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Bayesian Analysis Of Variable-Stress Accelerated Life Testing

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BAYESIAN ANALYSIS OF VARIABLE-STRESS ACCELERATED LIFE TESTING

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to my

MOTHER and FATHER

with love

BAYESIAN ANALYSIS OF VARIABLE-STRESS ACCELERATED LIFE TESTING

by

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THESIS

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Abstract

Several authors have over the years studied the art of modeling data from accelerated life testing and making inferences from such data. In this study, we consider a continuously varying stress accelerated life testing procedure which is the limiting case of the multiple stress-level discussed by Doksum and Hóyland [1]. We derive the likelihood function for the life distribution of the continuously increasing stress accelerated life testing model and consequently the Fisher's Information Matrix. We propose a Bayesian analysis for this distribution using the Gibbs Sampling Procedure. We conduct simulation studies and real data analysis to demonstrate the efficiency of the proposed Bayesian approach to parameter estimation over that of maximum likelihood estimation procedure.

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Chapter 1

Introduction

In this chapter, we provide a brief introduction on this study. Accelerated Life Testing (ALT) is widely used in the field of engineering and this study applies Bayesian analysis to make inference on the data collected on these life testing procedures. We will present some basic concepts related to Accelerated Life Testing (ALT) methods in this chapter. We will also present the objectives of this study and finally provide the outline of the entire thesis.

1.1 Background of Study

Reliability Life Data Analysis involves the study and modeling of observed product lives. For some products such as car tire, airplane parts, and telephone cables, obtaining times-to-failure data may be very difficult. This difficulty stems from the fact that, these products have median life of 15 years or more. With this, it is generally very expensive and impractical to complete reliability testing under normal conditions. Reliability practitioners have over the years devised methods and strategies to coerce these products to fail faster than they would under normal use conditions. In other words, they have accelerated the failure times of these products.

1.2 Accelerated Life Testing

Accelerated life testing is a process to shorten the testing period by subjecting the products to more severe conditions. These accelerated life testing conditions—which are mostly referred to as the "Stress variable"—include higher than usual temperature, voltage, pressure, humidity, vibration or any other stress that directly affects the life of the product(Dorp and Mazzuchi [13]). A model is then used to extrapolate the data obtained at these accelerated conditions to the normal use conditions to obtain an estimate of the life distribution under normal conditions. See Bhattacharyya(1987) [11] for an overview of the methodological approaches to ALT analysis. There are mainly three types of accelerated life testing designs. We have the constant stress accelerated life testing, step-stress accelerated life testing, and Progressive stress accelerated life testing.

If the stress is time-independent, the test units are often put under a constant stress. With this, the test units are run to failure at same stress level throughout the whole experiment. A model for the relationship between the life of the units and the constant stress is then fitted to the data. Then the relationship is extrapolated to estimate the life distribution under normal stress. The constant stress is illustrated in figure 1.1.

Step stress accelerated life testing(SSALT) may be used to reduce the times to failure still further, when constant-stress testing is considered too lengthy. With this, test units are initially subjected to a specified stress level. The remaining units that do not fail in a specified time continues to test at a higher stress level for another specified time. The stress is repeatedly increased and held this way until the end of the experiment. This is illustrated in figure 1.2

1.3 Objectives of Study

- Modeling variable stress using Bayesian inference.
- We compare the Bayesian approach to parameter estimation to that of MLE.

1.4 Outline of Thesis

The organization of the remaining parts of the thesis is as follows: Chapter two provides a literature review on some works already done on accelerated life tests. Chapter three

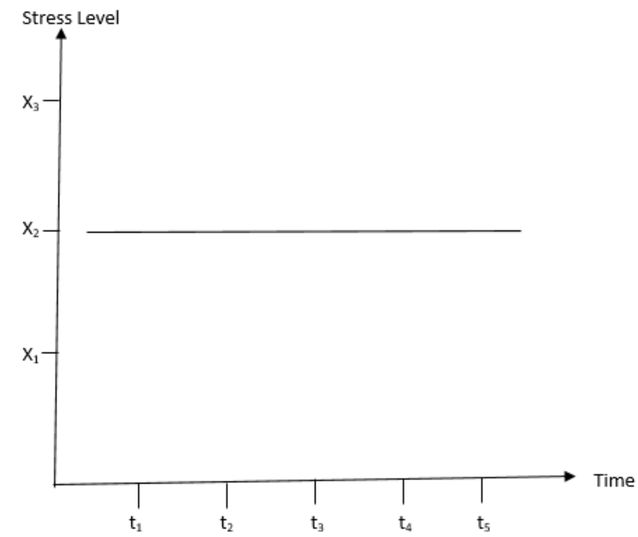

Figure 1.1: Constant stress accelerated life testing

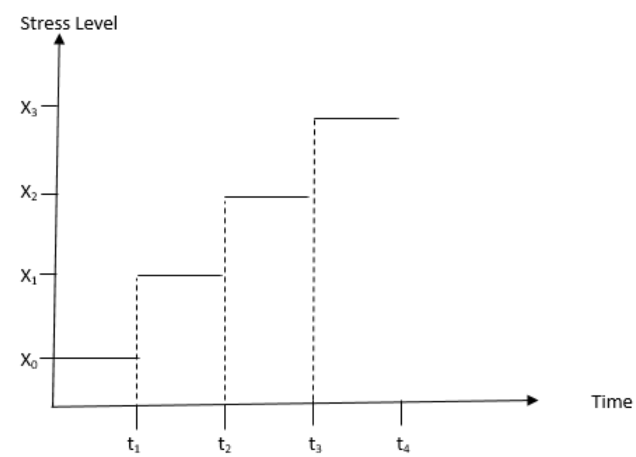

Figure 1.2: Step stress accelerated life testing

gives the methodology which comprises accelerated life test procedures, Frequentist and Bayesian methods of parameter estimation. Chapter four outlines the simulation studies and the results from the estimates of parameters using both Frequentist and Bayesian methods of parameter estimation.

Chapter 2

Literature Review

2.0 Introduction

Many authors have studied the art of modeling data from accelerated life testing and making inferences from such data. Tyoskin and Krivolapov [2] outlined a nonparameteric approach for making inferences for a step-stress accelerated life testing data. Bhattacharyya and Soejoeti (1989) [12]considered a two-step or simple SSALT where units are simultaneously put to a stress setting x_1. The experimenter records the failure times of units that fail in the specified time interval $[0, t]$. Starting at time t, the surviving units are subjected to a different (typically higher) stress setting x_2 and observed until they all fail.

The existing literature on analysis of SSALT centers around three types of models: The tampered random variable(TRV) model due to DeGroot and Goel(1979) [14], the cumulative exposure(CE) model due to Nelson(1980) [3] and the tampered failure rate(TFR) model due to Bhattacharyya and Soejoeti(1989) [12]. Nelson [3, 4, 5], Miller and Nelson [6], and Yin and Sheng [7] used the cumulative exposure model to model data from Step-stress accelerated life testing, SSALT. Miller and Nelson[6] introduced a simple step-stress accelerated life test plan in an exponential cumulative exposure model. Xiong and Milliken [8] considered the lifetime distribution for a Step-stress accelerated life testing, SSALT when the stress is changed according to some distribution. Thus, instead of increasing the stress at a pre-specified time, the stress is increased immediately after a certain number of test units fail. They studied an exponential cumulative exposure model with a threshold parameter in the simple step-stress accelerated life test. Khamis and Higgins [9] introduced the optimum three-step step-stress accelerated life test plan using quadratic stress-life relationship assuming that the failure time follows an exponential distribution.

Chapter 3

Methodology

3.0 Introduction

This chapter gives the details of the methodology used in analysis of this study. Here, we discuss the Wiener Process and the Inverse Gaussian Distribution, derive the likelihood function for the continuously increasing stress accelerated life testing and consequently the Fisher's Information Matrix. We also outline prior and posterior distribution and the Gibbs Sampling Procedure.

3.1 Variable Stress Accelerated Life Testing

Variable-stress accelerated life testing(VALT) trials are experiments in which each of the units in a random sample of units of a product is run under increasingly severe conditions to get information quickly on its life distribution.

Starting at t_0, test all products at stress level x_0 until t_1. At t_1, the remaining products continue to test at a more severe condition x_1 until t_1, and so on. See figure 1.2. Doksum and Hóyland [1] modeled time to failure in terms of accumulated decay reaching a critical level, ω. They considered a model in which accumulated decay is governed by a continuous Gaussian process $W(y)$ whose distribution depends on the stress $s(y)$ assigned to the experimental unit at each time point y.

3.2 The Wiener Process and the Inverse Gaussian Distribution

The basic idea is to model the accumulated decay in material subject to constant stress as a Wiener process $\{W_0(y); y \geq 0\}$ with drift $\eta > 0$ and diffusion constant $\delta^2 > 0$. The Wiener process $W_0(y)$ is defined to be an independent increment Gaussian process with $W_0(0) = 0$ and mean $E(W_0(y)) = \eta y$. Failure occurs when the decay process $W_0(y)$ crosses a critical boundary ω.

Given the first stress level, let fatigue failure time, Y be the first time the decay process $W_0(y)$ crosses the critical boundary ω, and let $\mu = \frac{\omega}{\eta}$ *and* $\lambda = \frac{\omega^2}{\delta^2}$; Then Y has has the inverse Gaussian distribution $IG(y|\mu, \lambda)$. See history and proof in the book by Chhikara and Folks(1989) [10]. The pdf of Y is given by

$f_0(y) = \sqrt{\frac{\lambda}{2\pi y^3}} \exp\left\{\frac{-\lambda(y-\mu)^2}{2\mu^2 y}\right\}, \quad y, \mu, \lambda > 0$

3.3 Multiple Step Stress

For $k+1$ stress levels $x_0, x_1, .., x_k$ over the $k+1$ intervals $[0, t_1), [t_1, t_2), ..., [t_k, \infty)$. (see figure 1.2)

Accumulated decay is modeled as the Gaussian process

$$\begin{aligned} W(y) &= W_0(y), \quad y \in [0, t_1) \\ &= W_i(y), \quad y \in [t_i, t_{i+1}), i = 1, ..., k \end{aligned} \tag{3.1}$$

Where $W_i(y) = W_{i-1}(t_i + \alpha_i[y - t_i]), \quad y \in [t_i, t_{i+1}), i = 1, ..., k$.

Here,

$$\begin{aligned} \tau(y) &= y, \quad y \in [0, t_1) \\ &= t_i + \alpha_i(y - t_i), \quad y \in [t_i, t_{i+1}), i = 1, ..., k \end{aligned} \tag{3.2}$$

$\tau(y)$ is a time transformation that converts the accelerated time to a non-accelerated or normal condition stress level time. Thus, $W(y)$ is a continuous Gaussian process with decay rate changing by the multiplicative factor α_i as y crosses the stress change point t_i. α_i is the decay effect of the increasing stress level from x_{i-1} to x_i as time y crosses the stress change point t_i. Let β_i also be the cumulative effect of increasing the stress from x_0 to x_1 and so on to x_i as time y crosses the stress change points $t_1, t_2, ..., t_i$. To re-express $W(y)$ in terms of $W_0(y)$, Doksum and Hóyland [1] introduced the notation

$$\begin{aligned}
\beta_i &= \prod_{j=1}^{i} \alpha_r, \quad \text{ where } \quad \alpha_0 = 1, \\
\beta(y) &= \beta_i, \quad y \in [t_i, t_{i+1}), i = 0, ..., k \\
\tau(y) &= y, \quad 0 \leq y < t \\
&= \sum_{r=0}^{i-1} \beta_r (t_{r+1} - t_r) + \beta_i (y - t_i), \\
& \quad y \in [t_i, t_{i+1}), i = 0, ..., k.
\end{aligned} \tag{3.3}$$

Just like in many applications, we assume $\alpha_i = 1 + \theta(x_i - x_{i-1})$, as the effect of the increasing stress level from x_{i-1} to x_i. With this $\beta_i = \prod_{j=1}^{i} \alpha_j$.
Thus,

$$\begin{aligned}
\beta_m &= \prod_{j=1}^{m} [1 + \theta(x_j - x_{j-1})] \\
&= 1 + \sum_{i} \theta(x_i - x_{i-1}) + \sum_{i,j} \theta^2 (x_i - x_{i-1})(x_j - x_{j-1}) + \ldots \\
&+ \theta^m (x_1 - x_0)(x_2 - x_1)(x_3 - x_2)(x_4 - x_3)...(x_m - x_{m-1})
\end{aligned} \tag{3.4}$$

Doksum and Hóyland [1] proposed that, for $k+1$- stress -levels case, the distribution of the failure time Y is $F(y) = F_0(\tau(y)|\mu, \lambda)$, where F_0 is the cdf of $IG(\mu, \lambda)$ life distribution at x_o.

Thus, in the continuous case, as the difference $(t_i - t_{i-1}) \to 0 \implies (x_i - x_{i-1}) \to 0$, $\beta_i \approx 1 + \theta(x_i - x_0)$. This is illustrated in figure 3.1

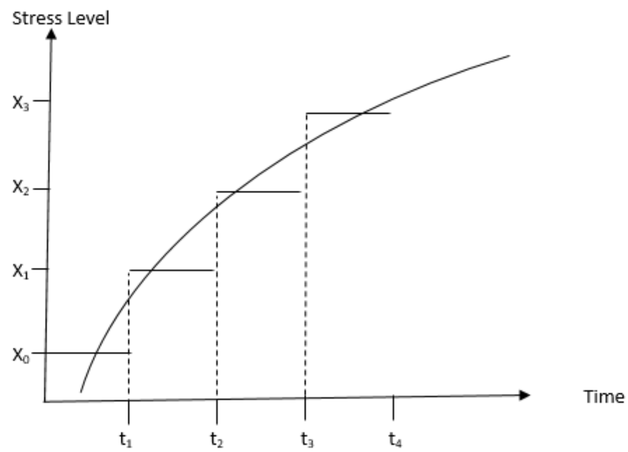

Figure 3.1: Continuously varying stress accelerated life testing

Suppose the experimenter imposes the known stress $x(y)$ at the time y. Then a reasonable model for the decay rate $\eta\beta(y)$ of the decay process $W(y)$ at time y would have

$\beta(y) = 1 + \theta(x(y) - x(0))$ where $x(0) = x_0$

It follows that

$$\tau(y) = \int_0^y [1 + \theta(x(s) - x(0))]ds \tag{3.5}$$

Nilsson and Uvell (1985) [15], assumed $x(y)$ to be linear, say $x(y) = x_0 + cy$ where c is a known constant set by the experimenter.

Now,

$$\beta(y) = 1 + \theta(x_0 + cy - x_0) = 1 + \theta y \tag{3.6}$$

where the constant c has been absorbed into θ .

Now equation (3.5) becomes

$$\tau(y) = \int_0^y (1 + \theta s)ds = y + \frac{1}{2}\theta y^2 \tag{3.7}$$

3.4 Likelihood Function

Doksum and Hóyland [1] defined the distribution of the failure time Y as $F(y) = F_0(\tau(y)|\mu, \lambda)$, for some non-negative increasing continuous function $\tau(y)$ with the property that $\tau(0) = 0$. Suppose $\tau(y)$ is differentiable, we have $\tau'(y) = \beta(y)$. Thus $f(y) = \beta(y) f_0(\tau(y))$. The likelihood and log-likelihood functions (for $y, \mu, \lambda > 0$) are given by:

$$L(\mu, \lambda, \theta) = \prod_{i=1}^{n} \beta(y_i) f_0(\tau(y_i)) = \prod_{i=1}^{n} (1 + \theta y_i) \sqrt{\frac{\lambda}{2\pi[\tau_\theta(y_i)]^3}} \exp\left\{ \frac{-\lambda[\tau_\theta(y_i) - \mu]^2}{2\mu^2 \tau_\theta(y_i)} \right\} \quad (3.8)$$

$$\ell(\mu, \lambda, \theta) = \sum_{i=1}^{n} \log(1+\theta y_i) + \frac{n}{2} \log \lambda - \frac{n}{2} \log 2\pi - \frac{3}{2} \sum_{i=1}^{n} \log \tau_\theta(y_i) - \sum_{i=1}^{n} \frac{\lambda[\tau_\theta(y_i) - \mu]^2}{2\mu^2 \tau_\theta(y_i)} \quad (3.9)$$

The partial derivatives with respect to the parameters are as follows:

$$\frac{\partial \ell}{\partial \mu} = \sum_{i=1}^{n} \frac{\lambda[\tau_\theta(y_i) - \mu]}{\mu^3} \quad , \quad \frac{\partial^2 \ell}{\partial \mu^2} = -\sum_{i=1}^{n} \frac{\lambda}{\mu^4} [3\tau_\theta(y_i) - 2\mu] \quad (3.10)$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{2\lambda} - \sum_{i=1}^{n} \frac{[\tau_\theta(y_i) - \mu]^2}{2\mu^2 \tau_\theta(y_i)} \quad , \quad \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{2\lambda^2} \quad (3.11)$$

$$\frac{\partial^2 \ell}{\partial \mu \partial \lambda} = \sum_{i=1}^{n} \frac{1}{\mu^3} [\tau_\theta(y_i) - \mu] \ , \ \frac{\partial^2 \ell}{\partial \lambda \partial \theta} = -\sum_{i=1}^{n} \frac{y_i^2 [\tau_\theta^2(y_i) - \mu^2]}{4\mu^2 \tau_\theta^2(y_i)} \ , \ \frac{\partial^2 \ell}{\partial \mu \partial \theta} = \frac{\lambda}{2\mu^3} \sum_{i=1}^{n} y_i^2 \quad (3.12)$$

From (3.7), $\tau_\theta(y) = y + \frac{1}{2}\theta y^2$, $\frac{\partial \tau_\theta(y)}{\partial \theta} = \frac{1}{2} y^2$, $\frac{\partial^2 \tau_\theta(y)}{\partial \theta^2} = 0$, and thus

$$\frac{\partial \ell}{\partial \theta} = \sum_{i=1}^{n} \frac{y_i}{1 + \theta y_i} - \frac{3}{4} \sum_{i=1}^{n} \frac{y_i^2}{\tau_\theta(y_i)} - \frac{\lambda}{4\mu^2} \sum_{i=1}^{n} \frac{y_i^2 [\tau_\theta^2(y_i) - \mu^2]}{\tau_\theta^2(y_i)} \quad (3.13)$$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\sum_{i=1}^{n} \frac{y_i^2}{(1 + \theta y_i)^2} + \frac{3}{8} \sum_{i=1}^{n} \frac{y_i^4}{\tau_\theta^2(y_i)} - \frac{\lambda}{4} \sum_{i=1}^{n} \frac{y_i^4}{\tau_\theta^3(y_i)} \quad (3.14)$$

$$E\left(\frac{\partial^2 \ell}{\partial \mu^2}\right) = -\frac{\lambda}{\mu^3} n \quad , \quad E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) = -\frac{1}{2\lambda^2} n \quad , \quad E\left(\frac{\partial^2 \ell}{\partial \mu \partial \lambda}\right) = 0 \quad (3.15)$$

3.5 Fisher's Information Matrix

Let the Fisher's Information matrix be given by:

$$I(\mu, \lambda, \theta) = \begin{pmatrix} w_{\mu\mu} & w_{\mu\lambda} & w_{\mu\theta} \\ w_{\lambda\mu} & w_{\lambda\lambda} & w_{\lambda\theta} \\ w_{\theta\mu} & w_{\theta\lambda} & w_{\theta\theta} \end{pmatrix} \tag{3.16}$$

Where $w_{\mu\mu} = -E\left(\frac{\partial^2 \ell}{\partial \mu^2}\right), w_{\lambda\lambda} = -E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right), etc.$

The determinant of $I(\mu, \lambda, \theta)$ is given by:

$det(I) = w_{\mu\mu}(w_{\lambda\lambda}w_{\theta\theta} - w_{\lambda\theta}^2) - w_{\mu\theta}^2 w_{\lambda\lambda}$, since $w_{\lambda\mu} = w_{\mu\lambda} = -E\left(\frac{\partial^2 \ell}{\partial \mu \partial \lambda}\right) = 0$

Thus,

$$I^{-1}(\mu, \lambda, \theta) = \frac{1}{c}\begin{pmatrix} (w_{\lambda\lambda}w_{\theta\theta} - w_{\lambda\theta}^2) & w_{\mu\theta}w_{\theta\lambda} & -w_{\mu\theta}w_{\lambda\lambda} \\ w_{\mu\theta}w_{\theta\lambda} & (w_{\mu\mu}w_{\theta\theta} - w_{\mu\theta}^2) & -w_{\mu\mu}w_{\lambda\theta} \\ -w_{\mu\theta}w_{\lambda\lambda} & -w_{\mu\mu}w_{\lambda\theta} & w_{\mu\mu}w_{\lambda\lambda} \end{pmatrix} \tag{3.17}$$

where $c = det(I) = w_{\mu\mu}(w_{\lambda\lambda}w_{\theta\theta} - w_{\lambda\theta}^2) - w_{\mu\theta}^2 w_{\lambda\lambda}$

3.6 Confidence Intervals

From the asymptotic normality of MLEs, $\hat{\theta} \sim N(\theta, var(\hat{\theta}))$ We use log transformation to obtain approximate confidence intervals, CIs, and the average length(AL) of these intervals for the parameters. For example, parameter θ and its MLE $\hat{\theta}$ will have the approximate normal distribution $\log(\hat{\theta}) \sim N(\log(\theta), var(\log(\hat{\theta})))$. We approximate the variance using the delta method as $\widehat{var(\log(\hat{\theta}))} = \frac{\widehat{var(\hat{\theta})}}{\hat{\theta}^2}$, where $\widehat{var(\hat{\theta})}$ is the corresponding diagonal element of the inverse Fisher's Information matrix, evaluated at MLEs $\hat{\mu}, \hat{\lambda}, \hat{\theta}$.

With this, a $(1-\gamma)100\%$ CI for θ is given by:

$$\left[\hat{\theta} \times \exp\left\{-\frac{z_{\gamma/2}\sqrt{var(\hat{\theta})}}{\hat{\theta}}\right\}, \hat{\theta} \times \exp\left\{\frac{z_{\gamma/2}\sqrt{var(\hat{\theta})}}{\hat{\theta}}\right\}\right] \tag{3.18}$$

where $z_{\gamma/2}$ is the upper $100 \times \gamma/2^{th}$ percentile of the standard normal distribution.

3.7 Bayesian Inference

3.7.1 Prior Distribution

Doksum(1992) [1] proposed that, if we let Y be the first time the process $W_0(\tau(y))$ crosses the critical boundary ω, and let $\mu = \frac{\omega}{\eta}$ *and* $\lambda = \frac{\omega^2}{\delta^2}$; Then Y has distribution $F(y) = F_0(\tau(y)|\mu, \lambda)$, where η is the drift and δ^2 is the diffusion constant of the wiener process. With this, we see that

$$\lambda = \frac{(\eta\mu)^2}{\delta^2} \Longrightarrow \lambda \propto \mu^2 \tag{3.19}$$

Therefore, we propose a joint prior

$$\pi(\mu, \lambda, \theta) = \pi(\mu)\pi(\lambda|\mu)\pi(\theta) \tag{3.20}$$

From the functional form of likelihood in (3.8), it can be deduced that for λ, a conjugate prior for the conditional likelihood $L(\lambda|\mu, \theta, Y)$ is a gamma distribution. With this, we choose $(\lambda|\mu) \sim Gamma(\frac{b_0}{2}, \frac{b_1}{2\mu^2})$, which gives us the prior density of $(\lambda|\mu)$ as:

$$\pi(\lambda|\mu) \propto \lambda^{\frac{b_0}{2}-1} e^{-\frac{b_1}{2\mu^2}\lambda} \tag{3.21}$$

Thus, $E(\lambda|\mu) = \frac{b_0}{b_1}\mu^2$. From the likelihood in (3.8), we see that μ *and* θ have no conjugate priors. We consider prior distributions that updated certain portions of their conditional likelihoods. Here, we choose

$\mu \sim$ Inverse Gamma(a_0, a_1) and $\theta \sim$ Gamma(c_0, c_1)

Thus, $\pi(\mu) \propto \mu^{-(a_0+1)} e^{-\frac{a_1}{\mu}}$ and $\pi(\theta) \propto \theta^{c_0-1} e^{-c_1\theta}$

3.7.2 Hyperparameter Settings

To determine the hyperparameters, we use the MLEs $\hat{\mu}, \hat{\lambda}$ and $\hat{\theta}$ as estimates for the expectations of their respective prior distributions.

3.7.3 Posterior Distribution

Given a sample data Y, the joint posterior distribution of parameters (μ, λ, θ) is given by:

$$\pi(\mu, \lambda, \theta|Y) \propto L(\mu, \lambda, \theta|Y)\pi(\mu)\pi(\lambda|\mu)\pi(\theta) \tag{3.22}$$

It follows that the full conditional posteriors are:

$$\begin{aligned}
\pi(\lambda|\mu, \theta, Y) &\propto L(\lambda|\mu, \theta, Y)\pi(\lambda|\mu) \\
&\propto \lambda^{\frac{n}{2}} \exp\left\{ - \frac{\lambda}{2\mu^2}\big[\sum_{i=1}^{n} \tau_\theta(y_i) - 2n\mu + \mu^2 \sum_{i=1}^{n} \tau_\theta^{-1}(y_i)\big]\right\} \times \lambda^{\frac{b_0}{2}-1} \exp\{-\frac{b_1}{2\mu^2}\lambda\} \\
&\propto \lambda^{\frac{n+b_0}{2}-1} \exp\left\{ - \frac{\lambda}{2\mu^2}\big[\sum_{i=1}^{n} \tau_\theta(y_i) - 2n\mu + \mu^2 \sum_{i=1}^{n} \tau_\theta^{-1}(y_i) + b_1\big]\right\}
\end{aligned} \tag{3.23}$$

Thus,

$$(\lambda|\mu, \theta, Y) \sim \text{ Gamma } \left(\frac{\eta_0}{2}, \frac{\eta_1}{2\mu^2}\right) \tag{3.24}$$

Where $\eta_0 = n + b_0$ *and* $\eta_1 = \sum_{i=1}^{n} \tau_\theta(y_i) - 2n\mu + \mu^2 \sum_{i=1}^{n} \tau_\theta^{-1}(y_i) + b_1$

$$\begin{aligned}
\pi(\mu|\lambda, \theta, Y) &\propto L(\mu|\lambda, \theta, Y)\pi(\mu)\pi(\lambda|\mu) \\
&\propto \exp\left\{ - \frac{\lambda}{2\mu^2}\big[\sum_{i=1}^{n} \tau_\theta(y_i) - 2n\mu + \mu^2 \sum_{i=1}^{n} \tau_\theta^{-1}(y_i)\big]\right\} \times \mu^{-(a_0+1)} \exp\{-\frac{a_1}{\mu}\} \\
&\quad \times \frac{(\frac{b_1}{2\mu^2})^{(\frac{b_0}{2})}}{\Gamma(\frac{b_0}{2})} \lambda^{\frac{b_0}{2}-1} \exp\{-\frac{b_1}{2\mu^2}\lambda\} \\
&\propto \mu^{-(a_0+b_0+1)} \exp\left\{ - \frac{\lambda}{2\mu^2}\big[\sum_{i=1}^{n} \tau_\theta(y_i) - 2n\mu + b_1\big] - \frac{a_1}{\mu}\right\}
\end{aligned} \tag{3.25}$$

$$
\begin{aligned}
\pi(\theta|\lambda,\mu,Y) &\propto L(\theta|\lambda,\mu,Y)\pi(\theta) \\
&\propto \prod_{i=1}^{n}(1+\theta y_i)\sqrt{\frac{1}{\tau_\theta^3(y_i)}}\exp\left\{\frac{-\lambda[\tau_\theta(y_i)-\mu]^2}{2\mu^2\tau_\theta(y_i)}\right\}\times\theta^{c_0-1}\exp\{-c_1\theta\} \\
&\propto \prod_{i=1}^{n}\left[(1+\theta y_i)\tau_\theta^{-\frac{3}{2}}(y_i)\right]\theta^{c_0-1}\exp\left\{-\frac{\lambda}{2\mu^2}\Big[\sum_{i=1}^{n}\tau_\theta(y_i)+\mu^2\sum_{i=1}^{n}\tau_\theta^{-1}(y_i)\Big]-c_1\theta\right\}
\end{aligned} \tag{3.26}
$$

3.7.4 Gibbs Sampling Procedure

To make posterior inference on the parameters $\mu, \lambda,$ and θ, we take posterior samples from their full conditional distributions.To do this, we implement the Gibbs sampling procedure as follows:

1. Sample μ_{t+1} from $\pi(\mu|\lambda_t,\theta_t,Y)$ using the Metropolis-Hastings algorithm. Here, our target density is $\pi(\mu|\lambda,\theta,Y)$. Let μ_t be a current value, and $q(\mu|\mu_t)$ be a proposal distribution. Thus

   - Sample $\mu^* \sim q(\mu|\mu_t)$. Where $q(\mu|\mu_t)$ is Inverse Gamma$(\alpha_\mu, \mu_t(\alpha_\mu - 1))$
   - We calculate the acceptance probability as

     $$\alpha(\mu_t,\mu^*) = \min\left\{1, \frac{\pi(\mu^*|\lambda_t,\theta_t,Y)}{\pi(\mu_t|\lambda_t,\theta_t,Y)}\frac{q(\mu_t|\mu^*)}{q(\mu^*|\mu_t)}\right\}$$

     Where $q(\mu_t|\mu^*)$ is Inverse Gamma$(\alpha_\mu, \mu^*(\alpha_\mu - 1))$ and $q(\mu^*|\mu_t)$ is Inverse Gamma$(\alpha_\mu, \mu_t(\alpha_\mu - 1))$
   - We set $\mu_{t+1} = \mu^*$ with probability $\alpha(\mu_t,\mu^*)$, otherwise, we set $\mu_{t+1} = \mu_t$

2. Sample $\lambda_{t+1}|\mu_{t+1},\theta_t \sim$ Gamma $\left(\frac{\eta_0}{2}, \frac{\eta_1}{2\mu_{t+1}^2}\right)$

3. Sample θ_{t+1} from $\pi(\theta|\mu_{t+1},\lambda_{t+1},Y)$ using the Metropolis-Hastings algorithm. We let $\pi(\theta|\mu,\lambda,Y)$ be our target density, θ_t be a current value, and $q(\theta|\theta_t)$ be a proposal distribution. Thus

- Sample $\theta^* \sim q(\theta|\theta_t)$. Where $q(\theta|\theta_t)$ is Gamma$(\theta_t\beta_\theta, \beta_\theta)$
- Calculate the acceptance probability as

$$\alpha(\theta_t, \theta^*) = \min\left\{1, \frac{\pi(\theta^*|\mu_{t+1}, \lambda_{t+1}, Y)}{\pi(\theta_t|\mu_{t+1}, \lambda_{t+1}, Y)} \frac{q(\theta_t|\theta^*)}{q(\theta^*|\theta_t)}\right\}$$

  Where $q(\theta_t|\theta^*)$ is Gamma$(\theta^*\beta_\theta, \beta_\theta)$ and $q(\theta^*|\theta_t)$ is Gamma$(\theta_t\beta_\theta, \beta_\theta)$

- Set $\theta_{t+1} = \theta^*$ with probability $\alpha(\theta_t, \theta^*)$, otherwise, we set $\theta_{t+1} = \theta_t$

Chapter 4

Simulation and Real Data Application

4.1 Simulation Study

To compare the performance of parameter estimation by ML and Bayesian methods, we conduct a simulation study. We take four settings of the parameters as $(\mu, \lambda, \theta) = (1, 1, 1), (0.5, 0.5, 1), (1, 0.5, 1.5), (3, 0.2, 1.5)$. We then generate 1000 datasets for each of these parameter settings with three sample sizes $n = 20, 30, 50$. For the Bayesian analysis, we run the gibbs sampling procedure with a burn-in of 2000 followed by 8000 iterations. The remaining 8000 samples are used to compute the parameter estimates, Mean square error(MSE), and average lengths(AL) of the 95% credible intervals(CI). We also compute maximum likelihood estimates, Mean square error(MSE), and average lengths(AL) of the 95% confidence intervals(CI) using frequentist approach. Table 4.1 and 4.2 display the results from the simulation study.

4.1.1 Remarks from simulation study.

- Just as expected, the estimates get closer to their true parameters as the sample size increases for both ML and Bayesian method.

- The MSE and AL decrease as sample sizes increase. This is also seen for both ML and Bayesian method.

- The estimation of all parameters from the Bayesian method is much better than from

Table 4.1: Results from Simulation Study -$(\mu, \lambda, \theta) = (1, 1, 1), (0.5, 0.5, 1)$

n	Para	ML Method			Bayesian		
		Estimate	MSE	AL	Estimate	MSE	AL
		True parameters: $\mu = 1, \lambda = 1, \theta = 1$					
20	μ	1.1444	0.3216	5.7743	1.0555	0.0677	0.6364
	λ	1.1589	0.2167	1.3831	1.0521	0.1018	0.9923
	θ	1.5138	2.9922	38.7825	1.1761	0.4780	0.0494
30	μ	1.0599	0.0985	4.3224	1.0244	0.0537	0.5286
	λ	1.1019	0.0956	1.1458	1.0377	0.0608	0.8239
	θ	1.2182	0.9518	28.9571	1.0583	0.3642	0.0465
50	μ	1.0369	0.0620	3.2827	1.0165	0.0466	0.4294
	λ	1.0634	0.0446	0.8681	1.0234	0.0320	0.6504
	θ	1.1492	0.5054	19.5924	1.0574	0.3400	0.0466
		True parameters: $\mu = 0.5, \lambda = 0.5, \theta = 1$					
20	μ	0.6271	0.0677	3.1111	0.5927	0.0291	0.3541
	λ	0.5592	0.0444	0.6607	0.5101	0.0221	0.4727
	θ	2.2627	6.2246	63.748	1.8657	1.7917	0.0621
30	μ	0.5899	0.0290	2.4069	0.5767	0.0228	0.298
	λ	0.5327	0.0191	0.55	0.5038	0.0131	0.3936
	θ	1.8230	2.0337	48.8791	1.6740	1.4230	0.0582
50	μ	0.5817	0.0240	1.8452	0.5723	0.0202	0.2458
	λ	0.5176	0.0095	0.4189	0.4984	0.0071	0.3123
	θ	1.7788	1.6956	33.6579	1.6692	1.3601	0.0584

the ML method in terms of Esimates, MSE and AL.

Table 4.2: Results from Simulation Study- for $(\mu, \lambda, \theta) = (1, 0.5, 1.5), (3, 0.2, 1.5)$

		ML Method			Bayesian		
n	Para	Estimate	MSE	AL	Estimate	MSE	AL
		True parameters: $\mu = 1, \lambda = 0.5, \theta = 1.5$					
20	μ	0.9616	0.2594	4.6081	0.8873	0.0713	0.6317
	λ	0.5745	0.0475	0.6696	0.5240	0.0223	0.5051
	θ	1.4460	1.7627	34.0673	1.2119	0.5059	0.0504
30	μ	0.9067	0.0818	3.8279	0.8846	0.0579	0.5575
	λ	0.5469	0.0219	0.5422	0.5160	0.0140	0.4177
	θ	1.2352	0.6092	26.6716	1.1319	0.4747	0.0487
50	μ	0.8455	0.0595	2.5778	0.8409	0.0587	0.4332
	λ	0.5281	0.0098	0.419	0.5091	0.0071	0.3272
	θ	1.0324	0.5506	18.4684	0.9862	0.5732	0.0451
		True parameters: $\mu = 3, \lambda = 0.2, \theta = 1.5$					
20	μ	2.2063	8.7398	10.2402	1.6871	3.0275	1.8883
	λ	0.2288	0.0065	0.2629	0.2130	0.0037	0.2138
	θ	1.1114	2.8959	17.6742	0.7809	1.0681	0.0386
30	μ	2.3001	5.9732	9.1405	1.9489	3.2603	2.1346
	λ	0.2201	0.0036	0.2133	0.2095	0.0024	0.175
	θ	1.1135	1.9362	13.5553	0.8734	1.1449	0.0406
50	μ	2.3205	4.8716	7.4968	2.0419	2.9737	2.0715
	λ	0.2136	0.0018	0.1644	0.2065	0.0013	0.1358
	θ	1.0723	1.5913	9.2299	0.8719	1.1001	0.0405

4.1.2 Trace Plots of Parameters from Simulated Data

The trace plots of our four settings of the parameters, $(\mu, \lambda, \theta) = (1, 1, 1), (0.5, 0.5, 1), (1, 0.5, 1.5), (3, 0.2, 1.5)$, from the posterior distribution of simulated data are shown in figures 4.1, 4.2, 4.3 and 4.5.

4.2 Real Data Analysis

In this study, we apply our analysis to a real data(Insulating fluid data), given by Nelson(2004) [16], to illustrate the usefulness of our method even further. This dataset is about the Oil breakdown time(seconds) in an accelerated test employed in an insulating oil. This data consists of 60 measured breakdown times(seconds).

4.2.1 Description of data

The data used in this study is the times to oil breakdown under high test voltages. Dataset is presented in Table 4.3. The higher than normal use voltages is to yield breakdown data quickly. At design voltages, time to breakdown runs thousands of years. The test to obtain this data employed a pair of parallel disk electrodes immersed in an insulating oil. The electrical stress is given as a voltage, since the electrode geometry was constant. Voltage, V across the pair was increased linearly with time, t at a specified rate R and breakdown time was recorded at a one square inch electrode. See Nelson(2004)[16].

Table 4.3: Insulating fluid data

3.4	3.4	3.4	3.5	3.5	3.5	3.6	3.8	3.8	3.8	3.8	3.9	3.9	3.9	4.0
4.0	4.0	4.0	4.1	4.1	4.1	4.1	4.1	4.1	4.2	4.2	4.2	4.2	4.2	4.3
4.3	4.3	4.3	4.3	4.4	4.4	4.4	4.4	4.4	4.4	4.4	4.5	4.5	4.6	4.6
4.6	4.6	4.6	4.7	4.7	4.7	4.7	4.7	4.8	4.9	4.9	4.9	5.0	5.1	5.2

Just like our simulation study, we run the gibbs sampling procedure with a burn-in of 2000 followed by 8000 iterations. The remaining 8000 samples are used to compute the parameter estimates, and average lengths(AL) of the 95% credible intervals(CI). We also

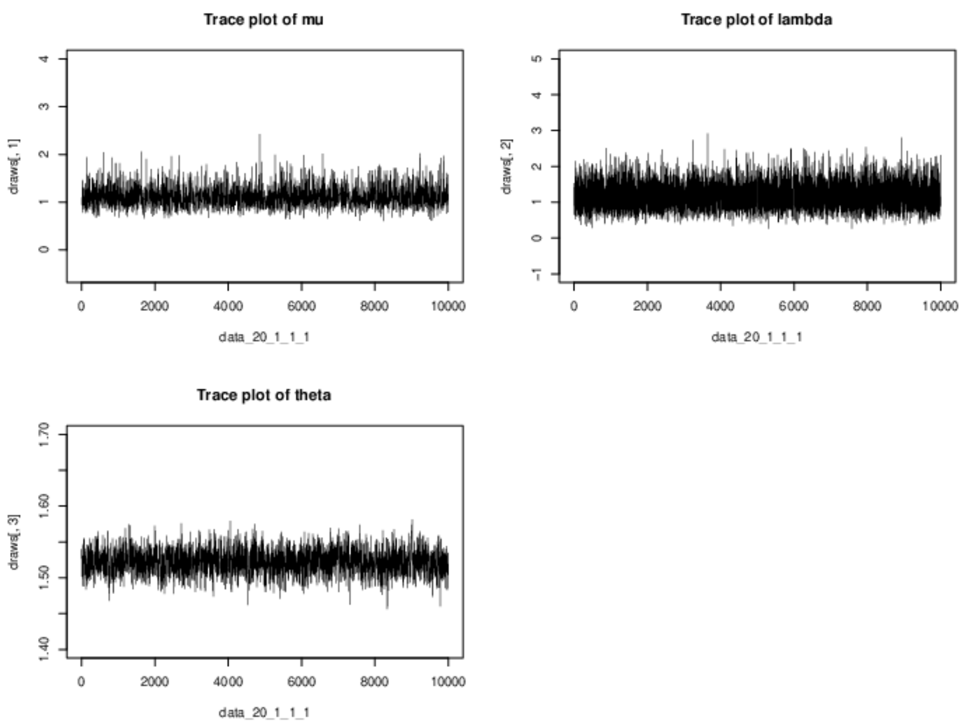

Figure 4.1: Trace plots for true parameters mu=1, lambda=1 and theta=1

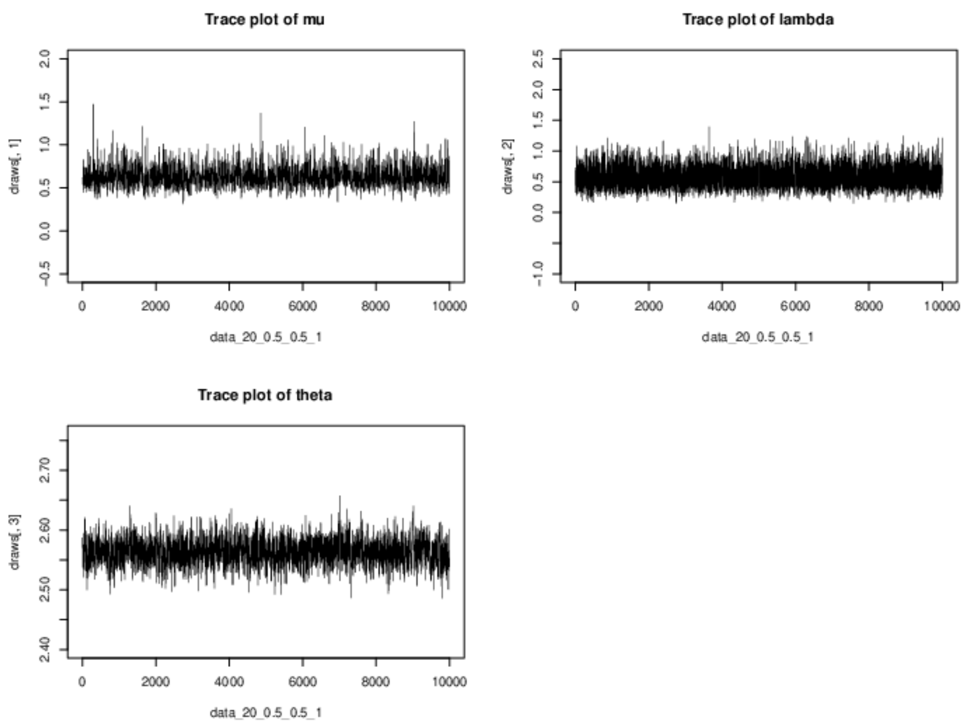

Figure 4.2: Trace plots for true parameters mu=0.5, lambda=0.5 and theta=1

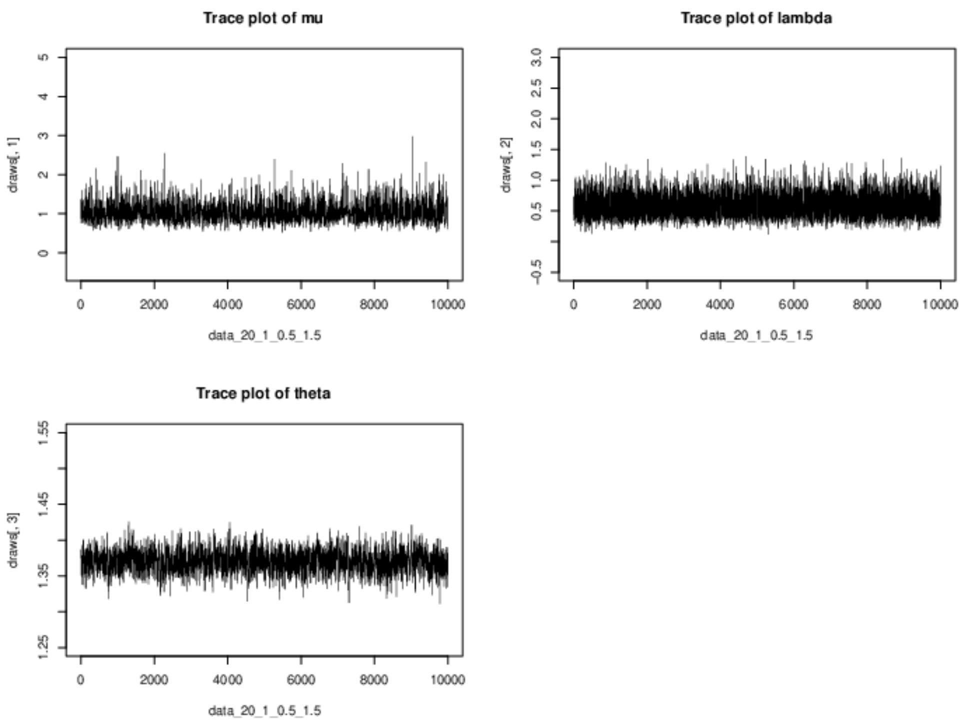

Figure 4.3: Trace plots for true parameters mu=1, lambda=0.5 and theta=1.5

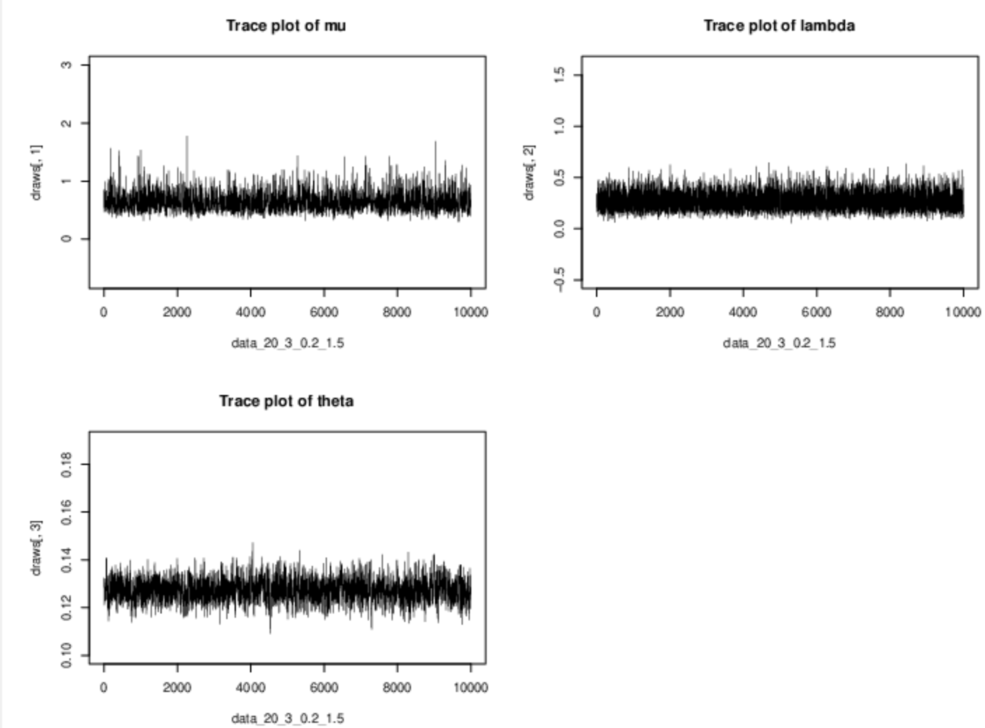

Figure 4.4: Trace plots for true parameters mu=3, lambda=0.2 and theta=1.5

compute maximum likelihood estimates, and average lengths(AL) of the 95% confidence intervals(CI) using frequentist approach. The results are given in table 4.4.

Table 4.4: Results from Real Data Analysis

	ML Method		Bayesian	
Para	Estimate	AL	Estimate	AL
μ	7.3767	142.8145	7.3654	0.5448
λ	332.5389	461.9926	328.7759	162.6450
θ	0.3408	378.0696	0.3400	0.0279

4.2.2 Trace Plots of Parameters from Insulating fluid Data

The trace plots of the parameters from the posterior distribution of Insulating fluid data are shown in figures 4.1.

4.2.3 Remarks from Real Data Analysis.

- The point estimates obtained in both ML and Bayesian methods are close to each other due to the relatively large sample size, $n = 60$.
- The ALs obtained under the Bayesian approach are however smaller than the ones of the ML approach.

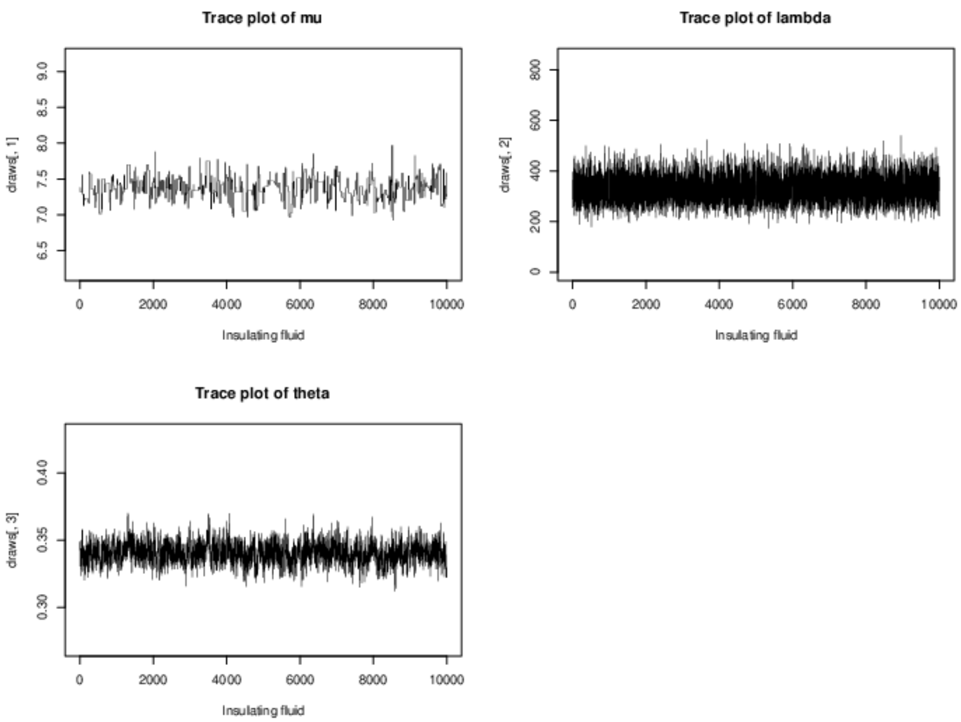

Figure 4.5: Trace plots of true parameters from Insulating fluid data

Chapter 5

Conclusion

In this study, a Bayesian inference approach to parameter estimation in continuously varying stress accelerated life testing which is the limiting case of the multiple stress-level discussed by Doksum and Hóyland [1] is presented. We derived the likelihood function of life testing model and studied it's Fisher's information used in the likelihood-based inference method. We calculated the MLEs of the parameters and also implemented the Gibbs sampling procedure to estimate the parameters from their full conditional posteriors. Our simulation study demonstrated the efficiency of our proposed Bayesian approach to parameter estimation over that of the likelihood-based inference method. With the application to a real dataset, we have illustrated that our Bayesian method can be readily applied for efficient, reliable, and precise inference.

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Curriculum Vitae

Richard Nii Okine is the first son of Mr. Seth Okine and Mrs. Josephine Okine. He graduated from Accra Academy High School, Accra, Ghana, in July 2012. He entered Kwame Nkrumah University of Science and Technology in September of 2012 to study Actuarial Science. He then traveled to the U.S to pursue his Master's degree in Statistics at The University of Texas at El Paso in the spring of 2018.

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