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The Isomorphisms Between the Upper and Lower Triangular Matrix Algebras

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THE ISOMORPHISMS BETWEEN THE UPPER

AND LOWER TRIANGULAR

MATRIX ALGEBRAS

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to my

MOTHER and FATHER

with love

THE ISOMORPHISMS BETWEEN THE UPPER AND LOWER TRIANGULAR MATRIX ALGEBRAS

by

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Abstract

Since matrix equations with triangular matrices are easier to solve, the triangular matrices are very important in mathematics. For instance, the LU decomposition gives an algorithm to decompose any invertible matrix A into two triangular factors: a lower triangle matrix L and an upper triangle matrix U. Moreover, the inverse of a triangular matrix is also triangular. Also the product of two lower triangular matrices produces a lower triangular matrix, the same apply for upper triangular matrices. A lot of important notions, such as the determinant, the eigenvalue problem and many others are easy to handle when we are working with triangular matrices. Each of the two classes – the class of the lower triangular matrices and the class of upper triangular matrices – form an algebra. This thesis studies an important relation between the algebras of the upper and lower triangular matrices, the isomorphism. As we know, isomorphisms are used in representation theorems, where an abstract structure is similar to a concrete structure. In our case we are going to find "THE" algebras isomorphisms between two algebras in question. The preliminaries provides basic definitions, theorems, and some incentives involved in adopting this topic. The following chapters will be dedicated for finding the isomorphism, the method used in solving the problem is a comparison of the dimensions of the right and left annihilators of some specific matrices.

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Chapter 1

Preliminaries.

The algebras we are considering are finite-dimensional algebras over an arbitrary field $\mathbb{F}$.

Definition 1 *An* algebra over a field *is a vector space equipped with a bilinear vector product. That means, it is an algebraic structure consisting of a vector space together with an operation, usually called multiplication, that combines any two vectors to form a third vector; to qualify as an algebra, this multiplication must satisfy certain compatibility axioms with the given vector space structure. In other words, an algebra over a field is a set together with operations of multiplication, addition, and scalar multiplication by elements of the field. Precisely,* $\mathcal{A}$ *is an algebra over a field* $\mathbb{F}$ *if* $\mathcal{A}$ *is a vector space over* $\mathbb{F}$*,* $(\mathcal{A}, \cdot)$ *is an associative ring and* $\alpha(xy) = (\alpha x)y = x(\alpha y)$ *for every* $\alpha \in \mathbb{F}$*, and* $x, y \in \mathcal{A}$*.*

Definition 2 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be any two algebras, and let* f *be a function from* $\mathcal{A}$ *to* $\mathcal{B}$*,*

$$f : \mathcal{A} \to \mathcal{B}.$$

We say that f *is a* linear transformation *if for all* x *and* y *in* $\mathcal{A}$ *and all scalars* α

$$f(x + y) = f(x) + f(y)$$

and

$$f(\alpha x) = \alpha f(x).$$

We say that f *is a* linear isomorphism *if* f *is a bijective linear transformation.*

Definition 3 *A* $\mathbb{F}$-homomorphism *between two algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *over a field* $\mathbb{F}$ *is a linear transformation* $f\colon \mathcal{A} \to \mathcal{B}$ *such that for* x, y *in* $\mathcal{A}$*,*

$$f(xy) = f(x)f(y).$$

Definition 4 *A function* $f\colon \mathcal{A} \to \mathcal{B}$ *is called an* algebra isomorphism *if and only if* f *is a bijective* $\mathbb{F}$*-homomorphism.*

Definition 5 *We define the* kernel *of a* $\mathbb{F}$*-homomorphism* $f : \mathcal{A} \to \mathcal{B}$ *as*

$$\ker f = \{x \in \mathcal{A} : f(x) = 0_{\mathcal{B}}\}.$$

Definition 6 *In an algebra* $\mathcal{A}$*, an element* $e \in \mathcal{A}$ *is called an* idempotent *if and only if:*

$$e^2 = e.$$

For any integer $k \geq 2$*, an element* $n \in \mathcal{A}$ *is called a* k-nilpotent *if and only if*

$$n^k = 0.$$

An element is called nilpotent *if it is* k*-nilpotent for some* k*.*

Theorem 1 *An algebra isomorphism preserves idempotency and* k*-nilpotency of elements.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be an algebra isomorphism. Let $e \in \mathcal{A}$ such that $e^2 = e$. We want to show that $f(e)^2 = f(e)$. Indeed,

$$f(e)^2 = f(e) \cdot f(e) = f(e \cdot e) = f(e^2) = f(e).$$

Now let $n \in \mathcal{A}$ such that $n^k = 0$ for some positive integer k. We want to show that $f(n)^k = 0$. Indeed,

$$f(n)^k = f(n) \cdot \ldots \cdot f(n) \ (k \text{ times}) = n \cdot \ldots \cdot n \ (k \text{ times}) = f(n^k) = f(0) = 0.$$

Definition 7 *A set of elements* $\{x_1, x_2 \ldots x_n\}$ *is called* linearly independent *if the only solution to the equation:*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$

is $a_1 = a_2 = \ldots = a_n = 0$*, where* $a_1, \ldots, a_n$ *are scalars.*

Definition 8 *Let $\mathcal{A}$ be an algebra, and let $\mathcal{S} = \{x_1, \ldots, x_n\}$ be a subset of $\mathcal{A}$. We say that $\mathcal{S}$ is a* spanning set *for $\mathcal{A}$, or simply that $\mathcal{S}$* spans *$\mathcal{A}$, if every element x in $\mathcal{A}$ can be expressed as a linear combination of vectors from $\mathcal{S}$:*

$$x = a_1 x_1 + \ldots + a_n x_n,$$

where $a_1, \ldots, a_n$ are scalars.

Definition 9 *Let $\mathcal{A}$ be an algebra. A* basis *for $\mathcal{A}$ is a linearly independent spanning set for $\mathcal{A}$.*

Definition 10 *Let $\mathcal{A}$ be an algebra, and $a \in \mathcal{A}$. We define the* set of all right annihilators *of a by $r(a) = \{x \in \mathcal{A} : ax = 0\}$. Dually, the* set of all left annihilators *of a is defined as $l(a) = \{x \in \mathcal{A} : xa = 0\}$. We say that x* annihilates *a on the right or left, respectively.*

Remark 1 *The set of all right annihilators forms a vector subspace of $\mathcal{A}$.*

Proof. Consider an algebra $\mathcal{A}$. Let $x \in \mathcal{A}$ and $y \in \mathcal{A}$ such that x and y are right annihilators of $a \in \mathcal{A}$. We have:

$$a(x + y) = ax + ay = 0$$

$$a(\alpha x) = \alpha(ax) = \alpha \cdot 0 = 0$$

for all scalars α. Therefore the set of all right annihilators forms a vector space. Similarly, the set of all left annihilators will form a vector space.

The annihilators are a powerful and useful tool in abstract and linear algebras. In particular, they were used in [3] which gave an inspiration for this work.

Theorem 2 *An algebra isomorphism preserves the dimensions of the right and left annihilators sets.*

Proof. Consider algebras $\mathcal{A}$ and $\mathcal{B}$, and let $a \in \mathcal{A}$. Let $f : \mathcal{A} \to \mathcal{B}$ be an algebra isomorphism. Our claim is that $\dim(r(f(a))) = \dim(r(a))$.

Indeed, let $\{e_1, e_2, \ldots, e_n\}$ be a basis of $r(a)$. We want to show that $f(e_1), f(e_2), \ldots, f(e_n)$ is a basis of $r(f(a))$.

Consider:

$$c_1 f(e_1) + c_2 f(e_2) + \ldots + c_n f(e_n) = 0$$

for some constants $c_1, \ldots, c_n$. Then:

$$f(c_1 e_1) + f(c_2 e_2) + \ldots + f(c_n e_n) = 0$$

$$f(c_1 e_1 + c_2 e_2 + \ldots + c_n e_n) = 0$$

Since f is an algebra isomorphism, f is injective and therefore

$$c_1 e_1 + c_2 e_2 + \ldots + c_n e_n \in \ker f = \{0\}.$$

This implies that

$$c_1 e_1 + c_2 e_2 + \ldots + c_n e_n = 0.$$

Thus $c_1 = c_2 = \ldots = c_n = 0$ since $e_1, \ldots e_n$ are linearly independent and hence

$$f(e_1), f(e_2), \ldots, f(e_n)$$

are linearly independent.

In order to prove that $f(e_1), f(e_2), \ldots, f(e_n)$ span the subspace $r(f(a))$, let $y \in \mathcal{B}$ such that $y \in r(f(a))$. So

$$f(a) \cdot y = 0.$$

Since f is onto, there exists $x \in \mathcal{A}$ such that $f(x) = y$. Hence

$$f(a) \cdot f(x) = 0$$

$$f(a \cdot x) = 0,$$

which implies that $a \cdot x \in \ker f = \{0\}$. So $a \cdot x = 0$ and, in conclusion, $x \in r(a)$. Then x can be written as:

$$x = c_1 e_1 + c_2 e_2 + \ldots + c_n e_n;$$

$$f(x) = f(c_1 e_1) + f(c_2 e_2) + \ldots + f(c_n e_n);$$

$$f(x) = c_1 f(e_1) + c_2 f(e_2) + \ldots + c_n f(e_n);$$

$$y = c_1 f(e_1) + c_2 f(e_2) + \ldots + c_n f(e_n).$$

Hence $f(e_1), f(e_2), \ldots, f(e_n)$ span $r(f(a))$, so they form a basis of $r(f(a))$.

Therefore the dimension is preserved. The proof is similar for the left annihilators.

Corollary 1 *$r(f(a)) = f(r(a))$.*

Proof. We showed in the previous theorem that if $y \in \mathcal{B}$ such that $y \in r(f(a))$, then for some $x \in r(a)$, $f(x) = y$, therefore $y \in f(r(a))$ hence $r(f(a)) \subseteq f(r(a))$.

Now let $y \in f(r(a))$, so $y = f(x)$ for some $x \in r(a)$, this means:

$$a \cdot x = 0$$

$$f(a) \cdot f(x) = 0$$

$$f(a) \cdot y = 0$$

So $y \in r(f(a))$, hence $f(r(a)) \subseteq r(f(a))$ and therefore

$$r(f(a)) = f(r(a)).$$

Definition 11 *An $n \times n$ matrix $U = u_{ij}$ is called* upper triangular *if $u_{ij} = 0$ whenever i is greater than j. And an $n \times n$ matrix $L = l_{ij}$ is called* lower triangular *if $l_{ij} = 0$ whenever j is greater than i. The set of all $n \times n$ upper triangular matrices is denoted by $\mathcal{U}_{n\times n}$, and the set of all $n \times n$ lower triangular matrices is denoted by $\mathcal{L}_{n\times n}$.*

Definition 12 *An* anti-isomorphism *between two algebras is a bijective linear map that reverses the order of multiplication. So if* φ*:* $\mathcal{A} \to \mathcal{B}$ *is an algebra anti-isomorphism, then:*

$$\varphi(xy) = \varphi(y)\varphi(x)$$

for all x*,* y *in* $\mathcal{A}$*.*

Definition 13 *If* $A = (a_{ij})$ *is an* $m \times n$ *matrix, then the* transpose *of* A*, denoted* A^t*, is the* $n \times m$ *matrix* $A^t = (b_{ij})$*, where* $b_{ij} = a_{ji}$ *for all* i *and* j*,* $1 \leq j \leq m$*, and* $1 \leq i \leq n$*.*

Example. The transpose of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$.

Remark 2 ([1]) *Let* $\varphi : \mathcal{U}_{n\times n} \to \mathcal{L}_{n\times n}$*;*

$$\varphi(A) = A^t.$$

Then φ *is an anti-isomorphism.*

Proof. While the transposition is a bijective linear mapping, we want to show that $\varphi(AB)$ $= \varphi(B)\ \varphi(A)$, in other words we want to show that $(AB)^t = B^tA^t$.

Note first that $(AB)^t$ and B^tA^t are both $n \times n$, so we only have to show that their corresponding entries are equal. From the definition of the transpose, the (i, j)-th entry of $(AB)^t$ is the jith entry of AB. Thus the (i, j)-th entry of $(AB)^t$ is given by:

$$\sum_{k=1}^{n} a_{jk}b_{ki}.$$

Next the (i, j)-th entry of B^tA^t is the inner product of the i-th row of B^t with the j-th column of A^t. In particular, the i-th row of B^t is $[b_{1i}, b_{2i}, \ldots, b_{ni}]$ (the i-th column of B), whereas the j-th column of A^t is

$$\begin{bmatrix} a_{j1} \\ a_{j2} \\ a_{j3} \\ \vdots \\ a_{jn} \end{bmatrix}$$

(the j-th row of A). Therefore, the (i, j)-th entry of $B^t A^t$ is given by :

$$b_{1i}a_{j1} + b_{2i}a_{j2} + \ldots + b_{ni}a_{jn} = \sum_{k=1}^{n} b_{ki}a_{jk}.$$

Finally, since

$$\sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki},$$

the (i, j)-th entries of $(AB)^t$ and $B^t A^t$ agree, and the matrices are equal.

With his ingenious proof, Šemrl [4] gave us a motivation to work on algebras isomorphisms. Moreover, since the theorem deals with the entire $n \times n$ matrix algebra, we chose to work on its most important subalgebras, the upper and lower triangular matrix algebras.

Theorem 3 *Let $\mathbb{F}$ be an arbitrary field, $\mathcal{M}_n(\mathbb{F})$ the algebra of all $n \times n$ matrices over $\mathbb{F}$ and $\varphi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ an algebra isomorphism. Then there exists an invertible matrix $T \in \mathcal{M}_n(\mathbb{F})$ such that*

$$\varphi(A) = TAT^{-1}$$

for every $A \in \mathcal{M}_n(\mathbb{F})$.

Proof. Choose and fix a pair of nonzero vectors u, v in $\mathbb{F}^n$. Since φ is injective we can find $z \in \mathbb{F}^n$ such that $\varphi(uv^t)z \neq 0$. Define $T : \mathbb{F}^n \to \mathbb{F}^n$ by $Tx = \varphi(xv^t)z$, $x \in \mathbb{F}^n$. The linearity of T follows from the linearity of φ. Moreover, T is nonzero since $Tu \neq 0$. For arbitrary $A \in \mathcal{M}_n(\mathbb{F})$ and $x \in \mathbb{F}^n$ we have:

$$TAx = \varphi((Ax)v^t)z = \varphi(A \cdot xv^t)z = \varphi(A)\varphi(xv^t)z = \varphi(A)Tx$$

and consequently,

$$TA = \varphi(A)T.$$

Let w be any vector in $\mathbb{F}^n$. Since $Tu \neq 0$ and because φ is surjective we can find $B \in \mathcal{M}_n(\mathbb{F})$ such that $\varphi(B)Tu = w = TBu$. Thus T is surjective, and therefore invertible. It follows that $\varphi(A) = TAT^{-1}$, $A \in \mathcal{M}_n(\mathbb{F})$, as desired.

Theorem 4 *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear isomomorphism between the algebras $\mathcal{A}$ and $\mathcal{B}$. Then φ is an algebra isomorphism if and only if for every basis $e_1, e_2, \ldots e_n$ of $\mathcal{A}$*

$$\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$$

for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$.

Proof. If φ is an algebra isomorphism, then the equality $\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$ holds for every two elements of $\mathcal{A}$, so in particular for e_i and e_j.

Now assume $\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$, and let f and g in $\mathcal{A}$. Then $f = \sum_{i=1}^{n} c_i e_i$ and $g = \sum_{j=1}^{n} d_j e_j$. So by linearity of φ and our assumption:

$$\varphi(fg) = \varphi\left(\sum_{i=1}^{n} c_i e_i \sum_{j=1}^{n} d_j e_j\right) = \varphi(\sum_{i,j=1}^{n} c_i d_j e_i e_j) =$$

$$\sum_{i,j=1}^{n} c_i d_j \varphi(e_i e_j) = \sum_{i,j=1}^{n} c_i d_j \varphi(e_i)\varphi(e_j) = \sum_{i=1}^{n} c_i \varphi(e_i) \sum_{j=1}^{n} d_j \varphi(e_j) =$$

$$\varphi\left(\sum_{i=1}^{n} c_i e_i\right) \varphi\left(\sum_{j=1}^{n} d_j e_j\right) = \varphi(f)\varphi(g).$$

Thus φ is an algebra isomorphism.

Chapter 2

The isomorphism between $\mathcal{U}_{2\times 2}$ and $\mathcal{L}_{2\times 2}$

Now we will work on showing the form of the algebras isomorphisms:

$$f : \mathcal{U}_{2\times 2} \to \mathcal{L}_{2\times 2},$$

where $\mathcal{U}_{2\times 2}$ and $\mathcal{L}_{2\times 2}$ are, respectively, the algebras of all two by two upper triangular matrices and lower triangular matrices over an arbitrary field, and will find all such isomorphisms.

2.1 Defining the basis of $\mathcal{U}_{2\times 2}$

Let us take the following basis of $\mathcal{U}_{2\times 2}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easy to observe that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are idempotents, denote them by I_1 and I_2 respectively.

As for $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, it is a nilpotent, denote it by N.

2.2 Idempotents in $\mathcal{L}_{2\times 2}$

Since by Theorem 1, an algebra isomorphism preserves idempotency and nilpotency of elements, let us study the form of the idempotents and nilpotents in $\mathcal{L}_{2\times 2}$.

For any $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in \mathcal{L}_{2\times 2}$, we have:

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 \\ ba + bc & c^2 \end{bmatrix}.$$

For A to be an idempotent these are the following conditions that have to be satisfied:

- $a^2 = a$ which implies that $a = 0$ or $a = 1$;
- $c^2 = c$ which implies that $c = 0$ or $c = 1$, and
- $ba + bc = b$.

Note that if $a = c = 0$, then A would be the zero matrix. Also if $a = c = 1$ this implies that $b = 0$ and therefore A would be the identity matrix. Therefore what we are left with is:

- $a = 1$, $c = 0$, which generates the matrix $I_1' = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}$, and
- $a = 0$, $c = 1$, which generates the matrix $I_2' = \begin{bmatrix} 0 & 0 \\ b' & 1 \end{bmatrix}$.

Therefore I_1 and I_2 can be mapped only to either I_1' or I_2' .

2.3 2-Nilpotents in $\mathcal{L}_{2\times 2}$

For A to be a 2-nilpotent these are the following conditions:

- $a^2 = 0$ which implies that $a = 0$;

- $c^2 = 0$ which implies that $c = 0$.

Therefore all the 2-nilpotents in $\mathcal{L}_{2\times 2}$, are of the form $\begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}$, where $d \in \mathbb{F}^*$ (used instead of b, since we reserve the letter b for the idempotent case).

2.4 Right annihilators in $\mathcal{U}_{2\times 2}$

Now let us study the dimension of the right annihilators of the idempotents in $\mathcal{U}_{2\times 2}$. Let

$$\begin{bmatrix} x & y \\ 0 & w \end{bmatrix} \in r\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

for some $x, y, w \in \mathbb{F}$. For I_1 we have:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & w \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}.$$

Therefore, in this case we must have $x = y = 0$. Moreover,

$$r\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix} : w \in \mathbb{F}\right\},$$

therefore $\dim r(I_1) = 1$. For I_2 we have:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix}.$$

In this case we must have $w = 0$. Moreover

$$r\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in \mathbb{F}\right\},$$

therefore $\dim r(I_2) = 2$.

2.5 Right annihilators in $\mathcal{L}_{2\times 2}$

Now let us study the dimensions of the right annihilators for the idempotents in $\mathcal{L}_{2\times 2}$. Let

$$\begin{bmatrix} x' & 0 \\ y' & w' \end{bmatrix} \in r\left(\begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}\right)$$

for some $x', y', w' \in \mathbb{F}$. For I_1' we have:

$$\begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}\begin{bmatrix} x' & 0 \\ y' & w' \end{bmatrix} = \begin{bmatrix} x' & 0 \\ bx' & 0 \end{bmatrix}.$$

In this case we must have $x' = 0$. Moreover

$$r\left(\begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 & 0 \\ y' & w' \end{bmatrix} : y', w' \in \mathbb{F}\right\},$$

therefore $\dim r(I_1') = 2$.

For I_2' we have:

$$\begin{bmatrix} 0 & 0 \\ b' & 1 \end{bmatrix}\begin{bmatrix} x' & 0 \\ y' & w' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b'x' + y' & w' \end{bmatrix}.$$

In this case we must have $w' = 0$ and $y' = -b'x'$. Moreover

$$r\left(\begin{bmatrix} 0 & 0 \\ b' & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} x' & 0 \\ -b'x' & 0 \end{bmatrix} : x' \in \mathbb{F}\right\},$$

therefore $\dim r(I_2') = 1$.

2.6 Observation

By Theorem 2, we can see that:

- $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is mapped onto $\begin{bmatrix} 0 & 0 \\ b' & 1 \end{bmatrix}$, and

- $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is mapped onto $\begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}$.

As for the nilpotents, there is only one in $\mathcal{L}_{2\times 2}$ and one in $\mathcal{U}_{2\times 2}$ therefore $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is mapped onto $\begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}$.

2.7 Verification

From the previous discussion, we conclude that $f : \mathcal{U}_{2\times 2} \to \mathcal{L}_{2\times 2}$ is defined by:

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ b' & 1 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}.$$

From this definition, since an algebra isomorphism preserve the identity, we conclude that

$$f(I_1 + I_2) = f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

$$f(I_1) + f(I_2) = f\left(\begin{bmatrix} 1 & 0 \\ b + b' & 1 \end{bmatrix}\right).$$

Therefore $b + b' = 0$, which implies that $b' = -b$.

To verify that f is an algebra isomorphism we use Theorem 4:

$$f(I_1 I_2) = f(0_{2\times 2}) = 0_{2\times 2} = f(I_1)f(I_2)$$

$$f(I_2 I_1) = f(0_{2\times 2}) = 0_{2\times 2} = f(I_2)f(I_1)$$

$$f(I_1N) = f(I_1)f(N) = f(N)$$

$$f(NI_1) = f(0_{2\times2}) = 0_{2\times2} = f(N)f(I_1)$$

$$f(I_2N) = f(0_{2\times2}) = 0_{2\times2} = f(I_2)f(N)$$

$$f(NI_2) = f(N)f(I_2) = f(N)$$

$$f(I_1^2) = f(I_1) = (f(I_1))^2$$

$$f(I_2^2) = f(I_2) = (f(I_2))^2$$

$$f(N_1^2) = f(N_1) = (f(N_1))^2.$$

2.8 Result

Theorem 5 *$f : \mathcal{U}_{2\times2} \to \mathcal{L}_{2\times2}$ is an algebra isomorphism if and only if it is defined in the following way:*

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ -b & 1 \end{bmatrix},$$
$$f\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ b & 0 \end{bmatrix},$$
$$f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix},$$

for some $b, d\mathbb{F}$ for which $d \neq 0$.

2.9 Example

An easy way to define this isomorphism $f : \mathcal{U}_{2\times2} \to \mathcal{L}_{2\times2}$ is by taking $b = 0$ and therefore we get $f : \mathcal{U}_{2\times2} \to \mathcal{L}_{2\times2}$ defined as follows:

$$f\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$f\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Chapter 3

The isomorphism between $\mathcal{U}_{3\times3}$ and $\mathcal{L}_{3\times3}$

Now we will work on showing the form of the algebra isomorphisms:

$$f : \mathcal{U}_{3\times3} \to \mathcal{L}_{3\times3}$$

3.1 Defining the basis of $\mathcal{U}_{3\times3}$

Let us take the following basis of $\mathcal{U}_{3\times3}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to observe that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are idempotents, we denote them respectively by I_1, I_2 and I_3.

As for

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

they are 2-nilpotents, we denote them respectively by N_1, N_2 and N_3.

3.2 Idempotents in $\mathcal{L}_{3\times 3}$

Let us study the form of the idempotents in $\mathcal{L}_{3\times 3}$

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & k \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 & 0 \\ ba+bc & c^2 & 0 \\ da+eb+kd & ec+ke & k^2 \end{bmatrix}.$$

For this matrix to be an idempotent, these are the conditions that have to be satisfied:

- $a^2 = a$ which imples that $a = 0$ or $a = 1$
- $c^2 = c$ which imples that $c = 0$ or $c = 1$
- $k^2 = k$ which imples that $k = 0$ or $k = 1$
- $ba + bc = b$
- $ec + ke = e$
- $da + eb + kd = d$.

Note that if $a = c = k = 0$, then $ba + bc = 0$ therefore $b = 0$, $ec + ke = e$ therefore $e = 0$, and $d = 0$. Hence if this is the case we will get the zero matrix. Since none of the basis matrices can be mapped to the zero matrix, each image of each element of the basis matrices has to have a single 1 on the main diagonal so that $f(I_1) + f(I_2) + f(I_3) = Id_{3\times 3}$. Thus each of the $f(I_1)$, $f(I_2)$, $f(I_3)$ has a single 1 in a distinct entry on the main diagonal. Therefore, we have three possible cases:

Case 1: In this case, $a = 1$, $c = 0$, and $k = 0$, which implies that:

- $ba + bc = b$,
- $ec + ke = e$ and hence $e = 0$
- $da + eb + kd = d$.

These conditions generate

$$I_1' = \begin{bmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & 0 & 0 \end{bmatrix}.$$

Case 2: In this case, $a = 0$, $c = 1$, and $k = 0$, which implies that:

- $ba + bc = b$,
- $ec + ke = e$,
- $da + eb + kd = d$, hence $d = eb$.

These conditions generate

$$I_2' = \begin{bmatrix} 0 & 0 & 0 \\ b & 1 & 0 \\ eb & e & 0 \end{bmatrix}.$$

Case 3: In this case, $a = 0$, $c = 0$, and $k = 1$, which implies that:

- $ba + bc = b$ and hence $b = 0$,
- $ec + ke = e$,
- $da + eb + kd = d$.

These conditions generate

$$I_3' = \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & e & 1 \end{bmatrix}.$$

To eliminate any notation confusion, we are going to denote:

$$I_1' = \begin{bmatrix} 1 & 0 & 0 \\ b & 0 & 0 \\ d & 0 & 0 \end{bmatrix}, \quad I_2' = \begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ ca & c & 0 \end{bmatrix}, \quad I_3' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & 1 \end{bmatrix}.$$

Thus I_1, I_2 and I_3 are mapped each to either I_1', I_2', or I_3'. Moreover since f is an isomorphism, f should preserve the identity, therefore:

$$f(I_1) + f(I_2) + f(I_3) = f(I_1 + I_2 + I_3) = f(Id_{3\times 3}) = Id_{3\times 3}.$$

But

$$f(I_1) + f(I_2) + f(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ b+a & 1 & 0 \\ g+ca+d & c+h & 1 \end{bmatrix}.$$

Hence:

- $b = -a$,
- $c = -h$,
- $g + ca + d = 0$.

Therefore we have:

$$I_1' = \begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ d & 0 & 0 \end{bmatrix}, \quad I_2' = \begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ -ha & -h & 0 \end{bmatrix}, \quad I_3' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & 1 \end{bmatrix},$$

with the condition $g - ha + d = 0$.

3.3 Right and left annihilators in $\mathcal{U}_{3\times 3}$

Now let us study the dimensions of the right and the left annihilators of the idempotents in $\mathcal{U}_{3\times 3}$. Let

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \in r\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), r\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right), \text{ or } r\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

for some $x, y, z, t, u, w \in \mathbb{F}$.

For I_1 we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} = \begin{bmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, in this case we must have $x = y = z = 0$. Hence,

$$r(I_1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} : t, u, w \in \mathbb{F} \right\}$$

and its dimension is 3.

Similarly, we consider the left annihilator of I_1:

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In this case, we must have $x = 0$. Hence

$$l(I_1) = \left\{ \begin{bmatrix} 0 & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} : y, z, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 5.

So we have proven that the dimensions of the right annihilator and the left annihilator of I_1 are 3 and 5 respectively.

For I_2 we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t & u \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have $t = u = 0$. Hence

$$r(I_2) = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & w \end{bmatrix} : x, y, z, w \in \mathbb{F} \right\},$$

and its dimension is 4.

Similarly for the left annihilator we have:

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & y & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have $y = t = 0$. Hence,

$$l(I_2) = \left\{ \begin{bmatrix} x & 0 & z \\ 0 & 0 & u \\ 0 & 0 & w \end{bmatrix} : x, z, u, w \in \mathbb{F} \right\},$$

and its dimension is 4.

So we have proven that the dimensions of the right annihilator and the left annihilator of I_2 are both 4.

For I_3 we have:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w \end{bmatrix}.$$

In this case we must have $w = 0$. Hence

$$r(I_3) = \left\{ \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & 0 \end{bmatrix} : x, y, z, t, u \in \mathbb{F} \right\},$$

and its dimension is 5.

Similarly for the left annihilator :

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & u \\ 0 & 0 & w \end{bmatrix}.$$

In this case $z = u = w$. Hence

$$l(I_3) = \left\{ \begin{bmatrix} x & y & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix} : x, y, t \in \mathbb{F} \right\},$$

and its dimension is 3.

So we have proven that the dimensions of the right annihilator and the left annihilator of I_3 are 5 and 3 respectively.

3.4 Right and left annihilators in $\mathcal{L}_{3\times3}$

Now let us study the dimensions of the right annihilators and the left annihilators for the idempotents in $\mathcal{L}_{3\times3}$. Let

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \in r\left(\begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ d & 0 & 0 \end{bmatrix}\right), r\left(\begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ -ha & -h & 0 \end{bmatrix}\right), \text{ or } r\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & 1 \end{bmatrix}\right)$$

for some $x, y, z, t, u, w \in \mathbb{F}$.

Then for I_1' we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ d & 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ -ax & 0 & 0 \\ dx & 0 & 0 \end{bmatrix}.$$

In this case we must have $x = 0$. Hence

$$r(I_1') = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} : y, z, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 5.

Similarly for the left annihilator:

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ d & 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ y - za & 0 & 0 \\ t - ua + dw & 0 & 0 \end{bmatrix}.$$

In this case we must have:

- $x = 0$,
- $y = za$,
- $t = ua - dw$.

Hence

$$l(I_1') = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ za & z & 0 \\ ua - dw & u & w \end{bmatrix} : z, u, w \in \mathbb{F} \right\},$$

and its dimension is 3.

So we have proven that the dimensions of the right and left annihilators of I_1' are 5 and 3 respectively.

For I_2' we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ -ha & -h & 0 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ ax + y & z & 0 \\ -hax - hy & -hz & 0 \end{bmatrix}.$$

In this case we must have $z = 0$ and $y = -ax$. Hence

$$r(I_2') = \left\{ \begin{bmatrix} x & 0 & 0 \\ -ax & 0 & 0 \\ t & u & w \end{bmatrix} : x, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 4.

Similarly for the left annihilator:

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ -ha & -h & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ za & z & 0 \\ ua - wha & u - wh & 0 \end{bmatrix}.$$

In this case we must have $z = 0$ and $u = wh$. Hence

$$l(I_2') = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & 0 & 0 \\ t & wh & w \end{bmatrix} : x, y, t, w \in \mathbb{F} \right\},$$

and its dimension is 4.

So we have proven that the dimensions of the right and left annihilators of I_2' are both 4.

For I_3' we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ gx + hy + t & hz + u & w \end{bmatrix}.$$

In this case we must have:

- $w = 0$,
- $u = -hz$,
- $t = -gx - hy$.

Hence

$$r(I_3') = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ -gx - hy & -hz & 0 \end{bmatrix} : x, y, z \in \mathbb{F} \right\},$$

and its dimension is 3.

Similarly for the left annihilator:

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ wg & wh & w \end{bmatrix}.$$

In this case we must have $w = 0$. Hence

$$l(I_3') = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & 0 \end{bmatrix} : x, y, z, t, u \in \mathbb{F} \right\},$$

and its dimension is 5.

So we have proven that the dimensions of the right and left annihilators of I_3' are 3 and 5 respectively.

3.5 Observation 1

If we compare the idempotents from $\mathcal{U}_{3\times 3}$ and the idempotents from $\mathcal{L}_{3\times 3}$, we can see that precisely the following pairs of idempotents have the same dimensions of the right and left annihilators:

$$(I_1, I_3'),$$

$$(I_2, I_2'),$$

$$(I_3, I_1').$$

Therefore our first conclusion is that:

$$f(I_1) = I_3',$$

$$f(I_2) = I_2',$$

$$f(I_3) = I_1'.$$

3.6 2-nilpotents and their right and left annihilators in $\mathcal{L}_{3\times 3}$

Now we investigate the form of the 2-nilpotents in $\mathcal{L}_{3\times 3}$.

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 & 0 \\ ba+bc & c^2 & 0 \\ da+eb+fd & ec+fe & f^2 \end{bmatrix}.$$

For this matrix to be a 2-nilpotent, the following conditions must be satisfied:

- a^2=0,
- c^2=0,
- f^2=0,
- $eb = 0$, therefore $e = 0$ or $b = 0$.

Depending on whether $e = 0$ and/or $b = 0$, we have three possible cases.

Case 1: In this case, $e = 0$, $b \neq 0$ and therefore, the first 2-nilpotent N_1' is of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ d & 0 & 0 \end{bmatrix}.$$

Case 2: In this case, $b = 0$, $e \neq 0$ and therefore the second 2-nilpotent N_2' is of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & e & 0 \end{bmatrix}.$$

Case 3: In this case, $e = b = 0$ and therefore the third 2-nilpotent N_3' is of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d & 0 & 0 \end{bmatrix}.$$

To eliminate the confusion we are going to denote:

- $N_1' = \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ j & 0 & 0 \end{bmatrix}$, where $i, j \neq 0$.

- $N_2' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m & n & 0 \end{bmatrix}$, where $m, n \neq 0$.

- $N_3' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{bmatrix}$ where $p \neq 0$.

Let us evaluate the dimensions of the right and left annihilators of these 2-nilpotents. Let

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \in r(N_1'), r(N_2'), \text{ or } r(N_3') \text{ for some } x, y, z, t, u, w \in \mathbb{F}.$$

Then, for N_1' we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ j & 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ ix & 0 & 0 \\ jx & 0 & 0 \end{bmatrix},$$

and so x must be equal to zero, since $i, j \neq 0$. Hence

$$r(N_1') = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} : y, z, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 5.

Similarly for the left annihilator:

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ j & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ zi & 0 & 0 \\ ui + wj & 0 & 0 \end{bmatrix}.$$

In this case, we must have $z = 0$ and $u = -\dfrac{wj}{i}$ since $i \neq 0$. Hence

$$l(N_1') = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & 0 & 0 \\ t & -\dfrac{wj}{i} & v \end{bmatrix} : x, y, t, w \in \mathbb{F} \right\},$$

and its dimension is 4.

So we have proven that the dimensions of the right and left annihilator of N_1' are 5 and 4 respectively.

For N_2' we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m & n & 0 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ mx + ny & nz & 0 \end{bmatrix}.$$

In this case we must have $z = 0$ and $x = -\dfrac{ny}{m}$ since $m, n \neq 0$. Hence

$$r(N_2') = \left\{ \begin{bmatrix} -\dfrac{ny}{m} & 0 & 0 \\ y & 0 & 0 \\ t & u & w \end{bmatrix} : y, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 4.

Similarly for the left annihilator:

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m & n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ wm & wn & 0 \end{bmatrix}.$$

In this case we must have $w = 0$ since $m, n \neq 0$. Hence

$$l(N_2') = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & 0 \end{bmatrix} : x, y, z, t, u \in \mathbb{F} \right\},$$

and its dimension is 5.

We have proven that the dimensions of the right and left annihilator of N_2' are 4 and 5 respectively.

For N_3' we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ px & 0 & 0 \end{bmatrix}.$$

In this case we must have $x = 0$ since $p \neq 0$. Hence

$$r(N_3') = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} : y, z, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 5.

Similarly for the left annihilator:

$$\begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ wp & 0 & 0 \end{bmatrix}.$$

In this case we must have $w = 0$ since $p \neq 0$. Hence

$$l(N_3') = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & z & 0 \\ t & u & 0 \end{bmatrix} : x, y, z, t, u \in \mathbb{F} \right\},$$

and its dimension is 5.

We have proven that the dimensions of the right and left annihilators of N_3' are both 5.

3.7 2-nilpotents and their right and left annihilators in $\mathcal{U}_{3\times 3}$

Now let us evaluate the dimension of the right and left annihilators of N_1, N_2 and N_3. Let

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \in r(N_1), r(N_2), \text{ or } r(N_3) \text{ for some } x, y, z, t, u, w \in \mathbb{F}.$$

For N_1, we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} = \begin{bmatrix} 0 & t & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have $t = u = 0$. Hence:

$$r(N_1) = \left\{ \begin{bmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & w \end{bmatrix} : x, y, z, w \in \mathbb{F} \right\},$$

and its dimension is 4.

Similarly for the left annihilator:

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have x=0. Hence:

$$l(N_1) = \left\{ \begin{bmatrix} 0 & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} : y, z, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 5.

We have proven that the dimensions of the right and left annihilators of N_1 are 4 and 5 respectively.

For N_2 we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have $w = 0$. Hence:

$$r(N_2) = \left\{ \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & 0 \end{bmatrix} : x, y, z, t, u \in \mathbb{F} \right\},$$

and its dimension is 5.

Similarly for the left annihilator:

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have $x = 0$. Hence:

$$l(N_2) = \left\{ \begin{bmatrix} 0 & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} : y, z, t, u, w \in \mathbb{F} \right\},$$

and its dimension is 5.

So we have proven that the dimensions of the right and left annihilators of N_2 are both 5.

For N_3 we obtain:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have $w = 0$. Hence:

$$r(N_3) = \left\{ \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & 0 \end{bmatrix} : x, y, z, t, u \in \mathbb{F} \right\},$$

and its dimension is 5.

Similarly for the left annihilator:

$$\begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & w \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & y \\ 0 & t & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case we must have $y = t = 0$. Hence:

$$l(N_3) = \left\{ \begin{bmatrix} x & 0 & z \\ 0 & 0 & u \\ 0 & 0 & w \end{bmatrix} : x, z, u, w \in \mathbb{F} \right\},$$

and its dimension is 4.

So we have proven that the dimension of the right and left annihilators of N_3 are 5 and 4 respectively.

3.8 Observation 2

If we compare the 2-nilpotents from $\mathcal{U}_{3\times 3}$ and the 2-nilpotents from $\mathcal{L}_{3\times 3}$ we can see that precisely the following pairs of 2-nilpotents have the same dimensions of the right and left annihilators:

$$(N_1, N_2'),$$

$$(N_2, N_3'),$$

$$(N_3, N_1').$$

We conclude that:

$$f(N_1) = N_2',$$

$$f(N_2) = N_3',$$

$$f(N_3) = N_1'.$$

3.9 Verification

$f : \mathcal{U}_{3\times 3} \to \mathcal{L}_{3\times 3}$ is defined by:

$$f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & 1 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ -ha & -h & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ d & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m & n & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ j & 0 & 0 \end{bmatrix},$$

with the condition established before: $g - ha + d = 0$.

To verify that f is an isomorphism, using Theorem 4, we check that $f(AB) = f(A)f(B)$ for the elements A, B of the basis. The following requirements are always satisfied:

$$f(I_1I_2) = f(0_{3\times3}) = f(I_1)f(I_2),$$

$$f(I_1I_3) = f(0_{3\times3}) = f(I_1)f(I_3),$$

$$f(I_2I_1) = f(0_{3\times3}) = f(I_2)f(I_1),$$

$$f(I_3I_1) = f(0_{3\times3}) = f(I_3)f(I_I),$$

$$f(I_2I_3) = f(0_{3\times3}) = f(I_2)f(I_3),$$

$$f(I_3I_2) = f(0_{3\times3}) = f(I_3)f(I_2),$$

$$f(N_1N_2) = f(0_{3\times3}) = f(N_1)f(N_2).$$

The requirement

$$f(N_1N_3) = f(N_2) = f(N_1)f(N_3)$$

leads to a new restriction $ni = p$. The following requirements are always satisfied:

$$f(N_2N_1) = f(0_{3\times3}) = f(N_2)f(N_1),$$

$$f(N_3N_1) = f(0_{3\times3}) = f(N_3)f(N_1),$$

$$f(N_2N_3) = f(0_{3\times3}) = f(N_2)f(N_3),$$

$$f(N_3N_2) = f(0_{3\times3}) = f(N_3)f(N_2),$$

$$f(I_1N_1) = f(N_1) = f(I_1)f(N_1),$$

$$f(I_1N_2) = f(N_2) = f(I_1)f(N_2),$$

The requirement

$$f(I_1N_3) = f(0_{3\times3}) = f(I_1)f(N_3)$$

leads to a new restriction $j = -hi$. The following requirements are always satisfied:

$$f(N_1I_1) = f(0_{3\times3}) = f(N_1)f(I_1),$$

$$f(N_2I_1) = f(0_{3\times3}) = f(N_2)f(I_1),$$

$$f(N_3I_1) = f(0_{3\times3}) = f(N_3)f(I_1),$$

$$f(I_2N_1) = f(0_{3\times3}) = f(I_2)f(N_1),$$

$$f(I_2N_2) = f(0_{3\times3}) = f(I_2)f(N_2),$$

$$f(I_2N_3) = f(N_3) = f(I_2)f(N_3).$$

The requirement

$$f(N_1I_2) = f(N_1) = f(N_1)f(I_2)$$

leads to a new restriction $m = na$. The following requirements are always satisfied:

$$f(N_2I_2) = f(0_{3\times3}) = f(N_2)f(I_2),$$

$$f(N_3I_2) = f(0_{3\times3}) = f(N_3)f(I_2),$$

$$f(I_3N_1) = f(0_{3\times3}) = f(I_3)f(N_1),$$

$$f(I_3N_2) = f(0_{3\times3}) = f(I_3)f(N_2),$$

$$f(I_3N_3) = f(0_{3\times3}) = f(I_3)f(N_3),$$

$$f(N_1I_3) = f(0_{3\times3}) = f(N_1)f(I_3),$$

$$f(N_2I_3) = f(N_2) = f(N_2)f(I_3),$$

$$f(N_3I_3) = f(N_3) = f(N_3)f(I_3),$$

$$f(I_1^2) = f(I_1) = (f(I_1))^2,$$

$$f(I_2^2) = f(I_2) = (f(I_2))^2,$$

$$f(I_3^2) = f(I_3) = (f(I_3))^2,$$

$$f(N_1^2) = f(0_{3\times3}) = (f(N_1))^2,$$

$$f(N_2^2) = f(0_{3\times3}) = (f(N_2))^2,$$

$$f(N_3^2) = f(0_{3\times3}) = (f(N_3))^2.$$

3.10 Result

Theorem 6 *$f : \mathcal{U}_{3\times3} \to \mathcal{L}_{3\times3}$ is an algebra isomorphism if and only if it is defined in the following way :*

$$f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & 1 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ a & 1 & 0 \\ -ha & -h & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ d & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ na & n & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ni & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ -hi & 0 & 0 \end{bmatrix},$$

for some values $a, d, g, h \in \mathbb{F}$ for which $g - ha + d = 0$, $i \neq 0$, and $n \neq 0$.

3.11 Examples

Example 1. An easy way to define this isomorphism $f : \mathcal{U}_{3\times 3} \to \mathcal{L}_{3\times 3}$ is by taking $a = d = g = h = 0$ and $i = n = 1$ and therefore we get

$$f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 2. There are infinitely many ways to define an isomorphism $f : \mathcal{U}_{3\times 3} \to \mathcal{L}_{3\times 3}$, and here is one more example. Let $g = 3$, $h = 4$, $a = 2$, $d = 5$, $i = 1$, $n = 2$, therefore:

$$f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 4 & 1 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ -8 & -4 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 2 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix},$$

$$f\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}.$$

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Curriculum Vitae

Zahi Fawaz was born in Beirut, Lebanon. The first son of Said Fawaz and Hala Fawaz, he graduated from College Protestant Francais, Beirut, Lebanon in the spring of 2006 and entered the American University of Beirut in Fall of 2006. While pursuing a bachelor degree in mathematical Science, he was the president of the math society for two years. In the spring of 2009, he graduated with a Bachelor of Science. He entered the University of Texas at El Paso in the spring of 2010 seeking a graduate degree in Mathematical sciences. For the entirety of his degree, he was employed there as a teaching assistant. He defended his masters thesis–which he plans to have published–entitled The Isomorphisms Between the Upper and Lower Triangular Matrix Algebras on May 10th, 2011, and will be graduating on May 14th, 2011. In the following fall 2011, he will be attending New Mexico State University in pursuit of his doctoral degree.

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