2010-01-01

Signature Matrices: The Eigenvalue Problem

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by

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THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Department of Mathematical Sciences

THE UNIVERSITY OF TEXAS AT EL PASO

May 2010

Abstract

Dealing with matrices can give us a hard time, especially when their dimension is too big, but we also well know how valuable information a matrix may carry, and that is why we study them. When a matrix has a significant number of zeroes we realize how much easier all calculations are. For instance, the product will be simpler to calculate, the determinant, the inverse and even the eigenvalue problem.

This thesis provides the description and behavior of a very special kind of matrices which we call *signature matrices*, definition that is due to Piotr Wojciechowski. A particular feature of these matrices lies in the fact that most of their elements are zeroes which makes significantly easier to work with them. The motivation that led us to analyze these matrices is that they play an important role in the study of partially-ordered algebras with the Multiplicative Decomposition Property. This topic will be briefly described in the Preliminaries chapter, while the formal definition and the properties of the signature matrices constitute the main part of this thesis. We will also give some possible applications and state some questions that still have no answers but seem to be very trackable.

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Chapter 1

Preliminaries

The following chapter gives an introduction to the matrix algebras with the *multiplicative decomposition property*, which is the area where the signature matrices first appeared and thus gave us significant reasons to study their behavior.

1.1 Partially-ordered algebras.

A partial order is a binary operation " $\leq$ " which is reflexive, antisymmetric and transitive. When we have a set with this relation, we say the set is **partially-ordered** and we usually call it a po-set. In particular, we may have a **partially-ordered vector space** V, i.e. a real vector space admitting a partial oder " $\leq$ " that is compatible with the linear operations, i.e. for all $u, v \in V$ with $u, v \geq 0$ and α a real number with $\alpha \geq 0$ we have $\alpha u + v \geq 0$. We will be interested in the vector spaces that are **directly-ordered**, i.e. where every pair of vectors $u, v \in V$ has an upper bound and a lower bound. Details of this topic may found in [1].

Now, if we have a partially-ordered real vector space that admits a product operation, forming a real algebra and it becomes a **partially-ordered algebra**, provided that the order relation is compatible with the product as well, i.e. if $u, v \geq 0$ then $uv \geq 0$, for any $u, v \in V$.

There are many examples of partially ordered algebras, such as all totally ordered algebras (e.g., $\mathbb{R}$, $\mathbb{R}[x]$ ordered lexicographically), $\mathbb{R}[x]$ with the lattice order given by $a_n x^n + \cdots + a_1 x + a_0 \geq 0$ if and only if $a_i \geq 0$ for all $0 \leq i \leq n$, etc. In this paper, however, we will focus on the real algebra of $n \times n$ matrices partially ordered entrywise.

Some further particular examples of the algebras will appear in the text.

1.2 Multiplicative Decomposition property.

A partially-ordered vector space V, satisfies the **Riesz Decomposition** property if for any vectors $u, v, w \in V$ with $0 \leq u, v, w$, such that $w \leq u + v$, there exist vectors u', v' with $0 \leq u' \leq u$ and $0 \leq v' \leq v$ such that $w = u' + v'$. If instead of working with addition in vector spaces we work with multiplication in algebras an analogous property arises and is called the Multiplicative Decomposition Property. It is due to Taen-Yu Dai [2] and is also studied in [5] by Julio Urenda and Piotr Wojciechowski.

Definition 1.1. *A partially-ordered algebra $\mathcal{A}$ has a **Multiplicative Decomposition** or* MD *property if for every $\mathbf{0} \leq \mathbf{u}, \mathbf{v} \in \mathcal{A}$ and $\mathbf{0} \leq \mathbf{w} \leq \mathbf{u}\mathbf{v}$, there exist $\mathbf{u}', \mathbf{v}' \in \mathcal{A}$ such that $\mathbf{0} \leq \mathbf{u}' \leq \mathbf{u}$, $\mathbf{0} \leq \mathbf{v}' \leq \mathbf{v}$ and $\mathbf{w} = \mathbf{u}'\mathbf{v}'$.*

The first example of an algebra having the MD-property can be found in [2]; it consists of all $n \times n$ matrices A for which $a_{ij} = 0$ if $i \neq j$ or $i \neq 1$. Also in [2] it is shown that the algebra of matrices of the form $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ does not have the MD property. In [5] Piotr Wojciechowski and Julio Urenda present the sufficient and necessary conditions for a matrix algebra to have the Multiplicative Decomposition property. Here we mention some of their main results.

Given a vector v in $\mathbb{R}^n$ we denote the number of nonzero elements of v by $\Xi(v)$. To denote the fact that for two vectors u, v we have either $\Xi(u) \leq 1$ or $\Xi(v) \leq 1$ we use $u \bowtie v$.

The following definition tells us when a pair of matrices is decomposable.

Definition 1.2. *Given two nonnegative matrices B and C, we say that the ordered pair (B, C) is a **decomposable pair** if for every nonnegative matrix A such that $A \leq BC$, there exist $0 \leq B' \leq B$ and $0 \leq C' \leq C$ such that $A = B'C'$.*

Let for a matrix A use A_s to denote its s^{th} column and A^s to denote its s^{th} row. Now we can state a theorem from [5] and [4]:

Theorem 1.1. *Given two matrices B and C, the following conditions are equivalent.*

(i) The pair (B, C) is decomposable.

(ii) For every $k = 1, \ldots, n$, $B_k \bowtie C^k$.

What this theorem is telling us is that we have a decomposable pair of matrices (B, C) if and only if for every $k = 1, ..., n$ the number of nonzero components of B_k is at most one, or of C^k is at most one. A detailed proof of this theorem is presented by Julio Urenda and Piotr Wojciechowski in [5].

By an ordered matrix algebra we understand here an entry-wise directly ordered subalgebra of $M_n(\mathbb{R})$ with the identity matrix I in it. From now on, $\mathcal{A}$ will denote a matrix algebra with MD property. Let us modify the Definition 1.2 so that the decomposition of matrices takes place inside a given matrix algebra.

Definition 1.3. *Let $\mathcal{A}$ be an ordered matrix algebra. We say that two nonnegative matrices $B, C \in \mathcal{A}$ form an **$\mathcal{A}$-decomposable pair** if for every nonnegative matrix $A \in \mathcal{A}$ such that $A \leq BC$, there exist nonnegative matrices $B', C' \in \mathcal{A}$ such that $B' \leq B$, $C' \leq C$ and $A = B'C'$.*

Consider the following lemma.

Lemma 1.2. *If for some $A \in \mathcal{A}$, and some $i = 1, \ldots, n$ we have $\Xi(A^i) > 1$, then for every $B \in \mathcal{A}$, $\Xi(B_i) \leq 1$. Similarly, if $\Xi(A_i) > 1$, then for every $B \in \mathcal{A}$, $\Xi(B^i) \leq 1$.*

The proof of this lemma is also given in [5]. The following corollary follows.

Corollary 1.3. *For every $A \in \mathcal{A}$ and for any $i = 1, ..., n$ we either have $\Xi(A^i) \leq 1$ or $\Xi(A_i) \leq 1$.*

Proof. Let $A \in \mathcal{A}$ such that $\Xi(A^i) > 1$. Then, by Lemma 1.2, we have $\Xi(A_i) \leq 1$. Conversely if $\Xi(A_i) > 1$ we have $\Xi(A^i) \leq 1$. □

The i^{th} row [j^{th} column respectively] of a matrix A will be called *diagonal* if $a_{ij} = 0$ for all $j \neq i$ [$i \neq j$ respectively].

Lemma 1.4. *[5] If $A \in \mathcal{A}$ and for some $1 \leq i \leq n$, $\Xi(A^i) > 1$, then for every $B \in \mathcal{A}$, B_i is a **diagonal row**. Similarly, if $\Xi(A_j) > 1$, then for every $B \in \mathcal{A}$, B^j is a diagonal column.*

If we take a look of these matrices we can realize that the majority of their nondiagonal entries are zeroes, and it turns out that somehow there is a way to describe how these zeroes appear.

1.3 Signature of a matrix.

We have come to the crucial definition on which this thesis is based, the signature of a matrix.

Definition 1.4. *We say that an $n \times n$ matrix has a **signature** $\sigma = (s_i)$ if (s_i) is an n-element sequence with $s_i \in \{R, C\}$, where $s_i = R$ means that for all $j \neq i$, $a_{ij} = 0$, similarly $s_i = C$ means that for all $j \neq i$, $a_{ji} = 0$.*

With a deep understanding of this definition we can see that the MD property on $\mathcal{A}$ forces every matrix in $\mathcal{A}$ to have a signature, in fact the following theorems have been proven in [5].

Theorem 1.5. *There exists a signature common to all matrices from $\mathcal{A}$.*

Theorem 1.6. *The collection of all $n \times n$ matrices with a given signature is an algebra with the MD property.*

Proof. Let the signature be $\sigma = (s_i)$ and suppose that for some $1 \leq i \leq n$, $s_i = R$. Let A and B have the signature σ. We will show that AB also has the i^{th} row diagonal. Since A^i is diagonal, the ij^{th} entry of AB, $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{ii}b_{ij}$. But B^i is also diagonal, so $b_{ij} = 0$ for $j \neq i$. Therefore the i^{th} row of AB is diagonal as well. In case $s_i = C$, a similar argument shows that the i^{th} column of AB is diagonal. Therefore AB has the same signature σ. Obviously the collection of all matrices with the signature σ forms a vector subspace of $M_n(\mathbb{R})$, so we have shown that the collection of all matrices with the same signature forms a subalgebra of $M_n(\mathbb{R})$. $\mathcal{A}_\sigma$. By Theorem 1.1 every two matrices from this algebra form a decomposable pair. Therefore the algebra has the MD property. $\square$

If all matrices in a given matrix algebra have the signature σ, we will say that the algebra has the signature σ. The algebra from Theorem 1.6 of all $n \times n$ matrices with the signature σ will be denoted by $\mathcal{M}_\sigma$.

One of the main result of this research is expressed in the following theorem which we call **The Embedding Theorem**.

Corollary 1.7. *Every matrix algebra with the MD property embeds into $\mathcal{M}_\sigma$ for some signature σ.*

Let us also stress out that this line of research leads to finding necessary and sufficient conditions for a subalgebra $\mathcal{A}$ of $M_n(\mathbb{R})$ to have the MD property. These are

(i) $\mathcal{A}$ is a subalgebra of $\mathcal{M}_\sigma$ for some signature σ,

(ii) There exist diagonal matrices $L, R \in \mathcal{A}$ such that

  (1) $L + R = I$

  (2) $L \perp R = 0$

  (3) $L\mathcal{A}_N = \mathcal{A}_N R = \{0\}$.

(iii) $\mathcal{A}$ has the Riesz Decomposition Property.

This thesis is concerned only with the condition (i), the other two conditions are discussed in [5].

The following four examples are also from [5] and give some glimpse into the MD problem. For us they simply are examples of directly ordered algebras with concrete signatures.

Example 1. *Let $\mathcal{A}$ be the collection of all matrices of the form:*

$$\begin{bmatrix} p & a_{12} & 0 \\ 0 & q & 0 \\ 0 & a_{32} & p \end{bmatrix}$$

where for some given α, β not simultaneously positive and not simultaneously negative $\alpha a_{12} + \beta a_{32} = 0$.

It can be easily checked that $\mathcal{A}$ is a directly-ordered algebra. The algebra A has the MD property. It satisfies the condition (i) since it has the signature CRC. It can also be seen that it satisfies the condition (ii) with the matrices $L = diag(1, 0, 1)$ and $R = diag(0, 1, 0)$, and the condition (iii) is also satisfied.

Example 2. *The algebra of matrices of the form:*

$$\begin{bmatrix} p & a_{12} & 0 \\ 0 & p & 0 \\ 0 & a_{32} & q \end{bmatrix}$$

with arbitrary p, q, a_{12} and a_{32} does not have the MD property, even though it has the same signature (CRC), but it does not satisfy condition (ii).

Example 3. *Let $n = 5$, $\sigma = (CCRCR)$ and $a_{13} - a_{15} - a_{23} - a_{25} - a_{43} - a_{45} = 0$. Then the collection of matrices of the form*

$$\begin{bmatrix} p & 0 & a_{13} & 0 & a_{15} \\ 0 & p & a_{23} & 0 & a_{25} \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & a_{43} & p & a_{45} \\ 0 & 0 & 0 & 0 & q \end{bmatrix}$$

is an algebra with the MD property.

Example 4. *Let $n = 5$, $\sigma = (crcrc)$. Then the collection of all matrices of the form*

$$\begin{bmatrix} p & a_{12} & 0 & a_{14} & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & a_{32} & 0 & a_{34} & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & a_{52} & 0 & 0 & p \end{bmatrix}$$

is an algebra with the MD property.

Some methods to construct signature matrix algebras with MD property are given in [5] by Julio Urenda and Piotr Wojciechowski.

Chapter 2

Signature matrices

In this chapter we give a detailed explanation of what it means for a matrix to have a signature, visualize the definition and describe its effect on the product, the determinant, the inverse, the eigenvalues and eigenvectors.

2.1 Understanding the definition.

The Definition 1.4 might be difficult to visualize, so we give the following example in order to have a better understanding of how a matrix with a signature looks like.

Example 5. *Let A be a 5×5 matrix with a signature. The signature has only five elements which may be R or C. Let us say that A has the signature $\sigma = (CRCCR)$. Since the first element of the signature is a C we will consider the first column. It may have a non-zero element only in the first entry and zeroes elsewhere. Next, since the second element of the signature is R, this tells us to look at the second row. We may have a non-zero element only in the second entry of the row and zeroes elsewhere. And so on following the elements of the signature. At the end the matrix A will have the following form:*

$$A = \begin{bmatrix} \bullet & \bullet & 0 & 0 & \bullet \\ 0 & \bullet & 0 & 0 & 0 \\ 0 & \bullet & \bullet & 0 & \bullet \\ 0 & \bullet & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet \end{bmatrix}$$

where the dots represent any real number. After filling out the dots with some particular *real numbers, A will look like:*

$$A = \begin{bmatrix} \mathbf{5} & \mathbf{2} & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{9} & 0 & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{7} & 0 & \mathbf{3} \\ 0 & \mathbf{4} & 0 & \mathbf{8} & \mathbf{0} \\ 0 & 0 & 0 & 0 & \mathbf{6} \end{bmatrix}$$

2.2 Some properties

Analyzing the signature matrices, we can see that they have a very nice behavior, they are easy to handle, all the calculations turn out to be simpler, preserving, of course, all the algebraic properties of $M_n(\mathbb{R})$. Note, however that they are not necessarily diagonal matrices and neither they are triangular.

From Theorem 1.4 we know that all the $n \times n$ matrices with a signature σ form an algebra with the MD property, we will denote this algebra by $\mathcal{M}_\sigma$. The product that this algebra enjoys is the usual multiplication of matrices, but the fact that its elements are signature matrices simplifies the multiplication process to the following rule:

Theorem 2.1. *Let A, B be signature matrices with the same signature, then their product AB is given by*

$$(AB)_{ij} = \begin{cases} a_{ii}b_{ii} & for \quad i{=}j \\ a_{ii}b_{ij} + a_{ij}b_{jj} & for \quad i \neq j. \end{cases} \tag{1}$$

Proof. Let $A, B \in \mathcal{M}_\sigma$ and $\sigma = (s_k)$ with $k = 1, ..., n$. Then as always

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

We have $s_k = C$ or $s_k = R$. If $s_k = C$ then $a_{ik} = 0$ for $k \neq i$ and if $s_k = R$ then $b_{kj} = 0$ for $k \neq j$. We have that when $a_{ik}b_{kj}$ may not be zero only in case when $i = k$ or when $j = k$.

Therefore $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{ii}b_{ij} + a_{ij}b_{jj}$ for $i \neq j$, and $(AB)_{ij} = a_{ii}b_{jj}$ for $i = j$. $\square$

The following example shows how this rule actually holds.

Example 6. *Consider the matrices A,B with signature $\sigma = (CRR)$, and let us find the product AB using the formula (1)*

$$AB = \begin{pmatrix} 3 & 4 & 6 \\ 0 & 8 & 0 \\ 0 & 0 & \frac{2}{7} \end{pmatrix} \begin{pmatrix} 7 & \frac{1}{4} & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} =$$

$$\begin{pmatrix} 3 \cdot 7 & 3 \cdot \frac{1}{4} + 4 \cdot 2 & 3 \cdot 3 + 6 \cdot 5 \\ 8 \cdot 0 + 0 \cdot 2 & 8 \cdot 2 & 8 \cdot 0 + 0 \cdot \frac{2}{7} \\ \frac{2}{7} \cdot 0 + 0 \cdot 7 & \frac{2}{7} \cdot 0 + 0 \cdot 2 & \frac{2}{7} \cdot 5 \end{pmatrix} = \begin{pmatrix} 21 & \frac{35}{4} & 39 \\ 0 & 16 & 0 \\ 0 & 0 & \frac{10}{7} \end{pmatrix}.$$

In this example we observe that AB has the same signature of A and B, that is to say that the zeroes are preserved and therefore when finding the product we have much less work to do than in a non-signature situation.

Now let us define the set $D_n \subseteq \mathcal{M}_\sigma$ as the set of all diagonal $n \times n$ matrices and let $N_\sigma \subseteq \mathcal{M}_\sigma$ be the set of matrices with zeroes in the diagonal entries. It is not hard to see that the elements of N_σ are nilpotent matrices, and the product of any two matrices in N_σ gives us the zero matrix, which is what we call a ***zero-ring***. The following theorem gives a proof of this fact.

Theorem 2.2. *N_σ is a zero-ring.*

Proof. Clearly N_σ is a subspace of $\mathcal{M}_\sigma$, so what is left to prove is the zero multiplication in N_σ. Let $A, B \in N_\sigma$. Then, by the multiplication rule we have:

$$(AB)_{ij} = \begin{cases} a_{ii}b_{jj} & \text{for } i = j \\ a_{ii}b_{ij} + a_{ij}b_{jj} & \text{for } i \neq j \end{cases} = 0,$$

since $a_{ii} = b_{jj} = 0$ for all i and j.

□

A signature matrix can be written as a sum of a diagonal matrix and a nilpotent matrix. We observe this by taking the diagonal elements of the matrix and forming a corresponding diagonal matrix, and we obtain the nilpotent matrix by replacing the diagonal entries of the original matrix by zeroes. Clearly the sum of these two matrices gives us the original matrix. The formal proof of this is given in the following theorem.

Theorem 2.3. *Every signature matrix is the sum of a diagonal matrix and a nilpotent matrix. Moreover $\mathcal{M}_\sigma = D_n \bigoplus N_\sigma$, a direct sum of vector spaces.*

Proof. It is clear that D_n and N_σ are subspaces of $\mathcal{M}_\sigma$. Let $A \in \mathcal{M}_\sigma$, define the matrices D and N by

$$D = (d)_{ij} = \begin{cases} a_{ij} & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \qquad N = (n)_{ij} = \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases},$$

clearly $A = D + N$, where D is diagonal and N is nilpotent. This proves the first part of the theorem. For the remaining part let $A \in D_n \cap N_\sigma$. Then, by Theorem 2.2, we have that $A^2 = 0$ where A is diagonal, thus $A^2 = (a_{ii}^2) = 0$ for $1 \leq i \leq n$. But then $a_{ii}^2 = 0$ which implies that $a_{ii} = 0$, and therefore A is the zero matrix. So we have that $\mathcal{M}_\sigma = D_n \bigoplus N_\sigma$. □

Example 7. *Consider the matrix A with the signature $\sigma = (RCR)$, defined as follows*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 5 & 8 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

we can see that the first term in this sum is a diagonal matrix and verify that the second addend is a nilpotent matrix.

2.3 Determinant and Inverse

As expected, the determinant and the inverse of a signature matrix are also easier to compute, due to, once more, the specific shape these matrices have. In this section we present a way to find the determinant and the inverse for the signature matrices.

In order to find the determinant of a signature matrix, let us recall the **Laplace Expansion** Theorem (also commonly known as the Cofactor Expansion). We need the following definitions.

Definition 2.1. *Let A be an $n \times n$ matrix, and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and the j-th column from A. M_{ij} is called a minor matrix of A and the determinant of this matrix is called the* ***minor*** *of the ij-th entry of A.*

Definition 2.2. *Let $A_{ij} = (-1)^{i+j} \det M_{ij}$. These numbers are called the* ***cofactors*** *of the matrix A.*

Then we defined the determinant of a matrix as follows.

Definition 2.3. *Let A be an $n \times n$ matrix then the* ***determinant*** *of A is*

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

where A_{ij} is the cofactor of a_{1j} with $1 \leq j \leq n$.

This is actually the **cofactor expansion along the first row**, but the Laplace Expansion Theorem tells us that we can find the determinant by expanding the cofactors along any row or any column of the matrix A.

Theorem 2.4. *(Laplace Expansion Theorem) Let A be an $n \times n$ matrix, then the determinant of A can be found by expanding by cofactors along the i-th row or the j-th column,*

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}.$$

When this theorem is cleverly applied to the signature matrices, we realized that it can be simplifies the computations to the following.

Theorem 2.5. *Let A be an $n \times n$ signature matrix, the **determinant** of A is given by*

$$\det A = \prod_{i=1}^{n} a_{ii}.$$

Proof. Proceeding by induction on n, consider a 2×2 matrix A, and let $\sigma = (s_1, s_2)$ be its signature. Suppose $\sigma = (CR)$, then A has the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

with $a_{11}, a_{1,2}, a_{22} \in \mathbb{R}$, so it is easy to see that $\det A = a_{11}a_{22}$. Similarly it holds for $\sigma = (RC)$ and $\sigma = (CC) = (RR)$. Now, suppose the theorem is true for $n = k - 1$, that is, for any $(k-1) \times (k-1)$ signature matrix with. Let A be a $k \times k$ matrix with the signature $\sigma = (s_1, s_2, s_3, ..., s_k)$. If $s_1 = R$ then $a_{1j} = 0$ for $j \neq 1$ and using the Laplace Expansion along the first row of A we obtain

$$\det A = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = a_{11}M_{11}$$

where M_{1j} is the determinant of the matrix A_{1j} that results from A by removing the first row and the j-th column. Then the matrix $A_{1,1}$ is a $(k-1)\times(k-1)$ matrix with signature $\sigma=(s_2,s_3,s_4,...,s_k)$ and by the inductive hypothesis we know that

$$\det A_{1,1}=\prod_{i=2}^{n} a_{ii}=a_{2,2}\cdot a_{3,3}\cdot a_{4,4}\cdot ...\cdot a_{k,k}=M_{1,1}$$

thus

$$\det A=a_{1,1}M_{1,1}=a_{1,1}\cdot a_{2,2}\cdot a_{3,3}\cdot a_{4,4}\cdot ...\cdot a_{k,k}=\prod_{i=1}^{n} a_{ii}.$$

If $s_1=C$ we proceed analogously using the Laplace Expansion along the first column.

□

From the theory of matrices we know that a matrix is invertible if and only if its determinant is not zero. Therefore in case of the signature matrices we have the following.

Corollary 2.6. *A signature matrix is invertible if and only if all its diagonal elements are different from zero.*

Proof. Let A be an invertible signature matrix. Then since A is invertible we have from Theorem 2.5 that $\det A=\prod_{i=1}^{n} a_{ii}\neq 0$, which implies that $a_{ii}\neq 0$ for all $1\leq i\leq n$. Conversely, if we have $a_{ii}\neq 0$ for all $1\leq i\leq n$ then clearly $\det A\neq 0$ and therefore A is invertible. □

Once we know when a matrix is invertible we can proceed to find its inverse. We denote the inverse of the matrix A by A^{-1}, and we know that $AA^{-1}=I$, where I is the identity matrix. The methods finding the inverses of a matrices are widely known and in particular can be found in [3], and they clearly can be used to find the inverse of a signature matrix. However in the next theorem we give a shorter way.

Theorem 2.7. *Let $A\in\mathcal{M}_\sigma$ with $\det A\neq 0$ and let B the matrix given by*

$$(B)_{ij}=\begin{cases} b_{ii}=\frac{1}{a_{ii}} & \textit{for}\ \ i=j\\ b_{ij}=-\frac{a_{ij}}{a_{ii}a_{jj}} & \textit{for}\ \ i\neq j.\end{cases} \tag{2}$$

then $B = A^{-1}$.

Proof. Let $A \in \mathcal{M}_\sigma$ and let B defined by (2). Then from Theorem 2.1 we obtain

$$(AB)_{ij} = \begin{cases} a_{ii}b_{jj} & \text{for } \ i = j \\ a_{ii}b_{ij} + a_{ij}b_{jj} & \text{for } \ i \neq j \end{cases} = \begin{cases} a_{ii}(\frac{1}{a_{ii}}) & \text{for } \ i = j \\ a_{ii}(-\frac{a_{ij}}{a_{ii}a_{jj}}) + a_{ij}(\frac{1}{a_{jj}}) & \text{for } \ i \neq j \end{cases}$$

$$= \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} = I.$$

Therefore B is the inverse of A. □

Corollary 2.8. *If A is a signature matrix and is invertible, then A^{-1} is also a signature matrix and has the same signature as A.*

Proof. Since A is an invertible signature matrix its inverse B is given by (2). If $a_{ij} = 0$ then $b_{ij} = -\frac{a_{ij}}{a_{ii}a_{jj}} = 0$ since $a_{ii} \neq 0$ for all $1 \leq i \leq n$, that is the "pattern of zeroes" of A is preserved in B. Therefore B has the same signature of A. □

Example 8. *Let A be a signature matrix defined as follows*

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 9 & 25 & 4 & 15 & 2 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 12 & 0 & 5 & 47 & 6 \end{pmatrix}.$$

Now let us construct B using the formula given in Theorem 2.7:

$$B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{9}{25\cdot 2} & \frac{1}{25} & -\frac{4}{25\cdot 7} & -\frac{15}{25\cdot 3} & -\frac{2}{25\cdot 6} \\ 0 & 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ -\frac{12}{6\cdot 2} & 0 & -\frac{5}{6\cdot 7} & -\frac{47}{6\cdot 3} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{9}{50} & \frac{1}{25} & -\frac{4}{175} & -\frac{1}{5} & -\frac{1}{75} \\ 0 & 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ -1 & 0 & -\frac{5}{42} & -\frac{47}{18} & \frac{1}{6} \end{pmatrix}.$$

Upon multiplication of A by B we obtain

$$AB = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 9 & 25 & 4 & 15 & 2 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 12 & 0 & 5 & 47 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{9}{50} & \frac{1}{25} & -\frac{4}{175} & -\frac{1}{5} & -\frac{1}{75} \\ 0 & 0 & \frac{1}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ -1 & 0 & -\frac{5}{42} & -\frac{47}{18} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, as expected, B is the inverse of A, $B = A^{-1}$.

Chapter 3

The eigenvalue Problem

The importance of the eigenvalue problem is well known in mathematics and its applications, in particular in solving systems of differential equations, diagonalization of linear transformations, analyzing data such us population growth, etc. It also plays an important role in areas as diverse as finance, physics, biology, statistics and many others. In this chapter we analyze how to solve the eigenvalue problem for a signature matrix, and obtain some interesting results.

Recall first **The Eigenvalue Problem** formulation:

For an $n \times n$ matrix A, find all the scalars λ such that the equation

$$A\mathbf{v} = \lambda\mathbf{v} \tag{3}$$

has a nonzero solution, $\mathbf{v}$. *Such a scalar λ is called an eigenvalue of A and any nonzero vector $\mathbf{v}$ satisfying the equation (3) is called an eigenvector associated with λ.*

To solve this problem we can rewrite (3) as $A\mathbf{v} - \lambda\mathbf{v} = 0$, which is the same as

$$(A - \lambda I)\mathbf{v} = 0 \tag{4}$$

where I is the $n \times n$ identity matrix. In order for (4) to have a nonzero solution we need λ to be chosen so that the matrix $A - \lambda I$ is singular, that is not invertible. Therefore the eigenvalue problem is reduced to two steps:

1. Find all scalars λ such that $A - \lambda I$ is singular, i.e. $\det(A - \lambda I) = 0$.

2. Given such a scalar λ, find all the nonzero vectors $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = 0$.

Let us then solve this problem for signature matrices.

3.1 Eigenvalues

The eigenvalues of a signature matrix can be determined by the method presented in [3] for any matrix, but with a little perspicacity it is not hard to see that the following theorem holds.

Theorem 3.1. *The **eigenvalues** of a signature matrix are precisely its diagonal entries.*

Proof. Let $A \in \mathcal{M}_\sigma$. Then $A - \lambda I$ is also in $\mathcal{M}_\sigma$ since $\mathcal{M}_\sigma$ is an algebra and I is in $\mathcal{M}_\sigma$. Then we use the fact that the determinant of a signature matrix is the product of its diagonal entries (Theorem 2.5). We obtain the characteristic polynomial

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)...(a_{nn} - \lambda).$$

Its roots are exactly $a_{11}, a_{22}, a_{33}, ..., a_{nn}$, the diagonal entries of A. By a theorem from [3] these are the eigenvalues of A. □

Therefore given a signature matrix its eigenvalues are transparent.

3.2 Eigenvectors

Once we know the eigenvalues of a matrix, in order to solve the second part of the eigenvalue problem, we need to find the associated eigenvectors with every eigenvalue. For a given eigenvalue λ, we have to solve the homogeneous system of equations:

$$(A - \lambda I)\mathbf{v} = 0.$$

Every nonzero vector $\mathbf{v}$ satisfying the system is an eigenvector. General methods of the solutions can be found, again in [3].

The subspace of $\mathbb{R}^n$ generated by all the eigenvectors associated with λ, is an **eigenspace** associated with λ.

The dimension of the eigenspace is called **geometric multiplicity** of the eigenvalue λ. The **algebraic multiplicity** of an eigenvalue, is defined as the multiplicity of the corresponding root of the characteristic polynomial. In this paper the multiplicity will mean the algebraic multiplicity unless otherwise stated.

For the signatures matrices, the eigenvectors can be found with almost no effort, provided that certain conditions on the multiplicities of the eigenvalues are satisfied. If we have non-repeating eigenvalues, i.e. if the multiplicity of every eigenvalue is one, then the following theorem describes the eigenvectors.

Recall from a theorem of [3], that the eigenvectors associated with pairwise different eigenvalues form a linearly independent set.

Theorem 3.2. *Let A be a signature matrix with pairwise distinct eigenvalues, let λ be an eigenvalue of A. Then the corresponding* ***eigenvector*** $\boldsymbol{v} = (v_1, v_2, ..., v_k, ..., v_n)^T$ *is given by:*

$$v_k = \begin{cases} \dfrac{a_{ki}}{a_{ii} - a_{kk}} & \text{if } k \neq i \\ 1 & \text{if } k = i. \end{cases} \tag{5}$$

Proof. Let A be an $n \times n$ signature matrix. By Theorem 3.1, $\lambda = a_{ii}$ for some $i = 1, ..., n$. Consider the vector $\mathbf{v} = (v_1, v_2, ..., v_i, ..., v_n)^T$ given by (5). Let B be an $n \times n$ matrix with the vector $\mathbf{v}$ in its i^{th} column and zeroes everywhere else. Now consider the product AB:

$$AB = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1i} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2i} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ii} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{ni} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{a_{1i}}{a_{ii}-a_{11}} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{a_{2i}}{a_{ii}-a_{22}} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \frac{a_{ni}}{a_{ii}-a_{nn}} & 0 & \dots & 0 \end{pmatrix}$$

Note that B has the same signature as A. Then, by the multiplication rule for signature matrices, this product is equal to an $n \times n$ matrix where only the elements of the i^{th} column are not zero, this column forms a vector $\mathbf{w} = (w_1, w_2, ..., w_i, ..., w_n)^T$ given by:

$$w_k = (AB)_{ki} = \begin{cases} a_{kk}\Big(\dfrac{a_{ki}}{a_{ii} - a_{kk}}\Big) + a_{ki}(1) & \text{if } k \neq i \\ a_{ii} & \text{if } k = i \end{cases}$$

$$= \begin{cases} \dfrac{a_{kk}a_{ki} + a_{ki}(a_{ii} - a_{kk})}{a_{ii} - a_{kk}} & \text{if } k \neq i \\ a_{ii} & \text{if } k = i \end{cases} = \begin{cases} \dfrac{a_{ki}a_{ii}}{a_{ii} - a_{kk}} & \text{if } k \neq i \\ a_{ii} & \text{if } k = i \end{cases}$$

$$= a_{ii} \begin{cases} \dfrac{a_{ki}}{a_{ii} - a_{kk}} & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases} = a_{ii}\ v_k.$$

Therefore $A\mathbf{v} = a_{ii}\mathbf{v} = \lambda\mathbf{v}$, i.e. $\mathbf{v}$ is the eigenvector corresponding to $\lambda = a_{ii}$. □

Note that if $a_{ki} = 0$ for some $k \neq i$ then $v_k = 0$, i.e. the zeroes of the eigenvectors associated with a_{ii} are in the same positions as the zeroes of the i^{th} column of the matrix A.

Corollary 3.3. *If A is a signature matrix with pairwise different eigenvalues, then all the eigenvectorsof A are given by the formula (5).*

Proof. By Theorem 3.2 the eigenvectors associated with the eigenvalues of A are given by the formula (5). Since the eigenvalues are pairwise different, there are precisely n linearly independent eigenvectors up to nonzero scalar multiples these are all eigenvectors of A. □

In the next example we illustrate this formula.

Example 9. *Let A be the following matrix with signature $\sigma = (CRCR)$. We will find its eigenvalues and eigenvectors.*

$$A = \begin{pmatrix} 6 & 2 & 0 & 4 \\ 0 & 5 & 0 & 0 \\ 0 & 7 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues of A are $6, 5, 2, 1$. The eigenvector associated with 6 is the vector $(1, 0, 0, 0)^T$, associated with 5 is $(\frac{2}{5-6}, 1, \frac{7}{5-2}, 0)^T = (-2, 1, \frac{7}{3}, 0)^T$, the eigenvector associated with 2 is $(0, 0, 1, 0)^T$ and the eigenvector associated with 1 is $(-\frac{4}{5}, 0, -3, 1)^T$.

We can check directly that they are indeed the eigenvectors. For example for $\lambda = 5$ we have:

$$\begin{pmatrix} 6 & 2 & 0 & 4 \\ 0 & 5 & 0 & 0 \\ 0 & 7 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{7}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -10 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & \frac{35}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The second column of the product matrix, which is clearly 5 times the second column of the second matrix.

For eigenvalues with multiplicity greater than 1 we yet have not identified any pattern that might simplify the way to find the eigenvectors but we have observed some interesting behaviors in specific cases. For instance, for some cases of signature matrices the claim from Theorem 3.2 holds regardless of the multiplicity of the eigenvalue. In the next chapter we will give some examples.

3.3 Diagonalization.

The diagonal matrices are particularly easy to deal with, especially in terms of computations. Everything would be easier if every matrix can be somehow equivalent or reduced to a diagonal matrix. In this section we recall the *diagonalization* process of a matrix and see how it works with the signature matrices whose eigenvalues are pairwise different.

First thing we need to recall *similarity* of matrices.

Definition 3.1. *Two $n \times n$ matrices A and B are* ***similar*** *if there is a nonsingular $n \times n$ matrix S such that $B = S^{-1}AS$.*

This definition comes from the fact that the matrices A and $B = S^{-1}AS$ have the same characteristic polynomial and hence the same eigenvalues with the same algebraic multiplicity. The proof of this is given in [3].

If A is similar to a diagonal matrix computations involving A may be significantly simplified. So we have the following definition.

Definition 3.2. *If a $n \times n$ matrix A is similar to a diagonal matrix, then A is* ***diagonalizable****.*

In [3] we can find the following theorem and its proof.

Theorem 3.4. *A $n \times n$ matrix A is diagonalizable if and only if A has a set of n linearly independent eigenvectors.*

So if A has n linearly independent eigenvectors $v_1, v_2, ..., v_n$ then the matrix $S = [v_1, v_2, ..., v_n]$ diagonalizes A and it is called the **transition** matrix.

From matrix theory in [3] we know that if an $n \times n$ matrix has n different eigenvalues then their associated eigenvectors are linearly independent.

Now, let us analyze the diagonalization process for the signature matrices.

A signature matrix A with pairwise different eigenvalues is diagonalizable, since all its eigenvectors are linearly independent. In fact, with the results that we have so far about this kind of matrices it is not hard to diagonalize a signature matrix. We illustrate this fact in the following example.

Example 10. *Consider the following signature matrix A with $\sigma = (CRRCRCRC)$*

$$A = \begin{pmatrix} -1 & 9 & 14 & 0 & -5 & 0 & 7 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 \\ 0 & -11 & 3 & 5 & 23 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 31 & -9 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & -4 & 6 & 0 & -7 & 0 & 3 & 8 \end{pmatrix}$$

In order to diagonalize A we need to find the transition matrix S and its inverse. So we need to find all the eigenvectors associated with the eigenvalues of A, for this we can use the formula in Theorem 3.2. So the transition matrix will look like this

$$S = \begin{pmatrix} 1 & \frac{9}{7} & \frac{7}{9} & 0 & -\frac{5}{4} & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -11 & \frac{1}{4} & 1 & -\frac{23}{2} & 0 & \frac{8}{9} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{31}{4} & -\frac{3}{5} & 0 & 3 & 1 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & \frac{2}{3} & 0 & \frac{7}{5} & 0 & -\frac{1}{4} & 1 \end{pmatrix}$$

Note that if we write the eigenvectors in the order that the eigenvalues appear, then the transition matrix has the same signature as A and it is invertible since all the diagonal entries are different from zero. Therefore we can find the inverse of S easily using the *formula (2) and we obtain*

$$S^{-1} = \begin{pmatrix} 1 & -\frac{9}{7} & -\frac{7}{9} & 0 & \frac{5}{4} & 0 & \frac{7}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & -\frac{1}{4} & 1 & \frac{23}{2} & 0 & -\frac{8}{9} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{31}{4} & \frac{3}{5} & 0 & -3 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -\frac{2}{3} & 0 & -\frac{7}{5} & 0 & \frac{1}{4} & 1 \end{pmatrix}$$

Now, let us find the product $S^{-1}A$ by the multiplication rule for signature matrices given in Theorem 2.1

$$S^{-1}A = \begin{pmatrix} 1 & -\frac{9}{7} & -\frac{7}{9} & 0 & \frac{5}{4} & 0 & \frac{7}{3} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & -\frac{1}{4} & 1 & \frac{23}{2} & 0 & -\frac{8}{9} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{31}{4} & \frac{3}{5} & 0 & -3 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -\frac{2}{3} & 0 & -\frac{7}{5} & 0 & \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} -1 & 9 & 14 & 0 & -5 & 0 & 7 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 \\ 0 & -11 & 3 & 5 & 23 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 31 & -9 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & -4 & 6 & 0 & -7 & 0 & 3 & 8 \end{pmatrix}$$

$$
= \begin{pmatrix}
-1 & \frac{9}{7} & \frac{7}{9} & 0 & -\frac{5}{4} & 0 & -\frac{7}{3} & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 \\
0 & 55 & -\frac{5}{4} & 5 & \frac{115}{2} & 0 & -\frac{40}{9} & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & -\frac{31}{2} & \frac{6}{5} & 0 & -6 & 2 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\
0 & -16 & -\frac{16}{3} & 0 & -\frac{56}{5} & 0 & 2 & 8
\end{pmatrix}
$$

Then the product $(S^{-1}A)S$ *is equal to*

$$
(S^{-1}A)S = \begin{pmatrix}
-1 & \frac{9}{7} & \frac{7}{9} & 0 & -\frac{5}{4} & 0 & -\frac{7}{3} & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 \\
0 & 55 & -\frac{5}{4} & 5 & \frac{115}{2} & 0 & -\frac{40}{9} & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & -\frac{31}{2} & \frac{6}{5} & 0 & -6 & 2 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\
0 & -16 & -\frac{16}{3} & 0 & -\frac{56}{5} & 0 & 2 & 8
\end{pmatrix}
\begin{pmatrix}
1 & \frac{9}{7} & \frac{7}{9} & 0 & -\frac{5}{4} & 0 & -\frac{7}{3} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -11 & \frac{1}{4} & 1 & -\frac{23}{2} & 0 & \frac{8}{9} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{31}{4} & -\frac{3}{5} & 0 & 3 & 1 & -\frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -2 & \frac{2}{3} & 0 & \frac{7}{5} & 0 & -\frac{1}{4} & 1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8
\end{pmatrix} = D
$$

as expected.

Chapter 4

Additional results and examples.

In this chapter we show some other interesting results and examples that we have observed while working with signature matrices.

4.1 Counting signatures.

The following theorem tells us how many different signatures an $n \times n$ matrix may have.

Theorem 4.1. *There are* $2^n - 1$ *different signatures for an* $n \times n$ *matrix.*

Proof. From the definition of a signature we know that it is an n-element sequence $\sigma = (s_i)$ with $s_i \in \{R, C\}$. Therefore for a given n we have 2^n possible signatures since for every s_i with $1 \leq i \leq n$ there are two choices, R or C. Note first that if $\sigma = (s_i)$ with $s_i = C$ for every $1 \leq i \leq n$ and $\sigma' = (s_i)$ with $s_i = R$ for every $1 \leq i \leq n$ yield the same algebras of matrices. More precisely $\mathcal{M}_\sigma = \mathcal{M}_{\sigma'} = D_n$ (the diagonal matrices). We count this class of matrices only once, so we subtract 1. On the other hand consider the signature σ_1 such that $s_i = C$ and $s_j = R$ and the signature σ_2 such that $s_i = R$ and $s_j = C$ with $i < j$. We will check that they yield different algebras: $\mathcal{M}_{\sigma_1} \neq \mathcal{M}_{\sigma_2}$. A matrix with signature σ_1 and a matrix with signature σ_2 will have the following forms, respectively:

$$
\begin{pmatrix}
 & & & 0 & & & & & \\
 & & & \vdots & & & & & \\
 & & & 0 & & & & & \\
 & & & \bullet_{ii} & & \bullet_{ij} & & & \\
 & & & 0 & & & & & \\
0 & 0 & \cdots & 0 & 0 & \bullet_{jj} & 0 & \cdots & 0 \\
 & & & 0 & & & & & \\
 & & & \vdots & & & & & \\
 & & & 0 & & & & &
\end{pmatrix},
\begin{pmatrix}
 & & & & & 0 & & & \\
 & & & & & 0 & & & \\
 & & & & & \vdots & & & \\
0 & \cdots & 0 & \bullet_{ii} & 0 & 0_{ij} & \cdots & 0 & 0 \\
 & & & & & 0 & & & \\
 & & & & & \bullet_{jj} & & & \\
 & & & & & 0 & & & \\
 & & & & & \vdots & & & \\
 & & & & & 0 & & &
\end{pmatrix}
$$

The ij-entries of both matrices may differ since the first one we may have a non-zero element while the second matrix may only have a zero. Therefore these two classes of matrices are different. □

4.2 Higher multiplicities of eigenvalues.

In the previous chapter we did not give a general formula describing the eigenvectors in case that the eigenvalue multiplicities are greater than 1. We have found, however, some particular cases when our formula for pairwise different eigenvalues holds. Here we present an example where the formula holds.

Example 11. *Consider the signature matrix A with $\sigma = (CRRC)$ defined as follows*

$$
A = \begin{pmatrix}
3 & -4 & 7 & 0 \\
0 & 8 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 6 & -1 & 3
\end{pmatrix}.
$$

We know that the eigenvalues are 3 *and* 8 *of multiplicity 3 and 1 respectively. Then the eigenvectors associated with 3 are the vectors* $(1,0,0,0)^T$ *and* $(0,0,0,1)^T$, *note that the geometric multiplicity of the eigenvalue 3 is less than its algebraic multiplicity. For the eigenvalue* 8 *the associated eigenvector is* $(\frac{-4}{8-3},1,0,\frac{6}{8-3})^T = (-\frac{4}{5},1,0,\frac{6}{5})^T = (-4,5,0,6)^T$. *It agrees with the calculations done by the formula (5).*

We have an example with a different signature where the formula for eigenvectors does not hold.

Example 12. *Consider the signature matrix* A *with* $\sigma = (CRRR)$ *defined as follows*

$$A = \begin{pmatrix} 3 & -4 & 7 & 1 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

We have the same eigenvalues with the same multiplicities of the previous example. According to formula (5), the expected eigenvector associated with 3 *is* $(1,0,0,0)^T$. *If, however, we find the eigenvectors with the usual method we can find out that they are* $(1,0,0,0)^T$ *and* $(0,0,-1,7)^T$. *The only eigenvector associated with 8 given by the formula (5), is* $(\frac{-4}{8-3},1,0,0)^T = (-\frac{4}{5},1,0,0)^T = (-4,5,0,0)^T$.

4.3 Transpose.

We also analyzed the transpose of a signature matrix. This will, in particular, enable us to tell when a signature matrix is symmetric.

Recall first the definition of a transpose matrix.

Definition 4.1. *Let* A *be an* $m \times n$ *matrix. Then* A^T, *the* ***transpose*** *of* A, *is the matrix* $A^T = (b_{ij})$, *where* $b_{ij} = a_{ji}$ *for all* i *and* j *with* $1 \leq i \leq n$ *and* $1 \leq j \leq m$.

In other words this definition tells us that we can obtain the transpose of a matrix A interchanging the rows and columns of A. Since we are working with the signature matrices we use the previous definition with $m = n$.

For any signature matrix A we can find A^T, and A^T is also a signature matrix. In fact we the following theorem holds.

Theorem 4.2. *If A has a signature $\sigma = (s_i)$, then A^T has a signature $\sigma^T = (z_i)$, where $z_i = R$ if and only if $s_i = C$.*

Proof. By the definition of $A^T = (b_{ij})$ we have that $b_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$. Define the signature $\sigma^T = (z_i)$ by letting $z_i = C$ if and only if $s_i = R$. If $s_i = C$ then $a_{ki} = 0$ for all $k \neq i$ with $1 \leq k \leq n$, which implies that $b_{ik} = 0$ for all $k \neq i$ with $1 \leq k \leq n$. Similarly if $s_i = R$ then $a_{ik} = 0$ for all $k \neq i$ with $1 \leq k \leq n$, which implies that $b_{ki} = 0$ for all $k \neq i$. Therefore A^T has a signature given by $\sigma^T = (z_i)$. □

Now, let us recall the definition of a symmetric matrix.

Definition 4.2. *A matrix A is **symmetric** if $A = A^T$.*

With this definition and the previous theorem we can conclude that in order for a signature matrix A to be symmetric it needs to have the signature $\sigma = (s_i)$ with $s_i = R$ for all $1 \leq i \leq n$ or $s_i = C$ for all $1 \leq i \leq n$, i.e. a signature matrix is symmetric if and only if it is diagonal.

We give the following example to illustrate the transpose matrix considerations.

Example 13. *Let A be a signature matrix with $\sigma = (RCCR)$ defined by*

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 6 & -1 & 0 & 7 \\ 8 & 0 & 5 & -4 \\ 0 & 0 & 0 & 16 \end{pmatrix}$$

A^T is given by

$$A^T = \begin{pmatrix} 3 & 6 & 8 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 7 & -4 & 16 \end{pmatrix}$$

we can clearly see that the signature of A^T is $(CRRC)$. Obviously A is not symmetric.

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Curriculum Vitae

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