mathbb{R})$ The (i, j) entry of the matrix M is denoted as $(M)_{ij}$ or m_{ij}.*

1.7 The Algebra Of Real Matrices

At last we now realize a particular collection of real matrices as a real-algebra.
Let $\mathbf{M}_n(\mathbb{R})$ be the set of all $n \times n$ real matrices. Now it is easy to verify the following:

- $\mathbf{M}_n(\mathbb{R})$ is closed under the standard addition of matrices
- $\mathbf{M}_n(\mathbb{R})$ is closed under the standard scalar multiplication of matrices
- $\mathbf{M}_n(\mathbb{R})$ is closed under the standard matrix multiplication.

Consequently, with little effort we have the following real-algebra of matrices.

Theorem 1 $\mathbf{M}_n(\mathbb{R})$ *together with the standard scalar multiplication, matrix addition, and matrix multiplication is a* $\mathbb{R}$*-algebra.*

Proof. Clearly $\mathbf{M}_n(\mathbb{R})$ together with the standard addition of matrices and standard scalar multiplication of matrices is a vector space over $\mathbb{R}$. Also clear, $\mathbf{M}_n(\mathbb{R})$ together with the standard addition of matrices and standard matrix multiplication is a ring with unity I. Then for all $r \in \mathbb{R}$ and $A, B \in \mathbf{M}_n(\mathbb{R})$

$$(r(AB))_{ij} = r\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right) = \sum_{k=1}^{n} r(a_{ik}b_{kj}) = \sum_{k=1}^{n}(ra_{ik})b_{kj} = ((rA)B)_{ij}$$

and

$$(r(AB))_{ij} = r\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right) = \sum_{k=1}^{n} r(a_{ik}b_{kj}) = \sum_{k=1}^{n} a_{ik}(rb_{kj}) = (A(rB))_{ij}.$$

Hence, $r(AB) = (rA)B = A(rB)$. □

Chapter 2

Subalgebra As A Subspace

In this chapter we now concern ourselves with subalgebras of $\mathbf{M}_n(\mathbb{R})$ as vector subspaces of $\mathbf{M}_n(\mathbb{R})$. We first recall that subalgebras over a field are vector subspaces and so, in anticipation of $\mathbf{M}_n(\mathbb{R})$ being a finite-dimensional vector space, we readily make use of the standard definitions and results about finite-dimensional vector spaces. Next, we seek vector subspaces of $\mathbf{M}_n(\mathbb{R})$ containing the identity matrix that are closed under matrix multiplication and then conclude that these subspaces are indeed subalgebras. Finally we conceive a method for constructing vector subspaces of $\mathbf{M}_2(\mathbb{R})$ that are in fact subalgebras.

2.1 Finite-Dimensional Vector Space

This section provides a brief overview of a finite-dimensional vector space and its basis and dimension. Only what is needed for our purposes is recorded here but further detail on the standard results mentioned in this section and later used can be revisited in an elementary Linear Algebra text. We only consider finite-dimensional vector spaces as it will become apparent that $\mathbf{M}_n(\mathbb{R})$ itself is finite-dimensional.

Definition 11 *Let $\mathbf{V}$ be a vector space over $\mathbf{F}$. A finite set of vectors $\{w_1, \ldots, w_n\}$ in $\mathbf{V}$ spans $\mathbf{W} \subset \mathbf{V}$ if*

$$span(w_1, \ldots, w_n) = \left\{ \sum_{i=1}^{n} a_i w_i \colon a_i \in \mathbf{F} \right\} = \mathbf{W}.$$

Recall that $span(w_1, \ldots, w_n)$ is itself a vector subspace. Now follows the definition of a finite-dimensional vector space.

Definition 12 *A vector space* $\mathbf{V}$ *is finite-dimensional if some finite set of vectors in* $\mathbf{V}$ *spans* $\mathbf{V}$*.*

A basis is a linearly independent set and so we must define linearly independent. We define linearly independent in the following way as to simplify our future investigations.

Definition 13 *Let* $\mathbf{V}$ *be a vector space over* $\mathbf{F}$*. A finite set of vectors* $\{v_1, \ldots, v_n\}$ *in* $\mathbf{V}$ *is linearly independent if the only choice of* $a_1, \ldots, a_n \in \mathbf{F}$ *that makes the sum* $\sum_{i=1}^{n} a_i v_i$ *equal to the identity vector is* $a_1 = \cdots = a_n = 0$*.*

Equivalently, vectors $v_1, \ldots, v_n$ are linearly independent if each vector $v \in span(v_1, \ldots, v_n)$ has a unique representation as a linear combination of the vectors $v_1, \ldots, v_n$. We remind of this result in passing as overture for the following definition.

Definition 14 *A basis of a vector space* $\mathbf{V}$ *is a finite set of vectors* $\{v_1, \ldots, v_n\}$ *in* $\mathbf{V}$ *that is linearly independent and spans* $\mathbf{V}$*. The vectors* $v_1, \ldots, v_n$ *are called basic vectors.*

Recall that every finite-dimensional vector space has a basis and that all bases have the same cardinality. It now follows the definition of dimension.

Definition 15 *Let* $\mathbf{V}$ *be a finite-dimensional vector space. The dimension of* $\mathbf{V}$ *is the cardinality of any basis of* $\mathbf{V}$*.*

Recall that an algebra over a field is a vector space. For clarity in the future we define the dimension of an algebra over a field as its dimension as a vector space.

Definition 16 *Let* $\mathbb{A}$ *be an algebra over a field. The dimension of the algebra* $\mathbb{A}$ *is the dimension of the vector space* $\mathbb{A}$*.*

Some standard results that are of particular importance we state here for record.

Lemma 1 *Every subspace of a finite dimensional vector space is finite dimensional.*

Lemma 2 *Every finite dimensional vector space has a basis.*

Lemma 3 *Every linearly independent set of a finite-dimensional vector space can be extended to a basis of the vector space.*

2.2 Foundation

We now start with some important groundwork for finding subalgebras of $\mathbf{M}_n(\mathbb{R})$ by way of finding vector subspaces that are also subalgebras. Then at the end of this section we realize that beginning to characterize all subalgebras of $\mathbf{M}_n(\mathbb{R})$ by first finding vector subspaces that are also subalgebras seems particularly formidable given the dimension of $\mathbf{M}_n(\mathbb{R})$.

First note that if a vector subspace of $\mathbf{M}_n(\mathbb{R})$ is closed under matrix multiplication then the needed property for an algebra involving the scalar and ring multiplication will automatically be satisfied as it holds in $\mathbf{M}_n(\mathbb{R})$. Then regarding the ring structure for an algebra we are only maybe missing the identity matrix I for the ring multiplication. We now record this in the following lemma.

Lemma 4 *Let* $\mathbf{V}$ *be a vector subspace of* $\mathbf{M}_n(\mathbb{R})$ *containing* I *that is closed under the usual matrix multiplication. Then* $\mathbf{V}$ *is a subalgebra.*

Proof. The needed ring multiplication properties hold in $\mathbf{M}_n(\mathbb{R})$ and so hold in $\mathbf{V}$. Therefore, $\mathbf{V}$ is a subring. Let $A, B \in \mathbf{M}_n(\mathbb{R})$ and $r \in \mathbb{R}$. Then $r(AB), (rA)B, A(rB) \in \mathbf{V}$ and the identity

$$r(AB) = (rA)B = A(rB)$$

holds in $\mathbf{M}_n(\mathbb{R})$ and therefore in $\mathbf{V}$. □

Note that not every vector subspace is closed under matrix multiplication. Consider the vector subspace $\mathbf{V}$ of $\mathbf{M}_3(\mathbb{R})$ with basis

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

Then

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \notin \mathbf{V}.$$

In this example, the identity matrix is not even an element of the subspace $\mathbf{V}$. This feature immediately discounts $\mathbf{V}$ as a subalgebra and brings us to our next point.

Every subalgebra must have unity. Namely, every subalgebra of $\mathbf{M}_2(\mathbb{R})$ must contain the identity matrix. Therefore with every vector subspace we are considering we can begin with a basis including the identity matrix and consequently focus on the remaining basic vectors. This leads us to the following.

Theorem 2 *The only one-dimensional subalgebra of* $\mathbf{M}_2(\mathbb{R})$ *is the subalgebra*

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in \mathbb{R} \right\} \cong \mathbb{R}.$$

Proof. Any one-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ must contain the identity matrix. Then trivially extend the linearly independent set $\{I\}$ to a basis. An isomorphism is given by

$$f\colon \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mapsto a.$$

□

Theorem 2 along with Lemma 4 is the foundation of our method of finding subalgebras of $\mathbf{M}_2(\mathbb{R})$.

Finally note that $\mathbf{M}_n(\mathbb{R})$ is finite-dimensional and of dimension n^2 since clearly the set $\{E_{ij} : 1 \leq i, j \leq n\}$ of n^2 vectors is linearly independent and spans $\mathbf{M}_n(\mathbb{R})$. Now a vector subspace will have dimension not exceeding the dimension of the vector space. Hence, subalgebras of $\mathbf{M}_n(\mathbb{R})$ can be particularly abundant as they will be of dimension ranging from 1 to n^2. So finding subalgebras of $\mathbf{M}_n(\mathbb{R})$ by constructing vector subspaces that are thereafter subalgebras can get quite unwieldy. This inspires us to begin with characterizing subalgebras of $\mathbf{M}_2(\mathbb{R})$.

2.3 Subalgebra From Subspace

We now describe our method for finding subalgebras of $\mathbf{M}_2(\mathbb{R})$ by way of first considering a vector subspace containing the 2×2 identity matrix and then forcing it to be a subalgebra.

Recall from the previous section that a subspace of $\mathbf{M}_2(\mathbb{R})$ containing the identity matrix that is closed under the matrix multiplication is a subalgebra. Therefore we only need to check that the product of any two basic vectors of the subspace is contained in the subspace in order to verify that the subspace is closed under matrix multiplication. In short, this verification is our method.

Suppose we have a vector subspace $\mathbf{V}$ of $\mathbf{M}_2(\mathbb{R})$ that contains the 2×2 identity matrix. Then for some positive integer d at most 4 we have a basis of $\mathbf{V}$, say $\{v_1 = I, \ldots, v_d\}$ of d matrices in $\mathbf{V}$. We then want the subspace to be closed under the matrix multiplication. In other words, we want that for all matrices $M_1, M_2 \in \mathbf{V}$ the product M_1M_2 is contained in $\mathbf{V}$. For $M_1, M_2 \in \mathbf{V}$, we have that

$$M_1 = \sum_{i=1}^{d} a_i v_i$$

for some constants $a_1, \ldots, a_d$ and

$$M_2 = \sum_{i=1}^{d} b_i v_i$$

for some constants $b_1, \ldots, b_d$. Then we have the product

$$M_1M_2 = \left(\sum_{i=1}^{d} a_i v_i\right)\left(\sum_{j=1}^{d} b_j v_j\right).$$

So

$$M_1M_2 = \sum_{i=1}^{d}\sum_{j=1}^{d} a_i b_j v_i v_j.$$

The terms of the sum involving the products I^2, Iv_i, and v_iI for $2 \leq i \leq d$ are all trivially contained in $\mathbf{V}$. Now if the remaining terms involving products v_iv_j for $2 \leq i, j \leq d$ were all contained in $\mathbf{V}$ then all these terms too would be contained in $\mathbf{V}$. Then, as $\mathbf{V}$ is closed under matrix addition, having the product M_1M_2 equal to a sum of matrices in $\mathbf{V}$ gives us $M_1M_2 \in \mathbf{V}$.

So in finding a d-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ we will start with a basis of d matrices including the identity matrix and suppose the product of any two basic vectors is equal to a linear combination of the basis vectors. We then solve for the constants of these linear combinations. These constants then might restrict what the basis vectors can be in order to allow d-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$.

Chapter 3

Two-Dimensional Subalgebras Of $\mathbf{M}_2(\mathbb{R})$

In this chapter we begin our search for subalgebras of $\mathbf{M}_2(\mathbb{R})$ beginning with the two-dimensional ones. We digress for a brief moment with the Cayley-Hamilton Theorem for $\mathbf{M}_2(\mathbb{R})$. Then we quickly realize that the Cayley-Hamilton Theorem effectively concludes that any two-dimensional subspace of $\mathbf{M}_2(\mathbb{R})$ containing the 2×2 identity matrix is in fact a two-dimensional subalgebra.

3.1 Preliminaries

In this section we go over some standard definitions and some of their relating results that are in order for the statement and proof of the Cayley-Hamilton Theorem for $\mathbf{M}_2(\mathbb{R})$. Although the contents of this section could be expressed in terms of $\mathbf{M}_n(\mathbb{R})$, for simplicity we only express the contents of this section in terms of $\mathbf{M}_2(\mathbb{R})$ as we are only concerned with $\mathbf{M}_2(\mathbb{R})$. We first define the adjugate, determinant, and trace of a 2×2. Then follows a standard result immediately suggesting their relevance. Lastly, the familiar definition of the characteristic equation for a 2×2 matrix is led by the standard definitions of eigenvalue and eigenvector.

Definition 17 *Let $M \in \mathbf{M}_2(\mathbb{R})$ be given by $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a, b, c, d \in \mathbb{R}$. Define the adjugate of M as the following matrix*

$$adj M = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Define the determinant of M as the following value

$$det(M) = ad - bc.$$

Define the trace of M as the following value

$$tr(M) = a + d.$$

Theorem 3 *If $M \in \mathbf{M}_2(\mathbb{R})$ then*

$$M(adjM) = (adjM)M = det(M)I.$$

Hence M is invertible iff $det(M) \neq 0$.

Proof. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then

$$\begin{aligned} M(adjM) &= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} det(M) & 0 \\ 0 & det(M) \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\ &= (adjM)M \end{aligned}$$

The rest immediately follows. □

The sudden presence of complex numbers may be alarming but is immediately justified with the characteristic equation that follows. In short, the eigenvectors are dependent on the eigenvalues and the eigenvalues are roots of a polynomial. Then of course real eigenvalues, and consequently real eigenvectors, may not exist.

Let $\mathbb{C}$ denote the field of complex numbers.

Definition 18 *Let $M \in \mathbf{M}_2(\mathbb{R})$. Let v be a nonzero vector in $\mathbb{C}^2$ such that*

$$Mv = \lambda v$$

for some $\lambda \in \mathbb{C}$. Then v is called an eigenvector of M with corresponding eigenvalue λ.

In search of eigenvalues the above equation can be rewritten as

$$(M - \lambda I)v = 0.$$

Then we are concerned with when $M - \lambda I$ is not invertible since otherwise $v = 0$. Therefore by previous we find λ such that $det(M - \lambda I) = 0$. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and we have that

$$\begin{aligned} 0 &= det(A - \lambda I) \\ &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc) \\ &= \lambda^2 - tr(M)\lambda + det(M). \end{aligned}$$

This is precisely the characteristic equation we wish to highlight.

Definition 19 *Let $M \in \mathbf{M}_2(\mathbb{R})$. The characteristic polynomial of M is the polynomial*

$$\begin{aligned} f(x) &= det(M - xI) \\ &= x^2 - tr(M)x + det(M). \end{aligned}$$

The characteristic equation of M is the equation

$$x^2 - tr(M)x + det(M) = 0.$$

Solutions are the eigenvalues of M.

Consequently, it is clear that eigenvalues can be complex and eigenvectors thereafter found to be complex as well. Also, note that since the characteristic equation is quadratic then eigenvalues of a real 2×2 matrix are either both real or both complex.

3.2 The Cayley-Hamilton Theorem

From here on out, when suitable, we freely write 0 to denote the 2×2 zero matrix whose entries are all the real number 0. We leave to context the distinction between the zero matrix 0 and the real number 0.

We are now able to state and prove the Cayley-Hamilton Theorem for $\mathbf{M}_2(\mathbb{R})$.

Theorem 4 *Every matrix $M \in \mathbf{M}_2(\mathbb{R})$ satisfies its characteristic equation. That is,*

$$M^2 - tr(M)M + det(M)I = 0.$$

Proof. By previous we have that

$$(M - xI)(adj(M - xI)) = det(M - xI)I.$$

Now $adj(M - xI)$ has entries that are linear polynomials since $M - xI$ has entries that are linear polynomials. So for some matrix coefficients $C_1, C_0 \in \mathbf{M}_2(\mathbb{R})$ we have that

$$adj(M - xI) = C_1x + C_0.$$

So

$$(M - xI)(C_1x + C_0) = (x^2 - tr(M)x + det(M))I.$$

Comparing matrix coefficients by the powers of x we have

$$\begin{aligned} -C_1 &= I \\ MC_1 - C_0 &= -tr(M)I \\ MC_0 &= det(M)I. \end{aligned}$$

Multiplying these equations on the left by M^2, M, I respectively gives us

$$\begin{aligned} -M^2C_1 &= M^2 \\ M^2C_1 - MC_0 &= -tr(M)M \\ MC_0 &= det(M)I. \end{aligned}$$

Then adding these equations up we get

$$0 = M^2 - tr(M)M + det(M)I.$$

□

3.3 Two-Dimensional Subalgebras

Returning to our pursuit of constructing subalgebras of $\mathbf{M}_2(\mathbb{R})$, we know that a two-dimensional subspace $\mathbf{V}$ of $\mathbf{M}_2(\mathbb{R})$ with basis $\{I, A\}$ such that $I^2, IA, AI, A^2 \in \mathbf{V}$ is, in fact, a subalgebra. Hence, the only condition of concern is $A^2 \in \mathbf{V}$. Equipped with the Cayley-Hamilton Theorem this is immediately addressed.

Theorem 5 *Any two-dimensional subspace of* $\mathbf{M}_2(\mathbb{R})$ *containing the identity is a subalgebra.*

Proof. Let $\mathbf{V}$ be a two-dimensional subspace of $\mathbf{M}_2(\mathbb{R})$ containing I. Extend the set $\{I\}$ to a basis $\{I, A\}$. Write the characteristic equation of A

$$x^2 - tr(A)x + det(A) = 0.$$

Then

$$A^2 = -det(A)I + tr(A)A \in \mathbf{V}.$$

□

In short, we now have that any two-dimensional subspace of $\mathbf{M}_2(\mathbb{R})$ is a subalgebra as long as it contains the identity matrix. This result is not to say that all two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ have the same algebraic structure. That is, some two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ may not be isomorphic. The next chapter focuses on this issue.

Chapter 4

Isomorphic Two-Dimensional Subalgebras Of $\mathrm{M}_2(\mathbb{R})$

In this chapter we realize that not all the two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ are isomorphic. We recall the definitions of an idempotent matrix and a nilpotent matrix. The presence of an idempotent matrix or a nilpotent matrix in a two-dimensional subalgebra leads to some conditions on isomorphisms between subalgebras.

4.1 Idempotent And Nilpotent Matrices

The following definitions are often expressed in terms of $\mathbf{M}_n(\mathbb{R})$ but we only express them in terms of $\mathbf{M}_2(\mathbb{R})$ as we are only concerned with $\mathbf{M}_2(\mathbb{R})$.

Definition 20 *A matrix $M \in \mathbf{M}_2(\mathbb{R})$ is idempotent if $M^2 = M$.*

Generally for $\mathbf{M}_n(\mathbb{R})$ one might recall the notion of index for nilpotent matrices but for $\mathbf{M}_2(\mathbb{R})$ there is no need for the index. Therefore, we present nilpotent matrices in the following way.

Definition 21 *A matrix $M \in \mathbf{M}_2(\mathbb{R})$ is nilpotent if $M^k = 0$ for some positive integer k.*

Lemma 5 *For any nilpotent matrix $M \in \mathbf{M}_2(\mathbb{R})$ $M^2 = 0$.*

Proof. Let $M \in \mathbf{M}_2(\mathbb{R})$ be a nilpotent nonzero matrix. Let k be the smallest positive integer such that $M^k = 0$. Note that $det(M) = 0$ since otherwise $I = MM^{-1} = M^k(M^{-1})^k = 0$.

Then

$$M^2 = -det(M)I + tr(M)M = tr(M)M.$$

Now $tr(M) = 0$ since otherwise

$$M^k = tr(M)M^{k-1}$$

gives us

$$M^{k-1} = 0,$$

which is not true. We conclude that $M^2 = 0$. $\square$

4.2 Remarks

First note that an algebra isomorphism f between any two subalgebras of $\mathbf{M}_2(\mathbb{R})$ must satisfy $f(I) = I$ by definition and $f(0) = 0$ since $f(0) + f(0) = f(0 + 0)$. Then given an algebra isomorphism $f : \mathbf{V} \to \mathbf{W}$ between two subalgebras of $\mathbf{M}_2(\mathbb{R})$ where $\mathbf{V}$ is given by the basis $\{I, A\}$ we have that for any two matrices $M_1, M_2 \in \mathbf{V}$

$$\begin{aligned} f(M_1 + M_2) &= f(M_1) + f(M_2) \\ &= f(a_1 I + a_2 A) + f(b_1 I + b_2 A) \\ &= (a_1 + b_1)I + (a_2 + b_2)f(A) \end{aligned}$$

and

$$\begin{aligned} f(M_1 M_2) &= f(M_1)f(M_2) \\ &= f(a_1 I + a_2 A)f(b_1 I + b_2 A) \\ &= a_1 b_1 I + (a_1 b_2 + a_2 b_1)f(A) + a_2 b_2 f(A)^2 \end{aligned}$$

for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. So given a basis where one basic vector is the identity matrix we are primarily concerned with the image, under an isomorphism, of the basic vector that is not the identity matrix. Now the interest in idempotent and nilpotent matrices stems from the fact that for an algebra isomorphism f we have $f(M^2) = f(M)^2$. So, if $M \in \mathbf{M}_2(\mathbb{R})$ is idempotent this becomes $f(M) = f(M)^2$. On the other hand, if M is nilpotent then $0 = f(0) = f(M)^2$. This observation leads to some useful equations and results. Our search for isomorphic two-dimensional subalgebras is led by the following lemmas.

Lemma 6 *Let $f : \mathbb{A}_1 \to \mathbb{A}_2$ be an algebra isomorphism between two subalgebras of $\mathbf{M}_2(\mathbb{R})$. If $M \in \mathbb{A}_1$ is idempotent then $f(M)$ is idempotent.*

Proof. If $M \in \mathbb{A}_1$ is idempotent then

$$f(M) = f(M^2) = f(M)^2.$$

Therefore $f(M)$ is idempotent. □

Lemma 7 *Let $f : \mathbb{A}_1 \to \mathbb{A}_2$ be an algebra isomorphism between two subalgebras of $\mathbf{M}_2(\mathbb{R})$. If $M \in \mathbb{A}_1$ is nilpotent then $f(M)$ is nilpotent.*

Proof. If $M \in \mathbb{A}_1$ is nilpotent then

$$0 = f(0) = f(M^2) = f(M)^2$$

Therefore $f(M)$ is nilpotent. □

4.3 Non-Isomorphic Two-Dimensional Subalgebras

Followed by this next lemma we have a case of two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ that are not isomorphic.

Lemma 8 *Let $f : \mathbb{A}_1 \to \mathbb{A}_2$ be an algebra isomorphism between two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, M\}$ and $\mathbb{A}_2$ be given by the basis*

$\{I, M'\}$. If M is idempotent and M' is nilpotent then $f(M) = c_1 I + c_2 M'$ for $c_1 \neq 0$ and $c_2 \neq 0$.

Proof. Let $f(M) = c_1 I + c_2 M'$ for some $c_1, c_2 \in \mathbb{R}$.
If $c_1 = 0$ then $M = 0$ since

$$f(M) = f(M)^2 = (c_2 M')^2 = 0.$$

If $c_2 = 0$ then $M = I$ since

$$c_1 I = f(M) = f(M)^2 = (c_1 I)^2.$$

By contradiction we conclude $c_1 \neq 0$ and $c_2 \neq 0$. □

Notice that in the above proof we get that $c_2 \neq 0$ without the need for M' being nilpotent.

Theorem 6 *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, M\}$ and let $\mathbb{A}_2$ be given by the basis $\{I, M'\}$. Suppose M is idempotent and M' is nilpotent. Then $\mathbb{A}_1$ and $\mathbb{A}_2$ are not isomorphic.*

Proof. Suppose $f : \mathbb{A}_1 \to \mathbb{A}_2$ is an isomorphism. Then for some $c_1 \neq 0$ and $c_2 \neq 0$ we have,

$$f(M) = c_1 I + c_2 M'.$$

Now,

$$\begin{aligned} f(M) = f(M)^2 &= c_1^2 I + 2c_1 c_2 M' + c_2^2 M'^2 \\ &= c_1^2 I + 2c_1 c_2 M'. \end{aligned}$$

Then by linear independence of I and M we have,

$$\begin{cases} c_1 = c_1^2 \\ c_2 = 2c_1 c_2. \end{cases}$$

Hence, $1 = 2$, a contradiction. $\square$

As an example, we have that the two-dimensional subalgebra

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{R} \right\}$$

is not isomorphic to the two-dimensional subalgebra

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{R} \right\}$$

since the former is given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

and the latter is given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is idempotent while $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is nilpotent.

In Theorem 6 M' being nilpotent played an important role. For a more general M' we only have the following necessary but not sufficient condition for isomorphisms of two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ that are given by a basis $\{I, M\}$ where M is idempotent.

Theorem 7 *Let $f : \mathbb{A}_1 \to \mathbb{A}_2$ be an algebra isomorphism between two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, M\}$ and let $\mathbb{A}_2$ be given by the basis $\{I, M'\}$. Suppose M is idempotent. Then $f(M) = xI + yM'$ for $x, y \in \mathbb{R}$ satisfying*

$$\begin{cases} x = x^2 - y^2 det(M') \\ 1 = 2x + ytr(M'). \end{cases}$$

Proof. Let $f(M) = xI + yM'$. Note that $y \neq 0$. Now

$$\begin{aligned} f(M)^2 &= x^2 I + 2xyM' + y^2 M'^2 \\ &= x^2 I + 2xyM' + y^2(-det(M')I + tr(M')M') \\ &= (x^2 - y^2 det(M'))I + (2xy + y^2 tr(M'))M'. \end{aligned}$$

Now $f(M)$ is idempotent and so $f(M) = f(M)^2$. That is,

$$xI + yM' = (x^2 - y^2 det(M'))I + (2xy + y^2 tr(M'))M'.$$

Then the linear independence of I and M gives us

$$\begin{cases} x = x^2 - y^2 det(M') \\ 1 = 2x + ytr(M'). \end{cases}$$

□

Then with convenient bases we have the following corollaries.

Corollary 1 *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, M\}$ and $\mathbb{A}_2$ be given by the basis $\left\{I, \left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)\right\}$. Suppose M is idempotent. Then $\mathbb{A}_1$ and $\mathbb{A}_2$ are not isomorphic for $bc \leq 0$.*

Proof. Take $\left\{I, \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)\right\}$ as a basis of $\mathbb{A}_2$. Then by Theorem 7, for an isomorphism $f : \mathbb{A}_1 \to \mathbb{A}_2$ we have that $f(M) = xI + y\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ for $x, y \in \mathbb{R}$ satisfying

$$\begin{cases} x = x^2 + bcy^2 \\ 1 = 2x. \end{cases}$$

Then y is a solution of

$$4bcy^2 - 1 = 0.$$

So we must have that $bc > 0$. □

Corollary 2 *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, M\}$ and $\mathbb{A}_2$ be given by the basis $\left\{I, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\}$ where $a \neq d$. Suppose M is idempotent. Then $\mathbb{A}_1$ and $\mathbb{A}_2$ are not isomorphic for $\frac{-bc}{(a-d)^2} \geq \frac{1}{4}$.*

Proof. Take $\left\{I, \left(\begin{smallmatrix} 1 & \frac{b}{a-d} \\ \frac{c}{a-d} & 0 \end{smallmatrix}\right)\right\}$ as a basis of $\mathbb{A}_2$. Then by Theorem 7, for an isomorphism $f : \mathbb{A}_1 \to \mathbb{A}_2$ we have that $f(M) = xI + y\left(\begin{smallmatrix} 1 & \frac{b}{a-d} \\ \frac{c}{a-d} & 0 \end{smallmatrix}\right)$ for x and y satsifying

$$\begin{cases} x = x^2 - y^2\left(\frac{-bc}{(a-d)^2}\right) \\ 1 = 2x + y. \end{cases}$$

Hence, x satisfies the quadratic equation

$$x^2 - (1-2x)^2\left(\frac{-bc}{(a-d)^2}\right) - x = 0.$$

Simplifying, we have the quadratic equation

$$\left(1 + \frac{4bc}{(a-d)^2}\right)x^2 + \left(\frac{-4bc}{(a-d)^2} - 1\right)x + \frac{bc}{(a-d)^2} = 0.$$

Now the equation has a real solution only provided that $\frac{-bc}{(a-d)^2} \neq \frac{1}{4}$ and the discriminant is non-negative. That is, we require that

$$\frac{-bc}{(a-d)^2} \neq \frac{1}{4}$$

and

$$\left(\frac{4bc}{(a-d)^2} + 1\right)^2 - 4\left(1 + \frac{4bc}{(a-d)^2}\right)\left(\frac{bc}{(a-d)^2}\right) = 4\left(\frac{bc}{(a-d)^2}\right) + 1 \geq 0.$$

So me must have that $\frac{-bc}{(a-d)^2} < \frac{1}{4}$. □

4.4 Isomorphic Two-Dimensional Subalgebras

A converse of sorts of Theorem 7 is true and is in order. The approach up until this section has been to assume some isomorphisms exist between some particular subalgebras and arrive at conclusions. In the previous section nearly identical results could have been found artlessly interchanging the use of idempotent and nilpotent matrices in the antecedents and consequents. This section, instead, constructs some isomorphisms and ceases any interest in rephrasing the previous section.

Suppose we have two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ say $\mathbb{A}_1$ with basis $\{I, A\}$ and $\mathbb{A}_2$ with basis $\{I, B\}$. Creating a bijective function $f : \mathbb{A}_1 \to \mathbb{A}_2$ that satisfies:

- $f(c_1M_1 + c_2M_2) = c_1f(M_1) + c_2f(M_2)$ for all $c_1, c_2 \in \mathbb{R}$ and for all $M_1, M_2 \in \mathbb{A}_1$
- $f(I) = I$

is not the difficulty. To see this, an immediately suitable f is given by

$$f\colon c_1I + c_2A \mapsto c_1I + c_2B.$$

But then wanting the f to be an algebra isomorphism we require that

$$f(M_1M_2) = f(M_1)f(M_2) \text{ for all } M_1, M_2 \in \mathbb{A}_1.$$

Here the difficulty arises but is relieved with the condition that A and B are both idempotent matrices or are both nilpotent matrices.

Suppose we have an $f\colon \mathbb{A}_1 \to \mathbb{A}_2$ such that

$$f(c_1M_1 + c_2M_2) = c_1f(M_1) + c_2f(M_2)$$

for all $c_1, c_2 \in \mathbb{R}$ and for all $M_1, M_2 \in \mathbb{A}_1$ and

$$f(I) = I.$$

Let $\{I, A\}$ be a basis of $\mathbb{A}_1$. We have that for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$

$$\begin{aligned} f(M_1M_2) &= f((a_1I + a_2A)(b_1I + b_2A)) \\ &= f(a_1b_1I + (a_1b_2 + a_2b_1)A + a_2b_2A^2) \\ &= a_1b_1I + (a_1b_2 + a_2b_1)f(A) + a_2b_2f(A^2) \end{aligned}$$

and

$$\begin{aligned} f(M_1)f(M_2) &= f(a_1I + a_2A)f(b_1I + b_2A) \\ &= (a_1I + a_2f(A))(b_1I + b_2f(A)) \\ &= a_1b_1I + (a_1b_2 + a_2b_1)f(A) + a_2b_2f(A)^2. \end{aligned}$$

Then for $f(A^2) = f(A)^2$ we have that f is an algebra isomorphism. Naturally, we now have the following theorem.

Theorem 8 *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, A\}$ where A is idempotent(nilpotent). $\mathbb{A}_1 \cong \mathbb{A}_2$ iff there exists idempotent(nilpotent) $B \in \mathbb{A}_2$ such that $\{I, B\}$ is a basis of $\mathbb{A}_2$.*

Proof. Suppose $f\colon \mathbb{A}_1 \to \mathbb{A}_2$ is an algebra isomorphism. Then $f(A)$ is idempotent(nilpotent). Then $\{I, f(A)\}$ is linearly independent and therefore a basis of $\mathbb{A}_2$. So we put $B = f(A)$.

Conversely, let $\{I, B\}$ be a basis of $\mathbb{A}_2$ where B is idempotent(nilpotent). Then an isomorphism $f\colon \mathbb{A}_1 \to \mathbb{A}_2$ is given by

$$f\colon c_1I + c_2A \mapsto c_1I + c_2B.$$

□

Corollary 3 *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, M\}$ and let $\mathbb{A}_2$ be given by the basis $\{I, M'\}$. Suppose both M and M' are idempotent(nilpotent). Then $\mathbb{A}_1 \cong \mathbb{A}_2$.*

An example regarding idempotent matrices is the algebra of all diagonal matrices.

$$D_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{R} \right\} \cong \left\{ \begin{pmatrix} a+b & b \\ 0 & a \end{pmatrix} | a, b \in \mathbb{R} \right\}$$

since the former is given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

and the latter is given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

An example regarding nilpotent matrices is

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} |a, b \in \mathbb{R} \right\} \cong \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} |a, b \in \mathbb{R} \right\}$$

since the former is given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

and the latter is given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

The following theorems address the existence of an idempotent or nilpotent matrix in a two-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$.

Theorem 9 *Let $\mathbb{A}$ be a two-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, M\}$. Then $\mathbb{A}$ contains an idempotent matrix different from I iff $tr(M)^2 > 4det(M)$.*

Proof. $xI + yM \neq I$ is idempotent iff

$$\begin{aligned} xI + yM &= (xI + yM)^2 \\ &= x^2 I + 2xyM + y^2 M^2 \\ &= x^2 I + 2xyM + y^2(-det(M)I + tr(M)M) \\ &= (x^2 - y^2 det(M))I + (2xy + y^2 tr(M))M. \end{aligned}$$

Since I and M are linearly independent, we have that $xI + yM \neq I$ is idempotent iff

$$\begin{cases} x = x^2 - y^2 det(M) \\ 1 = 2x + ytr(M). \end{cases}$$

From the given system we have that

$$\frac{1 - ytr(M)}{2} = \left(\frac{1 - ytr(M)}{2}\right)^2 - y^2 det(M).$$

So,

$$2 - 2ytr(M) = 1 - 2ytr(M) + y^2 tr(M)^2 - 4y^2 det(M).$$

That is,

$$1 = y^2(tr(M)^2 - 4det(M)).$$

We can then conclude that $xI + yM \neq I$ is idempotent iff $tr(M)^2 - 4det(M) > 0$. □

Theorem 10 *Let $\mathbb{A}$ be a two-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, M\}$. Then $\mathbb{A}$ contains a nilpotent matrix different from 0 iff $tr(M)^2 = 4det(M)$.*

Proof. $xI + yM \neq 0$ is nilpotent iff

$$\begin{aligned} 0 &= (xI + yM)^2 \\ &= (x^2 - y^2 det(M))I + (2xy + y^2 tr(M))M. \end{aligned}$$

Since I and M are linearly independent, we have that $xI + yM \neq 0$ is nilpotent iff

$$\begin{cases} 0 = x^2 - y^2 det(M) \\ 0 = 2x + ytr(M). \end{cases}$$

From the given system we have that

$$0 = \left(\frac{-ytr(M)}{2}\right)^2 - y^2 det(M).$$

So,

$$0 = y^2 tr(M)^2 - 4y^2 det(M).$$

That is,

$$0 = y^2(tr(M)^2 - 4det(M)).$$

We can then conclude that $xI + yM \neq 0$ is nilpotent iff $tr(M)^2 - 4det(M) = 0$. $\square$

An example of a two-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ with neither an idempotent matrix nor a nilpotent matrix is

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a, b \in \mathbb{R} \right\}$$

given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

where $tr(M)^2 = 0 < 4 = 4det(M)$.

Theorem 11 *Let $\mathbb{A}$ be a two-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$. Then one and only one of the following is true*

- *$\mathbb{A}$ contains an idempotent matrix different from I*
- *$\mathbb{A}$ contains a nilpotent matrix different from 0*
- *$\mathbb{A}$ contains a matrix M such that $M^2 = -I$.*

Proof. Let $\mathbb{A}$ be given by the basis $\{I, M\}$. Without loss of generality assume $tr(M) = 0$ since otherwise $M - \frac{tr(M)}{2} I \in \mathbb{A}$ and $tr\left(M - \frac{tr(M)}{2} I\right) = tr(M) - \frac{tr(M)}{2} tr(I) = 0$. Now $M^2 = -det(M)I + tr(M)M = -det(M)I$. If $det(M) < 0$ then $\left(\frac{1}{\sqrt{-det(M)}} M\right)^2 = I$. If $det(M) = 0$ then $M^2 = 0$. If $det(M) > 0$ then $\left(\frac{1}{\sqrt{det(M)}} M\right)^2 = -I$. That no two statements are true at once is covered by Theorem 9 and Theorem 10. $\square$

Corollary 4 *Up to an isomorphism there are only three two-dimensional subalgebras of* $\mathbf{M}_2(\mathbb{R})$.

Proof. Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two two-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbb{A}_1$ be given by the basis $\{I, A\}$ and let $\mathbb{A}_2$ be given by the basis $\{I, B\}$. In view of Theorem 11, we need only show that if $A^2 = B^2 = -I$ then $\mathbb{A}_1 \cong \mathbb{A}_2$. Suppose $A^2 = B^2 = -I$. Then we claim an isomorphism $f\colon \mathbb{A}_1 \to \mathbb{A}_2$ is given by

$$f\colon c_1 I + c_2 A \mapsto c_1 I + c_2 B.$$

Clearly

$$f(c_1 M_1 + c_2 M_2) = c_1 f(M_1) + c_2 f(M_2)$$

for all $c_1, c_2 \in \mathbb{R}$ and for all $M_1, M_2 \in \mathbb{A}_1$ and $f(I) = I$. Then as

$$f(A^2) = f(-I) = -I$$

and

$$f(A)^2 = B^2 = -I$$

it follows that

$$f(M_1 M_2) = f(M_1) f(M_2)$$

for all $M_1, M_2 \in \mathbb{A}_1$. We conclude $\mathbb{A}_1 \cong \mathbb{A}_2$ and we are done. $\square$

Chapter 5

Three-Dimensional Subalgebras Of $\mathbf{M}_2(\mathbb{R})$

In this chapter we search for three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. The approach of representing products of basis vectors as linear combinations of basis vectors is treated slightly different than as it was with two-dimensional subalgebras. Unlike with two-dimensional subalgebras, the allowed basis vectors for a three-dimensional subspace to be a subalgebra is not entirely clear. Ignoring the apparent need for conditions on the basis, a particular suspicion leads to the result that all three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ are isomorphic.

5.1 Initial Approach

Suppose we have a three-dimensional subspace $\mathbf{V}$ of $\mathbf{M}_2(\mathbb{R})$ with basis $\{I, A, B\}$. We already know that $A^2, B^2 \in \mathbf{V}$ since $M^2 = -det(M)I + tr(M)M$ for any $M \in \mathbf{M}_2(\mathbb{R})$ and so our primary concern is only of the products AB and BA. If $AB, BA \in \mathbf{V}$ then we obtain a three-dimensional subalgebra. As is shown below, we only need $AB \in \mathbf{V}$.

Lemma 9 *Let $\mathbf{V}$ be a three-dimensional vector subspace of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, A, B\}$. If $AB \in \mathbf{M}_2(\mathbb{R})$ then $\mathbf{V}$ is a subalgebra of $\mathbf{M}_2(\mathbb{R})$.*

Proof. By previous we know that $M^2 \in \mathbf{V}$ for all $M \in \mathbf{V}$. So $(A+B)^2, A^2, B^2 \in \mathbf{V}$. Then

$$BA = (A+B)^2 - A^2 - B^2 - AB \in \mathbf{V}.$$

□

In fact, if all the basis vectors of a three-dimensional subspace of $\mathbf{M}_2(\mathbb{R})$ commute then we have a subalgebra.

Lemma 10 *Let* $\mathbf{V}$ *be a three-dimensional vector subspace of* $\mathbf{M}_2(\mathbb{R})$ *given by the basis* $\{I, A, B\}$*. If* $AB = BA$ *then* $\mathbf{V}$ *is a subalgebra of* $\mathbf{M}_2(\mathbb{R})$*.*

Proof. By previous we know that $M^2 \in \mathbf{V}$ for all $M \in \mathbf{V}$. So $(A+B)^2, A^2, B^2 \in \mathbf{V}$. Then

$$AB = \frac{(A+B)^2 - A^2 - B^2}{2} \in \mathbf{V}.$$

□

We now commence our approach, as before, to find three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$. Let $\mathbf{V}$ be a three-dimensional subspace of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, A, B\}$. Suppose $AB \in \mathbf{V}$. We solve $AB = c_1 I + c_2 A + c_3 B$ for $c_1, c_2, c_3 \in \mathbb{R}$. From

$$\begin{aligned} c_1 I + c_2 A + c_3 B &= AB \\ &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{12} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \end{aligned}$$

we have the system

$$\begin{cases} c_1 + c_2 a_{11} + c_3 b_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_2 a_{12} + c_3 b_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_2 a_{21} + c_3 b_{21} = a_{21}b_{12} + a_{22}b_{21} \\ c_1 + c_2 a_{22} + c_3 b_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}.$$

This system may have no solution. An example of a three-dimensional subspace of $\mathbf{M}_2(\mathbb{R})$ containing I that is not a subalgebra is the subspace given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

since

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \mathbf{V}.$$

Of course there do exist three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$, for example the set of all upper-triangular matrices in $\mathbf{M}_2(\mathbb{R})$ given by

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{R} \right\}$$

is a subalgebra of $\mathbf{M}_2(\mathbb{R})$ given by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Similarly the set of all lower triangular matrices too is a three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$.

5.2 Associativity

In this section we use associativity in another attempt at writing the needed product of basis vectors as a linear combination of the basis vectors. Once more we get a system that has no solution but it leads to a particular feature of one of the constants. We record this attempt in the following lemma.

Lemma 11 *Let $\mathbb{A}$ be a three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, A, B\}$. Write*

$$AB = c_1 I + c_2 A + \lambda_A B$$

and

$$BA = d_1 I + d_2 B + \lambda_B A.$$

for unique constants $c_1, c_2, d_1, d_2, \lambda_A, \lambda_B \in \mathbb{R}$. *Then* λ_A *is an eigenvalue of* A *and* λ_B *is an eigenvalue of* B. *Furthermore, the eigenvalues of* A *and* B *are real.*

Proof. Recall that for a matrix $M \in \mathbf{M}_2(\mathbb{R})$ we have the following characteristic equation

$$\lambda^2 - tr(M)\lambda + det(M) = 0.$$

We show that

$$\lambda_A^2 - tr(A)\lambda_A + det(A) = 0.$$

and

$$\lambda_B^2 - tr(B)\lambda_B + det(B) = 0.$$

From $A(AB) = A^2 B$ we have that

$$c_1 A + c_2 A^2 + \lambda_A AB = -det(A)B + tr(A)AB.$$

So

$$\begin{aligned} c_1 A + c_2(-det(A)I + tr(A)A) + \lambda_A(c_1 I + c_2 A + \lambda_A B) \\ = -det(A)B + tr(A)(c_1 I + c_2 A + \lambda_A B). \end{aligned}$$

Rewriting both sides as linear combinations of I, A, and B we have

$$\begin{aligned} (c_1\lambda_A - c_2 det(A))I + (c_1 + c_2\lambda_A + c_2 tr(A))A + \lambda_A^2 B \\ = c_1 tr(A)I + c_2 tr(A)A + (\lambda_A tr(A) - det(A))B. \end{aligned}$$

Since $\{I, A, B\}$ is linearly independent we have the following system

$$\begin{cases} c_1\lambda_A - c_2 det(A) = c_1 tr(A) \\ c_1 + c_2\lambda_A = 0 \\ \lambda_A^2 = \lambda_A tr(A) - det(A) \end{cases}.$$

Hence

$$\lambda_A^2 - tr(A)\lambda_A + det(A) = 0.$$

Since λ_A satisfies the characteristic equation of A, conclude λ_A is an eigenvalue of A. Since $\lambda_A \in \mathbb{R}$, we conclude that A has only real eigenvalues.

Then from $B(BA) = B^2 A$ we have that

$$d_1 B + d_2 B^2 + \lambda_B BA = -det(B)A + tr(B)BA.$$

So

$$\begin{aligned} d_1 B + d_2(-det(B)I + tr(B)B) + \lambda_B(d_1 I + d_2 B + \lambda_B A) \\ = -det(B)A + tr(B)(d_1 I + +d_2 B + \lambda_B A). \end{aligned}$$

Rewriting both sides as linear combinations of I, A, and B we have

$$\begin{aligned} (d_1\lambda_B - d_2 det(B))I + \lambda_B^2 A + (d_1 + \lambda_B d_2 + d_2 tr(B))B \\ = d_1 tr(B)I + (\lambda_B tr(B) - det(B))A + d_2 tr(B)B. \end{aligned}$$

Since $\{I, A, B\}$ is linearly independent we have the following system

$$\begin{cases} d_1\lambda_B - d_2 det(B) = d_1 tr(B) \\ \lambda_B^2 = \lambda_B tr(B) - det(B) \\ d_1 + \lambda_B d_2 = 0 \end{cases} .$$

Hence

$$\lambda_B^2 - tr(B)\lambda_B + det(B) = 0.$$

We conclude λ_B is an eigenvalue of B and that B has only real eigenvalues. □

5.3 Simultaneous Triangularization

A standard result in linear algebra is that any two matrices that commute and whose entries are complex are simultaneously triangularizable[1]. This result, although not applicable here, does inspire us. The proof of the result for 2×2 matrices solely relies on the fact that the two matrices have an eigenvector in common. Upon imitating that proof we will be able to define a particularly interesting isomorphism for any three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$.

We now recall simultaneous triangularization for $\mathbf{M}_2(\mathbb{R})$ and elaborate on our interest in this topic.

Definition 22 *Matrices A and B in $\mathbf{M}_2(\mathbb{R})$ are simultaneously triangularizable if there exists an invertible matrix $S \in \mathbf{M}_2(\mathbb{R})$ such that $S^{-1}AS$ and $S^{-1}BS$ are both triangular(both upper-triangular or both lower-triangular).*

Now suppose we have a three-dimensional subalgebra $\mathbb{A}$ of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, A, B\}$ such that A and B are simultaneously triangularizable. We have an S such that $S^{-1}AS$ and $S^{-1}BS$ are both triangular, say upper-triangular. Then for any $M \in \mathbb{A}$ we have that for some $c_1, c_2, c_3 \in \mathbb{R}$

$$\begin{aligned} S^{-1}MS &= S^{-1}(c_1I + c_2A + c_3B)S \\ &= c_1I + c_2S^{-1}AS + c_3S^{-1}BS \end{aligned}$$

and therefore the product $S^{-1}MS$ is upper-triangular as well. Note that $\{I, S^{-1}AS, S^{-1}BS\}$ is linearly independent since $\{I, A, B\}$ is linearly independent. Then an isomorphism from $\mathbb{A}$ to the subalgebra of all upper-triangular matrices in $\mathbf{M}_2(\mathbb{R})$ is given by

$$f\colon M \mapsto S^{-1}MS.$$

[1]See [2], Theorem 3.10.

Clearly $f(I) = I$. Let $M_1, M_2 \in \mathbb{A}$. Now for any $c_1, c_2 \in \mathbb{R}$ we have that

$$\begin{aligned} f(c_1M_1 + c_2M_2) &= S^{-1}(c_1M_1 + c_2M_2)S \\ &= c_1S^{-1}M_1S + c_2S^{-1}M_2S \\ &= c_1f(M_1) + c_2f(M_2). \end{aligned}$$

Finally,

$$\begin{aligned} f(M_1M_2) &= S^{-1}M_1M_2S \\ &= S^{-1}M_1SS^{-1}M_2S \\ &= f(M_1)f(M_2). \end{aligned}$$

So if any A and B were simultaneously triangularizable then we could conclude that all three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ are isomorphic.

Indeed, in the next section we show that any three-dimensional subalgebra $\mathbb{A}$ of $\mathbf{M}_2(\mathbb{R})$ is given by a basis $\{I, A, B\}$ such that A and B are simultaneously triangularizable. Already having that both the A and B have only real eigenvalues we are able to show that the A and B have a real eigenvector in common. Subsequently little is left in the way.

5.4 Isomorphic Three-Dimensional Subalgebras

In this section we show that any three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ is isomorphic to the subalgebra of all upper-triangular matrices in $\mathbf{M}_2(\mathbb{R})$. We first begin by showing that given a basis $\{I, A, B\}$ of a three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$, A and B have a real eigenvector in common.

Lemma 12 *Let $\mathbb{A}$ be a three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, A, B\}$. Then A and B have a real eigenvector in common.*

Proof. Write

$$AB = c_1I + c_2A + \lambda_AB$$

and

$$BA = d_1 I + d_2 B + \lambda_B A.$$

for unique constants $c_1, c_2, d_1, d_2, \lambda_A, \lambda_B \in \mathbb{R}$. Let v_A be an eigenvector associated with λ_A and let v_B be an eigenvector associated with λ_B. Then from $B(Av_A) = (BA)v_A$ we have that

$$B(\lambda_A v_A) = (d_1 I + d_2 B + \lambda_B A)v_A.$$

So

$$\lambda_A B v_A = d_1 v_A + d_2 B v_A + \lambda_B \lambda_A v_A.$$

That is,

$$(\lambda_A - d_2)Bv_A = (d_1 + \lambda_A \lambda_B)v_A.$$

Then from $A(Bv_B) = (AB)v_B$ we have that

$$A(\lambda_B v_B) = (c_1 I + c_2 A + \lambda_A B)v_B.$$

So

$$\lambda_B A v_B = c_1 v_B + c_2 A v_B + \lambda_A \lambda_B v_B.$$

That is,

$$(\lambda_B - c_2)Av_B = (c_1 + \lambda_A \lambda_B)v_B.$$

Altogether we have the following equations:

- $(\lambda_A - d_2)Bv_A = (d_1 + \lambda_A \lambda_B)v_A$
- $(\lambda_B - c_2)Av_B = (c_1 + \lambda_A \lambda_B)v_B$.

If $\lambda_A \neq d_2$ then the eigenvector v_A of A is an eigenvector of B. If $\lambda_B \neq c_2$ then the eigenvector v_B of B is an eigenvector of A. Now if $\lambda_A = d_2$ and $\lambda_B = c_2$ then we have that

$$AB = c_1 I + \lambda_B A + \lambda_A B$$

and

$$BA = d_1 I + \lambda_A B + \lambda_B A.$$

So

$$AB - BA = c_1 I - d_1 I.$$

Then

$$0 = tr(AB - BA) = tr(c_1 I - d_1 I) = 2(c_1 - d_1).$$

So $c_1 = d_1$ and we get that $AB = BA$ and therefore A and B have an eigenvector in common[2]. We now conclude that A and B have a real eigenvector in common. $\square$

We now show that any three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ is given by a basis $\{I, A, B\}$ such that A and B are simultaneously triangularizable.

Theorem 12 *Let $\mathbb{A}$ be a three-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$ given by the basis $\{I, A, B\}$. Then A and B are simultaneously triangularizable.*

Proof. We construct a nonsingular matrix $S \in \mathbf{M}_2(\mathbb{R})$ such that $S^{-1}AS$ and $S^{-1}BS$ are both triangular. Let v_1 be a vector of unit length in $\mathbb{R}^2$ such that

$$Av_1 = \lambda_A v_1 \text{ and } Bv_1 = \lambda_B v_1.$$

Complete $\{v_1\}$ to a basis of $\mathbb{R}^2$, call it $\{v_1, w_2\}$. Then use the Gram-Schmidt process in $\mathbb{R}^2$ with the standard inner product to construct an orthonormal basis. As v_1 is already of unit length we leave it alone. Then let $v_2 = \frac{w_2 - (w_2, v_1)v_1}{|w_2 - (w_2, v_1)v_1|}$. So we have an orthonormal basis $\{v_1, v_2\}$. Now let S be the matrix whose j^{th} column is v_j. Then $S^{-1} = S^T$ since

$$\begin{aligned} S^T S &= \begin{pmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_2, v_1) & (v_2, v_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= I. \end{aligned}$$

[2]See [2], Theorem 3.9.

Then

$$S^{-1}AS = S^{-1}\begin{pmatrix}\lambda_A v_1 & Av_2\end{pmatrix}$$
$$= \begin{pmatrix}\lambda_A(v_1, v_1) & v_1 A v_2 \\ \lambda_A(v_2, v_1) & v_2 A v_2\end{pmatrix}$$
$$= \begin{pmatrix}\lambda_A & v_1 A v_2 \\ 0 & v_2 A v_2\end{pmatrix}.$$

Similarly,

$$S^{-1}BS = S^{-1}\begin{pmatrix}\lambda_B v_1 & Bv_2\end{pmatrix}$$
$$= \begin{pmatrix}\lambda_B(v_1, v_1) & v_1 B v_2 \\ \lambda_B(v_2, v_1) & v_2 B v_2\end{pmatrix}$$
$$= \begin{pmatrix}\lambda_B & v_1 B v_2 \\ 0 & v_2 B v_2\end{pmatrix}.$$

Therefore $S^{-1}AS$ and $S^{-1}BS$ are both upper-triangular and we are done. □

Finally as a corollary to Theorem 12 we have that all three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ are isomorphic.

Corollary 5 *All three-dimensional subalgebras of* $\mathbf{M}_2(\mathbb{R})$ *are isomorphic to the subalgebra of all upper-triangular matrices in* $\mathbf{M}_2(\mathbb{R})$.

Proof. Let $\{I, A, B\}$ be a basis. Let S be a nonsingular matrix such that $S^{-1}AS$ and $S^{-1}BS$ are real upper-triangular matrices. Then an isomorphism is given by $f\colon M \mapsto S^{-1}MS$. □

Chapter 6

Conclusion

6.1 Significance of the Result

Now that we have characterized a great deal of subalgebras of $\mathbf{M}_2(\mathbb{R})$, what can be taken away from our venture? The quadratic nature of the characteristic polynomial of a 2×2 matrix in conjunction with the Cayley-Hamilton Theorem was particularly useful for most of the work. The notion of eigenvalues and eigenvectors played a significant role in the realization that all three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ are isomorphic.

6.2 Future Work

Naturally, we are left to wonder what can be said about subalgebras of $\mathbf{M}_n(\mathbb{R})$. Perhaps there are analogous results for other n's different from 2. Being able to characterize subalgebras of $\mathbf{M}_n(\mathbb{R})$, given any n, would certainly be a particularly enthralling enterprise.

References

[1] W. A. Adkins and S. H. Weintraub, *Algebra: An Approach via Module Theory*, Springer-Verlag, New York, 1992.

[2] M. Marcus and H. Minc, *Introduction to Linear Algebra*, Dover Publications, New York, 1988.

Curriculum Vitae

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