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Introduction to Pseudo-ordered Groups

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Introduction to Pseudo-ordered Groups

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THESIS

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Preface

Transitivity and partial order have been like the guideline of the thought of the human being. Even more in science, being more specific, in mathematics is fundamental the idea of transitivity as partial order is in plenty of theories. However, there are some real-life situations in which the transitivity fails. For instance, a familiar relationship: Antony is brother of Bob sons of the same father. Bob is brother of Chris sons of the same mother, This does not necessarly imply that Antony is brother of Chris. Another example is the famous children game rock, paper, scissors, is a non-transitive game, in which we know that scissors beats paper and paper beats rock, but it is not true that scissors beats rock.

The concept of non-transivity gives us the idea to construct a mathematical structure without transitivity. Then we restricted ourselves in an algebraic structure such as pseudo-ordered group. We will develop the concept of an order that does not make use of transitive property on groups. In addition, we try to see the relation of an order as a set satisfying certain properties. Moreover, this special set allow us to talk about positive and negative elements of a group. Besides, this set establishes a relationship between positive elements and elements which are a conjugate of their inverses.

Moreover, we disscuss when this special set P is maximal. That is, what conditions P must have in order to be a maximal pseudo-order. In addition, we deal with the definition of pseudo chains. This is a new concept on groups. A pseudo-ordered group is a pseudo chain if every element can be connected with another element, then we say that the pseudo-order induces a pseudo chain on the pseudo-ordered group. This work is the beginning of a new theory related to groups. All the results written here more additional conditions or properties will give a rich source of researc on pseudo-ordered groups.

Abstract

We will give a definition, examples and basic properties of pseudo-ordered groups. These generalize partially-ordered groups. The pseudo-ordered groups have a compatible pseudo-order relation that may lack transitivity. Unlike partially-ordered groups, our groups can be finite and still admit a non-trivial pseudo-order. We will show necessary and sufficient conditions for a group to have a compatible pseudo-order. We also consider maximal pseudo-orders. We give a definition of a pseudo chain induced by a pseudo-order and characterize all pseudo chains on the group $\mathbb{Z}_n$. We will calculate the number of non-isomorphic pseudo-orders on $\mathbb{Z}_n$ for $n < 24$.

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Chapter 1

Some Preliminary Considerations

In this chapter we will recall necessary material from number theory and set theory. Of special significance are the topics of the greatest common divisorof numbers and so called pseudo-ordered sets. We show basic definitions, and prove some theorems. For more detailed examples, results, and proofs see [1] and [3].

1.1 The Greatest Common Divisor

Let us recall the g.c.d. and some of its properties.

Definition 1.1.1. An integer b is said to be *divisible by* an integer $a \neq 0$, written $a\,|b$, if there exists an integer c such that $b = ac$.

Definition 1.1.2. Let a and b be integers, with at least one of them different from zero. The greatest common divisor of a and b, denoted by gcd(a,b), is the positive integer satisfying the following:

1. $d\,|a$ and $d\,|b$.

2. If $c\,|a$ and $c\,|b$, then $c\,|d$.

One of the most important facts about the g.c.d of two numbers is the following

Theorem 1.1.1. *Let a and b be integer, not both of which are zero. Then there exist integers x and y such that*

$$gcd(a,b) = ax + by$$

The proof is in any number theory book, see eg [1] and [3]

Definition 1.1.3. Two nonzero integers a and b, are said to be *relatively prime* whenever $gcd(a, b) = 1$.

The following theorem characterizes relatively prime numbers in terms of their linear combinations.

Theorem 1.1.2. *Let a and b be integers, not both zero. Then a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$.*

Proof. ($\Rightarrow$) If a and b relatively prime then $gcd(a, b) = 1$, and Theorem 2.3 in [3] guarantees existence of x and y satisfying $1 = ax + by$.

($\Leftarrow$) Assume that $d = gcd(a, b)$ and $1 = ax + by$ for some integers x and y. Since d is a divisor of a and b, $d$is a divisor of any linear combination of a and b, so d is a divisor of 1 and since, d is a positive integer, $d = 1$.

$\square$

Some of the properties of the g.c.d may be easily extended to three or more numbers.

Definition 1.1.4. Let $n \geq 2$. By the $gcd(a_1, ..., a_n)$ we understand a positive d such that

1. $\forall i \; d \,|a_i$,
2. If $\forall i \; c \,|a_i$ then $c \,|d$.

Proposition 1.1.3. *For any $n \geq 2$ we have*

$$gcd(a_1, a_2, ..., a_n) = gcd(gcd(a_1, a_2, ..., a_{n-1}), a_n).$$

Proof. By (1) the Definition 1.1.4 for n numbers, $d = gcd(a_1, a_2, ..., a_n)$ must be a divisor of $a_1, a_2, ..., a_{n-1}$ so that d is a divisor of $gcd(a_1, a_2, ..., a_{n-1})$ by (2) Definition 1.1.4. Since d is also a divisor of a_n, d must be a common divisor of $gcd(a_1, a_2, ..., a_{n-1})$ and a_n. Then d is a divisor of $k = gcd((a_1, a_2, ..., a_{n-1}), a_n)$ by (2) the Definition 1.1.4. Now, by definition k

is a divisor of $gcd(a_1, a_2, ..., a_{n-1})$ and a_n. This implies that k is a divisor of $a_1, a_2, ..., a_{n-1}$. Hence, k is a divisor of $d = gcd(a_1, a_2, ..., a_n)$. Thus, $d = k$. □

Proposition 1.1.4. *If $d = gcd(a_1, a_2, ..., a_n)$, there exist n integers $x_1, x_2, ..., x_n$, for which*

$$d = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Proof. Induction on n. If $n = 2$, the result follows by Theorem 1.1.1. Suppose it holds for $n-1$. Since $gcd(a_1, a_2, ..., a_n) = gcd(gcd(a_1, a_2, ..., a_{n-1}), a_n)$, it follows that there exist two integers u and u_n such that

$$gcd(gcd(a_1, a_2, ..., a_{n-1}), a_n) = ugcd(a_1, a_2, ..., a_{n-1}) + u_na_n. \tag{1.1}$$

Then there exist $n-1$ integers, $u_1, u_2, ..., u_{n-1}$, for which $gcd(a_1, a_2, ..., a_{n-1}) = a_1u_1 + a_2u_2 + \cdots + a_{n-1}u_{n-1}$. Substituting in (1.1), we get $gcd(gcd(a_1, a_2, ..., a_{n-1}), a_n) = a_1uu_1 + a_2uu_2 + \cdots + a_{n-1}uu_{n-1} + u_na_n$, so that $gcd(a_1, a_2, ..., a_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $x_1 = uu_1$, $x_2 = uu_2$, ..., $x_{n-1} = uu_{n-1}$ $x_n = u_n$. □

In particular, if $gcd(a_1, a_2, ..., a_n) = 1$ then we say that $a_1, a_2, ..., a_n$ are relatively prime. If $a_1, a_2, ..., a_n$ are relatively prime then there are n integers, $x_1, x_2, ..., x_n$ such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1.$$

1.2 Pseudo-ordered Sets

In order to start the study of pseudo-ordered groups, we need to review the definition and examples of pseudo-ordered sets.

Definition 1.2.1. Let A be a set. A relation " $\leq$ " on A is a *pseudo order* if " $\leq$ " satisfies the two following conditions:

- For any $g \in A$, $g \leq g$ (Reflexive),

- If $g \leq h$ and $h \leq g$, then $h = g$ (Antisymmetric).

Then A is called a *pseudo-ordered set*.

Note 1. All partially-ordered sets are pseudo-ordered.

Example 1.2.1. A well-known pseudo-ordered set is the set of all real numbers $\mathbb{R}$ with the following relation: $x \preccurlyeq y$ iff $0 \leq y - x \leq a$, where a is a given positive number. The relation " $\preccurlyeq$ " is reflexive since $x \preccurlyeq x$ because $0 \leq x - x = 0 \leq a$. Antisymmetry holds because if $x \preccurlyeq y$ and $y \preccurlyeq x$ we have that $0 \leq y - x \leq a$ and $0 \leq x - y \leq a$. Then we conclude that $x = y$ since " $\leq$ " is the usual order on $\mathbb{R}$. Note this relation is not transitive. This happens in the situation when for example, $a = 4$, $x = 1$, $y = 2$, and $z = 6$.

For more examples and results see [10]. The next definition plays an important role in pseudo-ordered set theory since it is analogous to the definition of a totally-ordered set.

Definition 1.2.2. Let A be a pseudo-ordered set is said to be a *pseudo-ordered subset* if $B \subseteq A$ and B is pseudo-ordered set with the pseudo-order of A.

Definition 1.2.3. Let $(A, \leq)$ be a pseudo-ordered set, and $B \subseteq A$. We say that B is a *pseudo chain* if for every two elements $b, b' \in B$, there exists a finite subset of $b_1, b_2, ..., b_n \in B$, such that $b \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq b'$ or $b' \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq b$.

Chapter 2

Pseudo-ordered Groups

The foundation of the theory of pseudo-ordered groups can be found in [10]. We will develop this concept further here. We will show some basic examples and properties.

This theory is a generalization of the theory of partially-ordered groups. Basic concepts of the latter can be found in [5].

2.1 General Considerations

Definition 2.1.1. A pseudo-ordered group is a set G such that:

1. G is a group under multiplication.

2. G is a pseudo ordered set.

3. For $a, b, c \in G$ we have $a \leq b$ if and only if $ca \leq cb$ and $ac \leq bc$.

Note 2. Let G be a pseudo-ordered group. We say that G is *trivially pseudo-ordered* if and only if $g \leq h$ iff $g = h$. We show that this indeed defines a pseudo-ordered group for any group.

Proof. ($\Rightarrow$) It is clear.

($\Leftarrow$) We will show that this " $\leq$ " is a pseudo-order. Then for $g \in G$, $g \leq g$ iff $g = g$, so the reflexive condition is satisfied. Now, for $g, h \in G$ such that $g \leq h$ and $h \leq g$, so each inequality imples that $g = h$. Then the antisymmetric condition holds and therefore " $\leq$ " is a pseudo-order. It is called trivial since every group can be pseudo-ordered with this " $\leq$ ". □

Example 2.1.1. All partially-ordered groups are pseudo-ordered groups, since they hold with the Definition 2.1.1

The following example comes from [10].

Example 2.1.2. Let $\mathbb{Z}$ be a group. Then $(\mathbb{Z}, \preccurlyeq)$ is a pseudo-ordered group by setting

$$k \preccurlyeq m \text{ iff } k \leq m \text{ and either } m \equiv k \pmod 3 \text{ or } m \equiv k+1 \pmod 3$$

for $k, m \in \mathbb{Z}$. Suppose that $k \in \mathbb{Z}$ and $k \preccurlyeq k$ then $k \leq k$ and $k \equiv k \pmod 3$ we will show antisymmetry, let $k, m \in \mathbb{Z}$ and suppose that $k \preccurlyeq m$ and $m \preccurlyeq k$. Then $k \leq m$ and either $m \equiv k \pmod 3$ or $m \equiv k+1 \pmod 3$ and $m \leq k$ and either $k \equiv m \pmod 3$ or $k \equiv m+1 \pmod 3$ Since " $\leq$ " is the usual order on $\mathbb{Z}$, $m = k$. Condition (3) of Definition 2.1.1 is satisfied because if $k \preccurlyeq m$ and $c \in \mathbb{Z}$ then $k \leq m$. So multiplying by c on both sides we get $ck \leq cm$, by the usual order on $\mathbb{Z}$. Moreover, $cm \equiv ck \pmod 3$ or $cm \equiv ck + c \pmod 3$ since $c \equiv c \pmod 3$ Then $ck \preccurlyeq cm$.

We will see later more examples of pseudo-ordered groups. Similarly to partially ordered groups, we have the following:

Theorem 2.1.1. *G is a pseudo-ordered group if and only if for all $g \leq h$, $xgy \leq xhy$ for any $g, h, x, y \in G$.*

Proof. $(\Rightarrow)$ is clear.

$(\Leftarrow)$ Suppose that $g \leq h$ then $xge \leq xhe$. Therefore $xg \leq xh$. Similarly, $gx \leq hx$. □

Theorem 2.1.2. *Let G be a pseudo ordered group, $x \leq y$ iff $y^{-1} \leq x^{-1}$.*

Proof. $x \leq y$ if and only if $e \leq yx^{-1}$ if and only if $y^{-1} \leq y^{-1}yx^{-1} = x^{-1}$. □

The next proposition talks about existence of a special set P, analogous to the positive cone in partially-ordered groups.

Proposition 2.1.3. *A group G can be pseudo ordered if and only if there is a subset P of G such that:*

1. $P \cap P^{-1} = \{e\}$,

2. $gxg^{-1} \in P$ *for each* $g \in G$ *and* $x \in P$.

Proof. ($\Rightarrow$) If G is a pseudo ordered group, we define $P = \{g \geq e : g \in G\}$. Then $g, g^{-1} \in P$ implies that $g \geq e$, and $g^{-1} \geq e$. Then $gg^{-1} \geq ge$, so $e \geq g$ and we get $g = e$ by antisymmetry of " $\leq$ ". Thus $P \cap P^{-1} = \{e\}$. Let $g \in G$ and $x \in P$. Upon multiplying $x \geq e$ on the left by g^{-1} and on the right by g, we obtain $g^{-1}xg \geq g^{-1}eg = e$, hence $g^{-1}xg \in P$. Therefore, P is closed under conjugation.

($\Leftarrow$) Now, let P satisfy the conditions (1) and (2). Let us define a relation "$\leq$" on G by saying

$$g \leq h \text{ iff } hg^{-1} \in P.$$

Then $g \leq g$ because $gg^{-1} = e \in P$. Therefore, " $\leq$ " is reflexive. In order to prove antisymmetry let $g \leq h$ and $h \leq g$. This implies $hg^{-1} \in P$ and $gh^{-1} \in P$. But $gh^{-1} = (hg^{-1})^{-1} \in P^{-1}$. So $gh^{-1} \in P \cap P^{-1} = \{e\}$, so $gh^{-1} = e$. Therefore $g = h$. In order to prove that G is pseudo-ordered by " $\leq$ ", let $g, h, x, y \in G$ and $g \leq h$. Then $hg^{-1} \in P$. Therefore, $x(hg^{-1})x^{-1} \in P$ since P is closed under conjugation. So

$$(xhy)(y^{-1}g^{-1}x^{-1}) = xheg^{-1}x^{-1} = xhg^{-1}x^{-1} \in P.$$

Then $xgy \leq xhy$.

□

We will refer to the set $P = \{g \geq e : g \in G\}$ as the *positive part* of G.

A pseudo-ordered group G with the positive part P will be denoted by (G, P).

For convenience, if (G, P) is a pseudo-ordered group, we will refer to the set P as to the "the pseudo-order P".

Note 3. The trivial pseudo-order is equivalent to $P = \{e\}$.

Let us denote by $\widehat{G}$ the set $\{a \in G : a^{-1} = gag^{-1}$ for some $g \in G\}$. The following proposition shows the relationship between P and $\widehat{G}$.

Proposition 2.1.4. *Let (G, P) be a pseudo-ordered group,*

$$(P \cup P^{-1}) \cap \widehat{G} = \{e\}.$$

Proof. Assume $a \in (P \cup P^{-1}) \cap \widehat{G}$. Then $a \in (P \cup P^{-1})$ and $a \in \widehat{G}$. Suppose $a \in P \cap \widehat{G}$. Then there exists $g \in G$ such that $gag^{-1} = a^{-1}$, so $a^{-1} \in P$ by condition (2) of Proposition 2.1.3. Therefore, $a = e$ by Proposition 2.1.3. Similarly the case when $a \in P^{-1} \cap \widehat{G}$. □

Example 2.1.3. Let S_n be the group of permutations of n objects. Then S_n does not admit a non-trivial pseudo-order. By [6] every $\sigma \in S_n$ is a conjugate of its inverse, i.e, $S_n = \widehat{S_n}$. So by Proposition 2.1.4 if (S_n, P) is a pseudo-order then $P = \{e\}$.

Example 2.1.4. Let A_n be the alternating group of all even permutations of n objects. For $n = 1, 2, 5, 6, 10, 14$, A_n does not admit a nontrivial pseudo-order. Indeed, by theorem 1.2 of [6], $A_n = \widehat{A_n}$. So by Proposition 2.1.4, if (A_n, P) is pseudo-ordered then $P = \{e\}$.

Let us recall that an element $a \in G$ has order n if n is the least natural number such that $a^n = e$. The order of an element a is denoted by $o(a)$. The set of all elements of G of order two will be denoted by $o^2(G)$.

Note 4. For $a \in G$, $o(a) = 2$ if and only if $a = a^{-1}$. Since for any $a \in o^2(G)$ we have $eae^{-1} = a^{-1}$. It follows $o^2(G) \subseteq \widehat{G}$.

The following corollary makes use of the previous note.

Corollary 1. *Let (G, P) be a pseudo-ordered group. The pseudo-order P does not have elements of order two.*

Proof. Since $o^2(G) \subseteq \widehat{G}$ and by Proposition 2.1.4 $(P \cup P^{-1}) \cap \widehat{G} = \{e\}$, we obtain $o^2(G) \cap P = \{e\}$.

□

In the abelian case we will use the additive notation. If G is an abelian group, condition (2) in the Proposition 2.1.3 is always true, so in order to verify that (G, P) is a pseudo-ordered group it is enough to check if $P \cap -P = \{0\}$. So we have the following.

Corollary 2. *Let G be an abelian group. If $a \in G$ and $o(a) \neq 2$ then G can be pseudo-ordered by $P = \{0, a\}$.*

Proof. Let P be the set $P = \{0, a\}$, $a \in G \setminus o^2(G)$. Then we need to show that P is a pseudo order on G. Since $a \notin o^2(G)$, $a \neq -a$. By hypothesis, $a \in P$ so $-a \in -P$. So $-P = \{0, -a\}$ and therefore $P \cap -P = \{0\}$. □

Recall the following e.g. [7]:

Definition 2.1.2. Let a and b be elements of a group G. We say that a and b are *conjugate* in G if $xax^{-1} = b$ for some $x \in G$. The *conjugacy class* of a is the set

$$cl(a) = \{xax^{-1} : x \in G\}$$

.

Proposition 2.1.5. *Let G be any group. Then G can be pseudo-ordered by $P = \{e\} \cup cl(a)$, where $a \in G \setminus \widehat{G}$.*

Proof. Assume there is $e \neq x \in P \cap P^{-1}$. Then for some $g, h \in G$ $gag^{-1} = x = ha^{-1}h^{-1}$ (since $P^{-1} = \{e\} \cup cl(a^{-1})$). So

$$(h^{-1}g)a(h^{-1}g)^{-1} = h^{-1}gag^{-1}h = h^{-1}xh = a^{-1}.$$

But then $a \in \widehat{G}$, which contradicts the assumption about a. So $x = e$. We will show that this P is closed under conjugation. Let $x \in P$ be any element, then $gxg^{-1} = e$ or

$$gxg^{-1} = ghah^{-1}g^{-1} = (gh)a(h^{-1}g^{-1})$$

since $x \in cl(a)$. Since $(gh)a(gh)^{-1} \in cl(a)$, so

$$gxg^{-1} = gha(gh)^{-1} \in P.$$

$\square$

Example 2.1.5. Let A_4 be the alternating group of even permutations of $\{1, 2, 3, 4\}$. Then A_4 is a non-abelian pseudo-ordered group, with a non-trivial pseudo-order

$$P = \{\{(e)\} \cup cl((123))\}.$$

The condition $P \cap P^{-1} = \{e\}$ is satisfied because if $x \in P \cap P^{-1}$ then $x = e$ or $x \in cl((123))$ and $x \in cl((132))$. So if $x \in cl((123))$ and $x \in cl((132))$ then $(132) = g(123)g^{-1}$ for some $g \in A_4$. So $(132) \in cl((123))$, which is not true since $cl((123)) = \{(123), (134), (243), (142)\}$. Thus $x = e$. To show that P is closed under conjugations let $x \in P$ and $g \in A_4$. Then if $x = e$, $gxg^{-1} = e$. However, if $x \neq e$ then for some $h \in A_4$

$$gxg^{-1} = gh(123)h^{-1}g^{-1}$$

so $gxg^{-1} \in cl(123)$. Therefore, $gxg^{-1} \in P$.

Example 2.1.6. The dihedral group D_4 of eight elements does not have a non-trivial pseudo-order P. The elements of D_4 in the notation of [7] are

$$R_0, R_{90}, R_{180}, R_{270}, V, H, D, D'$$

The conjugacy classes are

$$cl(R_0) = \{R_0\}, cl(R_{90}) = cl(R_{270}) = \{R_{270}, R_{90}\},$$

$$cl(V) = \{V, H\} = cl(H), cl(D) = \{D, D'\} = cl(D'), cl(R_{180}) = \{R_{180}\}.$$

We can see that the only two elements that are not of order two are R_{270} and R_{90}. However, R_{270} is a conjugate of its inverse R_{90}. Therefore, by Proposition 2.1.4. the only pseudo-order on D_4 is the trivial one.

Example 2.1.7. The quaternion group Q is a group of order 8 which may be presented as a group of two generators a and b subject to the relations $a^4 = e$, $b^2 = a^2$, and $aba = b$. To be specific, the elements of Q are

$$e, a, a^2, a^3, b, ab, a^2b, a^3b.$$

So the conjugacy classes of Q are

$cl(a) = \{a, a^3\} = cl(a^3)$, $cl(a^2) = \{a^2\}$, $cl(e) = \{e\}$, $cl(b) = \{a^2b, b\} = cl(a^2b)$, $cl(ab) = \{a^3b, ab\} = cl(a^3b)$.

Therefore, it can be easily seen that $\widehat{Q} = cl(e) \cup cl(a) \cup cl(a^2) \cup cl(b) \cup cl(ab)$, which is all Q. Therefore, Q cannot be pseudo-ordered.

Example 2.1.8. The dihedral group D_5 of ten elements does not have a non-trivial pseudo-ordered P. If the elements of D_5 are denoted by:

$$e, A, B, C, D, E, F, G, H, I,$$

then by [7] the conjugacy classes are

$cl(e) = \{e\}, cl(B) = \{B, C\} = cl(C), cl(A) = \{A, D\} = cl(D), cl(E) = \{E, F, G, H, I\} = cl(F) = cl(G) = cl(H) = cl(I)$.

We can see that the only four elements that are not of order two are A, B, C, D . However, A is a conjugate of its inverse D and B is a conjugate of its inverse C. Therefore, by Proposition 2.1.4. the only pseudo-order on D_5 is the trivial one.

Theorem 2.1.6. *A non-abelian non-trivially pseudo-ordered group must have order at least 12.*

Proof. By Examples 2.1.3, 2.1.4, 2.1.6, 2.1.7, 2.1.8 all the non-abelian groups of order less than 12 do not admit a non-trivial pseudo-order. (See classification af all groups of small order in [9].) However, the Example 2.1.5 shows that A_4 can be non-trivially pseudo-ordered, and $o(A_4) = 12$. □

2.2 Maximal Pseudo-orders

In this section we will study pseudo-orders with maximal pseudo-orders, i.e. those with a the maximum number of elements.

Definition 2.2.1. Let (G, P) be a pseudo-ordered group. We say that the pseudo-order P is maximal if for any pseudo-order P' on G, the condition $P \subseteq P'$ implies $P = P'$.

Theorem 2.2.1. *Any pseudo-ordered group G contains at least one maximal pseudo-order.*

Proof. Let $\Im$ be the set of all pseudo-orders on G. $\Im \neq \varnothing$ since the trivial pseudo-order is in $\Im$ and $(\Im, \subseteq)$ is a partially-ordered set. Let $T \subseteq \Im$ be totally-ordered by inclusion. We will show that T has an upper bound, i.e. there exists $P' \subseteq G$ such that $P_i \subseteq P'$, for every $P_i \in T$. We totally order the index set of all $i's$, by a relation $\preccurlyeq$, so that $i \preccurlyeq k$ implies $P_i \subseteq P_k$.

Let $P' = \cup_i P_i$, $P_i \in T$. We must demonstrate that P' is a pseudo-order on G.

We show that $(\cup P_i) \cap (\cup (P_i)^{-1}) = \{e\}$. Assume there is $x \in (\cup P_i) \cap (\cup (P_i)^{-1})$. Then $x \in \cup P_i$ and $x \in \cup P_i^{-1}$. This means that $x \in P_j$ for some j and $x \in P_k^{-1}$ for some k. If $j \preccurlyeq k$ then $P_j \subseteq P_k$ and since $P_k \cap (P_k)^{-1} = \{e\}$, then $x = e$. Now if $k \preccurlyeq j$ then $(P_k)^{-1} \subseteq (P_j)^{-1}$ and $P_j \cap (P_j)^{-1} = \{e\}$. Then $x = e$. Therefore, $P' \in \Im$ is a pseudo-order on G and P' is an upper bound of T.

By Zorn's lemma, $\Im$ has a maximal element, i.e. G has at least one maximal pseudo-order. □

The following theorem characterizes the maximal pseudo-orders on pseudo-ordered groups with the help of the set $\widehat{G}$.

Theorem 2.2.2. *P is a maximal pseudo-order on G if and only if*

$$G = P \cup P^{-1} \cup \widehat{G}.$$

Proof. ($\Rightarrow$) Let P be a maximal pseudo-order on G and let $a \notin P$. Let further $S = P \cup G_a$, where $G_a = \{gag^{-1} : g \in G\}$.

Since P is maximal pseudo-order and $P \subseteq S$, we must have then $S \cap S^{-1} \neq \{e\}$. We show that S is closed under conjugations. Suppose $x \in S$ so $x \in P$ or $x \in G_a$. Then if $x \in P$, $gxg^{-1} \in P$. And if $x \in G_a$, then

$$gxg^{-1} = ghah^{-1}g^{-1} = (gh)a(gh)^{-1}.$$

Then $gxg^{-1} \in S$. So there exists $e \neq x \in S \cap S^{-1}$. By set theory,

$$S \cap S^{-1} = (P \cap P^{-1}) \cup (G_a \cap P^{-1}) \cup (P \cap (G_a)^{-1}) \cup (G_a \cap (G_a)^{-1}).$$

Then $x \notin P \cap P^{-1}$ because $x \neq e$ and P is a pseudo-order. So if $x \in G_a \cap P^{-1}$ then $x = gag^{-1} \in P^{-1}$ for some $g \in G$. So $(gag^{-1})^{-1} \in P$ hence $ga^{-1}g^{-1} \in P$. From Proposition 2.1.3 condition (2) we have $a^{-1} = g^{-1}ga^{-1}g^{-1}g \in P$, so $a \in P^{-1}$.

Now, if $x \in P \cap (G_a)^{-1}$, we similarly obtain that $a^{-1} \in P$ which implies that again $a \in P^{-1}$.

Finally, if $x \in G_a \cap (G_a)^{-1}$, we have that

$$gag^{-1} = x = ha^{-1}h^{-1}$$

for some $g, h \in G$. Upon multiplying on the left by h^{-1} and on the right by h we obtain $h^{-1}gag^{-1}h = a^{-1}$. Hence, $(h^{-1}g)a(h^{-1}g)^{-1} = a^{-1}$ so $a^{-1} \in G_a$ and $a \in G_a^{-1}$. Therefore,

$a \in \widehat{G}$. We conclude that either $a \in P$ or $a \in P^{-1}$ or $a \in \widehat{G}$.

($\Leftarrow$) Now suppose that $G = P \cup P^{-1} \cup \widehat{G}$, where P is a pseudo-order and that $P \subseteq M \subseteq G$ and M is a pseudo-order. Take $x \in M \backslash P$. By hypothesis $x \in P^{-1}$ or $x^{-1} = gxg^{-1}$ for some $g \in G$. If $x \in P^{-1}$, then $x^{-1} \in P$ and since M is a pseudo-order with $P \subseteq M$, $x = e$. The other option is $x^{-1} = gxg^{-1}$, but we know by Proposition 2.1.4 that $\widehat{G} \cap M = \{e\}$. Therefore, $x = \{e\}$

$\square$

Let us return to the Example 2.1.3

Example 2.2.1. Let A_4 the alternating group of even permutations of the elements {1,2,3,4}. The elements of the group are

$$e, (12)(34), (13)(24),\ (14)(23), (123), (134), (243),\ (142), (132), (143), (234), (124).$$

The pseudo-order P used in Example 2.1.3 is a maximal pseudo-order since

$$P \cup P^{-1} \cup \widehat{A_4} = \{e\} \cup cl((123)) \cup cl((132)) \cup \widehat{A_4} = A_4.$$

That is,

$$P = \{e\} \cup \{(123), (134), (243), (142)\}$$

and

$$P^{-1} = \{(e)\} \cup \{(132), (143), (234), (124)\}$$

and $\widehat{A_4} = \{(1), (12)(34), (13)(24), (14)(23)\}$.

The next corollary is about abelian groups.

Corollary 3. *P is a maximal pseudo-order in G if and only if*

$$G = P \cup -P \cup o^2(G).$$

Proof. Since G is abelian, $o^2(G) = \widehat{G}$. So the claim of the corollary follows inmediately from Theorem 2.2.2. $\square$

2.3 Homomorphisms

In this section, we will define a pseudo-ordered group homomorphism and pseudo-ordered group isomorphism. In addition, we shall discuss some theorems of pseudo-ordered homomorphisms.

Definition 2.3.1. Let G, H be pseudo-ordered groups and $f : G \to H$ be a function. Then f is called a *homomorphism* of pseudo-ordered groups if f is a group homomorphism and an isotone function, i.e, if $a, b \in G$ we have $a \leq b$ then $f(a) \leq f(b)$.

The function $f : (G, P) \to (H, Q)$ is an *isomorphism* of pseudo-ordered groups if f is a group isomorphism and both f and f^{-1} are isotone functions.

The following theorem involves the concept of pseudo-ordered isomorphisms.

Theorem 2.3.1. *$f : (G, P) \to (H, Q)$ is a pseudo-ordered group isomorphism if and only if f is a group isomorphism and $f(P) = Q$, where $P = \{g \in G : g \geq_G e\}$ and $Q = \{h \in H : h \geq_H e\}$.*

Proof. ($\Rightarrow$) For any element $g \in P$ such that $g \geq_G e$ we have $f(g) \geq_H f(e) = e$, so $f(P) \subseteq Q$. Now let $h \in Q$, then $h \geq_H e$ and since f is an isomorphism $h = f(g)$, for some $g \in G$. So we have $h = f(g) \geq_H e$ so $g \geq_G e$. Hence $Q \subseteq f(P)$, so $Q = f(P)$.

($\Leftarrow$) Let $f(P) = Q$ and let f be a group isomorphism. Suppose that $a \leq b$. Then $ba^{-1} \in P$ so we have $f(ba^{-1}) \in Q$. So $f(ba^{-1}) \geq_H e$ and so $f(b)f(a)^{-1} \geq_H e$. Therefore, $f(b) \geq f(a)$ so f is isotone Conversely, if $f(a) \leq f(b)$ then $f(b)f(a)^{-1} \in Q$, so $f(ba^{-1}) \in Q = f(P)$. Thus, $ba^{-1} \in P$ so $a \leq b$, therefore f^{-1} is isotone. $\square$

Proposition 2.3.2. *Let (G, P) be a pseudo-ordered group, and let $f : G \to H$ be a group isomorphism. Then $(H, f(P))$ is a pseudo-ordered group and f is a pseudo ordered group isomorphism.*

Proof. We show that $f(P)$ is a positive part of H. Suppose that there exists $e \neq x \in H$ such that $x \in f(P) \cap f(P)^{-1}$. So $x \in f(P)$ and $x \in f(P)^{-1}$ which implies that $x = f(p_1)$, for some $p_1 \in P$ and $x = f(p_2)$ for some $p_2 \in P^{-1}$. But we know that $p_1 \neq p_2$ since $P \cap P^{-1} = \{e\}$, and we have $f(p_1) = f(p_2)$. Therefore, f is not injective. This contradicts the hypothesis of f being an isomorphism. Moreover, let $h \in H$ and $p \in P$. We have $hf(p)h^{-1} = f(g)f(p)f(g)^{-1} = f(gpg^{-1}) \in f(P)$ where $g \in G$. Then $(H, f(P))$ is a positive part of a pseudo-ordered group. The isomorphism f is a pseudo-ordered group isomorphism by Theorem 2.3.1.

□

Corollary 4. *Let (G, P) be a pseudo-ordered group. Every group automorphism $f : G \to G$ is a pseudo-ordered group isomorphism.*

Proof. Inmediate from Proposition 2.3.2 with $H = G$. □

The next corollary shows the previous results to finite pseudo-ordered groups.

Corollary 5. *Let G be a finite group and P and Q be a pseudo-orders on G. Then $f : (G, P) \to (G, Q)$ is an isomorphism of pseudo-ordered groups if f is injective and isotone.*

Proof. If f is injective then f is surjective, since G is finite. By hipothesis f is isotone, so f is an isomorphism of pseudo-ordered groups by Theorem 2.3.1. □

Chapter 3

The Pseudo-ordered Group $\mathbb{Z}_n$

In this chapter we will concentrate on the structure of the pseudo-ordered group $\mathbb{Z}_n$. In particular, we will find the necessary and sufficient conditions for existence of induced pseudo chains on $\mathbb{Z}_n$.

3.1 Preliminaries on the Group $\mathbb{Z}_n$

Recall that $\mathbb{Z}_n = \{0, 1, ...n-1\}$ is a group under addition. The addition on $\mathbb{Z}_n$ is taken by modulo n, i.e, for a and b belonging to $\mathbb{Z}_n$ we have $a \oplus b = r$ where $a + b = np + r$ for $p, r \in Z$ and $0 \leq r < n$. Since $\mathbb{Z}_n$ is abelian, the conjugates of elements are trivial, so the only condition for the set $P \subseteq \mathbb{Z}_n$ to be the positive part of a pseudo-order is that $P \cap -P = \{0\}$.

Note that if $n = 2$ then by Corollary 1 of Theorem 2.1.4. the only pseudo-order is the trivial one. So let $n \geq 3$ and consider the pseudo-ordered group $(\mathbb{Z}_n, P)$

Lemma 3.1.1. *If n is even, $\mathbb{Z}_n$ has only one element of order two.*

Proof. Suppose that n is even. We have that for any $a \in \mathbb{Z}_n$ such that $a \in \{\frac{n}{2}+1, ..., n-1\}$ its inverse is $-a \in \{0, 1, ..., \frac{n}{2} - 1\}$. That is,

$$\mathbb{Z}_n = \{0, 1, ..., \frac{n}{2}, ..., n-1\}$$

if $m \in \mathbb{Z}_n$ is on the right side of $\frac{n}{2}$, then $(-m)$ is on the left side of $\frac{n}{2}$. the inverse of $\frac{n}{2}$ is $\frac{n}{2}$. So $\frac{n}{2}$ is the one element of order two.

□

Lemma 3.1.2. *If n is odd, $\mathbb{Z}_n$ has no elements of order two.*

Proof. Let n be an odd number. So

$$\mathbb{Z}_n = \{0, 1, ..., \frac{n+1}{2}, ..., n-1\}.$$

We have that if $m \in \{\frac{n+3}{2}, ..., n-1\}$, then $(-m) \in \{0, 1, ..., \frac{n-1}{2}\}$. The inverse of $\frac{n+1}{2}$ is $\frac{n-1}{2}$ and $\frac{n-1}{2} \in \{0, 1, ..., \frac{n-1}{2}\}$. So m, $m \neq -m$.

□

3.2 Pseudo Chains on $\mathbb{Z}_n$

In this section we will give a formal definition of a pseudo chain. Moreover, we will see the specific needs of a pseudo-order P to induce a pseudo chain.

Definition 3.2.1. Let (G, P) be a pseudo-ordered group, we say that P *induces a pseudo chain* on G if for each pair of $g, g' \in G$ at least one of

$$g \leq g_1 \leq \ldots \leq g_n \leq g' \text{ or } g' \leq g_1 \leq \ldots \leq g_n \leq g$$

holds, for some $g_1, g_2, ..., g_n \in G$.

The next theorems say what happens if the unity element of $\mathbb{Z}_n$ is in a pseudo-order on $\mathbb{Z}_n$.

Theorem 3.2.1. *Let $(\mathbb{Z}_n, P)$ be a pseudo-ordered group, if P contains 1 or -1 then P induces a pseudo chain on $\mathbb{Z}_n$.*

Proof. Since $1 \in P$ we have that

$$0 \leq 1 \leq 2 \leq 3 \leq \cdots \leq n-1.$$

This implies that we can connect any two elements of $\mathbb{Z}_n$ because every element of $\mathbb{Z}_n$ is in this chain. □

Example 3.2.1. Let us take $(\mathbb{Z}_4, P)$, a pseudo-ordered group with $P = \{0, 1\}$. We obtain the following pseudo chain

$$0 \leq 1 \leq 2 \leq 3.$$

we can clearly connect any number of $\mathbb{Z}_4$ with another of $\mathbb{Z}_4$.

The converse, however, is not true.

Example 3.2.2. Take $\mathbb{Z}_5$ and $P = \{0, 2\}$ this P induces a pseudo chain on $\mathbb{Z}_5$, since the only pseudo chain is

$$0 \leq 2 \leq 4 \leq 1 \leq 3,$$

so it is possible to connect every two elements of $\mathbb{Z}_5$, however $1 \notin P$ and $-1 \notin P$.

Corollary 6. *Let $(\mathbb{Z}_n, P)$ be a pseudo-ordered group. If P is a maximal pseudo-order on $\mathbb{Z}_n$ then P induces a pseudo chain on $\mathbb{Z}_n$.*

Proof. Since P is maximal, by Corollary 1 of Theorem 2.2.2 $\mathbb{Z}_n = P \cup -P \cup o^2(\mathbb{Z}_n)$. Since $n > 2$, $1 \notin o^2(\mathbb{Z}_n)$, so $1 \in P$ or $1 \in -P$, so $1 \in P$ or $-1 \in P$. By Theorem 3.2.1 P induces a pseudo chain on $\mathbb{Z}_n$. □

The converse, however, is not true, because again in Example 3.2.2 $P = \{0, 2\}$ induces a pseudo chain, but $P \subseteq P' = \{0, 1, 2\}$, and P' is a pseudo-order on $\mathbb{Z}_5$.

In case when a pseudo-ordered group of order infinite the previous corollary is not necessarly satisfied.

Example 3.2.3. Let $G = \mathbb{Z} \times \mathbb{Z}_2$. If we take,

$$P = \{\{(a, b) : a > 0\} \cup \{(0, 0)\}\}$$

G becomes a pseudo-order group.

Claim. There is no pseudo chain on G.

Proof. Let $0 < a \in Z$ and $x = (a, 0)$, $y = (a, 1)$ in G. We show there is no pseudo chain from x to y. Let $U(x)$ and $L(y)$ be the strict upper set and the strict lower set of x and y, respectively. Then $U(x) = \{(a_i, 0), (a_i, 1) : a_i > a \text{ for any } i\}$ and $L(y) = \{(a_j, 0), (a_j, 1) : a_j < a \text{ for any } j\}$. So assume that there exists $z \in U(x) \cap L(y)$. This means that $z = (a_z, 0)$ or $z = (a_z, 1)$ with $a_z > a$. Moreover, $z \in L(y)$, so $z = (a_z, 0)$ or $z = (a_z, 1)$ with $a_z < a$, which is impossible in a pseudo-order. Therefore, $U(x) \cap L(y) = \phi$. Thus, we cannot connect x and y.

□

The following is our main theorem about $\mathbb{Z}_n$.

Theorem 3.2.2. *Let $(\mathbb{Z}_n, P)$ be a pseudo-ordered group with $P = \{0, a_1, ..., a_r\}$. Then P induces a pseudo chain on $\mathbb{Z}_n$ if and only if $gcd(n, a_1, ..., a_r) = 1$.*

Proof. ($\Rightarrow$) Let us for convenience, denote $a_0 = n$. Assume that $\mathbb{Z}_n$ is a pseudo chain induced by P. By hypothesis there exist $b_1, ..., b_k \in \mathbb{Z}_n$ such that

$$0 = b_0 \leq b_1 \leq ... \leq b_k \leq b_{k+1} = 1,$$

where

$$b_1 = a_{i_1}$$

$$b_2 = b_1 + a_{i_2}$$

$$\vdots$$

$$b_j = b_{j-1} + a_{i_j}$$

$$\vdots$$

$$b_k = b_{k-1} + a_{i_k}$$

and $a_{i_j} \in P$ for all i, j. So every equality above has the form

$$b_j = \text{non-negative integer combination of elements of } P.$$

Therefore, as an integer

$$1 = m_0a_0 + m_1a_1 + ... + m_ra_r = m_0n + m_1a_1 + ... + m_ra_r,$$

for some integers $m_0, ..., m_r$. Therefore, we have

$$1 = m_0n + m_1a_1 + ... + m_ra_r.$$

We conclude that

$$gcd(n, a_1, ..., a_r) = 1$$

by Theorem 1.1.4.

($\Leftarrow$) Suppose that $gcd(n, a_1, ..., a_r) = 1$. Then by Theorem 1.1.4. we can write

$$m_0n + m_1a_1 + ... + m_ra_r = 1.$$

Then modulo n,

$$m'_1a_1 + ... + m'_ra_r = 1,$$

where $m'_i \equiv m_i \ (modn)$.

We want to construct a pseudo chain from any element of $\mathbb{Z}_n$ to another element of $\mathbb{Z}_n$. Let us take $g, h \in \mathbb{Z}_n$ with $c = g - h$. Upon multiplying both sides of the above equality by c in $\mathbb{Z}_n$ we get $m'_1a_1c + ... + m'_ra_rc = c$. Let us put $x_1 = m'_1c$, ... , $x_r = m'_rc$. Then we obtain

$$h \leq h + a_1 \leq h + a_1 + a_1 \leq \cdots \leq h + a_1 + ... + a_1 = h + x_1a_1 \leq h + x_1a_1 + a_2$$

$$\leq h + x_1a_1 + a_2 + a_2 \leq \cdots \leq h + x_1a_1 + x_2a_2$$

$$\leq h + x_1a_1 + x_2a_2 + a_3 \leq \cdots \leq h + x_1a_1 + x_2a_2 + x_3a_3.$$

Continuing this way we arrive at :

$$h \leq \cdots \leq h + x_1a_1 + x_2a_2 + ... + x_ra_r = h + c = g.$$

Therefore, we can connect any two elements, i.e, $\mathbb{Z}_n$ is a pseudo chain induced by P. $\square$

Chapter 4

Conclusions

We developed the concept of a pseudo-ordered group, showing some theorems and propositions. Moreover, we gave an equivalent result of the positive cone of partially-ordered groups, defining the positive part of a pseudo-ordered group similarly to the positive cone. We got a relationship between a pseudo-order P and the set $\widehat{G}$ whose elements are a conjugate of their inverses. Besides, we proved that some important groups can not be non-trivially pseudo-ordered such as the group of permutations of n elements S_n, the quaternion group Q, the dihedral groups D_4 and D_5, and finally, the alternating group A_n for some specific n.

In addition, we defined maximal pseudo-orders on pseudo-ordered groups. One of our main results was the characterization of maximal pseudo-orders, using the set $\widehat{G}$ as the main tool. We also gave analogous results for the abelian group.

We used the group $\mathbb{Z}_n$ to pseudo-order it in many different ways. Moreover, we described all the pseudo chains of the group $\mathbb{Z}_n$. Finally with computers help, we counted the number of non-isomorphic pseudo-orders on $\mathbb{Z}_n$ for $n < 24$.

References

[1] I . A. Barnett, *Elements of Number Theory*, Ohio University, University of Cincinnati, Boston, 1969.

[2] G. Birkhoff and R. Pierce, "Lattice-ordered rings," *Anais da Academia Brasileira de Ciencias*, 1956, Vol. 28, pp. 41-69.

[3] David M. Burton, *Elementary Number Theory*, University of New Hamspshire, Boston, 2002.

[4] Allan, Clark, *Elements of Abstract Algebra*, Purdue University , NY, 1935.

[5] M. Darnel, *Theory of Lattice-ordered Groups*, Marcel Dekker, NY, 1994.

[6] J. L. Berggren, *Finite Groups In Which Every Element is Conjugate to its Inverse*, Pacific Journal of Mathematics Vol 28, No. 2, 1969.

[7] Joseph A. Gallian, *Contemporary Abstract Algebra*, University of Minnesota Duluth, Minnesota, 2002.

[8] I. M. Isaacs, *Algebra a Graduate Course*, Brooks/Cole Publishing Company, Belmont CA, 1993.

[9] Joseph J. Rotman, *The Theory of Groups: An Introduction*, Allyn and Bacon, Inc., Boston, 1965.

[10] Helen Skala, *Trellis Theory*, 1973.

Appendix A

Algorithms

In this appendix we will present an algorithm to obtain the positive parts of $\mathbb{Z}_n$ to count the number of non-isomorphic ones. At the end of the appendix we will give particular examples.

A.1 Representations Positive Parts

This fragment of the algorithm gives the represantation of the positive parts of $\mathbb{Z}_n$.

```
Representation of the  cone (positive part) in java class:


/**
 * The class of cone implements the notion of a cone (positive part) of a psedo-p.o.
 * set in a finite cyclic group.
 * @author Lola
 */
public class Cone {


    private int groupCard; //the cardinality of the finite cyclic group
    private ArrayList<Integer> list; //the set of positive elements


    /**
     * Creates a cone given a consistent binary representation (refer to the
     * definition of ConesWinView.isConsistent) and the cardinality
```

```
 * @param binRep a consistent binary representation
 * @param grCard the cardinality of the finite cyclic group
 */
public Cone(String binRep, int grCard){
    groupCard = grCard;
    list = new ArrayList();
    list.add(new Integer(0));
    int strLen = binRep.length();
    int half = -1;
    boolean odd = grCard % 2 == 1;

    if(!odd){
        half = strLen/2;
    }//end if
    //create the set of the cone according to the binary
    //representation
    for(int i = 0; i < strLen; i++){
       if(binRep.charAt(i) == '1'){
           if(odd){
               list.add(new Integer(i+1));
           }//end if
           else{
               if( i < half){
                   list.add(new Integer(i+1));
               }
               else{
                  list.add(new Integer(i+2));
               }//end if-else
```

```
            }//end if-else
        }//end if
    }//end for
}//end Cone

/**
 * Creates a cone with the specified set of positive elemets and the given
 * cardinality
 * @param posElem a set of positive elemets that is a positive part for
 *        Z_{grCard}
 * @param grCard the cardinality of the finite cyclic group
 */
public Cone(ArrayList<Integer> posElems, int grCard){
    list = posElems;
    groupCard = grCard;
}//end Cone

/**
 * Produces the image of the cone under the given automorphism preserving
 * the total order of the correspondent integer elements
 * e.g. in Z_4 for C = {0,3} and f in Aut(Z_4) with f(x) = 3*x (mod 4)
 * comageCone(f) = {0,1}
 * @param automorphism represented as an integer b/w 1 and groupCard - 1
 *        with gcd(automotphism, groupCard) = 1.
 * @return the image of this cone.
 */
public Cone imageCone(int automorphism){
    Cone imCone; //the image of the cone
```

```
    int autOfElem; //the image of an element of the cone
    int insertionIndex = 0; //used to preserve the total ordering
    ArrayList<Integer> imageSet = new ArrayList();
    for(int i = 0; i < list.size(); i++){
       autOfElem = automorphism*(list.get(i).intValue()) % groupCard;
       //find insertion index
       insertionIndex = 0;
       while(insertionIndex < imageSet.size()
              && autOfElem > imageSet.get(insertionIndex).intValue()){
          insertionIndex++;
       }//end while
       imageSet.add(insertionIndex, new Integer(autOfElem));
    }// end for
    imCone = new Cone(imageSet, groupCard);
    return imCone;
}//end imageCone

/**
 * Retrives the integer that is at the position index
 * e.g. in C = {0,2,3} in Z_7 getElement(2) = 3
 * @param index an integer that is in b/w 0 and groupCard - 1
 * @return the integer at the given index
 */
public Integer getElement(int index){
    return list.get(index);
}//end getElement

/**
```

```
 * Retrives the cardinality of the cone
 * e.g. in C = {0,3} in Z_4 getConeCardinlity() = 2
 * @return the cardinality of the cone
 */
public int getConeCardinality(){
    return list.size();
}//end getConeCardinality

@Override
public int hashCode() {
    int hash = 7;
    hash = 11 * hash + (this.list != null ? this.list.hashCode() : 0);
    return hash;
}

@Override
public boolean equals(Object c){
    Cone cone;
    if(c instanceof Cone){
        cone = (Cone)c;
    }
    else{
        return false;
    }
    if(list.size() != cone.getConeCardinality()){
        return false;
    }
    for(int i = 0; i < list.size(); i++){
```

```
            if(!list.get(i).equals(cone.getElement(i))){
                return false;
            }

        }
        return true;
    }//end equals

    /**
     * Produces a string representation of the cone
     * in set notation e.i. {a_1, a_2,..., a_k}.
     * @return the string representation
     */
    public String toSetNotation(){
        int coneSize = list.size();
        String setRep = "{"; //string that represents the set
        for(int i = 0; i < coneSize-1; i++){
            setRep = setRep + list.get(i).toString() + ", ";
        }//end for
        if(coneSize > 0){
            setRep = setRep + list.get(coneSize-1);
        }//end if
        return setRep + "}";
    }//end toSetNotation

}//end class definition
```

A.2 Classification of Positive parts

This part of the programm classifies all positive parts according to the automorphisms of $\mathbb{Z}_n$. Then the algorithm counts the equivalence classes of the positive parts and gives the number of them.

```
private ArrayList<ArrayList<Cone>>[] generateClasses(int card){
    int size;
    if(card % 2 == 0){
        size = card / 2;
    }
    else{
        size = (card + 1) / 2;
    }

    this.jProgressBarClassGen.setMinimum(0);
    this.jProgressBarClassGen.setMaximum(100);

    //initialize collection
    ArrayList<Cone>[] collectorBySize = new ArrayList[size];
    for(int i = 0; i < size; i++){
        collectorBySize[i] = new ArrayList();
    }

    Cone tmpCone;
    String numRep;
    String flagStr = "";
    int maxLen, n;
```

```
if(card % 2 == 0){
    maxLen = card - 2;
}
else{
    maxLen = card - 1;
}
n = (int) Math.pow(2, maxLen);
int half = maxLen/2;
String tmp;

for(int i = 0; i < half; i++){
    flagStr = flagStr + "1";
}
for(int i = 0; i < n; i++){
    numRep = Integer.toBinaryString(i);

    if(numRep.length() == maxLen){
        tmp = (numRep.substring(0, half));
        if(tmp.equalsIgnoreCase(flagStr))
            i = n;
    }

    tmp = formatBinStr(numRep,maxLen);
    if(isConsistent(tmp,maxLen)){
        tmpCone = new Cone(tmp,card);
        //add cone by correspondent cardinality
        collectorBySize[tmpCone.getConeCardinality()-1].add(tmpCone);
    }
```

```
    }

    //once all cones are in the same size class collect them by
    //equivalence class;
    ArrayList<Integer> automorphisms = generateSetOfRelPrimes(card);
    ArrayList<ArrayList<Cone>> partition;
    ArrayList<ArrayList<Cone>>[] finalCollection = new ArrayList[size];

    this.jProgressBarClassGen.setValue(50);
    for(int i = 0; i < collectorBySize.length; i++){
        partition = generateEquivalenceClasses(collectorBySize[i],automorphisms);
        finalCollection[i] = partition;
    }


    this.jProgressBarClassGen.setValue(100);

    return finalCollection;
}

/**
 * Produces the set of all equivalence classes of
 * @param allCones
 */
private ArrayList<ArrayList<Cone>>
        generateEquivalenceClasses(ArrayList<Cone> allCones, ArrayList<Integer> aut
```

```
ArrayList<ArrayList<Cone>> equivClasses = new ArrayList();
Cone tmpCone, image;
ArrayList<Cone> tmpConeSet;
int removalIndex = 0;
boolean cont = true;




while(!allCones.isEmpty()){
    tmpCone = allCones.get(0);
    tmpConeSet = new ArrayList<Cone>();

    equivClasses.add(tmpConeSet);
    for(int i = 0; i < aut.size() && cont; i++){
        image = tmpCone.imageCone(aut.get(i).intValue());
        removalIndex = allCones.indexOf(image);
        if(removalIndex != -1){

            tmpConeSet.add(image);

            allCones.remove(removalIndex);

        }
        else{
            cont = false;

        }
```

```
        }//end for
        cont = true;
    }//end while


    return equivClasses;
}

private static ArrayList generateCones(int card){
    ArrayList allCones = new ArrayList();
    Cone tmpCone;
    String numRep;
    String flagStr = "";
    int maxLen, n;
    if(card % 2 == 0){
        maxLen = card - 2;
    }
    else{
        maxLen = card - 1;
    }
    n = (int) Math.pow(2, maxLen);
    int half = maxLen/2;
    String tmp;

    for(int i = 0; i < half; i++){
        flagStr = flagStr + "1";
    }
    for(int i = 0; i < n; i++){
```

```
        numRep = Integer.toBinaryString(i);

        if(numRep.length() == maxLen){
            tmp = (numRep.substring(0, half));
            if(tmp.equalsIgnoreCase(flagStr))
                i = n;
        }

        tmp = formatBinStr(numRep,maxLen);
        if(isConsistent(tmp,maxLen)){
            tmpCone = new Cone(tmp,card);
            allCones.add(tmpCone);
        }

    }
    return allCones;
}
```

Example A.2.1. Let $\mathbb{Z}_7$ be a group. On the following list are all the possible pseudo-orders of $\mathbb{Z}_7$ and the number of them.

$\{\{[\{0\}]\}, \{[\{0,1\}]\}, \{[\{0,1,2\}], [\{0,1,3\}]\}, \{[\{0,1,2,3\}], [\{0,3,5,6\}], [\{0,1,2,4\}]\}\}$

Number of classes of positive parts: 7

Example A.2.2. Let $\mathbb{Z}_{12}$ be a group. On the following list are all the possible pseudo-orders of $\mathbb{Z}12$ and the number of them. $\{\{[\{0\}]\}, \{[\{0,1\}], [\{0,2\}], [\{0,9\}], [\{0,4\}], [\{0,3\}]\}$,

{[{0, 1, 2}], {0, 1, 3}], {0, 2, 3}], {0, 1, 4}], {0, 2, 4}], {0, 3, 4}], {0, 7, 11}], {0, 2, 5}], {0, 4, 5}], {0, 1, 7}], {0, 3, 7}], [{0, 2, 8}], [{0, 1, 5}]}, {[{0, 1, 2, 3}], [{0, 1, 2, 4}], [{0, 1, 3, 4}], [{0, 2, 3, 4}], [{0, 1, 2, 5}], [{0, 7, 9, 11}], [{0, 2, 3, 5}], [{0, 1, 4, 5}], [{0, 2, 4, 5}], [{0, 3, 4, 5}], [{0, 1, 2, 7}], [{0, 1, 3, 7}], [{0, 2, 3, 7}], [{0, 1, 4, 7}], [{0, 3, 4, 7}], [{0, 1, 2, 8}], [{0, 2, 3, 8}], [{0, 2, 5, 8}], [{0, 1, 7, 8}], [{0, 3, 7, 8}], [{0, 3, 7, 11}], [{0, 2, 5, 9}], [{0, 1, 7, 10}], [{0, 1, 5, 9}], [{0, 1, 3, 5}]} , {[{0, 1, 2, 3, 4}], [{0, 1, 2, 3, 5}], [{0, 1, 2, 4, 5}], [{0, 1, 3, 4, 5}], [{0, 2, 3, 4, 5}], [{0, 1, 2, 3, 7}], [{0, 1, 2, 4, 7}], [{0, 1, 3, 4, 7}], [{0, 2, 3, 4, 7}], [{0, 1, 2, 3, 8}], [{0, 1, 2, 5, 8}], [{0, 2, 3, 5, 8}], [{0, 1, 2, 7, 8}], [{0, 1, 3, 7, 8}], [{0, 2, 3, 7, 8}], [{0, 1, 2, 5, 9}], [{0, 1, 4, 5, 9}], [{0, 2, 4, 5, 9}], [{0, 2, 5, 8, 9}], [{0, 1, 3, 7, 10}], [{0, 1, 4, 7, 10}], [{0, 1, 7, 8, 10}]} , {[{0, 1, 2, 3, 4, 5}], [{0, 1, 2, 3, 4, 7}], [{0, 1, 2, 3, 5, 8}], [{0, 1, 2, 3, 7, 8}], [{0, 1, 2, 4, 5, 9}], [{0, 1, 2, 5, 8, 9}], [{0, 1, 3, 4, 7, 10}], [{0, 1, 3, 7, 8, 10}]} }

Number of classes of positive parts: 74

We would be run out of space in order to write more examples including their positive parts. So the following list gives the name of the group and the number of positive parts.

For $\mathbb{Z}_3$ we have 2 positive parts. For $\mathbb{Z}_4$ we have 2 positive parts. For $\mathbb{Z}_5$ we have 3 positive parts. For $\mathbb{Z}_6$ we have 5 positive parts. For $\mathbb{Z}_7$ we have 7 positive parts. For $\mathbb{Z}_8$ we have 11 positive parts. For $\mathbb{Z}_9$ we have 17 positive parts. For $\mathbb{Z}_{10}$ we have 21 positive parts. For $\mathbb{Z}_{11}$ we have 26 positive parts. For $\mathbb{Z}_{12}$ we have 74 positive parts. For $\mathbb{Z}_{13}$ we have 67 positive parts. For $\mathbb{Z}_{14}$ we have 125 positive parts. For $\mathbb{Z}_{15}$ we have 315 positive parts. For $\mathbb{Z}_{16}$ we have 322 positive parts. For $\mathbb{Z}_{17}$ we have 411 positive parts. For $\mathbb{Z}_{18}$ we have 1121 positive parts. For $\mathbb{Z}_{19}$ we have 1111 positive parts. For $\mathbb{Z}_{20}$ we have 2563 positive parts. For $\mathbb{Z}_{21}$ we have 5204 positive parts. For $\mathbb{Z}_{22}$ we have 5913 positive parts. For $\mathbb{Z}_{23}$ we have 8055 positive parts.

Curriculum Vitae

Maria Dolores Cruz Quiones was born on November 20, 1984. The second daughter of Miguel Cruz and Maria Dolores Quiones, she graduated from Colegio de Bachilleres High school, Juarez city, Mexico, in the spring of 2001. She entered Universidad Autonoma de Ciudad Juarez in the fall of 2001 and in the fall of 2005, Maria received her bachelors degree in Mathematics. In the spring of 2006, Maria started to teach at the Universidad of Ciudad Juarez. While working as teacher, she studied a complete program of English at Language center in Juarez. In the spring of 2007, she entered El paso Comunity College to study English.

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