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Incidence Functions

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Dedicated to my parents,

Hsin-Fu and Chun-Chiao,

for their endless love and support.

INCIDENCE FUNCTIONS

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Abstract

In the mid 1960's, the incidence algebra was introduced in the seminal paper of Gian-Carlo Rota. He addressed the importance of the Möbius function in combinatorics. In particular, the incidence algebra of a locally finite poset plays an essentially unifying role in the theory of the Möbius function. One of the significant generalizations is the incidence algebra of a Möbius category introduced by Pierre Leroux. With the help from Möbius category, it was exciting to be able to extend the combinatorial results more broadly than just on posets. Before attempting to study this generalization of the Möbius function, we have to begin with the basic concepts needed to define the incidence algebra. In the first chapter, we will see some basic concepts and illustrations of incidence functions in posets. In the second chapter, we will introduce the decomposition-finite category $\mathscr{C}$, the incidence algebra of $\mathscr{C}$, and the Möbius function of the Möbius category $\mathscr{C}$.

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Chapter 1
Incidence Functions in Posets

Incidence algebra was introduced by Gian-Carlo Rota in his seminal paper: " On the Foundations of Combinatorial Theory I. Theory of Möbius Functions". This led to the creation of an unitary and systematic theory which unified results apparently without connections. Incidence algebras and the theory of Möbius functions play important roles in a variety of problems in combinatorics. In the first section, we will give a review of basic definitions and theorems about posets and the incidence algebra of a locally finite posets. There are some important examples of reduced incidence algebras, which will be addressed after the definition of the incidence algebra. One of them is the algebra of arithmetical functions with the Dirichlet product. Therefore, in the second section, we will mainly be concerned with the Dirichlet algebra of arithmetical functions. Those brief reviews in chapter one will be helpful in better understanding our attempt to make the bridge to the categorical language and the generalization of incidence algebras in Möbius categories from the second chapter.

1.1 Posets and Incidence Functions

A set P with a binary relation $\leq$ is a *poset* (or *partially ordered set*) if it satisfies the following three axioms:

- Reflexivity: $x \leq x$ for all $x \in P$
- Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$ for x, $y \in P$
- Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$ for x, y, $z \in P$.

We denote this partially ordered set by $(P, \leq)$. A subset C of $(P, \leq)$ is called a *chain* (or *totally ordered set*) of the poset P if for any x, $y \in C$, either $x \leq y$ or $y \leq x$. We will sketch the basic facts about posets. Let S be a subset of $(P, \leq)$. If there exists $u \in P$ and $x \leq u$ for every $x \in S$, then u is an *upper bound* of S. Dually, $l \in P$ and l is an *lower bound* of S if $l \leq x$ for every $x \in S$. Moreover, u is called a *supremum* (or a *least upper bound*) of S if u is an upper bound of S and $u \leq a$ for all upper bounds a of S. Similarly, l is called a *infimum* (or a *greatest lower bound* of S if l is an lower bound of S and $b \leq l$ for all lower bounds b of S. Also, a poset $(P, \leq)$ has a *maximal element* if there exists an element $m \in P$ such that for all $x \in P$, $x \leq m$. There is an important poset called a *lattice*. It is a poset in which every pair of elements (a, b) has a supremum and an infimum.

We extend the poset theory by further definitions needed later to define an incidence algebra without missing any details.

Definition 1.1. *A closed* ***interval*** $[a, b]$ *of* $(P, \leq)$ *is the set of all the elements* $x \in P$ *with the property that* $a \leq x \leq b$ *where* a, $b \in P$ *and* $a \leq b$.

Definition 1.2. *A poset* P *is* ***locally finite*** *if every closed interval of* $(P, \leq)$ *consists of finitely many elements.*

If $(P, \leq)$ is locally finite, then we denote by $Int(P)$ the set of all the intervals of P: $Int(P)= \{ \ [x, y] : \ x, \ y \in P$ and $x \leq y \ \}$. Let x, y be two elements of $(P, \leq)$ such that $x \leq y$. We say the element y *covers* x if there is no element z in P such that $x \leq z < y$. In other words, if $x < y$ and whenever $x \leq z < y$, then $z = x$.

Now, we need to define an R-algebra. First, let us recall the definition of a (left) R-module. Let R be a ring with unity 1. A (left) *R-module* is an abelian group $(A, +)$ with a scalar multiplication $R \times A \to A$, denoted by $(\alpha, x) \mapsto \alpha x$, such that the following axioms hold:

- $\alpha(x+y) = \alpha x + \alpha y$
- $(\alpha+\beta)x = \alpha x + \beta x$
- $(\alpha\beta)x = \alpha(\beta x)$
- $1x = x$ for any $x,\, y \in A$ and $\alpha,\, \beta \in R$.

Definition 1.3. *If R is a commutative ring with unity, then a ring A with unity is a R-**algebra** (or **an algebra over** R) if A is a R-module and for all $\alpha \in R$ and $x,\, y \in A$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$.*

Definition 1.4. *Let $(P, \leq)$ be a locally finite poset. A complex-valued function f, from $Int(P)$ to the field of the complex numbers $\mathbb{C}$, is called an **incidence function** of a locally finite poset. We denote the set of all incidence functions by $A(P)$. In $A(P)$, we define the following operations:*

- *Addition:* $(f+g)([x,y]) = f([x,y]) + g([x,y])$
- *Convolution:* $(f \cdot g)([x,y]) = \sum\limits_{x \leq z \leq y} f([x,z])g([z,y])$
- *Scalar multiplication:* $(\gamma f)([x,y]) = \gamma f([x,y])$

where $\gamma \in \mathbb{C}$, $f,\, g \in A(P)$.

One can observe that the convolution is well-defined because of the assumption that $(P, \leq)$ is locally finite. Now, consider $f,\, g,\, h \in A(P)$, we have

$$\begin{aligned}
\big[(f \cdot g) \cdot h\big]([x,y]) &= \sum_{x \leq z \leq y} (f \cdot g)([x,z])\; h([z,y]) \\
&= \sum_{x \leq z \leq y} \Bigg[\sum_{x \leq t \leq z} f([x,t])\; g([t,z])\Bigg] h([z,y]) \\
&= \sum_{x \leq t \leq z \leq y} f([x,t])\; g([t,z])\; h([z,y])
\end{aligned}$$

and

$$\begin{aligned}
\big[f \cdot (g \cdot h)\big]([x,y]) &= \sum_{x \le t \le y} f([x,t])\ (g \cdot h)([t,y]) \\
&= \sum_{x \le t \le y} f([x,t]) \Bigg[\sum_{t \le z \le y} g([t,z])\ h([z,y]) \Bigg] \\
&= \sum_{x \le t \le z \le y} f([x,t])\ g([t,z])\ h([z,y])
\end{aligned}$$

so the convolution is associative. The *convolution identity* is given by

$$\delta([x,y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

We have $f \cdot \delta = \delta \cdot f = f$ for any $f \in A(P)$. Then $\big(A(P), \cdot\big)$ is a monoid with the identity δ. Furthermore, for any f, g, $h \in A(P)$,

$$\begin{aligned}
\big[f \cdot (g + h)\big]([x,y]) &= \sum_{x \le z \le y} f([x,z])\ (g + h)([z,y]) \\
&= \sum_{x \le z \le y} f([x,z])\ \Big[g([z,y]) + h([z,y]) \Big] \\
&= \sum_{x \le z \le y} f([x,z])\ g([z,y]) + \sum_{x \le z \le y} f([x,z])\ h([z,y]) \\
&= f \cdot g([x,y]) + f \cdot h([x,y]).
\end{aligned}$$

Similarly, $(f + g) \cdot h = f \cdot h + g \cdot h$ is also true. Then we conclude the convolution distributes over addition.

Theorem 1.5. *$A(P)$ together with the above operations and with the convolution identity is a $\mathbb{C}$-algebra called the* ***incidence algebra*** *of P.*

Proof. Since every element in $A(P)$ is a complex-valued function, then $A(P)$ is closed under addition and the addition of incidence functions is associative and commutative. The *additive identity* (or called *zero element*) of $A(P)$ is given by $i_0([x,y]) = 0$ for all $[x,y] \in P$, so we have $(f + i_0)([x,y]) = f([x,y])$ for any $f \in A(P)$. Also, for each $f \in A(P)$, the *additive inverse* is defined by $g([x,y]) = -f([x,y])$ so $(f+g)([x,y]) = 0$ for all $[x,y] \in P$. Thus, $\big(A(P),+\big)$ is an abelian group. Moreover, $A(P)$ together with addition and scalar multiplication is a $\mathbb{C}$ vector space. In addition, since $\big(A(P),\cdot\big)$ is a monoid and the convolution is distributive over addition, so we conclude $\big(A(P),+,\cdot\big)$ is a ring with unity.

Since $f, g \in A(P)$ are complex-valued functions, we have:

$$\gamma(f \cdot g) = (\gamma f) \cdot g = f \cdot (\gamma g)$$

for all $\gamma \in \mathbb{C}$. Hence, $A(P)$ is a $\mathbb{C}$-algebra. □

Since we have shown $\big(A(P),\cdot\big)$ is a monoid, one may ask under what condition, an incidence function f will have a convolution inverse? What is the convolution inverse? In the following theorem, those questions will be answered.

Theorem 1.6. *An incidence function $f \in A(P)$ has a convolution inverse g if and only if $f([x,x]) \neq 0$ for all $x \in P$.*

Proof. ($\Rightarrow$) Assume that f has a convolution inverse g. For all $x \in P$, since $1 = \delta([x,x]) = (f \cdot g)([x,x]) = f([x,x])\ g([x,x])$, then $f([x,x]) \neq 0$.

($\Leftarrow$) Assume $f([x,x]) \neq 0$ for all $x \in P$. Let a function g be given inductively by

$$g([x,y]) = \begin{cases} \dfrac{1}{f([x,x])} & \text{if } x = y \\ \dfrac{-1}{f([y,y])} \displaystyle\sum_{x \leq z < y} g([x,z])\ f([z,y]) & \text{if } x < y \end{cases}$$

First, we claim $g \cdot f = \delta$

Case(1): if $x = y$, then

$$(g \cdot f)([x,x]) = g([x,x])\ f([x,x]) = \frac{1}{f([x,x])}\ f([x,x]) = 1 = \delta([x,x]).$$

Case(2): if $x < y$, first we have

$$\begin{aligned}(g \cdot f)([x,y]) &= \sum_{x \leq z \leq y} g([x,z])\ f([z,y]) \\ &= \sum_{x \leq z < y} g([x,z])\ f([z,y]) + g([x,y])\ f([y,y]) \\ &= -g([x,y])f([y,y]) + g([x,y])\ f([y,y]) \\ &= 0\end{aligned}$$

because $\sum_{x \leq z < y} g([x,z])\ f([z,y]) = -g([x,y])\ f([y,y])$. It implies $(g \cdot f)([x,y]) = \delta([x,y])$ whenever $x < y$. So we conclude $g \cdot f = \delta$.

Since $g([x,x]) = 1 \neq 0$ for all $x \in P$, there is a function $h \in A(P)$ such that $h \cdot g = \delta$ according to the proof we just show above. $\big(A(P), \cdot\ \big)$ is a monoid so we have

$$\begin{aligned}f \cdot g &= \delta \cdot (f \cdot g) = (h \cdot g) \cdot (f \cdot g) = h \cdot (g \cdot f) \cdot g \\ &= h \cdot \delta \cdot g = h \cdot g = \delta.\end{aligned}$$

Hence, g is a *convolution inverse* of f. □

Here we use f^{-1} as the convolution inverse of an incidence function f. We denote the set of all incidence functions f with the property that $f([x,x]) \neq 0$ for all $x \in P$ by $\mathcal{U}\big(A(P)\big)$. Next, we will introduce some useful theorems and classical incidence functions, such as the Möbius incidence function. We will need it later.

Observation 1.7. In general, the convolution of an incidence algebra is not commutative. In the locally finite poset $(\mathbb{N}, \leq)$, if we take

$$f([x,y]) = \begin{cases} 1 & \text{if } [x,y] = [2,2] \\ 2 & \text{if } [x,y] = [4,4] \\ 0 & \text{otherwise.} \end{cases}$$

and any $g([x,y]) \neq 0$ for all $[x,y] \in Int(\mathbb{N})$ in the incidence algebra $A(\mathbb{N}, \leq)$, then we have

$$\begin{aligned}(f \cdot g)([2,4]) &= f([2,2])\ g([2,4]) + f([2,3])\ g([3,4]) + f([2,4])\ g([4,4]) \\ &= g([2,4])\end{aligned}$$

and

$$(g \cdot f)([2,4]) = g([2,2])\ f([2,4]) + g([2,3])\ f([3,4]) + g([2,4])\ f([4,4]) = 2\ g([2,4]).$$

Since $g([x,y]) \neq 0$ for all $[x,y] \in Int(\mathbb{N})$, then $f \cdot g \neq g \cdot f$.

Definition 1.8. *([6]) The zeta function is an incidence function defined by $\zeta([x,y]) = 1$, for any $[x,y] \in P$. Since $\zeta([x,x]) \neq 0$, by Theorem 1.6, there exists a convolution inverse of ζ given inductively by*

$$\mu_P([x,y]) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu_P([x,z]) & \text{if } x < y \end{cases}$$

and μ_P is called the ***Möbius function*** *of the poset $(P, \leq)$.*

Theorem 1.9. *([6]) Let $(P, \leq)$ be a locally finite poset and μ_P be the Möbius function of the poset $(P, \leq)$. (i) Then we have*

$$\sum_{x \leq z \leq y} \mu_P([x,z]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $(P, \leq)$ is a chain, then

$$\mu_P([x,y]) = \begin{cases} 1 & \text{if } x = y \\ -1 & \text{if } y \text{ covers } x \\ 0 & \text{if } x < y \text{ but } y \text{ does not cover } x. \end{cases}$$

Proof. (i) If $x \leq y$, then

$$\sum_{x \leq z \leq y} \mu_P([x,z]) = \sum_{x \leq z \leq y} \mu_P([x,z])\ \zeta([z,y]) = (\mu_P \cdot \zeta)([x,y]) = \delta([x,y]) \tag{1.1.1}$$

In fact, this statement is equivalent to $\mu_P \cdot \zeta = \delta$.

(ii) By the result of (1.1.1), there are three cases we need to discuss. First, for case (1): if $x = y$, then $\mu_P([x,y]) = \delta([x,x]) = 1$.

Case(2): if y covers x, we have $\sum_{x \le z \le y} \mu_P([x,z]) = \delta([x,y]) = 0$. It implies $\mu_P([x,x]) + \mu_P([x,y]) = 0$. Hence, $\mu_P([x,y]) = -\mu_P([x,x]) = -1$.

Case(3): if $x < y$, then there is a sequence $\{z_i\}$ in $(P, \le)$ such that $x = z_0 < z_1 < z_2 < \ldots < z_{n-1} < z_n = y$ for the interval $[x,y]$ which is finite. If y does not cover x, then $n > 1$. We know $\sum_{x \le z \le y} \mu_P([x,z]) = \delta([x,y]) = 0$. Then we get $\mu_P([x,x]) + \mu_P([x,z_1]) + \mu_P([x,z_2]) = 0$. Since $x = z_0$ covers z_1, we know $\mu_P([x,z_1]) = -1$ from case (2). As a result, $\mu_P([x,z_2])$ $= -\mu_P([x,x]) - \mu_P([x,z_1]) = -1 - (-1) = 0$. By induction, it follows that $\mu_P([x,z_n]) = 0$. Hence, $\mu_P(x,y) = 0$ since $z_n = y$. □

Theorem 1.10 (Möbius inversion formula)**.** *Let $(P, \le)$ be locally finite poset with the least element element 0. For f_1, $g_1 \in A(P)$, we write $f(x)$ instead of $f_1([0,x])$ and $g(x)$ for $g_1([0,x])$. Then $f(x) = \sum_{0 \le z \le x} g(z)$ if and only if $g(x) = \sum_{0 \le z \le x} f(y)\ \mu_P([z,x])$.*

Proof. Assume $f(x) = \sum_{0 \le z \le x} g(z)$. Then

$$f_1([0,x]) = \sum_{0 \le z \le x} g_1([0,z]) = \sum_{0 \le z \le x} g_1([0,z])\ \zeta([z,x]) = (g_1 \cdot \zeta)([0,x]) \tag{1.1.2}$$

So $f_1 = g_1 \cdot \zeta$ for any $x \in P$. Since $\mu_P = \zeta^{-1}$, then

$$f_1 = g_1 \cdot \zeta \Longleftrightarrow g_1 = f_1 \cdot \mu_P. \tag{1.1.3}$$

Thus, $g_1([0,x]) = \sum_{0 \le z \le x} f_1([0,z])\ \mu_P([z,x])$. Therefore,

$$f(x) = \sum_{0 \le z \le x} g(z) \Longleftrightarrow g(x) = \sum_{0 \le z \le x} f(z)\ \mu_P([z,x]). \tag{1.1.4}$$

The proof is completed. □

Since we talk about the incidence algebras, it is natural to discuss some subalgebras of the incidence algebras. Here, we will only mention some special subalgebras called reduced incidence algebras. Let us define ρ to be an equivalence relation on $Int(P)$. Then, an incidence function $f \in A(P)$ is called a *constant on ρ-classes* if $[x, y]\rho[u, v]$ implies $f([x, y]) = f([u, v])$ where x, y, u, $v \in P$. Moreover, an equivalence relation ρ on $Int(P)$ is called *order-compatible* if for any $f, g \in A(P)$ which are constants on ρ-classes, then the convolution $f \cdot g$ is also a constant on ρ-classes. Let $A_\rho(P)$ be the collection of all constant on ρ-classes functions f in an incidence algebra $A(P)$. Since $A_\rho(P)$ is a subset of $A(P)$, one can verify that $A_\rho(P)$ is a subalgebra of $A(P)$. This subalgebra $A_\rho(P)$ is called a *reduced incidence algebra* of the poset P over $\mathbb{C}$.

In what follows we will give two examples of reduced incidence algebras.

Example 1.11. Let $(\mathbb{N}, \mid)$ be the poset of positive integers partially ordered by divisibility. Define the relation ρ on $Int(\mathbb{N})$ with divisibility by $[x, y]\rho[u, v] \Leftrightarrow \dfrac{y}{x} = \dfrac{v}{u}$ where x, y, u, $v \in \mathbb{N}$. It is straightforward to check that ρ is an order-compatible equivalence relation. Let f be the function of the reduced incidence algebra $A_\rho(\mathbb{N})$ of the poset $(\mathbb{N}, \mid)$ over $\mathbb{C}$. Then f satisfies $f([x, y]) = f([u, v])$ if $\dfrac{y}{x} = \dfrac{v}{u}$ where $x, y, u, v \in \mathbb{N}$. If we write $f(\dfrac{y}{x})$ for $f([x, y])$, then we can identify an element f with a function with only one variable on $\mathbb{N}$. As a result, this reduced incidence algebra $A_\rho(\mathbb{N}, \mid)$ becomes the *Dirichlet algebra* of arithmetical functions. In the next section, we will add more details of this Dirichlet algebra.

Now, let us see another example.

Example 1.12. Consider $(\mathbb{N} \cup \{0\}, \leq)$ the poset of non-negative integers with the standard ordering. The relation ρ on $Int(\mathbb{N} \cup \{0\})$ with the standard ordering is defined by $[x, y]\rho[u, v] \Leftrightarrow y - x = v - u$ where x, y, u, $v \in \mathbb{N} \cup \{0\}$. We can easily verify that ρ is an order-compatible equivalence relation. And the incidence function f in $A_\rho(\mathbb{N} \cup \{0\}, \leq)$ is determined by the values of $f(\ [0, n]\ )$. If $f([0, n])$ is written as $f(n)$, we can identify f with a function of one variable on $\mathbb{N} \cup \{0\}$. The reduced incidence algebra $A_\rho(\mathbb{N} \cup \{0\})$ with the standard ordering becomes the *Cauchy algebra* of arithmetical functions where the convolution is

$$(f * g)(n) = \sum_{k=0}^{n} f(k)\ g(n-k).$$

1.2 The Dirichlet Algebra of Arithmetical Functions

An *arithmetical function* is a mapping from the set of positive integers $\mathbb{N}$ into the set of complex numbers $\mathbb{C}$. In general, we denote $\mathcal{A}$ as the set of all the arithmetical functions. Recall the example 1.11 of the Dirichlet algebra of arithmetical functions. In this section we will define the Dirichlet convolution and address some important properties of the arithmetical functions.

Definition 1.13. *Let f, g be in $\mathcal{A}$. The **Dirichlet convolution** is defined by*

$$(f * g)(n) \ = \sum_{d|n} f(d)\ g(\frac{n}{d})$$

where $n \in \mathbb{N}$ and the summation is over the positive divisors d of n.

Let $n \in \mathbb{N}$ and d_1, d_2 the divisors of n such that $n = d_1 d_2$. For f, $g \in \mathcal{A}$, since f and g are complex valued functions,

$$(f * g)(n) = \sum_{n=d_1 d_2} f(d_1)\ g(d_2) = \sum_{d_1 d_2 = n} g(d_2)\ f(d_1) = (g * f)(n). \tag{1.2.1}$$

Then the Dirichlet convolution is commutative. The function $e(n)$ is defined by

$$e(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

For any $n \in \mathbb{N}$, $(f * e)(n) = \sum_{d|n} f(d)\ e(\frac{n}{d}) = f(n)\ e(1)$ because $e(\frac{n}{d}) \neq 0$ only when $d = n$. So $f * e(n) = f(n)$. Since $(\mathcal{A}, *)$ is commutative, $(e * f)(n) = f(n)$ is also true. Therefore, $e(n)$ is the (Dirichlet) *convolution identity.*

Recall that Dirichlet algebra is a subalgebra of $A(\mathbb{N})$ if we treat f as a function with only one variable on $\mathbb{N}$. Since in section 1.1, we have proved that $\big(A(P),\ +,\ \cdot\ \big)$ is a ring, then $\mathcal{A}$ together with usual addition and Dirichlet convolution is also a ring. Hence, $(\mathcal{A},\ +,\ *)$ is a commutative ring with unity $e(n)$.

Theorem 1.14. *The set of all the arithmetical functions $\mathcal{A}$ with the usual addition and the Dirichlet convolution is an integral domain.*

Proof. We have proved that $(\mathcal{A},\, +,\, *\,)$ is a commutative ring with unity. Now we want to show it has no zero divisors. Note that the zero element in $\mathcal{A}$ is defined by $\mathcal{O}(n) = 0$ for every $n \in \mathbb{N}$.

Let f, g be two nonzero elements in $\mathcal{A}$. Since any nonempty subset of the set of positive integers $\mathbb{N}$ is well-ordered, $f \neq \mathcal{O}$ implies there exists the least number $n \in \mathbb{N}$ such that $f(n) \neq 0$. Similarly, $g \neq \mathcal{O}$ implies there exists the least number $m \in \mathbb{N}$ such that $g(m) \neq 0$.

Let $d \in \mathbb{N}$ be a divisor of nm.

Case (1): if $d < n$, then $f(d) = 0$ because n is the least number such that $f(n) \neq 0$.

Case (2): if $d > n$, then $\dfrac{nm}{d} < m$ and then $g(\dfrac{nm}{d}) = 0$. So we have

$$\begin{aligned}(f * g)(nm) &= \sum_{d|nm} f(d)\; g(\frac{nm}{d}) \\ &= \sum_{\substack{d|nm \\ d<n}} 0\; g(\frac{nm}{d}) + f(n)\; g(m) + \sum_{\substack{d|nm \\ d>n}} f(d)\; 0 \\ &= f(n)\; g(m).\end{aligned}$$

Since neither $f(n)$ nor $g(m)$ is equal to zero, $(f * g)(nm) \neq 0$, so $f * g$ is not a zero element. Hence, $\mathcal{A}$ has no zero divisor in $(\mathcal{A}, +, *)$. Therefore, $\big(\mathcal{A},\, +,\, *\,\big)$ is an integral domain. □

Theorem 1.15. *The commutative ring $(\mathcal{A},\, +,\, *)$ with unity together with the usual scalar multiplication by elements from $\mathbb{C}$ is a $\mathbb{C}$-algebra.*

Proof. We know $(\mathcal{A},\, +,\, *)$ is a commutative ring with unity so $(\mathcal{A},\, +)$ is an abelian group. Let

f, $g \in \mathcal{A}$ be complex-valued functions and let γ be a complex number. Then

$$\begin{aligned}\big[\gamma(f*g)\big](n) &= \gamma\Big[\sum_{d|n} f(d)\ g(\frac{n}{d})\Big] \\ &= \sum_{d|n} \gamma\big[f(d)\ g(\frac{n}{d})\big] \\ &= \sum_{d|n} \big[\gamma f(d)\big]\ g(\frac{n}{d}) \\ &= \big[(\gamma f)*g\big](n).\end{aligned}$$

Similarly, $\gamma(f*g)(n) = f*(\gamma g)(n)$. Thus, $\gamma(f*g) = (\gamma f)*g = f*(\gamma g)$. Hence, the ring ($\mathcal{A}$, $+$, $*$) together with the scalar multiplication is a $\mathbb{C}$-algebra. □

In Theorem 1.6, we have proved that an incidence function $f \in A(P)$ has a convolution inverse if $f([x,x]) \neq 0$. In what follows we will prove there is a similar property for $\mathcal{A}$. Let $\mathcal{U}(\mathcal{A})$ denote the set of all arithmetical functions f which has a convolution inverse.

Theorem 1.16. *An arithmetical function f has a convolution inverse if and only if $f(1) \neq 0$.*

Proof. ($\Rightarrow$) If $f \in \mathcal{U}(\mathcal{A})$, then there exists the inverse $f^{-1} \in \mathcal{A}$. Since $(f*f^{-1})(1) = e(1)$, we have $\sum_{d|1} f(1)\ f^{-1}(1) = f(1)\ f^{-1}(1) = 1$. Therefore, $f(1)$ can not be zero.

($\Leftarrow$) Assume $f(1) \neq 0$ for some $f \in \mathcal{A}$. There is a function $\widetilde{f}$ given by

$$\widetilde{f}(n) = \begin{cases} \dfrac{1}{f(1)} & \text{if } n = 1 \\ \dfrac{-1}{f(1)} \displaystyle\sum_{\substack{d|n \\ d\neq 1}} f(d)\ \widetilde{f}(\frac{n}{d}) & \text{if } n > 1 \end{cases}$$

where $n \in \mathbb{N}$ and d is the divisor of n.

Case (1): if $n = 1$, since we know $f(1) \neq 0$,

$$(f*\widetilde{f})(1) = \sum_{d|1} f(1)\ \widetilde{f}(1) = f(1)\ \widetilde{f}(1) = f(1)\ \frac{1}{f(1)} = 1.$$

Then $(f * \widetilde{f})(n) = e(n)$ when $n = 1$.

Case (2): if $n > 1$, then we have

$$\begin{aligned}(f * \widetilde{f})(n) &= \sum_{d|n} f(d)\ \widetilde{f}(\frac{n}{d}) \\ &= f(1)\ \widetilde{f}(n) + \sum_{\substack{d|n \\ d\neq 1}} f(d)\ \widetilde{f}(\frac{n}{d}) \\ &= f(1)\ \widetilde{f}(n) - f(1)\ \widetilde{f}(n) \\ &= 0.\end{aligned}$$

Then, we conclude $(f * \widetilde{f})(n) = e(n)$ for all $n \in \mathbb{N}$. Also, $\widetilde{f} * f = e$ is true because Dirichlet convolution is commutative. Hence, $\widetilde{f} = f^{-1}$. Therefore, $f \in \mathcal{U}(\mathcal{A})$. □

We can see that $\big(\mathcal{U}(\mathcal{A}),\ *,\ e\ \big)$ is a group.

Definition 1.17. *An arithmetical function f is called **multiplicative** if*

$$f(nm) = f(n)\ f(m) \text{ whenever } gcd(n, m) = 1. \tag{1.2.2}$$

We denote by $\mathcal{M}$ the set of all non-zero arithmetical functions which are multiplicative.

Theorem 1.18. *The set of all multiplicative arithmetical functions $\mathcal{M}$ together with the Dirichlet convolution is a subgroup of $\big(\mathcal{U}(\mathcal{A}),\ *,\ e\ \big)$.*

Proof. If an arithmetical function f is in $\mathcal{M}$, then $f \neq \mathcal{O}$. It implies there is a positive number $k \in \mathbb{N}$ such that $f(k) \neq 0$. Since f is multiplicative, we have $f(k) = f(1k) = f(1)\ f(k)$. So $f(1)$ can not be zero because $f(k) \neq 0$. By Theorem 1.16, $f \in \mathcal{U}(\mathcal{A})$. Hence, $\mathcal{M} \subseteq \mathcal{U}(\mathcal{A})$.

(i) We show that if $f,\ g \in \mathcal{M}$, then $f * g \in \mathcal{M}$.

Let $n,\ m \in \mathbb{N}$, $gcd(n, m) = 1$ and d be the divisor of nm. There exist $d_1, d_2 \in \mathbb{N}$ such that $d = d_1 d_2$, $d_1|n$, $d_2|m$. Then $gcd(d_1, d_2) = 1$ as well as $gcd(\frac{n}{d_1}, \frac{m}{d_2}) = 1$. For any $f,\ g \in \mathcal{M}$, we have

$$\begin{aligned}
(f * g)(nm) &= \sum_{d|nm} f(d)\ g(\frac{n}{d}) \\
&= \sum_{\substack{d_1|n \\ d_2|m}} f(d_1 d_2)\ g(\frac{n}{d_1}\frac{m}{d_2}) \\
&= \sum_{\substack{d_1|n \\ d_2|m}} f(d_1)\ f(d_2)\ g(\frac{n}{d_1})\ g(\frac{m}{d_2}) \\
&= \Big[\sum_{d_1|n} f(d_1)\ g(\frac{n}{d_1})\Big]\ \Big[\sum_{d_2|m} f(d_2)\ g(\frac{m}{d_2})\Big] \\
&= \big[(f * g)(n)\big]\big[(f * g)(m)\big].
\end{aligned}$$

So $f * g$ is multiplicative.

(ii) We show that if $f \in \mathcal{M}$, then $f^{-1} \in \mathcal{M}$.

For any $f \in \mathcal{M} \subseteq \mathcal{U}(\mathcal{A})$, there exists a convolution inverse f^{-1}. Now let us consider the function g defined by

$$g(n) = \begin{cases} 1 & \text{if } n = 1 \\ \prod_{i=1}^{k} f^{-1}(p_i^{\alpha_i}) & \text{if } n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \end{cases}$$

where $p_1^{\alpha_1}, \cdots, p_k^{\alpha_k}$ are distinct primes, $\alpha_1, \cdots, \alpha_k$ are positive integers. In what follows, we will show that g is the inverse of f and g is multiplicative.

Let $n,\ m > 1$, $gcd(m,n) = 1$. And $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $m = p_{k+1}^{\alpha_k+1} \cdots p_s^{\alpha_s}$ where p_i, $i = 1, \cdots, s$ are distinct primes, α_j, $j = 1, \cdots, s$ are positive integers. By the definition of g, we have

$$g(nm) = g(p_1^{\alpha_1} \cdots p_s^{\alpha_s}) = \prod_{i=1}^{s} f^{-1}(p_i^{\alpha_i}) = \prod_{i=1}^{k} f^{-1}(p_i^{\alpha_i}) \prod_{j=k+1}^{s} f^{-1}(p_j^{\alpha_j}) = g(n)\ g(m).$$

Thus, $g \in \mathcal{M}$.

It remains to show that $g = f^{-1}$. By (i), $f * g$ is also multiplicative.

Case (1): if $n = 1$, then $(f * g)(1) = f(1)\ g(1) = e(1)$.

Case (2): if $n > 1$ and $n = p^\alpha$, then

$$(f * g)(p^\alpha) = \sum_{i=0}^{\alpha} f(p^i)\ g(p^{\alpha-i}) = \sum_{i=0}^{\alpha} f(p^i)\ f^{-1}(p^{\alpha-i}) = (f * f^{-1})(p^\alpha) = e(p^\alpha). \tag{1.2.3}$$

Therefore, $(f * g)(p^\alpha) = e(p^\alpha)$.

Case (3): if $n > 1$ and $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then

$$(f * g)(n) = (f * g)(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = (f * g)(\prod_{i=1}^{k} p_i^{\alpha_i}) = \prod_{i=1}^{k} (f * g)(p_i^{\alpha_i}) \tag{1.2.4}$$

because $f * g$ is multiplicative. By equation (1.2.3), we obtain $(f * g)(n) = \prod_{i=1}^{k} e(p_i^{\alpha_i})$. Since $e \in \mathcal{M}$, then $\prod_{i=1}^{k} e(p_i^{\alpha_i}) = e\Big(\prod_{i=1}^{k} p_i^{\alpha_i}\Big) = e(n)$. As a result, we have $(f * g)(n) = e(n)$ for all n. By (i) and (ii), we conclude $(\mathcal{M}, *\ )$ is a subgroup of $\big(\mathcal{U}(\mathcal{A}), *, e\ \big)$. □

At the end of this section, we want to show when and how the ordinary multiplication distributes over Dirichlet convolution. In the following, we use the ordinary multiplication $(fg)(n) = f(n)g(n)$ and the ordinary addition $(f + g)(n) = f(n) + g(n)$ whenever $f, g \in \mathcal{A}$. Two well-known arithmetical functions will appear in the next propositions. One is $\tau(n)$, the number of positive divisors of $n \in \mathbb{N}$. Another one is the zeta function, $\zeta(n) = 1$ for all $n \in \mathbb{N}$. Note that $\tau(n) = (\zeta * \zeta)(n)$.

Definition 1.19. *For all $n, m \in \mathbb{N}$, an arithmetical function f is called **completely multiplicative** if*

$$f(nm) = f(n)\ f(m). \tag{1.2.5}$$

*Also, f is called **completely additive** if*

$$f(nm) = f(n) + f(m). \tag{1.2.6}$$

Proposition 1.20. *([8]) Let f be an arithmetical function. The following statements are equivalent:*

(i) f is completely multiplicative;

(ii) $f(g * h) = fg * fh$ *for any* $g,\ h \in \mathcal{A}$*;*

(iii) $f(g * g) = fg * fg$ *for any* $g \in \mathcal{A}$

(iv) $f\tau = f * f$.

Proof. $(i) \Rightarrow (ii)$ Assume an arithmetical function f is completely multiplicative. For $n \in \mathbb{N}$, let d be the divisor of n and let $d_1,\ d_2 \in \mathbb{N}$ be such that $n = d_1 d_2$. For any $g,\ h \in \mathcal{A}$,

$$\begin{aligned} f(g * h)(n) &= f(n)\ (g * h)(n) \\ &= f(d_1 d_2) \left[\sum_{n=d_1 d_2} g(d_1)\ h(d_2) \right] \\ &= f(d_1)\ f(d_2) \left[\sum_{n=d_1 d_2} g(d_1)\ h(d_2) \right] \\ &= \sum_{n=d_1 d_2} f(d_1)\ f(d_2)\ g(d_1)\ h(d_2) \\ &= \sum_{n=d_1 d_2} \left[f(d_1)\ g(d_1) \right] \left[f(d_2)\ h(d_2) \right] \\ &= \sum_{n=d_1 d_2} fg(d_1)\ fh(d_2) \\ &= fg * fh. \end{aligned}$$

$(ii) \Rightarrow (iii)$ $f(g * g) = fg * fg$ for any $g \in \mathcal{A}$ is just an application of (ii) with $h = g$.

$(iii) \Rightarrow (iv)$ Since $\tau = \zeta * \zeta$, then $f\tau(n) = f(\zeta * \zeta)(n)$. By the assumption of $f(\zeta * \zeta) = f\zeta * f\zeta$, we have

$$\begin{aligned} f\tau(n) &= (f\zeta * f\zeta)(n) \\ &= \sum_{n=d_1 d_2} f\zeta(d_1)\ f\zeta(d_2) \\ &= \sum_{n=d_1 d_2} f(d_1)\ \zeta(d_1)\ f(d_2)\ \zeta(d_2) \\ &= \sum_{n=d_1 d_2} f(d_1)\ f(d_2) \\ &= (f * f)(n). \end{aligned}$$

Thus, $f\tau = f * f$.

$(iv) \Rightarrow (i)$ Assume (iv) holds. We shall discuss following two cases:

Case (1): if $f(1) = 0$. $f(2)\ \tau(2) = (f\tau)(2) = (f * f)(2) = 0$, so $f(2) = 0$. Similarly, $f(3)\ \tau(3) = (f\tau)(3) = (f * f)(3) = 0$ because of $f(2) = 0$, so $f(3) = 0$. By induction, we have $f(n) = 0$ for all $n \in \mathbb{N}$. So f is equal to the zero function which is completely multiplicative.

Case (2): if $f(1) \neq 0$. We get $f(1) = f(1)\ \tau(1) = (f\tau)(1) = (f * f)(1) = f(1)\ f(1)$. Since $f(1) \neq 0$, then we obtain $f(1) = 1$.

If $n = p$ where p is a prime, then

$$f(p)\ \tau(p) = (f\tau)(p) = (f * f)(p) = f(1)\ f(p) + f(p)\ f(1).$$

It implies $f(p) \cdot 2 = 2f(1)\ f(p)$. Hence, $f(p) = f(1)\ f(p)$.

Similarly, $f(p^2) \cdot 3 = f(p^2)\ \tau(p^2) = (f\tau)(p^2) = (f * f)(p^2) = 2f(p^2) + f(p)\ f(p)$, then $f(p^2) = f(p)\ f(p)$. By induction, we obtain

$$f(p^k) = \left[f(p)\right]^k \tag{1.2.7}$$

for all $k = 1, 2, \cdots$.

Now, let $n = p_1p_2p_3 \cdots p_k$ where $p_1, \cdots, p_k$ are primes. For $k = 1$, we can use the above result. Now, for $k = 2$, we have

$$f\tau(p_1p_2) = (f * f)(p_1p_2) = 2f(p_1p_2) + 2f(p_1)\ f(p_2).$$

It implies $3f(p_1p_2) = 2f(p_1p_2) + 2f(p_1)\ f(p_2)$. Thus, $f(p_1p_2) = f(p_1)\ f(p_2)$. Similarly, for $k = 3$, we will obtain $f(p_1p_2p_3) = f(p_1)\ f(p_2)\ f(p_3)$. By induction, we get

$$f(\prod_{i=1}^{k} p_i) = \prod_{i=1}^{k} f(p_i) \tag{1.2.8}$$

for all $n = p_1p_2p_3 \cdots p_k$.

By (1.2.7) and (1.2.8), if $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, then

$$f\Big(\prod_{i=1}^{k} p_i^{\alpha_i}\Big) = \prod_{i=1}^{k} \big[f(p_i)\big]^{\alpha_i}.$$

So f is completely multiplicative. $\square$

The statement (iv) is known as *Carlitz's characterization* ([1]). Also $(i) \Leftrightarrow (ii)$ is widely known (see [10], page 114).

Proposition 1.21. *([8]) Let f be an arithmetical function. The following statements are equivalent:*

(i) f is completely additive;

(ii) $f(g * h) = fg * h\ +\ g * fh$ for any $g,\ h \in \mathcal{A}$;

(iii) $f(g * g) = 2(fg * g)$ for any $g \in \mathcal{A}$

(iv) $f\tau = 2(f * \zeta)$.

Proof. $(i) \Rightarrow (ii)$ Assume that the arithmetical function f is completely multiplicative. For $n \in \mathbb{N}$, by the definition, $[f(g * h)](n) = f(n)\ (g * h)(n) = f(n)\Big[\sum_{d|n} g(d)\ h(\frac{n}{d})\Big]$. Since f is completely additive, $f(n) = f(d) + f(\frac{n}{d})$. Then we have

$$\begin{aligned}
\big[f\ (g * h)\big](n) &= \Big[f(d) + f(\frac{n}{d})\Big]\ \sum_{d|n} g(d)\ h(\frac{n}{d}) \\
&= \sum_{d|n} \Big[f(d)\ g(d)\Big]\ h(\frac{n}{d}) + \sum_{d|n} g(d)\ \Big[f(\frac{n}{d})\ h(\frac{n}{d})\Big] \\
&= (fg * h)(n) + (g * fh)(n).
\end{aligned}$$

Hence, $f(g * h) = fg * h + g * fh$ as desired.

$(ii) \Rightarrow (iii)$ By applying (ii) to $g = h$, we obtain $f(g * g) = 2(fg * g)$.

$(iii) \Rightarrow (iv)$ From (iii) and $f\tau = f(\zeta * \zeta)$, we have $f\tau = 2\big[(f\zeta) * \zeta\big] = 2(f * \zeta)$.

$(iv) \Rightarrow (i)$ Since we have $(f\tau)(p) = 2(f * \zeta)(p)$ for any arbitrary positive prime p, then $f(p) = f(1) + f(p)$. Hence $f(1) = 0$. Now let $n \in \mathbb{N}$, $n > 1$, with $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ the canonical representation of n.

We shall prove that

$$f(n) = \alpha_1 f(p_1) + \cdots + \alpha_k f(p_k) \tag{1.2.9}$$

by induction on $m = \alpha_1 + \cdots + \alpha_k$. If $M_i = \{ 0, 1, 2, \cdots, \alpha_i \}$ for $i = 1, 2, \cdots, k$ and $M = M_1 \times M_2 \times \cdots \times M_k$, then by (iv), we have

$$\frac{1}{2}\, f(n)\tau(n) = \sum_{(\beta_1,\cdots,\beta_k)\in M} f(p_1^{\beta_1} \cdots p_k^{\beta_k}) = \sum_{\substack{(\beta_1,\cdots,\beta_k)\in M \\ \beta_1+\cdots+\beta_k\neq m}} f(p_1^{\beta_1} \cdots p_k^{\beta_k}) + f(n).$$

It follows by induction:

$$\frac{1}{2}\, f(n)\tau(n) = \sum_{\substack{(\beta_1,\cdots,\beta_k)\in M \\ \beta_1+\cdots+\beta_k\neq m}} \Big[\beta_1 f(p_1) + \cdots + \beta_k f(p_k)\Big] + f(n). \tag{1.2.10}$$

Furthermore, we find that

$$\sum_{\substack{(\beta_1,\cdots,\beta_k)\in M \\ \beta_1+\cdots+\beta_k\neq m}} \Big[\beta_1 f(p_1) + \cdots + \beta_k f(p_k)\Big] = \frac{1}{2}\left[\prod_{i=1}^{k}(\alpha_i + 1)\right]\left[\sum_{i=1}^{k} \alpha_i f(p_i)\right] - \sum_{i=1}^{k} \alpha_i f(p_i). \tag{1.2.11}$$

By equations (1.2.10) and (1.2.11), we get

$$f(n) = \sum_{i=1}^{k} \alpha_i f(p_i).$$

□

Chapter 2
Incidence Functions in Möbius Categories

In this section we will talk about Möbius Categories introduced by P. Leroux([3]) in 1980 as a generalization of the study of incidence algebras which include the case of posets.

2.1 Definitions and Examples of Categories

The main idea of the category theory is to construct the links among different areas in mathematics. This algebraic structure can reveal some hidden relationship or similarities in distinct mathematical systems.

Definition 2.1. *A* ***category*** $\mathscr{C}$ *consists of the following components:*

(i) A class $Ob\mathscr{C}$ *whose elements are called* ***objects*** *of* $\mathscr{C}$*;*

(ii) For each A*,* $B \in Ob\mathscr{C}$*, a set* $Hom_{\mathscr{C}}(A, B)$ *whose elements are called* ***morphisms*** *(or* ***arrows****) from* A *to* B*. We write* $f : A \to B$ *if* f *is a morphism from* A *to* B*;*

(iii) A mapping μ_{ABC} *from* $Hom_{\mathscr{C}}(A, B) \times Hom_{\mathscr{C}}(B, C)$ *to* $Hom_{\mathscr{C}}(A, C)$*, denoted by* $(f, g) \longmapsto \mu_{ABC}(f, g) = gf$*, called a* ***composition*** *(or a* ***product****) of* g *by* f*.*

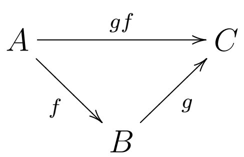

Figure 2.1: Objects A, B, C and morphisms f, g, gf

The category $\mathscr{C}$ *is required to satisfy the following axioms:*

(C1) *For any* $(A, B) \neq (C, D)$*, where* (A, B)*,* $(C, D) \in Ob\mathscr{C} \times Ob\mathscr{C}$*,* $Hom_{\mathscr{C}}(A, B)$ *and* $Hom_{\mathscr{C}}(C, D)$ *are disjoint.*

(C2) *Associativity: For any* $f \in Hom_{\mathscr{C}}(A,B)$*,* $g \in Hom_{\mathscr{C}}(B,C)$ *and* $h \in Hom_{\mathscr{C}}(C,D)$*, we have* $(hg)f = h(gf)$*.*

(C3) *Existence of identity: For each* $A \in Ob\mathscr{C}$*, there exists an element* $1_A : A \to A$ *such that* $f1_A = f$ *and* $1_A g = g$ *whenever* $f \in Hom_{\mathscr{C}}(A,X)$*,* $g \in Hom_{\mathscr{C}}(X,A)$*, and* $X \in Ob\mathscr{C}$*. We call* 1_A *the* ***identity morphism*** *of* A*.*

In the following, we will give some examples of categories.

Example 2.2. The category *Ens* whose objects are sets and morphisms are functions, where the composition is the usual composition of functions, is called the *category of sets*.

Example 2.3. The *category of groups*, denoted by *Grp*, is a category whose objects are groups and morphisms are group homomorphisms. The composition is the usual composition of functions.

Similar to *Grp* are the *category of finite groups FGp*, the *category of semigroups SGp*, and the *category of abelian groups Ab*. Here we will not go over the details of these categories.

Example 2.4. The category *Rng* is called the *category of rings* if its objects are rings, morphisms are ring homomorphisms, and the composition is the usual composition of functions.

Similar one is the *category of rings with unity*, denoted by Rng_u, if the class of all rings is replaced by the class of all rings with unity and the ring homomorphisms are replaced by the unitary ring homomorphisms.

Example 2.5. The *category of (left) R-modules*, denoted by R^{Mod}, is a category whose objects are R-modules and morphisms are R-modules homomorphisms. The composition is usual composition of functions.

Example 2.6. The category *Top* whose objects are topological spaces and morphisms are continuous functions, where the composition is the usual composition of functions, is called the *category of topological spaces.*

Example 2.7. The category *Ord* is called the *category of ordered sets* if its objects are posets, morphisms are isotone (or order-preserving) functions, and the composition is the usual composition of functions.

Example 2.8. The category *Latt* whose objects are lattices, the morphisms are homomorphisms of lattices, and the composition is the usual composition of functions, is called the *category of lattices.*

Example 2.9. The category *Rel* is the *category of relations* if its objects are sets, morphisms are binary relations from a set A to a set B, and the composition is the composition of relations.

Example 2.10. The *category of real matrices*, denoted by $\mathscr{M}_{\mathbb{R}}$, is a category whose class of objects is the set of positive integers $\mathbb{N}$ and for n, $m \in \mathbb{N}$,

$$Hom_{\mathscr{M}_{\mathbb{R}}}(n, m) = \{ \ (a_{ij})_{m\times n} \mid a_{ij} \in \mathbb{R} \ \}.$$

The composition is $\mu_{\mathscr{M}_{\mathbb{R}}}(A, B) = BA$ for morphisms $A : n \to m$ and $B : m \to k$, n, m, $k \in \mathbb{N}$.

Example 2.11. The *ordered category*, denoted by $\mathcal{O}_P$, is a category whose class of objects are elements of a poset P, and for x, $y \in P$,

$$Hom_{\mathcal{O}_P}(x, y) = \begin{cases} \{(x, y)\} & \text{if } x \le y \\ \emptyset & \text{if } x \not\le y. \end{cases}$$

The composition is $\mu_{xyz}\ \big(\ (x, y)\ (y, z)\ \big) = (x, z)$ for x, y, $z \in P$, $x \le y$ and $y \le z$.

Definition 2.12. *A category $\mathscr{C}$ is called **small** if its class of objects is a set.*

Among previous examples, we can see that $\mathscr{M}_{\mathbb{R}}$ and $\mathcal{O}_P$ are small categories. Once we have given many examples of categories and indicated their objects and morphisms, now we will consider some special morphisms and objects in categories.

Definition 2.13. *Let $\mathscr{C}$ be a category. A morphism $f : A \to B$ is called a* ***monomorphism*** *if for any u, $v \in Hom_{\mathscr{C}}(X, A)$, $fu = fv$ implies $u = v$. A morphism $f : A \to B$ is called an* ***epimorphism*** *if for any u, $v \in Hom_{\mathscr{C}}(B, X)$, then $uf = vf$ implies $u = v$. We call a morphism f a* ***bimorphism*** *if it is both a monomorphism and an epimorphism.*

Theorem 2.14. *In Ens, let f be a morphism from a set A to a set B. Then we have:*

(i) f is a monomorphism if and only if f is injective.

(ii) f is an epimorphism if and only if f is surjective.

(iii) f is a bimorphism if and only if f is bijective.

Proof. (i) ($\Rightarrow$) Suppose that f is not injective. Then there exist two elements x_1, $x_2 \in A$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$. Let $X = \{x_1, x_2\}$ and u, $v : X \to A$ such that $u(x) = x_1$, $v(x) = x_2$ for any $x \in X$. Clearly, $u \neq v$. Then, we have

$$(fu)(x) = f(u(x)) = f(x_1) = f(x_2) = f(v(x)) = (fv)(x)$$

for all $x \in X$. Thus, $fu = fv$ but $u \neq v$. Therefore, f is not a monomorphism and it is a contradiction with the hypotheses.

($\Leftarrow$) Assume that f is injective. Let u, $v \in Hom_{Ens}(X, A)$, $X \in ObEns$, such that $fu = fv$. For each $x \in X$, since $(fu)(x) = (fv)(x)$, then $u(x) = v(x)$ because f is injective. So, $u = v$. Therefore, f is a monomorphism.

(ii) ($\Rightarrow$) Suppose that f is not surjective. Then there exists an element $b_0 \in B$ such that b_0 is not an element of the image of f (denoted by Imf). Let $Y = \{*, b_0\}$ where $*$ is a symbol different than the elements of B.

Let u, $v : B \to Y$ where

$$u(y) = *$$

and

$$v(y) = \begin{cases} * & \text{if } y \neq b_0 \\ b_0 & \text{if } y = b_0 \end{cases}$$

for any $y \in B$. Clearly, $u \neq v$. Since $f(x) \in Imf$, $f(x) \neq b_0$, then

$$(uf)(x) = u(f(x)) = * = v(f(x)) = (vf)(x)$$

for all $x \in A$. So, $uf = vf$ but $u \neq v$. So f is not an epimorphism, contradiction with the hypotheses.

($\Leftarrow$) Assume that f is surjective. For each $y \in B$, there exists an element $x \in A$, $f(x) = y$. Let u, $v \in Hom_{Ens}(B, Y)$, $Y \in ObEns$, such that $uf = vf$. For any $y \in B$,

$$u(y) = u(f(x)) = (uf)(x) = (vf)(x) = v(f(x)) = v(y).$$

Then, $u = v$. Hence, f is an epimorphism.

(iii) If f is a *bimorphism*, then it is both a monomorphism and an epimorphism. By the proofs of (i) and (ii), f is bijective. □

Next, we will give a useful theorem. For convenience, we denote

$$Mor\mathscr{C} = \bigcup_{(A,B)\in Ob\mathscr{C}\times Ob\mathscr{C}} Hom_{\mathscr{C}}(A, B).$$

Theorem 2.15. *Let $\mathscr{C}$ be a category and f, $g \in Mor_{\mathscr{C}}$, and let f, g be morphisms of $Hom_{\mathscr{C}}(A, B)$ and $Hom_{\mathscr{C}}(B, C)$. Then*

(i) if f, g are monomorphisms, then gf is a monomorphism,

(ii) if f, g are epimorphisms, then gf is an epimorphism,

(iii) if gf is a monomorphism, then f is a monomorphism,

(iv) if gf is an epimorphism, then g is an epimorphism.

Proof. (i) Let h, $k \in Mor\mathscr{C}$, $(gf)h = (gf)k$. By the associativity of morphisms, then $g(fh) = g(fk)$. It implies $fh = fk$ because g is a monomorphism. Since f is a monomorphism, then $h = k$. Thus, gf is also a monomorphism.

(ii) Let h, $k \in Mor\mathscr{C}$ such that $h(gf) = k(gf)$. By the associativity, $(hg)f = (kg)f$. It implies $hg = kg$ because f is an epimorphism. We know that g is an epimorphism so $h = k$. Therefore, gf is also an epimorphism.

(iii) Let h, $k \in Mor\mathscr{C}$, $fh = fk$. Then, $g(fh) = g(fk)$ implies $(gf)h = (gf)k$ by the associativity. So, $h = k$ because gf is a monomorphism. Thus, f is a monomorphism.

(iv) Let h, $k \in Mor\mathscr{C}$, $hg = kg$. It follows $(hg)f = (kg)f$. Then, $h(gf) = k(gf)$. Since gf is an epimorphism, $h = k$. Hence, g is an epimorphism. □

At the beginning of this section, we have defined an identity morphism. Now we will introduce some other special morphisms (left and right inverses) in categories.

Definition 2.16. *Let $\mathscr{C}$ be a category, $f \in Hom_{\mathscr{C}}(A, B)$. If there exists a morphism $g : B \to A$ such that $gf = 1_A$, then f is called a* ***coretraction****. If there exists a morphism $g : B \to A$ such that $fg = 1_B$, then f is called a* ***retraction****. We call f an* ***isomorphism*** *if f is both a coretraction and a retraction.*

If f is both a coretraction and a retraction, then there exist morphisms $g_1, g_2 : B \to A$ such that $g_1 f = 1_A$ and $fg_2 = 1_B$. Then, $g_1 = g_1 1_B$ because 1_B is the identity morphism of B. Since 1_A is the identity morphism of A, then we have

$$g_1 = g_1 1_B = g_1(fg_2) = (g_1 f)g_2 = 1_A g_2 = g_2.$$

Hence, there is an alternative definition of an isomorphism $f \in Hom_{\mathscr{C}}(A, B)$: there exists a morphism $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$. We call g the *inverse* of f and denote it by f^{-1}.

In addition, an *endomorphism* is a morphism whose domain is the same as its codomain, and an endomorphism which is an isomorphism is called an *automorphism*.

Theorem 2.17. *Let $\mathscr{C}$ be a category, $f \in Hom_{\mathscr{C}}(A, B)$. Then we have*

(i) if f is a coretraction, then f is a monomorphism,

(ii) if f is a retraction, then f is an epimorphism,

(iii) if f is an isomorphism, then f is a bimorphism.

Proof. (i) Assume that f is a coretraction. Then there exists a morphism $g : B \to A$ such that $gf = 1_A$. For any u, $v \in Hom_{\mathscr{C}}(X, A)$, if $fu = fv$, then $g(fu) = g(fv)$. It implies $(gf)u = (gf)v$. So, $u = v$. Therefore, f is a monomorphism.

(ii) Assume that f is a retraction. Then there exists a morphism $g : B \to A$ such that $fg = 1_B$, $1_B : B \to B$. For any u, $v \in Hom_{\mathscr{C}}(B, X)$, if $uf = vf$, then $(uf)g = (vf)g$. It follows $u(fg) = v(fg)$. Then, $u = v$. Hence, f is an epimorphism.

(iii) Suppose that f is an isomorphism, then we know that f is both a coretraction and a retraction. By (i) and (ii), f is both a monomorphism and an epimorphism. Thus, f is a bimorphism. □

Theorem 2.18. *In Ens, let f be a morphism from a set A to a set B. Then we have*

(i) f is a coretraction if and only if f is a monomorphism,

(ii) f is a retraction if and only if f is an epimorphism,

(iii) f is an isomorphism if and only if f is a bimorphism.

Proof. (i) ($\Rightarrow$) By Theorem 2.17, f is a monomorphism.

($\Leftarrow$) Assume that f is a monomorphism. For any u, $v \in Hom_{\mathscr{C}}(X, A)$, if $fu = fv$ then $u = v$. Let a morphism $g : B \to A$ be given by

$$g(y) = \begin{cases} f^{-1}(y) & \text{if } y \in f(A) \\ a & \text{if } y \notin f(A) \end{cases}$$

where $a \in A$. If $y \in f(A)$, there exists an element $x \in A$ such that $y = f(x)$. By the definition of g, we have $g(y) = f^{-1}(y)$. Then, $g(f(x)) = f^{-1}(f(x))$. It follows $(gf)(x) = (f^{-1}f)(x) = 1_A(x)$. So, $gf = 1_A$. Hence, f is a coretraction.

(ii) ($\Rightarrow$) By Theorem 2.17, f is an epimorphism.

($\Leftarrow$) Assume that f is an epimorphism. By Theorem 2.14 (*ii*), f is surjective. Then, for all $y \in B$, $f^{-1}(\{y\})$ is a nonempty subset of A. Let $g : B \to A$ be a morphism such that $g(y) = x$ whenever $x \in f^{-1}(\{y\})$, $y \in B$. Since $x \in f^{-1}(\{y\})$, then

$$(fg)(y) = f(g(y)) = f(x) = y = 1_B(y) \text{ for all } y \in B.$$

Hence, $fg = 1_B$. Therefore, f is a retraction.

(iii) ($\Rightarrow$) By Theorem 2.17, f is a bimorphism.

($\Leftarrow$) If f is an bimorphism, by Theorem 2.14 (*iii*), f is bijective. Hence, f is an isomorphism because of (i) and (ii). □

Definition 2.19. *A category $\mathscr{C}$ is* ***balanced*** *if each bimorphism is also an isomorphism.*

Definition 2.20. *Let $\mathscr{C}$ be a category. An object A is called* ***initial*** *if for all $X \in Ob\mathscr{C}$, the cardinality of $Hom_{\mathscr{C}}(A, X)$ is 1. An object A is called* ***terminal*** *if the cardinality of $Hom_{\mathscr{C}}(X, A)$ is 1.*

Example 2.21. In *Ens*, the empty set $\emptyset$ is an initial object and each singleton (a set with exactly one element) is a terminal object.

Example 2.22. In *Grp*, the trivial group, a group consisting of only one element, is both an initial object and a terminal object.

Example 2.23. In Rng_u, the set of all integers $\mathbb{Z}$ together with usual addition and multiplication is an initial object.

2.2 Möbius Categories and Incidence Functions

In this section, we will use the version of Schwab ([7] Section 3).

Definition 2.24. *A small category $\mathscr{C}$ is* ***decomposition-finite*** *if for each morphism f, the set*

$$< f > = \{ \ (f_1, f_2) \mid f = f_1 \ f_2 \text{ where } f_1, \ f_2 \in Mor\mathscr{C} \ \}$$

is finite.

If let a morphism f, $f = f_1 f_2 \cdots f_n$, where $f_1, \cdots, f_n \in Mor_{\mathscr{C}}$, we call n the *degree* of the decomposition $f_1 f_2 \cdots f_n$. The *length* of a morphism f, denoted by $l(f)$, is the supremum of the degrees of the decompositions of f where $f_1, \cdots, f_n$ are non-identity morphisms. In this section, we will concentrate our attention on decomposition-finite categories with finite length (i.e. $l(f)$ is finite for each morphism f).

Definition 2.25. *A morphism f is called* ***indecomposable*** *if it has no decomposition without identities of the degree more than 2.*

Definition 2.26. *The* ***incidence algebra*** *$A(\mathscr{C})$ of a decomposition-finite category $\mathscr{C}$ is a $\mathbb{C}$-algebra of all complex valued functions $\alpha : Mor\mathscr{C} \to \mathbb{C}$ (called* ***incidence functions*** *of a decomposition-finite category) together with the following operations:*

- *Addition:* $(\alpha + \beta)(f) = \alpha(f) + \beta(f)$
- *Convolution:* $(\alpha * \beta)(f) = \sum_{(f_1, f_2) \in < f >} \alpha(f_1) \ \beta(f_2)$
- *Scalar multiplication:* $(a\alpha)(f) = a\alpha(f)$

where $a \in \mathbb{C}$, $\alpha, \ \beta \in A(\mathscr{C})$.

The function $\delta : Mor\mathscr{C} \to \mathbb{C}$ is defined by

$$\delta(f) = \begin{cases} 1 & \text{if } f \text{ is an identity morphism} \\ 0 & \text{otherwise.} \end{cases}$$

Let $f \in Hom_{\mathscr{C}}(A, B)$, A, $B \in Ob\mathscr{C}$. By the axiom $(C3)$ of the definition 2.1, there exist the identity morphisms 1_A and 1_B such that $f1_A = f$, $1_Bf = f$. Since $(f_1, f_2) \in < f >$ implies $f = f_1\ f_2$, then we have

$$(\alpha * \delta)(f) = \sum_{(f_1,f_2)\in<f>} \alpha(f_1)\ \delta(f_2) = \alpha(f)\ \delta(1_A) = \alpha(f) \tag{2.2.1}$$

for any incidence function $\alpha \in A(\mathscr{C})$. Similarly,

$$(\delta * \alpha)(f) = \sum_{(f_1,f_2)\in<f>} \delta(f_1)\ \alpha(f_2) = \delta(1_B)\ \alpha(f) = \alpha(f). \tag{2.2.2}$$

Therefore, δ is the *convolution identity* of $A(\mathscr{C})$.

Let $f \in Mor\mathscr{C}$. An incidence function $\xi_f : Mor\mathscr{C} \to \mathbb{C}$ is given by

$$\xi_f(g) = \begin{cases} 1 & \text{if } g = f \\ 0 & \text{otherwise.} \end{cases}$$

For any $g \in Mor\mathscr{C}$, we have

$$(\xi_{f_1} * \xi_{f_2})(g) = \sum_{(g_1,g_2)\in<g>} \xi_{f_1}(g_1)\ \xi_{f_2}(g_2).$$

If $f_1 \neq g_1$, then $\xi_{f_1}(g_1) = 0$. Similarly, $\xi_{f_2}(g_2) = 0$ if $f_2 \neq g_2$. Therefore, $(\xi_{f_1} * \xi_{f_2})(g) = 1$ if $g = f_1\ f_2$; otherwise, $(\xi_{f_1} * \xi_{f_2})(g) = 0$. Hence, we conclude

$$(\xi_{f_1} * \xi_{f_2}) = \xi_{f_1\ f_2}. \tag{2.2.3}$$

Definition 2.27. *A* ***Möbius category*** *is a decomposition-finite category $\mathscr{C}$ with finite length; that is, $l(f)$ is finite for each morphism f of $\mathscr{C}$.*

One can observe that the identity morphisms in a Möbius category are indecomposable morphisms of length 0. The length of a non-identity indecomposable morphism is 1. A decomposition $f_1 f_2 \cdots f_n = f$ of a morphism f in a Möbius category is called *elementary* if $f_1, \cdots, f_n$ are non-identity indecomposable morphisms.

Theorem 2.28. *Let $\mathscr{C}$ be a decomposition-finite category. The following statements are equivalent:*

(i) $\mathscr{C}$ is a Möbius category.

(ii) $\mathscr{C}$ has no non-identity retractions and coretractions; and for f, $g \in Mor\mathscr{C}$, if $fg = f$, then g is an identity.

(iii) $\mathscr{C}$ has no non-identity retractions and coretractions; and for f, $g \in Mor\mathscr{C}$, if $gf = f$, then g is an identity.

(iv) An incidence function $\alpha \in A(\mathscr{C})$ has a convolution inverse if and only if $\alpha(1_A) \neq 0$ for all $A \in Ob\mathscr{C}$.

Proof. We will break this proof into three parts: $(i) \iff (ii)$, $(i) \iff (iii)$, and $(i) \iff (iv)$.

Part (1): $(i) \iff (ii)$. First, we will show that $(i) \Rightarrow (ii)$.

Assume that $\mathscr{C}$ is a Möbius category. Then, $\mathscr{C}$ is a decomposition-finite category with finite length. Let $g \in Hom_{\mathscr{C}}(B, A)$, $h \in Hom_{\mathscr{C}}(A, B)$, where A, $B \in Ob\mathscr{C}$, such that $gh = 1_A$. If g is an identity, then $gh = h$ is automatically an identity. Similarly, if h is an identity, then $gh = g$ is also an identity. Suppose that g, h both are not identities, we have

$$1_A = (1_A)^n = \underbrace{(gh)\ (gh) \cdots (gh)}_{n\ times}$$

is a decomposition without identities. By the definition of the length, $l(1_A) = 2n$ for any $n \in \mathbb{N}$. It is a contradiction with the fact that $\mathscr{C}$ is a decomposition-finite category with finite length. Therefore, if gh is an identity, g and h are both identities. Thus, $\mathscr{C}$ has no non-identity retractions and coretractions.

For the second part of the proof, if there exist two morphisms f, g such that

$$fg = f, \text{ then } fg^n = f \text{ for all } n \in \mathbb{N}. \tag{2.2.4}$$

Since $\mathscr{C}$ is a decomposition-finite category with finite length, then g has to be an identity.

Now, we shall prove $(i) \Leftarrow (ii)$.

Suppose that there exists a morphism $f \in Hom_{\mathscr{C}}(A, B)$ such that f has no finite length. If the cardinality of $< f >$ is m, then $m \geq 2$ since $(f, 1_A)$, $(1_B, f) \in < f >$. For every natural number $n > 2$, there are non-identity morphisms $f_1, f_2, \cdots, f_n$ such that $f = f_1 f_2 \cdots f_n$. In particular, let $n > m$. Since $\mathscr{C}$ has no non-identity retractions and coretractions, $f_p f_{p+1} \cdots f_q$ are non-identities for any p, $q \in \mathbb{N}$, $1 \leq p < q \leq n$.

Because $n > m$, there exist i, $j \in \mathbb{N}$, $1 \leq i < j \leq n$, such that

$$(f_1 \cdots f_i, f_{i+1} \cdots f_n) = (f_1 \cdots f_j, f_{j+1} \cdots f_n) \tag{2.2.5}$$

where $(f_1 \cdots f_i,\ f_{i+1} \cdots f_n)$, $(f_1 \cdots f_j,\ f_{j+1} \cdots f_n) \notin \{(f, 1_A),\ (1_B, f)\}$. Therefore, we obtain

$$f_1 \cdots f_i = f_1 \cdots f_i\ f_{i+1} \cdots f_j. \tag{2.2.6}$$

By the assumption that if $fg = f$, then g is an identity, we obtain that $f_{i+1} \cdots f_j$ is an identity. It is a contradiction with the result above: $f_p f_{p+1} \cdots f_q$ are non-identities for any p, $q \in \mathbb{N}$, $1 \leq p < q \leq n$. Hence, it follows that $l(f)$ is finite for each morphism f of $\mathscr{C}$.

Part (2): $(i) \Longleftrightarrow (iii)$. The proof of $(i) \Rightarrow (iii)$ is similar to $(i) \Rightarrow (ii)$.

We just need to replace (2.2.4) by

$$\text{if } gf = f, \text{ then } g^n f = f \text{ for all } n \in \mathbb{N}. \tag{2.2.7}$$

Then the conclusion that g is an identity will follow.

The proof $(i) \Leftarrow (iii)$ is also similar to $(i) \Leftarrow (ii)$.

The only difference is that instead obtaining the equation (2.2.6), we will have:

$$f_{i+1} \cdots f_j \; f_{j+1} \cdots f_n = f_{j+1} \cdots f_n \tag{2.2.8}$$

from the equation (2.2.5). Then, we can conclude that $f_{i+1} \cdots f_j$ is an identity. It is again a similar contradiction. Therefore, $l(f)$ is finite for each morphism f of $\mathscr{C}$.

Part (3): $(i) \iff (iv)$. We will show $(i) \Rightarrow (iv)$ first.

Let $\mathscr{C}$ be a Möbius category.

$(\Rightarrow)$ Assume that an incidence function $\alpha \in A(\mathscr{C})$ has the convolution inverse $\beta \in A(\mathscr{C})$, so $\alpha * \beta = \beta * \alpha = \delta$. For each $A \in Ob\mathscr{C}$, we have $1 = \delta(1_A) = (\alpha * \beta)(1_A) = \alpha(1_A)\ \beta(1_A)$, so $\alpha(1_A) \neq 0$ for all $A \in Ob\mathscr{C}$.

$(\Leftarrow)$ Assume that $\alpha(1_A) \neq 0$ for all $A \in Ob\mathscr{C}$. A function $\beta \in A(\mathscr{C})$ is given inductively by

$$\beta(f) = \begin{cases} \dfrac{1}{\alpha(f)} & \text{if } f \text{ is an identity morphism} \\ \dfrac{-1}{\alpha(1_B)} \displaystyle\sum_{\substack{(f_1,f_2) \in <f> \\ f_2 \neq f}} \alpha(f_1)\ \beta(f_2) & \text{otherwise} \end{cases}$$

for any $f \in Hom_{\mathscr{C}}(A, B)$, A, $B \in Ob\mathscr{C}$.

We should verify that $\beta(f)$ is defined for every f. If f is an identity, then $f = 1_A$ for some $A \in Ob\mathscr{C}$. By assumption $\alpha(f) \neq 0$, so $\beta(f) = \dfrac{1}{\alpha(f)}$ is defined.

Next, if $f \in Hom_{\mathscr{C}}(A, B)$, where A, $B \in Ob\mathscr{C}$, is a non-identity morphism, we shall prove that $\beta(f)$ is defined by induction on $l(f)$. If $l(f) \geq 1$, assume that all $\beta(\bar{f})$ has been defined for all $\bar{f} \in Mor\mathscr{C}$, $l(\bar{f}) < l(f)$. Now, if $(f_1, f_2) \in <f>$ and $f_2 \neq f$, then $l(f_2) < l(f)$. Therefore, the function $\beta(f) = \dfrac{-1}{\alpha(1_B)} \displaystyle\sum_{\substack{(f_1,f_2) \in <f> \\ f_2 \neq f}} \alpha(f_1)\ \beta(f_2)$ is also defined.

To complete the proof, we need to show that β is the convolution inverse of α.

Case (1): if $f = 1_A$ for some $A \in Ob\mathscr{C}$. Since $\alpha(f) \neq 0$, we have

$$(\alpha * \beta)(f) = \alpha(f)\ \beta(f) = \alpha(f)\ \frac{1}{\alpha(f)} = 1.$$

Then, $(\alpha * \beta)(f) = \delta(f)$ for any identity morphism f.

Case (2): if $f \in Hom_{\mathscr{C}}(A, B)$, where A, $B \in Ob\mathscr{C}$, is a non-identity morphism, then $l(f) > 1$. Now, it follows that

$$\begin{aligned}(\alpha * \beta)(f) &= \sum_{(f_1,f_2)\in<f>} \alpha(f_1)\ \beta(f_2) \\ &= \alpha(1_B)\ \beta(f) + \sum_{\substack{(f_1,f_2)\in<f> \\ f_2\neq f}} \alpha(f_1)\ \beta(f_2) \\ &= 0.\end{aligned}$$

Therefore, $(\alpha * \beta)(f) = \delta(f)$ for any non-identity morphism f. Hence,

$$\alpha * \beta = \delta. \tag{2.2.9}$$

Since $\alpha(1_A) \neq 0$, it is clear that $\beta(f) = \dfrac{1}{\alpha(f)} \neq 0$ for each identity morphism $f = 1_A$ where $A \in Ob\mathscr{C}$. Then, it follows from (2.2.9) that there exists a morphism $\bar{\beta} \in A(\mathscr{C})$ such that $\beta * \bar{\beta} = \delta$. So, it gives us that

$$\beta * \alpha = \beta * \alpha * \delta = \beta * \alpha * (\beta * \bar{\beta}) = \beta * (\alpha * \beta) * \bar{\beta} \tag{2.2.10}$$

Because of $\alpha * \beta = \delta$, the last expression in equation (2.2.10) becomes

$$\beta * (\alpha * \beta) * \bar{\beta} = \beta * \delta * \bar{\beta} = \beta * \bar{\beta} = \delta. \tag{2.2.11}$$

Therefore, $\beta * \alpha = \delta$. With the result (2.2.9), we know that β is a convolution inverse of α.

Finally, now we shall prove $(i) \Leftarrow (iv)$.

Since $(ii) \Leftrightarrow (i)$, if we can show that $(iv) \Rightarrow (ii)$, then the proof will be completed.

Assume that an incidence function $\alpha \in A(\mathscr{C})$ has a convolution inverse if and only if $\alpha(1_A) \neq 0$ for all $A \in Ob\mathscr{C}$. We prove by contradiction that $\mathscr{C}$ has no non-identity retractions and coretractions. Let $f \in Hom_{\mathscr{C}}(A, B)$, where A, $B \in Ob\mathscr{C}$, be a non-identity morphism. If there exists a morphism $g \in Hom_{\mathscr{C}}(B, A)$ such that

$$fg = 1_B \text{ for some } B \in Ob\mathscr{C}, \tag{2.2.12}$$

then clearly, g is not an identity. For any identity morphism h, we have

$$(\delta + \xi_f)(h) = \delta(h) + \xi_f(h) = 1 + 0 = 1$$

and

$$(\delta - \xi_g)(h) = \delta(h) + \xi_g(h) = 1 - 0 = 1.$$

Thus, $(\delta + \xi_f)(h) \neq 0$ and $(\delta - \xi_g)(h) \neq 0$ for any identity morphism h. Hence, $\delta + \xi_f$ and $\delta - \xi_g$ have the convolution inverses by our assumption. Therefore, it is straightforward that $(\delta+\xi_f)*(\delta-\xi_g)$ will also have the convolution inverse. In other words, $\big[(\delta+\xi_f)*(\delta-\xi_g)\big](h) \neq 0$ for any identity morphism h. However,

$$\begin{aligned}
\big[(\delta + \xi_f) * (\delta - \xi_g)\big](1_B) &= (\delta + \xi_f)(1_B)\ (\delta - \xi_g)(1_B) \\
&= \big[\delta(1_B) + \xi_f(1_B)\big]\ (\delta - \xi_g)(1_B) \\
&= \delta(1_B)\big[(\delta - \xi_g)(1_B)\big] + \xi_f(1_B)\big[(\delta - \xi_g)(1_B)\big] \\
&= [\delta * (\delta - \xi_g)](1_B) + \xi_f(1_B)\ \delta(1_B) - \xi_f(1_B)\ \xi_g(1_B) \\
&= \delta(1_B) - \xi_g(1_B) + (\xi * \delta)(1_B) - (\xi_f * \xi_g)(1_B)
\end{aligned}$$

By (2.2.3) and (2.2.12), the last expression above becomes

$$\begin{aligned}
\delta(1_B) - \xi_g(1_B) + (\xi * \delta)(1_B) - (\xi_f * \xi_g)(1_B) &= \delta(1_B) - \xi_g(1_B) + \xi(1_B) - \xi_{fg}(1_B) \\
&= 1 - 0 + 0 - 1 \\
&= 0
\end{aligned}$$

because $\xi_{fg}(1_B) = \xi_{fg}(fg) = 1$. It is a contradiction with that $\big[(\delta+\xi_f)*(\delta-\xi_g)\big](h) \neq 0$ for any identity morphism h. It follows that f and g are identities. Therefore, $\mathscr{C}$ has no non-identity retractions and coretractions.

Now, consider any two morphisms f, $g \in Mor\mathscr{C}$ such that $fg = f$. Then

$$\begin{aligned}(\delta - \xi_f) * (\delta - \xi_g) &= \delta * \delta - \xi_f * \delta - \delta * \xi_g - \xi_f * \xi_g \\ &= \delta - \xi_f - \xi_g - \xi_{fg} \\ &= \delta - \xi_g\end{aligned}$$

because $\xi_f * \xi_g = \xi_{fg} = \xi_f$. Similarly, we have

$$\begin{aligned}(\delta + \xi_f) * (\delta - \xi_g) &= \delta * \delta + \xi_f * \delta - \delta * \xi_g - \xi_f * \xi_g \\ &= \delta + \xi_f - \xi_g - \xi_{fg} \\ &= \delta - \xi_g.\end{aligned}$$

Therefore, it shows that

$$(\delta - \xi_f) * (\delta - \xi_g) = (\delta + \xi_f) * (\delta - \xi_g). \tag{2.2.13}$$

Suppose that β is non-identity. Since for any $A \in Ob\mathscr{C}$,

$$(\delta - \xi_g)(1_A) = \delta(1_A) - \xi_g(1_A) = 1 - 0 = 1 \neq 0,$$

then $\delta - \xi_g$ has a convolution inverse by assumption. Then, the equation (2.2.13) will lead us to obtain

$$\delta - \xi_f = \delta + \xi_f \text{ for any morphism } f. \tag{2.2.14}$$

However, we know that $(\delta - \xi_f)(f) = -1$ and $(\delta + \xi_f)(f) = 1$ for any non-identity morphism f. It is a contradiction with (2.2.14). Therefore, β is an identity morphism. The proof is completed. $\square$

If $\mathscr{C}$ is a Möbius category, the *Möbius function* $\mu_{\mathscr{C}}$ of a Möbius category $\mathscr{C}$ is the convolution inverse of the *zeta function* ζ, $\mu_{\mathscr{C}} = \zeta^{-1}$, where $\zeta(f) = 1$ for any morphism $f \in Mor\mathscr{C}$. Furthermore, the *Möbius inversion formula* of a Möbius category $\mathscr{C}$ is equivalent to the statements: if α, β, $\gamma \in A(\mathscr{C})$, we have

$$\alpha = \beta * \zeta \text{ if and only if } \beta = \alpha * \mu_{\mathscr{C}}$$

and

$$\gamma = \zeta * \beta \text{ if and only if } \beta = \mu_{\mathscr{C}} * \gamma.$$

Let us see some examples of the Möbius categories.

Example 2.29. A Möbius category $\mathscr{C}$ whose set of objects is the set of nonnegative integers and for m, $n \in \mathbb{N} \cup \{0\}$,

$$Hom_{\mathscr{C}}(m, n) = \{\ (a, b, m) \mid a, b \in \mathbb{N} \cup \{0\} \text{ and } a + b + m = n\ \}.$$

The composition is defined by

$$\mu_{\mathscr{C}}\ \big[\ (a, b, m)\ (c, d, n)\ \big] = (a + c, b + d, m)$$

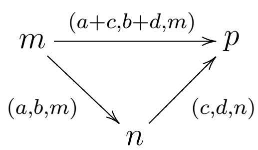

Figure 2.2: Objects m, n, p and morphisms (a, b, m), (c, d, n), $(a + c, b + d, m)$

where $(a, b, m) \in Hom_{\mathscr{C}}(m, n)$, $(c, d, n) \in Hom_{\mathscr{C}}(n, p)$ and m, n, $p \in \mathbb{N}$.

Here, we will verify that the category defined in above example is a Möbius category. For any $f = (a, b, m) \in Hom_{\mathscr{C}}(m, n)$, there is a morphism $1_m = (0, 0, m) \in Hom_{\mathscr{C}}(m, m)$ such that $f1_m = f$. So, $1_m = (0, 0, m)$ is an identity morphism.

Let $f = (a, b, m) \in Hom_{\mathscr{C}}(m, n)$. It follows $a+b+m = n$, therefore $m \leq n$. If $f = gh$ is a factorization of f, then $gh \in < f >$ and $g \in Hom_{\mathscr{C}}(p, n)$, $h \in Hom_{\mathscr{C}}(m, p)$ with $m \leq p \leq n$. Since $Hom_{\mathscr{C}}(p, n)$, $Hom_{\mathscr{C}}(m, p)$ are finite and $m \leq p \leq n$, it follows $< f >$ is finite. Therefore, $\mathscr{C}$ is decomposition-finite.

Let $f = (a, b, m)$, $g = (c, d, n)$. The identity morphism of m is $1_m = (0, 0, m) \in Hom_{\mathscr{C}}(m, m)$. We shall verify that $\mathscr{C}$ has non-identity retractions and coretractions. If $gf = 1_m$ such that f and g are both non-identities, then $gf = (a + c, b + d, m) = (0, 0, m)$. It follows that $a + c = 0$, $b + d = 0$, and $m = p$. So, $a = c = b = d = 0$ because a, b, c, d are nonnegative integers. It is a contradiction with the fact that f and g are non-identities. Thus, $\mathscr{C}$ has no non-identity retractions and coretractions.

If $gf = f$, then $gf = (a + c, b + d, m) = (a, b, m)$. Then, $a + c = a$ and $b + d = b$. It implies that $c = d = 0$. So, g is an identity. From the characterization (iii) in Theorem 2.28, we conclude that $\mathscr{C}$ is a Möbius category.

Example 2.30. A triangular category $\triangle$ whose set of objects is the set of positive integers $\mathbb{N}$ and 0 as the initial object. For s, $t \in \mathbb{N}$,

$$Hom_{\triangle}(s, t) = \{\ f : \{1, 2, \cdots, s\} \to \{1, 2, \cdots, t\} \mid f \text{ is injective and isotone}\}.$$

The composition is given by

$$\mu_{\triangle}(fg) = g \circ f$$

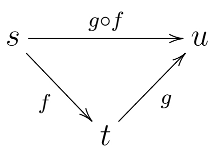

Figure 2.3: Objects s, t, u and morphisms f, g, $g \circ f$

where $f \in Hom_{\triangle}(s, t)$, $g \in Hom_{\triangle}(t, u)$, s and t, $u \in \mathbb{N}$.

Now, we shall verify that $\triangle$ is a Möbius category. Let the mapping $i : k \to k$ be defined by $i(k) = k$ for any $k \in \mathbb{N}$. It is straightforward that i is the identity morphism.

Let $f \in Hom_{\triangle}(s,t)$. It follows $s \leq t$. If $f = g \circ h$ is a factorization of f, then $g \circ h \in< f >$ and $g \in Hom_{\triangle}(u,t)$, $h \in Hom_{\triangle}(s,u)$ with $s \leq u \leq t$. Since $Hom_{\triangle}(u,t)$, $Hom_{\triangle}(s,u)$ are finite and $s \leq u \leq t$, it follows $< f >$ is finite. Therefore, this triangular category is decomposition-finite.

First, we shall verify that this triangular category has no non-identity retractions and coretractions. Suppose that f and g are both non-identities and $g \circ f = i$. Since f is not an identity, there exist an element k, $1 \leq k \leq s$, such that $f(k) = p$, $s < p < t$. Then, $g(p) = g\big[f(k)\big] = (g \circ f)(k) = i(k) = k$. Therefore, $g(p) \leq s$. Since g is isotone, $g\big(\{1, \cdots, p\}\big) \subseteq \{1, \cdots, s\}$ but then g cannot be injective. It is a contradiction with the hypotheses. Then, the category $\triangle$ has no non-identity retractions and coretractions.

If $g \circ f = f$ where $f \in Hom_{\triangle}(s,t)$, $g \in Hom_{\triangle}(t,u)$, then $u = t$ because of the axiom (C1) of a category. Then, $g : \{1, 2, \cdots, t\} \to \{1, 2, \cdots, t\}$. Since g is injective and isotone, then g is an identity. By Theorem 2.28 (iii), we conclude that triangular category $\triangle$ is a Möbius category.

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Curriculum Vitae

Yi-Yu Liao (also known as Amanda Liao) was born in Taipei city, Taiwan. She is the third daughter of Hsin-Fu Liao and Chun-Chiao Liao.

After completing her work at Taipei Wego Private Senior High School, she entered Tunghai University in Taichung city, Taiwan. She received the degree of Bachelor of Science from Tunghai University in the summer of 2003. During the following years she was employed as a product planner of an e-commerce business company. Later she became a product manager in Monday Tech Co., Ltd. (later it was emerged by Yahoo! Taiwan Inc. in 2008). In the spring of 2007, she came to El Paso, Texas, and entered the English Language Institute of the University of Texas at El Paso (UTEP) to prepare applying the Graduate School of UTEP. She was a teaching assistant in the Department of Mathematical Science at UTEP from 2008 to 2009. Since September 2009, she started teaching undergraduate courses in UTEP as a graduate student instructor. She received the STEM Scholarship of Workforce Solutions Upper Rio Grande at El Paso on July 2009.

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