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Metaphors, Metonymies, Modes and Linear Algebra

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Dedication

To the only person who never gave up on me and whose memory will always be in my heart, my mom

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by

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Abstract

The analysis focused on the presence of different thinking modes, metonymies, and metaphors found on the interview responses to questions related to linear independence, span, and spanning sets of four students taking their first linear algebra course at the college level. The findings provide insight of how first year linear algebra students move from one thinking mode to another and what kind of metonymies and metaphors are used to construct new knowledge. The main purpose of this research was to document and determine the main characteristics that categorize the four students' reasoning.

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Chapter 1: Introduction

College students are required to take a wide variety of courses; all these courses are aimed to provide the students with a well-rounded education. Although some of these courses may seem irrelevant to the vast majority of the student population, each one of them plays a crucial part in helping them understand the world and become better, more critical, thinkers. One of the courses that students often perceive as irrelevant and even as somewhat of a nuisance is matrix algebra.

One, if not the main reason, contributing to this negative perception can be attributed to a "high level of formalism and the axiomatic approach" (Dogan, 2010; 2006) and the lack of familiarity with the elementary set theory, algebraic manipulation, and logic. This combination of factors creates a generalized feeling of frustration in the students, leading them to lose interest in the topic.

This paper is part of an ongoing research partially funded by the National Science Foundation (NSF; CCLI:0737485) that focuses on the cognitive constructs displayed by students who have been exposed to teaching methods used in first year linear algebra courses at university level. Furthermore, the study investigates the role that different visual representations have in the development of knowledge formed by those students.

The purpose of this paper is to explore the different thinking modes, metonymies, and metaphors shown by four interviewed students. By doing so, I intent to further document their understanding and misconceptions in the context of their previous exposure to graphical, algebraic, and abstract representations of linear independence, span, spanning set, and vector spaces. In order to accomplish this, the study will address the following research problem:

What are the thinking modes, metonymies, and metaphors displayed by four students in their responses to interview tasks on linear independence?

1.1 Learning Theories

This paper will mainly focus on two specific cognitive entities: modes of thinking and metonymy/metaphors. Let's now shortly provide description of these entities.

1.1.1 Modes of Thinking

The operational definition of modes of thinking will be based on the descriptions given by Anna Sierpinska in the paper "On some aspects of students' thinking in linear algebra" (2000). Students' modes of thinking will be documented through the qualitative analysis of the students' interview responses. Thinking modes identified by Sierpinska are: Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural.

1.1.1.1 Synthetic-Geometric

The Synthetic-Geometric mode is characterized by the use of geometric representations and the lack of concrete definitions. A good example of this is the ability of a student to describe the qualitative characteristics of a plane, while being unable to provide a comprehensive definition of it (Sierpinska, 2000). This mode of thinking can be best described as a geometry-based practical way of thinking used by students to create an understanding of mathematical concepts.

1.1.1.2 Analytic-Arithmetic

The Analytic-Arithmetic mode consists of using a predetermined formula to carry out a series of computations (Sierpinska, 2000). It takes into consideration the numerical and algebraic aspects of the objects. An example of this thinking mode is a student referring to the linear combination of vectors to imply linear independence.

1.1.1.3 Analytic-Structural

This mode is characterized by describing objects by their properties and by synthesizing them into compact structural wholes (Sierpinska, 2000). Students using this thinking mode rely on theorems and definitions to prove their understanding without the use of arithmetical

procedures to prove their understanding. An example of this could be the use of dimension arguments to prove linear independence (Dogan-Dunlap, 2010).

It is important to mention that these thinking modes are not mutually exclusive and that a student may move from one mode to another in order to maximize their understanding of an object (Sierpinska, 2000).

1.1.2 Metonymy and Metaphor

Metonymies and metaphors help us connect concepts that facilitate our understanding and the construction of meaning. They are key components in the representation of mathematical concepts since they help individuals to understand their representations by using the relationship with learning processes and structures.

1.1.2.1 Metonymy

Metonymy is a figure of speech where a concept is called by a word associated with it. The words associated can represent certain attributes or characteristics of the object and part of the object can be used to represent a whole (Presmeg, 1998). The use of metonymies can be beneficial but also problematic to some students learning new concepts, since they can lead to different misinterpretations regarding the object as a whole. The function of this concept can be illustrated by utilizing "The Pentagon" to represent "the United States Department of Defense". The attributes attached to the object are subject to the interpretation of each individual. For example, "OK, slope? During my second year in high school? Yeah, it was basically said some about, you know, lines, depend how tilt it is. That is the slope". In this example, the student is using the concept of a tilted line to explain what a slope is, by using one of the physical attributes of a line without a zero or undefined slope.

1.1.2.2 Metaphors

Webster's dictionary defines metaphor as "A figure of speech in which an expression is used to refer to something that it does not literally denote in order to suggest the similarity". The

use of metaphors while learning mathematics has been researched as part of the learning mechanisms utilized by students learning new concepts. They are an implicit form of analogy that compares one domain or the elements that belong to it to another domain or its elements by stating that both domains are alike (Presmeg, 1998). Presmeg refers to this domains as ground and tension the ground is comprised by the similar elements of the entities being compared and the tension refers to their different elements (1998) As explained by Dogan-Dunlap (2007), a student who refers to a function as "something that would help something to flow or to run properly" is using a metaphor comparing functions of machinery to mathematical function ideas. Dogan-Dunlap suggests that teachers carefully choose their analogies, especially while introducing a new concept, since what may be tension for the teacher could become ground for a student.

1.1.2.3 The relationship between metonymy and metaphor

Metonymies and metaphors are closely related to the understanding and creation of new knowledge while learning mathematical concepts. The main difference between them is that metaphors replace words by similarity and metonymy by association.

1.2 Research Questions

The purpose of this thesis is to identify and analyze the responses given by four students from the different modular and traditional groups to the set of questions asked during the semi-structured interviews. Our goal will be achieved by addressing the following questions:

- What are the thinking modes displayed by the students in their responses to interview tasks on linear independence?
- What are the metonymies and metaphors displayed by students in such responses?

1.3 Methodology

For the purpose of this thesis, we will analyze the responses of four undergraduate students, enrolled in the Matrix algebra course, to a set of eight questions asked during their one-

on-one interviews scheduled toward the end of the Spring 2009 semester. Each of these students was selected at random from a list of volunteers interviewed at the end of the semester and belonging to three different courses. Our main goal is to analyze the different aspects of learning shown by each student through the presence of the distinct modes of reasoning and the use of metaphors and metonymy as part of their responses.

1.3.1 Participants

The study utilized a convenience sample comprised of four college students between the ages of 19 and 23 enrolled in a matrix algebra course at the University of Texas at El Paso during the Spring semester of 2009. For the purpose of this study, a sample consisting of four students' responses were analyzed. Two of those students were interviewed at the same time, and they are twin brothers.

Two of the groups were enrolled in what was referred to as a modular matrix algebra course (non-traditional course), while the other was called a non-modular course (traditional course). The modular matrix algebra courses enforced the use of interactive internet based mathematical modules that were introduced as part of the class. Students enrolled in these classes used the mathematical modules to complete homework assignments. On the other hand, the non-modular course had a traditional approach, where the professor lectured and assigned homework, but the mathematical modules were not included or even mentioned as part of the course.

1.3.2 Modular and Non-modular characteristics

In the modular versions of the course, topics were presented during class to students through a formal definition, row reduced echelon operations, algebraic manipulations, and, regularly, through the graphical representations of the topics being taught. Homework from the required textbook was often assigned (but not collected) and an assignment, to be answered through the use of the computer modules, was assigned every other week. Sometimes, professors

would introduce new topics by using the computer modules through an overhead projector and would explain the characteristics of the new topic and the relationship with past topics.

In the non-modular version of the course, topics were presented during the class through formal definition, row reduced echelon operations and, depending on the questions asked by students, rarely the professor would provide a graphical representation of the topic. Homework was assigned (but not collected) and consisted on problems taken mainly from the required textbook.

The official course description for both, the modular and the non-modular sections of this class during the spring 2009 semester is as follows:

MATRIX ALGEBRA 3323: Systems of linear equations, matrices, determinants, eigenvalues and eigenvectors, diagonalization, vector spaces and linear transformations.

However, the topics chosen for this thesis have been limited to:

1. The definition of linear dependence or independence in a set of vectors; identification of linear dependency in particular sets of vectors.
2. Characteristics of linearly independent/dependent set of vectors in R^2 and R^3.

1.4 Analysis

A qualitative approach, namely the constant comparison method (Glaser, 1992), will be used to analyze student responses on the interviews. The qualitative analysis will focus on the presence of thinking modes and metaphors/metonymy in students' responses to questions about linear independence, span, and spanning sets.

1.1.4 Qualitative Analytical Procedures

The interviews of 4 students – one from each of the classes available during the Spring 2009 semester— will be transcribed and summarized. The qualitative analysis of the transcripts will be conducted by the author of this thesis, her advisor, and an additional graduate student.

The analysis will consist in the identification and classification of the presence of the cognitive constructs defined previously: thinking modes and metaphors/metonymy. An inter-reliability test will be conducted for each interview to rate the consensus existent within the raters. Discussion among the raters will be done continuously to discuss the different categories identified in each interview and will stop when no additional categories emerge. Once all possible categories are listed, student responses and category descriptions will be included in each as identifiers. Afterwards, the frequency and types of thinking modes and metaphors/metonymy identified will be recorded for each student to address our research questions.

Chapter 2: Literature Review

2.1 Epistemological Aspects of Linear Algebra

Many students attempt to learn mathematics by memorization. Unfortunately, as the complexity level increases, they find that it does not work. Memorization only provides with a superficial knowledge, enough to pass a class, but problematic when moving to a higher level. Since many college students have been relying on memorization throughout their elementary, middle, and high school years, they are not able to make connections at the mathematics college level courses. Linear algebra is one of them, with a high level of abstraction, it is one of the classes that seem to be extremely difficult to most students. Some students become frustrated in not being able to understand the material of mathematics college courses, as a consequence, their knowledge structures become fragmented and lack logic. Researchers McGowen (2010) and Dogan-Dunlap (2009) mention that some of the problems students experience in linear algebra courses are the "high level of formalism" and the "axiomatic approach" for which students are not prepared to deal with.

2.1.1 The Conceptual Aspects of Linear Algebra

Dorier and Sierpienska (2001) believe there is a "necessity of cognitive flexibility" for a profound understanding of linear algebra concepts. Many students have trouble connecting different visual representations used to represent linear algebra concepts due to their lack of logic and set theory knowledge (Dogan-Dunlap, 2006). As mentioned in Dogan-Dunlap (2009, pp. 2), Dubisky and Harrel (1997) explained that students are capable of achieving abstraction if the flexibility between the representations of the same concept is instituted. Abstraction is established if concept images and concept definitions are not contradicting each other (Dogan-Dunlap, 2009).

After 1930, a theoretical reconstruction of the methods to solve linear algebra problems initiated a new axiomatic central theory (Dorier et al, 2001). According to Dorier and Sierpinska (2001) the new axiomatic central theory gave linear algebra a more universal approach and

language to be used in different contexts. This new theory also involved the use of concepts and tools that were not explicitly formulated or unified, and it marked a new level in abstraction (Dorier et al, 2001). The different perspective brought a more sophisticated level in mental operations that as a result, manifested in difficulties associated with the pre-existing related elements of knowledge from lower levels (Dorier et al, 2001).

To have a solid understanding of linear algebra concepts, students need to 'concretize' these abstract objects and their representations (Dorier et al, 2001). Most linear algebra students are overwhelmed by the amount of new definitions and theorems and with the high level of formalism students seem to have a lack of connection to what they already know (Dogan-Dunlap, 2006, 2010).

According to Hillel (2000) linear algebra can be represented with the use of three basic languages, such as geometric, algebraic, and abstract. Abstract language refers to abstract theory, such as vector spaces, linear transformations of vector spaces, and the eigen value theory. The algebraic language of the R^n space, includes n-tuples, matrices, and rank, and the geometric language of two and three dimensional spaces includes line segments, points, geometric transformations, and planes (Dorier et al, 2001). From his research, Hillel (2000) found that the way instructors used to shift from one language to the other, without any pause or attempt to alert students of the change, deprived the students of the time needed to assimilate the relationships among the concepts being learned.

2.1.2 The Cognitive Characteristics of Linear Algebra

Semiotic representations, as defined by Duval (1995) are "productions made by the use of signs belonging to a system of representation which has its own constraints of meaning and functioning". According to Duval, these representations are 'absolutely necessary' in mathematics because some objects cannot be directly recognized and must be represented (Dorier et al, 2001). Semiotic representations play an important role in the development of mental representations, accomplishment of cognitive functions, and production of knowledge

(Dorier et al, 2001). Duval (1995) stated that semiosis and noesis -the highest cognitive process- are two acts that cannot be separated from each other, but they differ in that the first refers to "the comprehension or production of a representation by a sign" while the second refers to "the conceptual comprehension of an object". Duval identified three types of cognitive activities related to semiosis, the formation of a representation, the processing and transformation of a representation, and the conversion of a semiotic representation from one register to another (Dorier et al, 2001).

Pavlopoulou (see Dorier 2000, pp. 247-252) was able to differentiate between three registers of semiotic representation of vectors; arrows as the graphical register, columns of coordinates as the table register, and finally the axiomatic theory of vector spaces as the symbolic register. On her research, Pavlopoulou also found confusion among the students with respect to an object and its representation and difficulty in converting from one register to another (Dorier et al, 2001). As reported by Dorier and Sierpinska (2001), Alves-Dias was able to "generalize the necessity of conversions from one semiotic register to another for the understanding of linear algebra to the necessity of cognitive flexibility". Registers of semiotic representation requires the student to be able to move from one to another (Dorier et al, 2001).

Another cognitive requirement in linear algebra students is the need for background knowledge in areas such as, set theory, logic, and proofs (Zamora, 2010). According to Dogan-Dunlap (2006), Bogomolny (2007), and Rogalski (2000) some of the problems that linear algebra students face manifest due to the lack of background knowledge in the those areas.

2.2 Principles of Teaching Linear Algebra

A movement in the United States to reform the learning and teaching of linear algebra established in 1990 the Linear Algebra Curriculum Study Group (LACSG) to address concerns involving the teaching and learning of linear algebra. The LACSG, composed by sixteen mathematics educators from across the country, created a list of recommendations based on a

combination of three major sources, research-based knowledge done on students' learning processes and the optimal teaching methods of linear algebra, individual teaching experience of LACSG members, and the input of consultants from various disciplines who explained how linear algebra was related to their field and what kind of changes in the curriculum could benefit them (Harel, 2000).

The LACSG members made five major recommendations to improve the teaching and learning of linear algebra (Harel, 1997).

- The first course in linear algebra should not be entirely focused on proofs
- A second course of linear algebra should be part of every mathematics curriculum
- The incorporation of technology
- The introduction of linear algebra concepts in high school
- A core syllabus that included concepts such as matrix addition and multiplication, Gaussian elimination, echelon and reduced echelon form, matrix inverses, determinants, linear combinations, linear dependence and independence, subspaces of R^n, bases of R^n, matrices as linear transformations, rank, inner products, eigen vectors, eigen values, in between others (Harel, 2000).

Following these recommendations Harel (2000) developed a theoretical framework based on the three learning-teaching principles: the Concreteness Principle, the Necessity Principle, and the Generalizibility Principle.

2.2.1 Concreteness Principle

After working on experiments with high school and beginning college students, Harrel (2000) found the assumption of students being able to deal with abstract structures without extensive preparation to be unjustified. His discovery led to the formulation of the Concreteness Principle (Harel, 1987) which states:

For students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student's eyes; that is to say, the student has mental procedures that can take these objects as inputs. (page 180)

This principle advises us that students build onto their understanding of concepts more, if the context is concrete to them (Harel, 2000). Concreteness of the abstract concepts may be achieved through technology activities providing the initial mental structure needed for successful learning of topics (Dogan-Dunlap, 2010). My thesis in fact uses data from a pool of students some of whom were provided initial mental construct for basic linear algebra concepts via online web activities.

2.2.2 Necessity Principle

The main idea behind the second principle formulated by Harel (2000) is that instructors must include problem solving activities in which students can reflect abstract conceptions and apply them to solve mathematical problems that are realistic and appreciated by them, it states:

For students to learn, they must see a need for what they are intended to be taught. By 'need' it is meant an intellectual need, as opposed to a social or economic need. (page 185)

Harel (1998) believes the way to transform the Necessity Principle into a more concrete teaching setting is by recognizing and identifying the intellectual need of students, allowing the interactions of students with the problems corresponding to their intellectual needs, and by guiding students in the processes of transferring their knowledge to find a solution.

2.2.3 Generalizibility Principle

The third and last principle formulated by Harel (2000) is a complement of the Concreteness Principle and the Necessity Principle, and it states:

When instruction is concerned with a 'concrete' model, that is a model that satisfies the Concreteness Principle, the instructional activities within this model should allow and encourage the generalizibility of concepts. (page 187)

This principle intends to aid students summarize concepts learned in a specific model in order to make generalizations (Harel, 2000).

2.3 The Use of Geometry in the Teaching and Learning of Linear Algebra

The use of geometry has been somehow controversial, and its use depends on the instructor's preferences. Some instructors believe that geometrical representations are beneficial and necessary to develop understanding; others argue that the excessive use of geometry while introducing a new concept could be harmful since its geometrical representation might be taken in too metaphorically (Gueudet-Chartier, 2004). Linear algebra appears as an abstract subject to many students, some of these students find it hard to relate algebraic statements to geometric statements. As reported by Dogan-Dunlap (2010), the use of geometrical representations helps students consider the different representations of a concept flexibly and allows them to move from one thinking mode to another. Pecuch-Herrero (2000) reported that geometrical interpretations of certain linear algebra concepts, such as the Grand-Schmidt orthogonalization process, prevented students from getting lost in their computations.

Gueudet-Chartier (2004) stated that "linear algebra cannot appear as a generalization of geometry alone; it rather must be grounded in several mathematical domains" and concluded that geometry must be used carefully in linear algebra courses. Marc Rogalsky (2000) acknowledges geometry as an important background support for language and meaning in linear algebra and explains how it can provide images of concepts, such as subspaces, linear combinations, direct sum, solutions of systems of linear equations, etc. Geometric representations used to illustrate general situations in linear algebra are useful when used carefully and concurrently with other representations (Rogalsky, 2000) and as long as students understand how ideas can be represented symbolically, numerically, and graphically they will be able to move back and forth from one thinking mode to another (Zamora, 2010), which will facilitate their understanding.

It was Harel's recommendation to utilize geometrical representations and technology in linear algebra courses to aid students summarize concepts learned in a specific models in order to make generalizations and acquire a higher level of understanding of abstract concepts. The intention of this thesis is primarily to document the thinking modes, metaphors and metonymies between two different sections of a matrix algebra class. One of these sections being exposed to concrete geometric representations providing initial mental constructs to aid successful learning. The results coming from this thesis may be considered as a comparison major to further understand the effect of geometric representations in learning. Now, I will provide a short discussion on the frameworks used in my analysis of data.

2.4 Sierpinska's Modes of Thinking

The analysis on the thinking modes used by the students interviewed for the purpose of this thesis followed the Sierpinska's framework on student thinking modes, where three kinds of thinking modes were documented Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural. In her paper (Sierpinska, 2000) "On some aspects on students' thinking in linear algebra" Sierpinska identified these modes of reasoning in linear algebra based on their interacting language, the visual geometric, the arithmetic, and the structural language (Sierpinska, 2000). These three thinking modes co-exist in linear algebra, and the use of one of them does not imply the elimination of the other (Sierpinska, 2000). The main difference between the 'synthetic' and 'analytic' modes is that in the first, objects are given directly to the students mind, and their mind then tries to describe them, while in the second mode objects are given indirectly to the student, so the student tries to make sense of them by the definition of properties of their elements (Sierpinska, 2000). Dogan-Dunlap (2010) reported that the view of the geometric representations does not replace one's arithmetic or algebraic modes, but encourages students to utilize multiple modes of reasoning interchangeably.

2.4.1 Synthetic-Geometric

The synthetic-geometric mode utilizes the language of geometric figures: planes, lines, intersections, and their graphical representations. Synthetic-geometric arguments are not part of linear algebra proper, but they are heuristic tools used for the visualization that leads to the understanding of an idea (Sierpinska, 2000). Students in geometric-synthetic mode tend to describe objects without defining them (Dogan-Dunlap, 2010). It was reported by Dogan-Dunlap (2009) that students' geometric modes incorporated multiple aspects of vectors such as vector's magnitude, direction, dimension, and position within space.

2.4.2 Analytic-Arithmetic

In the analytic-arithmetic mode, geometric figures become sets of numbers satisfying its written conditions (Sierpinska, 2000) and students consider objects with respect to their processes and procedures (Dogan-Dunlap, 2010). Dogan-Dunlap (2009) stated that students' arithmetic and algebraic modes included processes such as row reduced echelon form of matrices, the use of linear combination, and the reference to theorems seen in class.

2.4.3 Analytic-Structural

The analytic-structural thinking mode synthesizes the algebraic elements of the representation into structural wholes (Sierpinska, 2000). Students tend to consider objects in systems and ignore processes and procedures (Dogan-Dunlap, 2010). An example could be a student referring to a theorem to imply linear independence. Furthermore, (Dogan-Dunlap, 2010) stated "if a student considers the characteristics of an object in the context of a system with geometric features then he/she may be applying both the structural and geometric modes".

The following table summarizes the different thinking modes identified by Sierpinska, examples of their representations, and the levels of competency associated with each mode.

Table 2.1. Thinking Modes Modified from Sierpinska (obtained from Dogan-Dunlap, 2010)

Mode of Thinking	Representations/Definition	Student Competency
Synthetic-Geometric	Graphical representations Provide properties of objects readily. It describes an object but not define it.	Student is be able to determine whether vectors whose graphs are provided in R^2 or R^3 are linearly independent or dependent.
Analytic-Arithmetic	Numerical Representations. Defines objects.	Student is able to construct a matrix from vectors, compute its row-reduced echelon form and relate the reduced matrix to linear dependence and independence.
	Linear Combination.	Student is able to provide/refer to linear combination of vectors and determine linear independence.
Analytic-Structural	Objects are considered in a system. Defines objects.	Use of the dimension of vector spaces in determining the linear independence of vectors.

These thinking modes are not mutually-exclusive, and they can all be used interchangeably in order to maximize understanding.

This thesis used the framework of metonymy and metaphors to further understand student cognition of abstract linear algebra concepts. Now, I will provide a short description of the framework.

2.5 Metonymy and Metaphor

Presmeg (2008) suggests that metaphors, metonymies, imagery, and symbolism are key elements in the representation of mathematical concepts since they help individuals make sense of these representations. “A representation does not represent by itself – it needs interpreting and, to be interpreted, and it needs an interpreter.” (Presmeg, 1998). Metonymies and metaphors

are necessary components that aid students in making sense of ambiguities represented in mathematical concepts (Dogan-Dunlap et al, 2010, 2011). Metaphors originate from everyday experiences and metonymies are formed as a result of classroom instructions; both can be used to construct an understanding of mathematical concepts, such as slope (Dogan-Dunlap, 2011). Metonymies and metaphors are used by students and mathematicians, and contribute to “an epistemology of mathematics” that enables students to form personal meanings and relationships (Presmeg, 1998). They can also be both powerful and problematic for students in understanding complicated concepts (Zandieh, 2006).

2.5.1 Metonymy

The word metonymy is from the Greek word metonymia- denoting change of name (Presmeg, 1998). Metonymy is a figure of speech where one concept or thing is not called by its own name, therefore its name is replaced by a word that is associated with it. However, metonymies are not just part of language; they are extensively used in thinking (Panther et al, 2004). The words associated can represent certain attributes or characteristics of the thing or concept, and part of the concept can be used to represent the whole. Whenever we used a letter to represent any set of numbers, this is referred to as the *Fundamental Metonymy of Algebra* by Lakoff & Nunez (2000).

Panther & Thornburg (2004) defined conceptual metonymy as “a contingent relation within one conceptual domain between a source meaning and a target meaning”. In a conceptual metonymy, the source meaning provides mental access to the target meaning, which might be created on the spot, but with frequent use, it may become part of our own lexicon (Panther et al, 2004). Panther & Thornburg (2004) claimed that metonymies should not be seen as a plain substitution relation, but as a ‘reference point’ that triggers meaning, and they argued that metonymies help us determine explicit and implicit meaning.

2.5.2 Metaphor

Metaphor comes from the Greek word metophora- to transfer or to carry over (Presmeg, 1998). According to Presmeg (1998) a metaphor is an implicit analogy that compares one domain or the elements belonging to it, to another domain or its elements by stating that both domains are the same, for example ‘Domain A is like domain B’. Metaphors are composed of ground and tension, ground refers to similarities and tension refers to the differences implied in the analogy (Dogan-Dunlap, 2007). Mathematics educators must be cautious with the use of metaphors because what might be tension for an instructor may become ground for students (Dogan-Dunlap, 2011). Dogan-Dunlap (2011) suggests educators explicitely cover similarities and difference between the examples of the source and the domain, so that students do not adopt any irrelevant aspect to form new understanding. As suggested by Presmeg (1998), metaphors should not be used to help students learn new concepts, but only to reinforce and relate ones previously learned by making connections. Example of a metaphor given by Presmeg: Domain A is Domain B.

2.5.3 The Relationship between Metonymy and Metaphor

Metonymies and metaphors are both literacy devices used in the process of constructing new knowledge while learning mathematical concepts (Zamora, 2010). Presmeg (1998) points out that metonymies are in contrast used to refer to an element or attribute of a class to stand for another element or the whole class, while metonymies link similarities of one domain with another domain to create a meaning of connection. The main difference between metonymies and metaphors is that the first replaces words by association and the second by similarity, where the metonymy is like a horizontal chain of signifiers and the metaphor a vertical descent into meaning (Presmeg, 1998). According to Kovecses (2002), in a metaphor “the relationship is based on the similarity of the two domains”, while in a metonymy “the relationship is based on the contiguity or correctness of the two entities”.

Chapter 3: Methodology

The purpose of this thesis is to analyze the answers of four undergraduate students while taking their first linear algebra course, to a set of questions asked during interviews at the end of the spring 2009 semester. Two of those students had a one-on-one interview, and the other two were interviewed at the same time. Our main goal is to analyze, by following the same modes of reasoning presented by Sierpinska (2000), the different aspects of learning shown by each student and their use of metonymies and metaphors as described by Presmeg (1998). The main idea, for documentation purposes, is to determine the main characteristics that categorize students' reasoning enrolled in their first linear algebra course at the university level.

This investigation was executed with students from a four-year southwestern university attending a junior level linear algebra class. This course, named Matrix Algebra, is the first linear algebra course offered at the undergraduate level to students that satisfy the pre-requisites with a minimum grade of C in Calculus II, Calculus III, or Differential Equations. For the Spring 2009 semester, this course was offered in three different sections –two in the afternoon and one in the morning. At the beginning of this semester, there were a total of 95 students registered.

Two of the sections mentioned above, had a modular format, and the third had a traditional one. In the modular courses, aside of lecture and homework assignments, series of constructivist assignments were completed with the aid of graphical tools available online –these tools were created by a group of instructors with extensive knowledge of the course and will be referred as modules from now on. Student in these two sections were given about a week to complete the assignments. The third and last section had a more traditional setting where students were lectured and had homework exercises assigned from the textbook. To this particular section of students, the modules were not even mentioned nor used in the classroom.

The main purpose of the constructivist assignments utilized in the modular sections was to make available the graphical representation of the topics included in the matrix algebra course

before introducing a new concept through theoretical representations. The constructivist module assignments are not part of the scope of this thesis.

At the end of the Spring 2009 semester students from each of the three sections volunteered to participate in an interview conducted by a professor and/or a graduate student from a southwestern university. For the purpose of this thesis, the interview responses of four students are reported and analyzed. The three interviews for the thesis were selected at random from a list of volunteers interviewed. Each interview belongs to a different section of the matrix algebra course –traditional and modular.

The results found in this study were obtained by performing a qualitative analysis to the transcripts and videos obtained from the interviews conducted with the volunteered students. The Grounded Theory, introduced by Glaser and Strauss (1967), was applied to conduct the analysis of the interviews in order to capture the thinking modes, metonymies, and metaphors revealed by students while responding to the questions asked with respect to linear algebra concepts.

A more detailed description of each class section, including the students, lecture style, assignments, and the analysis of the data is presented in this chapter.

3.1 Participants

As explained above, the students who participated in this study belonged to three different sections of the first linear algebra course offered at a four-year southwestern university during the Spring 2009 semester. Due to the location of the institution where this study took place, a significant percentage of the students from each section are of Hispanic origin, and a substantial percentage has English as a second language.

The two sections in which the online modules were implemented are referred to as modular course (non-traditional course), and the third one is referred to as non-modular course (traditional course). In the modular matrix algebra courses, the use of the online modules was

enforced and the integration of these was encouraged through the constructivist homework assignments. The non-modular course was more traditional. The instructor assigned homework strictly from the book, and the computerized modules were not even introduced to the students registered in this section.

Some of the main characteristics of the students belonging to each section of the matrix algebra course in the Spring 2009 semester are summarized in the following tables.

Table 3.1 Demographics of Group A; modular section (obtained from Zamora, 2010)

Section A	
Question	
Gender	-Males:25 -Females: 9
Ethnicity	-Hispanic/Hispanic American: 79.4% -White/Caucasian/American: 17.64% -American-Asian/Asian: 2.94%
Classification	-Freshman: 0% -Sophomore: 35.29% -Junior: 41.18% -Senior: 23.53%
Major	-Mathematics: 20.59% -Computer Science: 44.12% -Electrical Engineering: 23.53% -Industrial Engineering: 8.82% -Computer Engineering: 2.94%
Courses this semester	-Mean: 4.4 -Standard Deviation: 0.86 -Mode: 4
Have a job? **For how long?** **Hours/week?**	-No: 20.59% -Yes: 79.41% -Less than a year: 62.96% -1 to 3 years: 25.93 -No answer: 11.11% -Less than 20: 51.85% -Exactly 20: 18.52% -More than 20: 29.63%
English first language	-No: 44.12% -Yes: 55.88% 100% of the students that answered no to this question, reported to have Spanish as their first language (15 students)
Fluency	-10: 33.33% -9: 6.67% -8.5: 6.67% -8: 46.66% -5: 6.67% -Standard Deviation: 1.35 -Mean: 8.60

There were 35 students registered in Section A at the beginning of the Spring 2009 semester. Of these 35 students, only 34 students attended class the day the pre-survey was administered during one of the first class meetings (Zamora, 2010).

Below Table 3.2 summarizes the demographics of Section B (modular course)

Table 3.2 Demographics of Group B; modular section (obtained from Zamora, 2010)

Section B	
Question	
Gender	-Males:22 -Females: 6
Ethnicity	-Hispanic/Hispanic American: 75% -White/Caucasian/American: 17.86% -Mexican/Chicano: 3.57% -American-Asian/Asian: 3.57%
Classification	-Freshman: 0% -Sophomore: 7.14% -Junior: 50% -Senior: 42.86%
Major	-Mathematics: 3.57% -Computer Science: 35.7% -Electrical Engineering: 37.5% -Industrial Engineering: 7.14% -Mechanical Engineering: 7.14% -Philosophy: 3.57% -Physics: 3.57% -Multidisciplinary Studies: 3.57%
Courses this semester	-Mean: 4.04 -Standard Deviation: 1.04 -Mode: 5
Have a job? **For how long?** **Hours/week?**	-No: 32.14% -Yes: 67.86% -Less than a year: 26.32% -1 to 3 years: 52.63% -More than 3 years: 21.05% -Less than 20: 5.26% -Exactly 20: 42.11% -More than 20: 52.63%
English first language	-No: 64.29% -Yes: 35.71% 77.78% of the students who answered no to this question, reported to have Spanish as their first language; 5.56% had Thai as their first language, and 16.66% gave no answer.
Fluency	-10: 16.67% -9: 55.567% -8: 22.22% -7: 5.55% -Standard Deviation: 1.35 -Mean: 8.60

There were 35 students registered in Section B at the beginning of the Spring 2009 semester. Of these 35 students, only 28 students attended class the day the pre-survey was administered during one of the first class meetings (Zamora, 2010).

Below Table 3.3 summarizes the demographics of the non-modular section, Section C. All 35 students registered for this section attended class and were able to answer the pre-survey administered at the beginning of the Spring 2009 semester (Zamora, 2010).

Table 3.3 Demographics of Group C; Non-modular section (obtained from Zamora, 2010)

Section C	
Question	
Gender	-Males:26 -Females: 9
Ethnicity	-Hispanic/Hispanic American: 80% -White/Caucasian/American: 11.43% -Mexican/Chicano: 2.86% -American-Asian/Asian: 5.71%
Classification	-Freshman: 2.86% -Sophomore: 17.14% -Junior: 62.86% -Senior: 17.14%
Major	-Mathematics: 17.14% -Computer Science: 22.86% -Electrical Engineering: 51.43% -Industrial Engineering: 5.71% Physics: 2.86%
Courses this semester	-Mean: 4.54 -Standard Deviation: 1.20 -Mode: 4
Have a job? **For how long?** **Hours/week?**	-No: 37.14% -Yes: 62.86% -Less than a year: 31.82% -1 to 3 years: 50% -No answer: 18.18% -Less than 20: 50% -Exactly 20: 27.27% -More than 20: 22.73%
English first language	-No: 37.14% -Yes: 62.86% 100% of the students that answered no to this question, reported to have Spanish as their first language (13 students)
Fluency	-10: 30.77% -9: 30.77% -8: 23.08% -7: 50.37% -Standard Deviation: 1.09 -Mean: 8.77

The students attending all three sections were pursuing degrees from a wide variety of disciplines, including electrical engineering, industrial engineering, mechanical engineering, mathematics, computer science, computer engineering, physics, philosophy, and multidisciplinary studies. The nature of this matrix algebra class introduced students from all three sections to proofs; in some cases students expected to have the same format as previous courses with less theoretical framework, such as precalculus and calculus. The percentage of students who had experienced proofs in previous courses from Section A was 35.29%, from Section B 60.71%, and from Section C was 40% (Zamora, 2010).

Students from sections A and C attended class twice per week for an hour and twenty minutes, while students from Section B attended class three times per week for 50 minutes. The average age among the university population was reported to be 26 years old (University of Texas at El Paso- UTEP, 2009).

In order to avoid bias in our analysis, the name of students who participated in the surveys and interviews was replaced by a code composed of a letter corresponding to the section

they belonged to and a number. The codes from Section A ranged from A1 through A35, for Section B from B1 through B34, and finally for Section C from C1 through C36. Students enrolled in this course were asked to sign a consent form, at the beginning of the Spring 2009 semester, that allowed the researchers to use their information for the purpose of this investigation.

Students interviewed were volunteers from each section of the course. The interviews were videotaped for informational purposes, so they could be transcribed by the author of this thesis. One interview from each section was randomly chosen for the purpose of this thesis from the group of students who volunteered to be interviewed. No preconceptions were placed on the base of race, gender, age, or socioeconomic status of any of the students while interviewing and analyzing the interview transcripts. The interview transcripts are available upon request.

3.2 Modular and Non-modular Section Characteristics

Students from the two modular sections of the course had topics presented in class through a formal definition, elementary row operations, algebraic manipulations, and static graphical representations of the topics being taught (Zamora, 2010). In these two sections, students were assigned homework exercises from the textbook (these were not collected) regularly, and assignments in which the use of the computer modules was required were assigned every other week. In some class sessions, instructors would introduce a new topic with the aid of the computer modules and an –LCD projector to explain some of its characteristics and relationships with other topics previously studied.

In Section C, the non-modular section, topics were presented through formal definitions, elementary row operations, and in limited occasions the instructor would provide a static graphical representation of the topic, only if asked by a student (Zamora, 2010). Homework assigned was strictly from the textbook and was never collected.

In the non-modular and modular sections of the course, quizzes were given at the beginning of the class period. There were no major differences in the content of the modular and non-modular sections, with the exception of the use of the computer modules and the constructivist assignments.

The official description of the matrix algebra course at the institution is as follows:

MATRIX ALGEBRA 3323: Systems of linear equations, matrices, determinants, eigenvalues and eigenvectors, diagonalization, vector spaces, and linear transformation.

However, the topics chosen for the purpose of this thesis are:

1. Definition of linear independence and dependence of a set of vectors, including the identification of linear dependency among the vectors of a specific set.
2. Characteristics of linearly dependent and independent sets of vectors in R^2 and R^3.

The instructors teaching the matrix algebra course during the Spring 2009 semester differ not only on their teaching style, but also on the level of abstraction used in introducing topics. Due to this situation, summaries of observations of each section, conducted by graduate student and research assistant Zamora (2010), will be provided. It is important to mention that Zamora (2010) conducted these observations during unannounced visits to the sections of the matrix algebra course.

3.2.1 Section A Observations

We include observations obtained from Zamora (2010). -These observations were conducted while Zamora was an active research assistant during this investigation in the Spring 2009 semester. The observations reported below were obtained from Zamora (2010) pp. 31-39.

Classroom Observation 1. *02/04/2009:*

- *The presentation of the material is done in an overhead projector.*
- *Students are given a quiz on the question "Find all values of a for which the system $x_1+ax_2=6$ and $ax_1+2ax_2=4$ has no solutions. Show your work." Time assigned for this activity: 20 minutes.*
- *One of the students shared his answer with the class; extra points were awarded to him for the presentation.*
- *Instructor asks questions to students looking for feedback while explaining different answers for the quiz.*
- *Instructor encourages class participation by asking students feedback about previous concepts and lectures.*
- *Some of the topics covered this day are elementary row operations, row reduced echelon form (e.g. how do you know if a matrix is in row reduced echelon form?), and Gauss-Jordan elimination process.*
- *Class participation/discussion on the following topic: Let A be an nx(m+1) matrix where A is the augmented matrix of a system. After the discussion, the instructor re-explained the definitions and properties that made the answers acceptable.*
- *Discussion; is it possible to get* $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \dots & 0 & a \\ 0 & 1 & 0 & \dots & 0 & b \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & z \end{array}\right]$*? Not possible to get a unique solution (since we don't have enough rows to get the values of the variables). Is this an inconsistent system, with infinitely many solutions (as indicated by the row of only zeroes) or no solutions (as indicated by the last row of all zeroes and a z at the end)?*

- *An additional problem is posted by the instructor: Given that A represents a consistent system and RREF of A has r non-zero rows where r<m, then does the system have infinitely many solutions. If r=m then there is one unique solution or no solution at all?*
- *Instructor uses computer module to illustrate the idea that parallel planes do not intersect and, hence, have no solution.*
- *What other types of answers can you have? What's the geometrical representation? E.g.* $z=f_1=x+y-5$; $z=f_2=2x+2y+1$. *This system would have no solution. How does it look like?*
- *Question: Give a system of planes that has a different answer.*

  - Not possible for a unique solution

  - Is it possible to have infinitely many solutions? Yes. Same planes— equations are multiples of one another.

  Example: $z=x+y-5$; $2z=2x+2y-10$: *infinitely many solutions (same plane). At the end you'd get* $\left[\begin{array}{ccc|c} 1 & \dots & 0 & \dots \\ 0 & \dots & 0 & 0 \end{array}\right]$ *as the RREF.*

- Infinitely many solutions: $z=x+y-5$; $z=x-2y$ → *infinitely many solutions (they intersect at a line)* → $\left[\begin{array}{ccc|c} 1 & 0 & \dots & \dots \\ 0 & 1 & \dots & \dots \end{array}\right]$

- *Instructor continues discussion on the computer module named Linear Systems developed in Geometer's Sketch pad (GSP) and explains its functionality.*
- *Equivalent system of linear equations:*

  System A→ ERO (Elementary Row Operations)→ System B

  System A≠ System B, but both have the same set of solutions.
- *Definition: Let A, B be equivalent systems of equations. Then A and B have the same solution sets. This is explained by the instructor through the use of a module in GSP.*
- *Instructor goes over the proofs of this theorem; for simplicity purposes only 3x4 systems are considered.*

Classroom Observation 2. *02/18/2009:*

- *A quiz is given to students at the beginning of class (20 minutes allowed) on the question:*

  a) Define 'consistent system'

  b) Given an example of a consistent system. Explain why your example is a consistent system.

- *Group Work. A theorem is introduced to the class; students had to work in groups to come up with a proof. Theorem given: "Let A, B, C be compatible matrices. Then A(B+C)=AB+AC." Distributive property of matrices. Two students provide an explanation on their reasoning to come up with the proofs.*
- *Instructor goes over the proofs provided by the students and analyzes the strengths and weaknesses of each one of them and goes over the definitions and properties previously introduced in class that make the proofs acceptable.*
- *Abstract ideas on the concepts of multiplicative inverse and identity matrix in* R^n *are presented by instructor through different representations of the concepts (through the use of algebraic expressions, matrices, and a numerical example with matrices).*
- *Homework form textbook is assigned to students for the next class.*

Classroom Observation 3. *04/01/2009:*

- *A quiz was given to students at the beginning of the class (20 minutes were assigned for this activity). The question was: "Let AB be a non-singular matrix. Use the fact that B is also a nonsingular matrix to prove that A is a nonsingular matrix."*
- *There is a class discussion on the quiz problem.*
- *Instructor uses computer module called Vector Spaces to demonstrate linear combinations of vectors graphically.*
- *Instructor uses overhead projector as a tool to teach the class.*

Classroom Observation 4. *04/06/2009:*

- *A quiz was given to students at the beginning of the class (20 minutes were assigned for this activity). The question was: "Determine whether the given set of vectors is linearly independent or dependent. If the set is linearly dependent, express one vector in the set as a linear combination of the others.* $S=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\1\\-3\end{bmatrix},\begin{bmatrix}-1\\4\\3\end{bmatrix},\begin{bmatrix}4\\4\\0\end{bmatrix}\right\}$*"*
- *Two students presented their answers to the class by explaining their reasoning. One answer was numerical (algebraic, by row reducing the matrix and looking at the resulting matrix and its elements) while the other skipped this part by using his calculator and analyzing the results using more abstract ideas and concepts seen in previous classes. The instructor explained why both answers were acceptable.*
- *Instructor goes over the concept of linear independence graphically by using a module (called Vector Spaces) in the computer.*
- *Students realized that all the possible linear combinations of the 4 vectors would give* R^3 *(through the use of the computer module).*
- *Instructor uses module to show that all possible linear combinations of the first 3 vectors in the set given in the quiz would also give* R^3*; instructor uses 2 out of the first 3 vectors to show that* R^3 *can't be obtained by obtaining all possible linear combinations.*
- *Group discussion on: "Can you get* R^3 *from any set of three vectors?"*
- *Numerical examples are introduced on the use of the computer module for students to see, graphically, if the sets were dependent or independent and how they would look like.*
- *Examples of sets of vectors in* R^2 *are used to illustrate dependent and independent sets of vectors through the use of computer module.*
- *Instructor goes over the geometrical representation of ideas in* R^2 *and then moves to* R^n*.*

Section A Observations Summary

The instructor of this section used a wide variety of teaching techniques and tools in class, such as quizzes, computer modules, constructivist assignments, individual, and group work. Proofs were introduced during lecture by allowing students to discuss and attempt to find a solution. The instructor often encouraged students to participate and always validated students' ideas. Different representations of the same topic were often seen in class with the aid of the computer modules, and ideas were then generalized into abstract concepts and theorems (Zamora (2010) pp. 31-39).

3.2.2 Section B Observations

These observations were conducted by Zamora (2010) while attending Section B during the Spring 2009 semester in three different unannounced visits. The observations reported below were obtained from Zamora (2010) pp. 40-45.

Classroom Observation 1. *02/06/2009*

- *Notes for test 1: calculators allowed; work must be included for credit.*
- *Instructor goes over the vector form of the solution.* $x_1=-4t-s-3$, $x_2=2t+s+1$, $x_3=t$, $x_4=2$, $x_5=s$, *where t and s are free variables. Solution can be written as a vector. Vector Solution:*

$$\begin{bmatrix} -4t-s-3 \\ 2t+s+1 \\ t \\ 2 \\ s \end{bmatrix} = t\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

- *Instructor mentions that the answer is in 5 dimensional space.*
- *Vector solution:* $\left\{ t\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} : t, s \in \Re \right\}$

- *Instructor assigns homework from book (not to be collected).*
- *Instructor goes over the homework questions in order to guide students through the process of answering them.*
- *General questions from students are answered at the beginning of the class.*
- *Instructor emphasizes the need of knowing how to solve systems of linear equations with elementary row operations since the rest of the class is based on that topic.*
- *Question answered in class: For what value of a is the system consistent?* $\begin{cases} x_1 + ax_2 = 6 \\ ax_1 + 2ax_2 = 4 \end{cases}$.
- *Instructor answers the question as follows:* $\left[\begin{array}{cc|c} 1 & a & 6 \\ a & 2a & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & a & 6 \\ 0 & a^2 - 2a & 6a - 4 \end{array}\right]$.
- *Instructor asks questions to students throughout the class about what the answer would be and what is the reason behind the answer.*
- *Possibilities: If a^2-2a≠0, then the system is consistent. If a^2-2a=0→ a(a-2)=0→ a=2 OR a=0. For a=0: 6a-4=6(0)-4=-4≠0; for a=2: 6(2)-4=12-4=8≠0. Therefore, for a=0 and a=2, 6a-4≠0→ system is inconsistent if and only if a=0 OR a=2.*
- *Review for test during next course.*
- *Problem from book (similar to one from homework). X represents 1's digit, y represents ten's digit, and z represents 100's digit. Four equations involved: N=z*100+y*10+x; N=15(x+y+z); 100*x+10y+z=N+396; x=1+y+z. System with four equations and four unknowns which can be represented as:*

$$\left[\begin{array}{cccc|c} 15 & 15 & 15 & -1 & 0 \\ 100 & 10 & 1 & -1 & 396 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 10 & 100 & -1 & 0 \end{array}\right] = \begin{cases} 15x + 15y + 15z - N = 0 \\ 100x + 10y + z - N = 396 \\ x - y - z = 1 \\ x + 10y + 100z - N = 0 \end{cases}$$

- *Instructor states that this system can also be represented with three equations and three unknowns (by eliminating N at the beginning).*

Classroom Observation 2. *02/20/2009*

- *Quiz is given at the beginning of the class (30 minutes of the class were used for this) on the following question: "An nxn matrix is called diagonal if for every* $1 \leq i \leq n$ *and* $1 \leq j \leq n$ *such that* $i \neq j$, $a_{ij}=0$. *Prove that the sum of two diagonal nxn matrices is a diagonal matrix. Proof (provided by instructor): Let* $A, B \in M_{nxn}(R)$ *both be diagonal matrices. Let C=A+B. Let* $i \neq j$, $C_{ij}=a_{ij}+b_{ij}=0+0$; *since* $a_{ij}=0$ *and* $b_{ij}=0$ *then* $C_{ij}=0$. *Therefore, C is a diagonal matrix.*
- *Instructor provided an additional less abstract proof to students (with matrices instead of abstract ideas) such that they could see the different representations that this answer could have. E.g.* $A=\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$, $B=\begin{bmatrix} b_{11} & 0 & \dots & 0 \\ 0 & b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix}$;

  $C=A+B=\begin{bmatrix} a_{11}+b_{11} & 0 & \dots & 0 \\ 0 & a_{22}+b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn}+b_{nn} \end{bmatrix}$ *which is also a diagonal matrix.*
- *Instructor goes over project 2 (from modules) during the class by introducing the functionality of the module and going over the questions on the project.*
- *The concept of non-commutativity of matrix multiplication is explained graphically with the use of a project module.*
- *Instructor goes over some examples that can be applied to the module and the results that can be obtained.*
- *Instructor goes over the concepts that are related to the ideas seen in the module.*
- *Students ask questions about what to do in a proof. For example: non-commutativity- show one where test fails (one that is non-commutative then you can't say all are commutative), commutativity- show condition is true for all elements.*

- *For homework, prove that matrix multiplication of nxn matrices is non-commutative. Hint: Look for the easier example for 2x2 matrices— using only zeroes and ones and as many ones as possible. Then generalize your idea for nxn.*
- *Instructor shows students what they may be able to do in order to understand what is being asked and what they need in order to generalize the idea (e.g. start with a 2x2 matrix with numbers in it; see its behavior— with respect to multiplication with other matrices— then try to generalize the idea for nxn matrices).*
- *Instructor goes over the theorem: if* $A \in M_{mxn}(R), B \in M_{nxr}(R)$ *then* $(AB)^T = B^T A^T$. *Proof: First look at the dimensions* $B^T = rxn, A^T = nxm \rightarrow B^T A^T = (rxn)(nxm) = rxm$ *Now, let A=*a_{ij} *0≤ i ≤m, 0≤ j ≤n; B=*b_{jk} *0≤ j ≤n, 0≤ k ≤r. Then (*a_{ij}*) (*b_{jk}*)= (*ab_{ik}*), ((*ab_{ik}*))'=*ab_{ki}*…*

Classroom Observation 3. *04/06/2009*

- *Student asks questions about homework problems: A=(1,1,-1), B=(0,1,2), C=(3,0,1);*
- *a(x-*x_o*)+b(y-*y_o*)+c(z-*z_o*)=0, ax+b(y-1)+c(z-2)=0. At point A: a(1)+b(0)+c(-3)=0→ a-3c=0; at point C: a(3)+b(-1)+c(-1)=0→ 3a-b-c=0.*
- *Instructor goes over the matrix representation of the concepts above:*

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 3 & -1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -8 & 0 \end{array}\right] = \begin{cases} a = 3t \\ b = 8t \\ c = t \end{cases} \rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} t$$

. Now choosing t=1, would yield: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$ *. Hence, 3x+8(y-1)+(z-2)=0 is the equation of the plane containing the points A, B, and C.*

- *Instructor introduces the topic of determinants. He states that this is the most important numerical aspect of nxn matrices.*

- *Question. Given an nxn matrix A, we define its determinant det(A) (or |A|) as a number obtained in the following manner for n=1,2,3. For n=1: det[a11]=a11;*

- *n=2:* $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$

- *n=3:* $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} * \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} * \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} * \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

  . In order to calculate the determinant of 3x3 matrices, instructor mentions the use of a recursive definition in which the definition of the determinant of 2x2 matrices is used.

- *Numerical example provided by instructor:* $\det\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1(1) - 1(2) + 1(0) = -1$.

- *Instructor mentions the fact that calculators do give you the determinant of the matrix.*

- *Topic introduced by instructor. Orthogonal unit vectors in a plane; instructor provides a graphical representation of the vectors represented by* $i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

- *Definition: Given two vectors in R^3,* $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$*, their cross-product is defined as:*

  $\vec{u} \mathrm{x} \vec{v} = \det\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$*. Remember that i, j, and k are vectors!!*

  $$\vec{u} \mathrm{x} \vec{v} = \det\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} * \vec{i} - \det\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} * \vec{j} + \det\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} * \vec{k}$$

- *Instructor states that the cross product of two vectors is a linear combination of the three basic vectors i, j, and k with scalars given by the determinants.*

- *Instructor states that calculators are useful to obtain the cross product of the vectors u and v.*
- *Instructor introduces the topic of equations of a plane and its relationship with the normal vector.*
- *Instructor provides two numerical examples of the cross product of two vectors; using the vectors* $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ *and calculating the cross products* $\vec{i} \text{x} \vec{j} = \vec{k}$ and $\vec{j} \text{x} \vec{i} = -\vec{k}$. *One of the students mentions the fact that these cross products are not equal and hence this operation is not commutative.*
- *Instructor goes over the properties of the cross product and re-assigns homework from textbook.*

Section B Observations Summary

The instructor of this section introduced abstract ideas and then illustrated them through the use of examples written on the board. The use of calculators was allowed during class and tests. The use of the computer modules was minimal, usually to provide students with a basic idea of the controls needed to complete the constructivist assignments. Students were free to ask questions about previous homework assignments, for these questions the instructor spent a considerable amount of time of class answering them. The participation and interaction levels among students were low, despite the instructor's encouragement. The instructor used the blackboard to draw geometrical interpretations of the topics covered in class sporadically. There was a high level of abstraction in this section; however the instructor provided interpretations of these abstract ideas through matrices, systems of linear equations, and numerical examples (Zamora (2010) pp. 40-45).

3.2.3 Section C Observations

Observations conducted by Zamora (2010) while attending Section C during the Spring 2009 semester in three different unannounced visits are reported below. The observations reported below were obtained from Zamora (2010) pp. 46-51.

Observation 1. *02/05/2009*

- *Instructor goes over a homework problem as requested by students (these problems were due the next class).*
- $\begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ -1 & 1 & 7 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -13 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right]$ *where*

  x_4 is a free variable.
- *Instructor posted the following question: is it a consistent or inconsistent system based on this method? Students expressed their opinions about this question.*
- *Instructor continues with the explanation of the solution of this problem: Assign a parameter to the free variable by using the information of the matrix:*

  $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -13 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array}\right] = \begin{cases} x1 + 6x4 = -13 \\ x2 - 2x4 = 17/3 \\ x3 + x4 = -8/3 \end{cases}$. *Now with x4 as the free variable, substitute*

  x4=t in the equations: $\begin{cases} x1 + 6x4 = -13 \\ x2 - 2x4 = 17/3 \\ x3 + x4 = -8/3 \end{cases} = \begin{cases} x1 + 6t = -13 \\ x2 - 2t = 17/3 \\ x3 + t = -8/3 \end{cases}$ and $x4 = t, t \in R$
- *Instructor then talks about the geometrical representation (planes, lines, etc) but does not show it on the blackboard or computer. Instructor just mentions verbally how it would be seen in three dimensions.*

- *Instructor goes over the vector form of a solution to a problem. He goes over the last example and states the solution as being:* $\begin{pmatrix} -6t-13 \\ 2t+17/3 \\ -t-8/3 \\ t \end{pmatrix} = t\begin{pmatrix} -6 \\ 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -13 \\ 17/3 \\ -8/3 \\ 0 \end{pmatrix}$.
- *Instructor states verbally that a geometrical interpretation of this is not possible since the answer is in four dimensions.*
- *Instructor states that students are required to provide answers in vector form for quizzes and tests.*
- *Student asks question on homework problem. Instructor goes over it. Question: "For what value of a is the system consistent:* $x_1+ax_2=6$ *and* $ax_1+2ax_2=4$*"*
- *Instructor goes over problems assigned for homework and guides students on how to solve them (by providing hints).*
- *Example solved in class by instructor: N is a three digit number; it equals 15 times the sum of its digits. If digits are reversed, the resulting number exceeds N by 396. One's digit is one larger than the sum of the other 2. Give a linear system of 3 equations and state what N is. Four equations involved: N=z*100+y*10+x; N=15(x+y+z); 100*x+10y+z=N+396; x=1+y+z. System with four equations and four unknowns which can be represented as:*

$$\begin{cases} 1x+10y+100z=N \\ 15(x+y+z)=N \\ 100x+10y+1z=1x+10y+100z+396 \\ x=y+z+1 \end{cases} = \begin{cases} 1x+10y+100z=15x+15y+15z \\ 100x+10y+z=x+10y+100z+396 \; OR \\ x=y+z+1 \end{cases}$$

$$\begin{cases} x+10y+100z=N \\ 15(x+y+z)=N \\ 100x+10y+1z=N+396 \\ x=y+z+1 \end{cases}$$

Classroom Observation 2. *02/26/2009*

- *Instructor lectures on the distributive law theorem for matrices. Let* $A \in M_{mxn}(R); B,C \in M_{nxr}(R)$ *then A(B+C)=AB+AC. Instructor proves the theorem on blackboard for the group.*
- *Instructor goes over a homework problem. Question: Suppose* $A^2=AB$, $A^2-AB=0$, $A(A-B)=0$. *Then A=0 or A-B=0, but since A cannot be equal to 0, then A=B, which is not true, since you can actually multiply two non-zero matrices and get the zero matrix as a result. Counter-example:* $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- *Instructor advices students to try to generalize the ideas of the example to get a conclusion for nxn matrices if possible.*
- *An example of two nxn non-zero matrices whose product is the zero matrix (n>1) is given by the instructor with the help of students who provide ideas on how to come up with the proof after working on an example for 2x2 matrices. The matrices given are:*

$$\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{nxn} \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{nxn} = 0 \text{ (the zero matrix).}$$

- *Instructor multiplies matrices in reversed order to show that multiplication of matrices is non-commutative, AB≠BA.*
- *Instructor pinpoints to students that there is an easier pair of matrices that when multiplied give the 0 matrix as a result:* $$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{nxn} \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{nxn} = 0$$
- *Instructor does a small review for test 2.*
- *Instructor states that a vector is an nx1 matrix; vectors are matrices.*

- *Instructor explains the definitions of vector multiplication, and norm, length, and magnitude of vectors.*
- *Instructor talks about three elements of vectors: orientation, direction, and length, and explains them by drawing static version of a two dimensional plane.*

Classroom Observation 3. *04/09/2009:*

- *Instructor goes over the following problem at the beginning of the class: Given points $A_1=(1,2)$, $B_1=(0,4)$, $A_2=(0,1)$, $B_2=(-1,1)$. Find a point of intersection of lines* $\overline{A_1B_1}$ and $\overline{A_2B_2}$.
- *Process:* $$\begin{array}{l} \overline{A_1B_1}: \text{point}\,(0,4);\ \text{vector}\,(1,\text{-}2)^{\mathrm{T}} \rightarrow \begin{cases} x = t \\ y = -2t+4 \end{cases} \\ \overline{A_2B_2}: \text{point}\,(0,1);\ \text{vector}\,(\text{-}1,0)^{\mathrm{T}} \rightarrow \begin{cases} x = -s \\ y = 1 \end{cases} \end{array}$$ *. Intersection point: (3/2,1).*
- *Student asks how this result was obtained; instructor goes over the procedure and formulas again:* $\text{pt}: (x_o, y_o), \text{vector}: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \begin{cases} x = u_1 t + x_o \\ y = u_2 t + y_o \end{cases} \text{in 3D, add } z = u_3 t + z_o$.
- *Instructor goes over the general equation $a(x-x_o)+b(y-y_o)+c(z-z_o)=0$ with a point: (x_o,y_o,z_o) and a normal vector* $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
- *Instructor mentions how problems can be solved and what type of problems they can encounter involving these topics (e.g. intersection of a plane and a line).*
- *Instructor goes over an additional problem a student requested.*
- *Instructor encourages students to use calculators during class and tests to come up with the row reduced echelon forms of matrices.*

- *Instructor illustrates statically an example of finding an angle between two planes geometrically (first, by using hands and the desk and then by drawing a plane on the blackboard) to show students how the intersection of a line and a plane would look like.*
- *Instructor goes over how to calculate a cross product and its properties.*
- *Instructor introduces the Parallelogram method to add vectors and provides a geometrical explanation.*
- *Instructor goes over a procedure for students to follow in order to check for linear independence or dependence in a set of vectors.*

1. *Recall the definition of linear independence:*

$$\vec{v}_1, \vec{v}_2, ..., \vec{v}_k \text{ are linearly independent vectors} \Leftrightarrow$$
$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + ... + \alpha_k\vec{v}_k = \vec{0} \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_k$$

2. *Use a matrix to check for independence by stacking the vectors as columns of a matrix.*

$$A = [\vec{v}_1, \vec{v}_2, ..., \vec{v}_k]$$

3. *Reduce matrix A (instructor recommends, again, to use a calculator to do this).*
4. *Look for free variables (columns that are not represented by a leading 1). If there is any free variable, the set* $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ *is linearly dependent. Otherwise,* $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ *is a linearly independent set of vectors.*

- *Instructor provides an example on this procedure. Take vectors* $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 5 \\ 2 \end{pmatrix} \right\}$ *which form the matrix* $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *In this example, there is no leading one in the third*

column, and the number 2 in position (1,3) of the matrix is considered to be the free variable. Therefore, this set of vectors is considered to be linearly dependent.

- *Instructor goes back to explain why this set of vectors is dependent and what it means.*
- *Instructor solves the same problem in a different way:*

$$x\begin{pmatrix}1\\0\\2\\1\end{pmatrix}+y\begin{pmatrix}2\\1\\1\\0\end{pmatrix}+z\begin{pmatrix}4\\1\\5\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Rightarrow\left[\begin{array}{ccc|c}1&2&4&0\\0&1&1&0\\2&1&5&0\\1&0&2&0\end{array}\right]\Rightarrow\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}\Rightarrow\begin{cases}x+2z=0\\y+z=0\end{cases}$$

$$\Rightarrow\begin{cases}x=-2z\\y=-z\\z=z\end{cases}\Rightarrow \text{Let } z=t:\begin{cases}x=-2t\\y=-t\\z=t\end{cases}$$ *. Therefore, this is a dependent set of vectors. For example, when* $t=1\Rightarrow\begin{cases}x=-2\\y=-1\\z=1\end{cases}\Rightarrow -2\begin{bmatrix}1\\0\\2\\1\end{bmatrix}-\begin{bmatrix}2\\1\\1\\0\end{bmatrix}+\begin{bmatrix}4\\1\\5\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$*. Therefore, the definition of linear independence is not met.*

- *Instructor assigns homework from textbook.*

Section C Observations Summary

The class environment of this section was somehow similar to Section B. Students were allowed and encouraged to use their calculators in order to minimize the time spent in calculations. Students were encouraged to share their ideas, and the participation and interaction level in this section was considerably high. The instructor spent a considerable amount of time of class answering questions asked by students about the homework problems from the textbook. New concepts were first introduced in the form of theorems followed by a proof, and then graphical and geometric representations were used to illustrate those concepts. The graphic representations were only given in the form of drawings on the blackboard. The use of the computer modules was not included as part of the class (Zamora (2010) pp. 46-51).

3.2.4 Comparison of Sections

The main difference between sections B and C is that the use of the computer modules was not included as part of the class in Section C. Students in Section B had to complete assignments during their own time by using the computer modules; however, the use of the modules was limited. The most noticeable difference between the students from those two sections mentioned above was the level of participation and involvement during class, with the level of students from Section C being higher. Sections B and C were examples of a traditional classroom where students were lectured and assigned homework; on the other hand, students from Section A had a greater variety of activities and involvement (Zamora, 2010). Even though the use of the computer modules was present in sections A and B, their use was more evident in Section A with the instructor heavily relying on them to connect ideas and to transition from geometrical to algebraic representation of abstract concepts.

3.3 Procedure

At the beginning of the spring 2009 semester, a pre-survey and a consent form that would allow the data collected to be used for informational purposes were administered by the instructors of each section. Copies of the pre-survey and consent form are included in the appendix.

During the months of April and May, instructors from all sections invited students to participate, as volunteers, in the interviews conducted as part of the research to document students' reasoning while taking their first linear algebra class at the university level. To encourage the participation of all students, including the ones who wouldn't normally volunteer, a few credit points were offered (Zamora, 2010). Interviews initiated about a week after topics such as linear independence, span, and spanning set were covered in class.

Students received a schedule date and time to be interviewed by a professor and a research assistant. The interviews conducted were video-taped from two different angles for the

purpose of capturing written responses, anxiety levels, and gestures made while orally responding to a specific set of questions. During the interviews, additional questions were added and some were slightly changed in order to understand and capture the student thinking and reasoning processes. Each interview lasted between 60 and 120 minutes.

From the videos of the three interviews randomly selected for the purpose of this thesis, a transcript was generated by the author of this thesis. Each interview transcript was then analyzed independently by the author of this thesis, her mentor, and an additional graduate student. Following the description and classification of Sierpinska (2000) on thinking modes and the description of metonymies and metaphors provided by Presmeg (1998), independent analyzes were completed by the three researchers mentioned above. More details on the categorization are provided in the following chapter.

For the purpose of this thesis, the codes of the students randomly selected from the list of interviewed students are A18, A33, B6, and C7. These students belonged to sections A, B and C respectively. A measure of reliability was obtained in order to determine the percentage of agreement among raters while analyzing each interview. Two raters analyzed the interviews belonging to A18 and A33 (the only two students who were interviewed at the same time) and two raters analyzed the interview belonging to student C7. The interview transcript belonging to student B6 was only analyzed by the author of this thesis. After comparing the information obtained from the different raters, there was a 75% of agreement among the opinions for student A18 and 50% of agreement among the opinion for student A33 with respect to the thinking modes. This was calculated by comparing the number of categories created by the two raters, their appropriate descriptions, and the number of matches among categories. The final result is determined by obtaining the percentage of categories created by the second rater that matched those of the author of this thesis. With respect to student C7, the percentage of agreement between the author of this thesis and the additional rater was 100 %.

Looking at the analysis of metaphors and metonymies, for student A18 there was a 28% of agreement between the raters, for student A33 there was a 32%, and finally for student C7 the agreement

was 52.72% of the time.. The percentage of agreement with respect to the finding of metonymies and metaphors for students A18, A33, and C7 is significantly low, but since the analysis of these aspects is subjective, these percentages are significant enough for our purpose, which is documenting the cognitive constructs present.

3.4 Instruments

A list of assignments and surveys is presented in table 3.4 below available to instructors from modular and non-modular sections (Zamora, 2010).

Table 3.4. Data Collected by section (Obtained from Zamora, 2010).

Section A	Section B	Section C
Pre-survey	Pre-survey	Pre-survey
Quiz 1	HW1-System Module	Test 1
Quiz 2	Test 1	Test 2
Quiz 3	HW2- Matrix Product	Test 3
HW1-System Module	HW3- Vectors	HW Linear Independence
HW2- Matrix Product	Test 2	Post-Survey
Test 1	Test 3	3 Class Observations
HW3- Vectors	HW Linear Independence	
HW4- Linear Independence	Post-Survey	
Test 2	3 Class Observations	
HW Linear Independence		
Post-Survey		
4 Class Observations		

As reported above, the homework on linear independence (HW Linear Independence) was assigned to students from all three sections, the use of the computer modules was not required to complete this assignment. -The purpose of this assignment was to familiarize students with algebraic and numerical representations of vectors and matrices so they could be able to identify the necessary conditions that would make a set of vectors in R^3 linearly independent. For the purpose of this thesis, the only items that would be consider are the pre-survey, post-

survey, and the analysis of the interview transcripts that focused on the students' use of metonymies and metaphors and the presence of different thinking modes while responding to the questions asked.

A total of 16 students were interviewed towards the end of the Spring 2009 semester, from those interviews, one from each section, was randomly selected to be analyzed by the writer of this thesis. Those interviews corresponded to students with the codes A18, A33, B6, and C7 (students A18 and A33 were interviewed at the same time). Interviews lasted between 60 and 120 minutes, and an average of 7 questions was answered by each student. The transcripts of each interview included in the analysis of this thesis are available upon request.

The set of questions used while conducting the interview are presented below (obtained from Zamora, 2010). The number of questions each student answered varied among students. Minor modifications were made to each interview and some questions were added in order to gather as much as possible information from the students' perspective. A copy of the actual page used by the researcher during the interview is attached as an appendix.

1. Provide a definition of linear independence.
2. Provide an example of a linearly dependent set of vectors.
3. Given the set {u1, u2, u3, u4} where vectors u1, u2, u3, are on the same plane and u4 is not, determine if the set is linearly dependent.
4. Given a linearly independent set {u1, u2, u3, u4} in R^n. Prove or disprove that the set {u1, u2+5u1, u3, u4} is linearly independent.
5. Given an nxm matrix A where ai2 = ai4+ai5, for all $0 \leq i \leq n$. Determine if the set {A1, A2, A3, …Am} (here Aj is the jth column of A) is linearly independent.
6. Given a singular 3x3 matrix A, determine if the vectors of the set {A1, A2, A3}, where Aj is the jth column of A) are all on the same plane. Explain your answer.
7. Given that the vector equation xu+yv+zw=0 has infinitely many solutions, determine if the vectors u, v, w are on the same plane. Explain your answer.

8. Given the vector equation a1u1+a2u2+a3u3=0 with a solution a1= 1, a2 = -2 and a3=0, determine linear independence of the set u1, u2, u3.
9. Given that dim(Span(u, v, w)) = 1, determine the linear independence of the set.

3.5 Online Modules Available

The instructors of the modular sections of the matrix algebra course during the spring 2009 semester had a list of online tools available to utilize during their lectures. During this particular semester, students from the modular section were required to complete the following guided questions assignments (Zamora, 2010).

1. Linear Systems Module (Equivalent Systems and Solution Sets Activity): This tool was developed with the aid of Geometer's Sketch Pad (GSP) program. The module provides a graphical representation of linear functions and solution sets in R^2.
2. Matrix Operations Module (Matrix Product Activity): This tool was developed with the aid of Geometer's Sketch Pad (GSP) program. The module provides a graphical representation of matrices in R^2.
3. Vector Spaces Module (Linear Combination Activity and Linear Independence Investigation): This tool was developed with the aid of Mathematica. The module provides a graphical representation of matrices in R^3.
4. Linear Transformation Modules: The tools included in this section were developed with the aid of GPS and Mathematica. Their main goal was to provide a graphical representation of vector spaces and linear transformations in R^2 and R^3.

In each assignment, students were asked to access the online modules in order to answer the required questions. Copies of the assignment questions are attached to this thesis in the appendix.

Chapter 4: Results

The students' responses to a set of questions were analyzed following a constant comparison method (Glaser, 1992) in which a qualitative approach was taken. The qualitative analysis focused on the presence of different thinking modes, metonymies, and metaphors found on the students' interview responses to questions related to linear independence, span, and spanning sets.

4.1 Qualitative Analysis

Three interviews were randomly selected for the purpose of this thesis; each interview contained the responses of one or two students from each section of the Matrix Algebra class during the Spring 2009 semester. Two of those interviews were conducted individually, while in the third one, two students were interviewed at the same time. Each interview was recorded, each video was transcribed, and then the transcripts were independently analyzed by the author of this thesis, her advisor, and an additional graduate student. -The analysis focused on the identification and classification of cognitive constructs –thinking modes, metaphors, and metonymies- found in the students' responses.

The results reported in this chapter were found by applying the Grounded Theory introduced by Glaser and Strauss (1967). By following this theory's ideas the way thinking modes were found and categorized is the following: raters identified key ideas present in the students' responses with codes that were created in order to classify their cognitive constructs. Some codes found to be similar were merged into the same category in order to minimize the amount of codes. After codes were obtained, they were grouped for further analysis (Glaser and Strauss, 1967). The interview transcripts were first checked for similarities and differences present in the student responses in order to establish the terminology used by each student. Categories and codes were individually created for each student.

After all thinking modes, metaphors, and metonymies were identified, a table of category descriptions and examples was created, as well as another table with metonymies and metaphors

used by each student. The frequency of types of thinking modes, metaphors and metonymies was recorded separately to address the research questions.

4.2 Classification of Responses

On thinking modes, after the individual categories were formed, representative quotations were obtained from the transcript to stand for each category. Once the students' responses were categorized by the author of this thesis, two independent researchers performed the same analysis. These independent analyses were then compared to the findings of the author of this thesis in order to establish credibility. The same process was followed in order to document metonymies and metaphors found in the students' responses.

The following subsections present the results found while analyzing the data obtained from the interview transcripts of the four volunteered students. The following example of a student's argument explains what aspects were taken into account and how it was categorized:

Student A18: "...we know it's definitely dependent…linearly dependent…but I don't think there is ...for certainty we can say they all exist on…on the same plane…because these {first and second columns…} two are actually a line…and this {third column} last one is a different line…oh…wait…so that does mean…yeah that means they are all on the same plane…"

It is clear that student A18 was focusing on the geometrical aspects of a set of vectors, he knew the first and second vectors were dependent, but instead of mentioning the fact that they might have been scalar multiples of each other, he chose to focus on the fact that they were on the same line to then determine that all three vectors were on the same plane, which came from the fact that it was a dependent set in R2. The category created for this argument was LCPL – Linear Combination Plane- and because of the use of geometrical attributes and the lack of definitions and/or numerical computations LCPL was classified into the Synthetic-Geometric thinking mode.

4.2.1 Modes of Thinking

The description of thinking modes identified by Sierpinska (2000) was followed in the analysis of the interview transcripts; this category was explained in section 2.4 of chapter 2. See chapter 2 of the thesis for detailed discussion on the particular framework. The thinking modes found in the responses of students A18, A33, B6, and C7 and the frequency of these are presented in the subsections below.

4.2.1.1 Student A18

Students A18 and A33 are twin brothers who belonged to Section A of the matrix algebra course. They were both interviewed at the same time by a graduate student in the spring 2009 semester and answered to the set of 9 questions alternately. Student A18 responded to questions 2, 4, 6, and 8; representative interview responses are reported in table 4.1 below. Table 4.1 summarizes the thinking modes identified by the author of this thesis; each category (thinking mode) has its own code, a description, and an example taken from the transcript of Student A's interview.

Table 4.1. Categories for responses corresponding to student A18.

STUDENT A18			
Code	**Category**	**Description**	**Example**
BASIS	Basis	Student constructs a dependent set from the basis	"...this one is actually a basis for R2 {referring to [1; 0] and [0; 1]}, which means that there is a unique combination of these two vectors to create every vector in R2… so with the additional vector in this set…it's already a set made up with this two…"
LCST	Linear Combination Stated	Student states that vectors are dependent because there exists a linear combination among them	"...if I can find a linear combination….of a …. set of vectors, then… I don't know…it's dependent…"
DIMENV	Dimension exceeds number of vectors	Student believes the system will be independent when the number of vectors is less than n, in R^n.	"… I think… in this case… this is a high probability that it is independent because…there is…this is R4 and we have three vectors, so there is a good chance that it is independent…"
IDM	Identity Matrix	Student refers to identity matrix to represent independence	"…if a… we had the zeros instead…{student writes [1, 0, 0; 0, 1, 0; 0, 0, 1]}…and a one…then I know that…this two vectors {[1; 0; 0] and [0; 1; 0]} will have no impact

			on this vector {[0; 0; 1]}… but…a….because there is no way that anyone of these will make this last one…but there might be a possibility that…in this case…uhm… these vectors {referring to v1, v2, and v3}…have some sort of relationship with this last one…"
ZVEC	Zero Vector	Student believes that having the zero vector in the system does not affect dependence	"…uhm…I can …uhm…most likely this one is the zero matrix {referring to fourth column of [1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 0]} …or… it didn't start a zero matrix, but it goes to the zero matrix…uhm…that means this one is practically irrelevant….is like adding the zero to the set… "
LCNUM	Linear Combination Numerically	Student states a linear combination numerically	"…something that I can see in here…that's still… uhm…then it becomes dependent…but…u1…u2 plus 5u1…{student writes [0; 1; 0; 0] + 5[1; 0; 0; 0] = [5; 1; 0; 0]} …uhm… doesn't make anything that's else that it could {inaudible}…uhm…[5; 1; 0; 0]… none of these vectors…or combinations of these vectors…other than this one…is gonna make this last vector…"
SING	Singularity	Student uses the concept of singularity to find dependence	"…for a nonsingular the matrix it's…uhm…the only solution for this equation {referring to Ax=0…} is…uhm…trivial solution…therefore if singular we know that there is at least one other…"
LCPL	Linear combination Plane	Student determines independence by picturing vectors on a plane or viceversa	"...we know it's definitely dependent…linearly dependent…but I don't think there is …for certainty we can say they all exist on…on the same plane…because these {first and second columns…} two are actually a line…and this {third column} last one is a different line…oh…wait…so that does mean…yeah that means they are all on the same plane…"

Based on Sierpinska's ideas, shown in section 2.4, the categories reported in table 4.1 can be separated into three different thinking modes recognized by Sierpinska (2000) as the ways linear algebra students think while solving problems, answering questions. The thinking modes identified by Sierpinska (2000) are Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural.

Revisiting the information provided in table 2.1, we can conclude that the categories omitting exact descriptions that based their explanation on graphical representations can be classified into the Synthetic-Geometric thinking mode. Categories based on numerical and algebraic representations of objects requiring manipulation of data to arrive to conclusions, are classified into the Analytic-Arithmetic thinking mode, and the categories in which objects are analyzed with the use of theorems and definitions are classified as part of the Analytic-Structural thinking modes. Some of the categories found -contain ideas that can would allow each category to be classified into more than one thinking mode, categories are not mutually exclusive (Sierpinska, 2000) since the two students tended to use a combination of tools to arrive to conclusions and provide an answer.

The categories belonging to student A18 –summarized in table 4.1- can be classified as follows: only 1 category, LCPL can be classified as a Synthetic–Geometric thinking mode. For this category, student A18 referred to the mental picture of vectors on a plane to determine dependence among the vectors in the set. There were a total of 6 categories that fit into the Analytic-Arithmetic thinking mode, ZVEC, LCNUM, LCST, DIMENV, BASIS, and IDM. With these categories student A18 referred to numerical representations and manipulated numerical computations to determine linear dependence or independence. Lastly, the categories that can be classified into the Analytic Structural thinking mode are SING, BASIS, and IDM. Student A18 made use of definitions and key words that led to these thinking modes. Two categories, BASIS and IDM, were classified as Analytic-Arithmetic and Analytic-Structural since student A18 reasoning included definitions in some cases and numerical computations in others.

There were a total of 8 different categories created for student A18, during the interview, a total of 20 uses of those categories were identified. It is important to mention that student A18 and student A33 were interviewed at the same time, therefore each student did not answer to all the questions from the set. Student A18 only responded to questions 2, 4, 6, and 8. Table 4.2 contains the frequency of each category used by student A18.

Table 4.2. Frequency of used categories for student A18.

STUDENT A18							
	Category	***Description***	***Frequency***				***Totals***
			Q2	***Q4***	***Q6***	***Q8***	
1	**BASIS**	Basis	1				1
2	**LCST**	Linear Combination Stated	1	6	1		8
3	**DIMENV**	Dimension exceeds number of vectors	1				1
4	**IDM**	Identity Matrix	3	1	1		5
5	**ZVEC**	Zero Vector	1				1
6	**LCNUM**	Linear Combination Numerically		1			1
7	**SING**	Singularity			1	1	2
8	**LCPL**	Linear combination Plane			1		1

Looking at table 4.2, we can see that the categories most frequently used by student A18 are LCST (frequency 8) and IDM (frequency 5). The frequency of each thinking mode used by student A18 can be distributed as follows:

- Synthetic-Geometric- LCPL (1), with this being the only category in this graphical mode.
- Analytic-Arithmetic- BASIS (1), LCNUM (1), LCST (8), IDM (5), DIMENV (1), and ZVEC (1). A total of 6 categories with a combined frequency of 17.
- Analytic-Structural- BASIS (1), SING (2), and IDM (5). A total of 3 categories with a combined frequency of 8

It appears that the thinking mode most frequently used by student A18 in his reasoning was Analytic-Arithmetic, and the thinking mode used the least was Geometric-Synthetic.

4.2.1.2 Student A33

Student A33 was interviewed along with his brother, as reported above. He responded to questions 1, 3, 5, 7, and 9; his responses are reported in table 4.3 below. Table 4.3 summarizes

the thinking modes used by Student A33, these are represented by a code, category description, and an example.

Table 4.3. Categories for responses corresponding to student A33.

STUDENT A33			
Code	**Category**	**Description**	**Example**
LCALG	Linear Combination Algebraically	Student states a linear combination algebraically	"there's that formula where…you have the au +bv = c for…say this is u, v, and c…or w, ups…w…{student labels vectors from previous set u, v, and w respectively}… so you multiply that by a scalar, you multiply that by another scalar, you should end up with this one or you shouldn't if they are independent"
SCMLT	Scalar Multiple	Student states that one vector is a scalar multiple of another	"…then these two…since a…[2; 1] where you multiply 2 by 2 and you get 4, and you multiply 1 by 2 and you get this one…so these two are the same… they are just multiplied by a scalar…"
LCNUM	Linear Combination Numerically	Student states a linear combination numerically	"but two minus zero is not gonna give you zero… so then each of these will probably be linearly independent… "
IDM	Identity Matrix	Student refers to identity matrix to represent independence	"oh, then you only end up with just I2, so that means that you only have two linearly independent…"
SYSEQ	System of Equations	Student bases independence on a system of equations	"… okay, normally when we see this … it means.. okay well… you should do Gauss Jordan… it will be like a system of equations… if you have an a and a b, or x1, x2… and say well…and use your constants… that's what you usually…that happens when my result gets like that… "
NVED	Number of vectors exceeds dimension	The number of vectors exceeds the dimension	"…where the dimension is two means it's indepe…a…dependent…already that these three {referring to u1, u2, u3} are somehow dependent…"
VCPL	Vectors in a plane	Student determines dependence by the number of vectors inside or outside a plane	"…1, 2, 3 {counting the vectors on the plane}…and even if …uhm… this one made another plane {student draws a fourth vector outside the plane, but inside another plane}…if R4 made another plane…like a cube or something…those three will still be part of a R2…and it will still be dependent…"
LCST	Linear Combination Stated	Student states that vectors or columns are dependent because there exists a linear combination among them	… so each …uhm…this column {referring to column A2…} it's dependent on these {referring to columns A4 and A5…} two columns…so then this is not independent…it's dependent…

SING	Singularity	Student uses the concept of singularity to find dependence	"…then if they are infinitely many that means it's singular…uhm…which means they are dependent…"
LCPL	Linear combination Plane	Student determines independence by picturing vectors on a plane or viceversa	"…but…dependent can also mean that these {referring to u, v, w…} three are a line…"

Referring back to Sierpinska's (2000) description of thinking modes, after analyzing the responses of student A33, every category was associated with a thinking mode. These categories –summarized in table 4.3- can be classified as follows: 3 categories, VCPL, LCPL, and NVED can be classified as a Synthetic–Geometric thinking mode. The categories belonging to this thinking mode were identified when student A33 was making reference of geometrical representations as part of his reasoning. There were a total of 6 categories that fit into the Analytic-Arithmetic thinking mode, LCALG, SCMLT, SYSEQ, LCST, LCNUM, and IDM. With these categories student A33 referred to numerical representations and manipulated numerical computations to determine linear dependence or independence. Lastly, the categories that can be classified into the Analytic Structural thinking mode are SING, IDM, LCALG, and NVED. Student A33 made use of definitions and key words that led to the categories classified as the Analytical-Structural mode. Two categories, LCALG and IDM, were classified as Analytic-Arithmetic and Analytic-Structural since student A33's reasoning included definitions in some cases and numerical computations in others. Category NVED being the most frequent used category was classified as Synthetic-Geometric and Analytic-Structural because student A33 referred to the number of vectors exceeding the dimension several times while mentioning definitions and making connections and in some occasions made the conclusion after looking at a graphical representation of vectors.

There were a total of 10 different categories created for student A33, during the interview 19 uses of those categories were identified. -It is important to mention that student A18 and student A33 were interviewed at the same time, therefore each student did not answer to all the questions from the set. Student A33 only responded to questions 1, 3, 5, 7, and 9. Table 4.4 contains the frequency of each category used by student A33.

Table 4.4. Frequency of used categories for student A33.

STUDENT A33								
	Category	***Description***	***Frequency***					***Totals***
			Q1	***Q3***	***Q5***	***Q7***	***Q9***	
1	**LCALG**	Linear Combination Algebraically	1					1
2	**SCMLT**	Scalar Multiple	1					1
3	**LCNUM**	Linear Combination Numerically	1					1
4	**IDM**	Identity Matrix	4					4
5	**SYSEQ**	System of Equations	1					1
6	**NVED**	Number of vectors exceeds dimension		2			3	5
7	**VCPL**	Vectors in a plane		3				3
8	**LCST**	Linear Combination Stated			1			1
9	**SING**	Singularity				1		1
10	**LCPL**	Linear combination Plane				1		1

Looking at table 4.4, we can see that the categories most frequently used by student A33 are NVED (frequency 5) and IDM (frequency 4). The frequency of each thinking mode used by student A33 can be distributed as follows:

- Synthetic-Geometric- LCPL (1), VCPL (3), and NVED (5). A total of 3 categories with a combined frequency of 9.
- Analytic-Arithmetic- LCALG (1), SCMLT (1), LCNUM (1), IDM (4), SYSEQ (1), LCST (1). A total of 6 categories with a combined frequency of 9.
- Analytic-Structural- IDM (4), LCALG (1), SING (1), and NVED (5). A total of 4 categories with a combined frequency of 11.

It appears that the thinking mode most frequently used by student A33 in his reasoning was Analytic-Structural. Student A33 appeared to be able to use more than one thinking mode in repeated occasions to provide answers to the questions and to make connections.

4.2.1.3 Student B6

Student B6 belonged to Section B of the matrix algebra course, the other modular section during the spring 2009 semester. She participated in a one to one interview and answered to questions 1, 2, 3, 4, 7, 8, and 9. Table 4.5 includes some of her responses used as examples of the categories created to represent the thinking modes used during the interview.

Table 4.5. Categories for responses corresponding to student B6.

STUDENT B6			
Code	**Category**	**Description**	**Example**
INVM	Invertible Matrix	Student refers to an invertible matrix to imply independence of a set of vectors	…well, I know that when a vector…well, not the vector...when a matrix...I mean a matrix is to be defined as a set of vectors, right?...and then when…when the matrix is invertible, that means it is linearly independent…
FRVAR	Free Variable	Student uses the term free variable to imply a unique solution or linear dependence	…when you have a…a matrix and it has a bunch of numbers and you can reduce it, and if at the end you have any free variable…that means it is not a unique solution because that free variable can be whatever…
ZROW	Zero Row	Student uses a row of zeros to imply dependence	SB6: …I don't know, one…{Student writes down matrix [1, 2, 3, 4; 3, 3, 3, 3; 2, 2, 2, 2]}… actually this is going to be dependent anyway…I1: …why is that?...SB6: …because you can do this and it will give you a row of zeros…
UNSOL	Unique Solution	Student relates a unique solution to a lineraly independent set	I1: …you don't know?...okay…but when you said it doesn't have a unique solution…that's why it is…it is linearly…SB6: …dependent…I1: …dependent?...okay. If it has a unique solution, what do you say?...SB6: …that it is independent…
IDM	Identity Matrix	Student refers to the identity matrix or a the matrix with in row reduced form to imply independece	I1: …what are you thinking?...SB6: …and it is independent…I1: …why did you say that?...SB6: …like if you write the set, it is going to be something like…{student writes down matrix {1, 0; 0, 1; 0, 0; 0, 0]}…
TRIV	Trivial Solution	Student uses the fact that x1=0 and x2=0 to imply independece	SB6: …this is…it's x…equals to zero?...{student writes x=0, 0=0, and 0=0, next to the first, third, and fourth rows}…I1: …okay…SB6: …and then 0 equals zero…something like that…I1: …how about his one?...{referring to the entry in the second row, second column}…SB6: …oh, yeah…{student writes x2 = 0}…

MINSPT	Minimal Spanning Tree	Student uses the concept of a minimal spanning tree to express the lower cost of connecting a set of points	I1: …and branches…wow…what do you mean by the minimal spanning?...SB6: …I think the minimal spanning tree is like…if it is like…you can put a value…to these vertex…like a three, a two…so it is like the minimal…like you connect all the vectors with the minimal cost…
LCST	Linear Combination Stated	Student refers to a linear combination among the vectors of a set to imply dependece	SB6: …yeah, I said something like this…this were…what I thought about it is, this is a set of vectors and these two…I mean this vector was a combination of another two vectors…then I think since it was a combination I said it was dependent…
NVED	Number of Vectors Exceeds Dimension	Student mentions that if the number of vectors is greater than n in Rn, then the system is dependent	…because like this is in R2 and there is three of them…but yeah, I don't know if you can…yeah, I think it is…

Referring back to Sierpinska's (2000) description of thinking modes, after analyzing the responses of student B6, every category was associated with a thinking mode. These categories –summarized in table 4.5- can be classified as follows: only 1, the MISPT category was found to belong to the Synthetic–Geometric thinking mode. Student B6 made use of this category in 2 occasions by making reference to the graphical representation of a spanning tree in a Hamiltonian cycle. There were a total of 6 categories that fit into the Analytic-Arithmetic thinking mode, FRVAR, ZROW, UNSOL, IDM, MINSPT, and LCST. With these categories student B6 referred to numerical representations and manipulated numerical computations to determine linear dependence or independence. Lastly, the categories that can be classified into the Analytic Structural thinking mode are FRVAR, UNSOL, TRIV, IDM, INVM, and NVED. Student B6 made use of definitions and key words that led to the categories classified as the Analytical-Structural mode. Three categories, FRVAR, UNSOL, and IDM, were classified as Analytic-Arithmetic and Analytic-Structural since student B6's reasoning included definitions in some cases and numerical computations in others. Category MINSPT was classified as Synthetic-Geometric and Analytic-Structural because student B6 made use of the graphical representation of Hamiltonian cycles and attempted to use her reasoning and transferred to the idea of span of a set. There were a total of 9 different categories created for student B6, and 22

uses of those categories were identified. Table 4.6 contains the frequency of each category used by student B6.

Table 4.6. Frequency of used categories for student B6.

STUDENT B6									
	Category	Description	Frequency						Totals
			Q1 & Q2	Q3	Q4	Q7	Q8	Q9	
1	**INVM**	Invertible Matrix	1						1
2	**FRVAR**	Free Variable	3					1	4
3	**ZROW**	Zero Row	1						1
4	**UNSOL**	Unique Solution	1			1	1		3
5	**IDM**	Identity Matrix	1		1	1		3	6
6	**TRIV**	Trivial Solution	1						1
7	**MINSPT**	Minimal Spanning Tree	1					1	2
8	**LCST**	Linear Combination Stated			1			2	3
9	**NVED**	Number of Vectors Exceeds Dimension						1	1

Looking at table 4.6, we can see that the categories most frequently used by student B6 are IDM (frequency 6) and FRVAR (frequency 4). The frequency of each thinking mode used by student B6 can be distributed as follows:

- Synthetic-Geometric- MINSPT (2). This was the only category classified into this thinking mode.
- Analytic-Arithmetic- FRVAR (4), ZROW (1), UNSOL (3), IDM (6), MINSPT (2), and LCST (3). A total of 6 categories with a combined frequency of 19.
- Analytic-Structural- IDM (6), INVM (1), FRVAR (4), UNSOL (3), NVED (1), and TRIV (1). A total of 6 categories with a combined frequency of 16.

It appears that the thinking mode most frequently used by student B6 in her reasoning was Analytic-Arithmetic. Student B6 appeared to be able to use more than one thinking

mode in repeated occasions by providing definitions or statements to arrive to conclusions after making some numerical computations or looking at the graphical representation of certain sets of vectors.

4.2.1.4 Student C7

Student C7 belonged to Section C of the matrix algebra course, the only non-modular section during the spring 2009 semester. She participated in a one to one interview and answered to questions 1, 2, 3, 4, 5, 6, 7, and 8 (see appendix for questions) Table 4.7 includes some of her responses used as examples of the categories created to represent the thinking modes used by student C7 during the one on one interview.

Table 4.7. Categories for responses corresponding to student C7.

STUDENT C7			
Code	**Category**	**Description**	**Example**
IDM	Identity Matrix	Student refers to identity matrix to represent independence	SC7: …and once reducing it…if…let's say we get the identity matrix…which is the one, zero, all the zeros…zero, one, zero, all the way down, etcetera…until you get over here, for where the bottom is one…I1: …okay…SC7: …if you get that then it is linearly independent…
FRVAR	Free Variables	Student uses the concept of free variables to imply linear dependence	SC7: …okay, so if you get this {referring to the identity matrix} then they are linearly independent…however if you get some matrice where you have what is known as free variables…I1: …uhm, uhm…SC7: …so if you have like a bunch of zeroes here…one here…but over here you had something like a two or a…and then a three…a bunch of zeroes…and then once again you come down here…then it is not linearly independent…
LCST	Linear Combination Stated	Student states that vectors are dependent because there exists a linear combination among them	SC7: …right…right, then the set of vectors is not linearly…it is linearly dependent because of this…because this vector right here can be expressed in terms of the other vectors…it depends…I1: …oh, I see…SC7: …the vectors depend on it…
SYSEQ	System of Equations	Student bases independence on a system of equations	SC7: …and then because of…because of the parameters then

			obviously the first vector depends on x3…in order to be itself…I1: …in order to be itself, what do you mean?...SC7: …in order to make the…to…sorry…in order to make the…uhm…basically to keep the system of equations…what they are without changing the system of equations…
TRIV	Trivial Solution	Student states that having the tivial solution implies independence	SC7: …I believe it is, as long as x1 is equal to x2 is equal to x3 etcetera …is equal to xn… as long as that is the only…the only way for this system to be equal to zero…I1: …oh, okay…SC7: …for the only way for this to be equal to zero vector is this…the trivial solution…
SCMLT	Scalar Multiple	Student mentions a scalar multiple of a vector to represent a dependent set of vectors	I1: …can you also give me an example of a linearly dependent set?...SC7: …sure…{ }…this is a whole different thing…so v1…let's call it v4…let me see if this works…one, two, zero…oh…two, four, zero…zero, zero, one…so let me double check, to make sure that this is correct…I believe it is because this is a scalar multiple of this one…I1: …oh okay…SC7: …zero, zero, one…I1: …okay, can we just stop right there?...without doing this…SC7: …yes, actually I believe you can because, because the fact that you look automatically and you… this can be a scalar multiple because it obviously depends on the other one already…and it becomes a row of zeroes…
ZVEC	Zero Vector	Student believes that having the zero vector in the system does not affect dependence	…because of the fact that the zero vector doesn't really, I guess, affect it…if you put a bunch of zeroes here, when you reduce it…it would just be the same…
REF	Reduced echelon form	Student reduces system to check for dependence	…ok, well..uhmm…because when you reduce a matrice, or matrix…sorry, I said matrice…when you reduce a matrix you have to be able to have…you essential goal is to get it down to the identity matrix…so when you…do you want me to just reduce it? Because I can…

VCPL	Vectors in a plane	Student determines dependence based on the amount of vectors in the same plane	…and u4 is not…okay…determine if the set u1, u2, u3, u4 is linearly independent and explain your answer…okay, automatically knowing just from right here…just from knowing that these three are coplanar…then they are automatically linearly dependent…because of the fact that if they are coplanar, then you can express any of them in terms of the others…so you can express u1 in terms of u2 and u3…so they are linearly dependent on each other, so adding another vector is not going to change the fact that they are linearly dependent…
LCPL	Linear Combination Plane	Student refers to the vectors in the same plane to imply a linear combination among vectors	…because when you have vectors and they are coplanar, then you can express…like, lets say this is u1, u2, and u3…you can express u2…you can create u2 by some combination of u1 and u3…
NVED	Number of vectors exceeds dimension	The number of vectors is greater than than the vector dimension	…because if you have four vectors, and you only have three in R…you have R3…then it is automatically dependent…
ZROW	Zero Row	Student refers to a row containing zeros to imply independence	…meaning that…you can…I guess unless the last row is a bunch of zeros, that will be the only way that this will be linearly independent…if the last row was a row of zeros…
LCVE	Linear combination Vector Equation	Student writes vector equation and tries to use it to represent independence	SC7: …uhm…let me see…so u1x1…{student writes equation u1x1 + u2x2 + 5u1x2+ u3x3 + u4x4 = 0}…okay, well let me try this…I1: …yeah, go ahead…SC7: …I am just going to see if I do like this…u2x2 plus 5u1x2 plus u3x3 plus u4x4 is equal to zero…but the only thing that I see here is the fact that this is still a…a scalar of that…{ }…but you are adding something to it…like another vector to it…
EQPL	Equation of a Plane	Student tries to use the equation of a plane to determine dependence	…right, well what I was…no, what I am saying is the equation of plane would help direct me…I thought… I thought so, but I didn't get any other insight from it…
SING	Singular matrix	Student uses the concept of singularity to determine dependence	SC7: …I remember that…but…sorry I didn't even see that…uh…okay, so a singular matrix is not invertible…which means that…when you augment it, you can't put it into the identity matrix…I1: …okay…SC7: …so that probably means that they are dependent…

Referring back to Sierpinska's (2000) description of thinking modes, after analyzing the responses of student C7, every category was associated with a thinking mode. These categories

–summarized in table 4.7- can be classified as follows: 3 categories, VCPL, LCPL, and EQPL can be classified as a Synthetic–Geometric thinking mode. The categories belonging to this thinking mode were identified when student C7 was making reference of geometrical representations as part of her reasoning. There were a total of 9 categories that fit into the Analytic-Arithmetic thinking mode, IDM, FRVAR, LCST, REF, SYSEQ, LCVE, SCMLT, ZVEC, and ZROW. With these categories student C7 referred to numerical representations and performed numerical computations while trying to provide answers. Lastly, the categories that can be classified into the Analytic Structural thinking mode are IDM, LCST, NVED, TRIV, and SING. Student C7 made use of definitions and key words that led into these categories to be classified as the Analytical-Structural mode. Two categories, LCST and IDM, were classified as Analytic-Arithmetic and Analytic-Structural since the since student C7's reasoning included definitions in some cases and numerical computations. It is important to mention that these two categories were the most frequently used during this student's interview. There were a total of 15 different categories created for student C7; during the interview 43 uses of those categories were identified. Table 4.8 contains the frequency of each category used by student C7.

Table 4.8. Frequency of used categories for student C7.

STUDENT C7										
	Category	***Description***	***Frequency***							***Totals***
			Q1 & Q2	***Q3***	***Q4***	***Q5***	***Q6***	***Q7***	***Q8***	
1	**IDM**	Identity Matrix	4		2		2			8
2	**FRVAR**	Free Variables	2		1					3
3	**LCST**	Linear Combination Stated	2		1	1	2		1	7
4	**SYSEQ**	System of Equations	1							1
5	**TRIV**	Trivial Solution	1		1			1	1	4
6	**SCMLT**	Scalar Multiple	1		1		2			4
7	**ZVEC**	Zero Vector	1							1
8	**REF**	Reduced echelon form	1		2	1				4
9	**VCPL**	Vectors in a plane		1			3			4
10	**LCPL**	Linear Combination Plane		1						1

11	**NVED**	Number of vectors exceeds dimension			1					1
12	**ZROW**	Zero Row			1					1
13	**LCVE**	Linear combination Vector Equation			1		1			2
14	**EQPL**	Equation of a Plane					1			1
15	**SING**	Singular matrix					1			1

Looking at table 4.8, we can see that the categories most frequently used by student C7 are IDM (frequency 8) and LCST (frequency 7). The frequency of each thinking mode used by student C7 can be distributed as follows:

- Synthetic-Geometric- LCPL (1), VCPL (4), and EQPL (1). A total of 3 categories with a combined frequency of 6.
- Analytic-Arithmetic- IDM (8), FRVAR (3), REF (4), LCST (7), SYSEQ (1), LCVE (2), SCMLT (4), ZVEC (1), ZROW (1). A total of 9 categories with a combined frequency of 31.
- Analytic-Structural- IDM (8), LCST (7), NVED (1), TRIV (4), and SING (1). A total of 5 categories with a combined frequency of 21.

It appears that the thinking mode most frequently used by student C7 in her reasoning was Analytic-Arithmetic. Student C7 appeared to be able to use more than one thinking mode in repeated occasions to provide answers.

4.2.2 Metonymy and Metaphor

The second part of the analysis of the responses belonging to students A18, A33, B6, and C7 was conducted in order to find the metonymies and metaphors used during their interviews. The identification of these metonymies and metaphors is similar to the one reported by Presmeg (1998) and is explained in section 2.5 of chapter 2.

4.2.2.1 Student A18

The transcript analysis done by the author of this thesis yielded a total of 47 metonymies used by student A18. Table 4.9 summarizes the metonymies used and the part of the transcript where they were found.

Table 4.9. Metonymies and Metaphors displayed by student A18 (in order of appearance).

	Sample Response	***Metonymy***	***used to represent***
	Q2. Given an example of a linearly dependent set of vectors.		
1	"...if we are in R2, then all we need is an additional vector in R2 so…{student writes down vectors} [1; 0] and [0; 1] which is the basis, then we throw in a [1; 1]… and we already no matter have a dependent set of vectors…because these two were already linearly independent…after adding another one… it's gotta be a combination of these two somehow…"	three vectors in R2	linear combination
2		linear combination of vectors	linearly dependent set
3	"….this one is actually a basis for R2 {referring to [1; 0] and [0; 1]}, which means that there is a unique combination of these two vectors to create every vector in R2… so with the additional vector in this set…it's already a set made up with this two…"	basis plus another vector in the set	linearly dependent set
4	"….there is a….if I can find a linear combination….of a …. set of vectors, then… I don't know…it's dependent…so his way is just Gauss Jordan…the quickest way….yeah the calculator is the quickest…"	linear combination	linear dependence
5		gauss jordan	quickiest way to find dependence of a set
6	"...so the set is a…. linearly dependent because if we have a combination, a linear combination of two vectors… that makes three… that makes a third…or it can even be just a scalar multiplication of one of the vectors…"	linearly dependent set	linear combination of vectors
7	"… in this case… this is a high probability that it is independent because…there is…this is R4 and we have three vectors, so there is a good chance that it is independent…"	less vectors than components	linearly independent set

8	I2: how do you know if it is in R4?...what do you look at in the set of vectors or…in the vectors…to know that it is in R4?... you say this is in R4, right?...SA18: I just….just…the number of rows…	number of rows	dimension
9	"…if this was a system then this last equation for the system is independent on anything inside…"	pivot row	independence among equations
10	"…and these right here {referring to numbers inside the matrix [1, 2, 0; 0, 1, 0; 0, 0, 1; 0, 0, 0]}…would be the scalars for the unknowns…if we are talking about a system, but in here this is …how many of each vector to use and to solve for the system…"	scalars for the unknowns	how much of a vector to use to solve a system
11	"…if we have a { student writes matrix [1, 0, 4; 0, 1, 1; 0, 0, 2] and labels columns v1, v2, and v3} …for instance…uhm…then this vector plus this vector equals….ah… this vector…{implying that v1+v2=v3}…"	matrix not on row reduced echelon form	one vector equals a combination of other two
12	"…if a… we had the zeros instead…{student writes [1, 0, 0; 0, 1, 0; 0, 0, 1]}…and a one…then I know that…this two vectors {[1; 0; 0] and [0; 1; 0]} will have no impact on this vector {[0; 0; 1]}… but… a….because there is no way that anyone of these will make this last one…"	identity matrix	vectors not being a combination of others
13	I2: …so whenever…you are saying that whenever you see this {pointing to 3x3 identity matrix} you know that it is independent…SA18: yeah…right…	identity matrix	linear independence
14	"… I know that this one {3x3 identity matrix} is independent…and a…this one is just, this matrix {referring to [1, 0, 0; 0, 1, 0; 0, 0, 1; 0, 0, 0]} is this… matrix {referring to 3x3 identity matrix} moved into another dimension…"	identity matrix	linear independence
15		row of zeroes	3x3 identity matrix moved into another dimension
16	"…most likely this one is the zero matrix {referring to fourth column of [1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 0]} …or… it didn't start a zero matrix, but it goes to the zero	having the zero vector	adding the empty set to a set

17	matrix…uhm…that means this one is practically irrelevant….is like adding the zero to the set…"	zero vector	zero matrix
18	"…adding the empty set again to a…to any set is …it's already contained, so it's useless…it's… it doesn't make a difference in…"	having the zero vector	not useful to determine linear independence
Q4. Given a linearly independent set $\{u_1, u_2, u_3, u_4\}$ in R^n. Determine the linear independence of the set $\{u_1, u_2 +5u_1, u_3, u_4\}$.			
19	"…if there is…if there is a linear combination… the two vectors to make a third… to make a third one… in this set or a scalar multiple of one makes another vector in the set, then the set is linearly dependent…"	linear combination	linear dependence
20	"…in this case we are just subtracting one of the vectors from another…we are not necessarily…it doesn't actually necessarily say that we are making any of the other vectors in the set…"	addition/subtraction	not a linear combination
21	"…and actually I don't think it would either because…u2 plus 5u1 exist in the…in the set… in the original set…but it's independent, so this doesn't make any of the other vectors… but that's given..so I'm gonna say this set it's still linearly independent…	u2 and 5u1 not making any other vectors	linearly independent set
22	"… they are independent…even adding one of these vectors to the other…uhm…if they were in R4…I can say that this set is equal to the standard basis of R4…{student writes the 4x4 identity matrix}…"	linearly independent set in R4	row equivalent to standard basis for R4
23		4x4 identity matrix	standard basis for R4
24	"…well this is the row equivalence…the…standard basis…because…ah…I know that if it was linearly independent then it's row equivalent to this…{referring to the 4x4 identity matrix }…"	linear independence	row equivalence to identity matrix

25	"…something that I can see in here…that's still… uhm…then it becomes dependent…but…u1…u2 plus 5u1…{student writes [0; 1; 0; 0] + 5[1; 0; 0; 0] = [5; 1; 0; 0]} …uhm… doesn't make anything that's else that it could {inaudible}…uhm…[5; 1; 0; 0]… none of these vectors…or combinations of these vectors…other than this one…is gonna make this last vector…"	not a linear combination	linear independence
26	"…uhm…this is unique…it's only a combination of these, so that means that there isn't another combination…it doesn't make…something that already exists in the set…if that exists elsewhere in the set…it makes its only part…unique vector that is in the set…so, like the original one…it's independent because there only exists one combination for every vector…	unique combination	linearly independent set
27	"…if we can't have this {4x4 identity matrix}…, then this…uhm process doesn't work out…because we can't prove that the…it doesn't exist still…it doesn't…this combination doesn't create a vector that didn't already existed in the linear one…	identity matrix	unique linear combination

28	"…well, if I can't use this {4x4 identity matrix}…, then uhm…oh right, I think it still holds true because…uhm…even if we don't know if it is independent, it starts with, so that any combination…any linear combination of…these vectors is not gonna make another vector in the set…so…uhm…actually, that will be the first thing I said…we are in …that's right…so even if I knew it was in R4, or I didn't know it was in R4..as long as I knew where the original one was independent…and then I changed one of the vectors just to be a modified version, so this one is u2 plus 5u1…it doesn't necessarily mean that this vector…uhm…is a combination of the other three…because is not…this one has u2 and that doesn't exist elsewhere…"	linear combination	independent vectors in a set

By looking at table 4.9, we can conclude that a high percentage of metonymies used by student A18 dealt with his reasoning of linear independence and linear dependence. Student A18 had a tendency of referring to a linear combination being present to imply linear dependence and concluded several times that the absence of a linear combination meant linear independence among a set of vectors. The excerpt below (obtained from the original transcript of the interview) indicates the student's reasoning based on linear combinations.

SA18: Let's say we have a set x, and if we are in R2, then all we need is an additional vector in R2 so…{student writes down vectors} $\begin{bmatrix}1\\0\end{bmatrix}$ *and* $\begin{bmatrix}0\\1\end{bmatrix}$ *which is the basis, then we throw in a* $\begin{bmatrix}1\\1\end{bmatrix}$*… and we already no matter have a dependent set of vectors…because these two were already linearly independent…after adding another one… it's gotta be a combination of these two somehow…*

I2: …how do you know that?…

SA18: ….this one is actually a basis for R2 {referring to $\begin{bmatrix}1\\0\end{bmatrix}$ *and* $\begin{bmatrix}0\\1\end{bmatrix}$*}, which means that there is a unique combination of these two vectors to create every vector in R2… so with the additional vector in this set…it's already a set made up with this two…*

I2: …okay….that is for….you said R2?…

SA18: uhm, uhm…

I2: so if I have two….these two vectors…and then I add another one…then it's going to be automatically dependent…you said…

SA18: …yes…

Even after being asked to think about a different set of vectors, student A18 goes back to his reasoning of finding a linear combination, but this time he mentions the Gauss-Jordan method in hope to find a linear combination present based on the existence of the identity matrix.

I2: …okay why don't we change these two vectors?...instead of [1; 0] and [0; 1] let it be…I don't know… [4; 5], [6; 7]… and then the [1; 1]… is it dependent or independent?...

SA18: …{student writes down vectors [4; 5], [6; 7], [1; 1]} ….ah….still dependent

I2: ….how do you know that?... what do you… when you… if you were asked in your test, for example… if it is dependent or independent… what would you do?...

SA18: ….ah….ah….there is a….if I can find a linear combination….of a …. set of vectors, then… I don't know…it's dependent…so his way is just Gauss Jordan…the quickest way….yeah the calculator is the quickest…

I2: …can you do it?...

SA18: yeah, sure…

I2: …do you also know how to use the calculator?...

SA18: uhm… actually the person who taught us was doing his master's thesis on calculator dependency…

I2: oh…{giggles}…so are you dependent or independent of the calculator?....

SA18: …. I tried to do things by hand…. but if I really need accuracy…. I just use the calculator…but actually I could do this one by hand…a…. this is…there we go…actually yeah I can already see that.. if we take this vector and this vector… and subtract this one from this one, we get this one…plus six, minus..oh wait….uhm… minus two of this, and seven minus two of this… basically is two times {student writes the rref } so the set is a…. linearly dependent because if we have a combination, a linear combination of two vectors… that makes three… that makes a third…or it can even be just a scalar multiplication of one of the vectors… make a…

I2: …okay…so in…in that case….uhm…you said the set will be dependent?...

SA18: yeah…

Another metonymy identified in table 4.9 used in several arguments was the identity matrix, it is important to mention that student A18 used the identity matrix and linear combination arguments interchangeably to arrive to a conclusion.

SA18: …then…I would know that this last vector {v3 being a combination of v1 and v2}… is a combination of some scalars of these vectors, not of the unknowns…

I2: …can you do an example?...

SA18: …uhm…I can try…{student writes down matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *}…*

I2: …okay…so….that one tells you it is…the set is dependent or independent?....or…

SA18: …yeah…it doesn't tell me much {student erases second and third columns}…uhm…if we have a { student writes matrix $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ *and labels columns v1, v2, and v3} …for instance…uhm…then this vector plus this vector equals….ah… this vector…{implying that v1+v2=v3} you know what I mean?...*

I2: …okay…this…the first vector…plus this one {v2} could…would give you this one {v3}?...

SA18: …yeah…right…

I2: …are those zeros at the bottom?... or what are those numbers?...

SA18: …uhm, zeros…

I2: …oh…and how do you know that… uhm… this plus this would give you this?...v1 plus v2 would give you v3…

SA18: …uhmm…

I2: …how do you know from… that… what do you do?...do you see the relationship in there?... or…

SA18: …uhmm…I am trying…to clear my thoughts here…no… it's… uhm…because this one isn't a one…or a zero, zero, one I should say {referring to vector $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ *}… that means that…ah…it has no relation….it has a relationship with these two vectors over here {v1 and v2]…uhm… so…*

from some combination, these vector will make this last one…if a… we had the zeros instead…{student writes $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ *}…and a one…then I know that…this two vectors {* $\begin{bmatrix}1\\0\\0\end{bmatrix}$ *and* $\begin{bmatrix}0\\1\\0\end{bmatrix}$*} will have no impact on this vector* $\begin{bmatrix}0\\0\\1\end{bmatrix}$ *but… a….because there is no way that anyone of these will make this last one…but there might be a possibility that…in this case…uhm… these vectors {referring to v1, v2, and v3}…have some sort of relationship with this last one…*

I2: …okay…so you are saying that because you can't … because this {referring to matrix $\begin{bmatrix}1 & 0 & 4\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix}$ *not being equal to matrix* $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ *}… is not equal to this…then it has to be dependent…*

SA18: …right…

Based on the statements provided by student A18 reported above, it is clear that student A18 used the fact the matrix did not reduce into the identity matrix to exemplify that a linear combination existed and that the set was linearly dependent.

4.2.2.2 Student A33

The transcript analysis done by the author of this thesis yielded a total of 56 metonymies and 1 metaphor (see number 2) used by student A33. Table 4.10 summarizes the metonymies and metaphor used and the part of the transcript where they were found.

Table 4.10. Metonymies and Metaphors displayed by student A33 (in order of appearance).

	Sample Response		
		Metonymy	***used to represent***
	Q1:Define the linear independence of a set of vectors		
1	I2: yeah…okay, now… oh you were saying that if you have a linearly dependent set... SA33: …it will be in three, then it will get a line or a plane…	line or plane	linearly dependent set in R3
2		(metaphor) three	R3
3	"…the way we were looking at these two earlier {referring to a two dimensional space represented by the drawing of a plane and a one dimensional space represented by a line}, then this will just be a plane… with the x and y."	dependence of 3 vectors in R2	plane
4		x and y	components of each vector in R2 forming a plane

5	"… so you multiply that by a scalar, you multiply that by another scalar, you should end up with this one or you shouldn't if they are independent."	linear combination	dependence
6	"then these two…since a…[2; 1] where you multiply 2 by 2 and you get 4, and you multiply 1 by 2 and you get this one…so these two are the same… they are just multiplied by a scalar…"	one vector multiple of another	the same vector, multiplied by a scalar
7	"but two minus zero is not gonna give you zero… so then each of these will probably be linearly independent"	not a linear combination	linear independence
8	"… I still have my calculator… but to do reduced row and then if it is identity, then that means each of those probably {are} independent…"	identity after row operations	linear independence
9	"…and if it's identity or if you get {inaudible…}, then we probably know to add these two together {referring to first and second columns of the matrix previously written}… to form some sort of combination…"	identity	linear combination
10	I2: okay, so if it reduced to the identity…then it is…SA33: … independent	identity	linear independence
11	"oh, then you only end up with just I2, so that means that you only have two linearly independent, so that will give you a plane…	2x2 Identity inside a 3x3 matrix	two linearly independent vectors in R3
12		two linearly independent vectors in R3	plane
13	"then it is just tied down to these…because… you can just multiply this by zero and {get} that…"	any vector times zero	zero vector being a linear combination of other vectors
14	"… okay, normally when we see this … it means… okay well… you should do Gauss Jordan… it will be like a system of equations… if you have an a and a b, or x1, x2… and say well…and use your constants… that's what you usually…that happens when my result gets like that… so it just means you can result in this from these two…{referring to the third column}.	gauss jordan	proves one column is a combination of others

Q3. Given the set {u1, u2, u3, u4} where the vectors u1, u2, u3 are on the same plane and u4 is not. Determine if the set {u1, u2, u3, u4} is linearly independent. Explain your answer.			
15	"…oh right… Given the set {u1, u2, u3, u4} where the vectors u1, u2, u3 are on the same plane and u4 is not. Determine if the set {u1, u2, u3, u4} is linearly independent and explain your answer. So…u1, u2, and u3 are on one plane…that means they are on R2, right?..."	vectors on a plane	vectors in R2
16	"…uhm…well… because…it would be a…more of a …. space or 3D, but… uhm… the model of those are in R3, so R2 means it's a plane…"	R2	plane
17	SA33: …and that would be a line… uhm… so … three vectors in … the dimension is…where the dimension is two means it's indepe…a…dependent…already that these three {referring to u1, u2, u3} are somehow dependent…I2: …how do you know that?...SA33: …three vectors in…three vectors in two…uhm…I guess…what was that there?...it was in discrete …pigeon hole…	three vectors in a plane	three vectors in R2
18		number of vectors exceeds space	linearly dependent set
19		three vectors in R2	pigeon hole
20	"…that.. where…one of these will fit as… at…or two of these will be independent and the third one, will just be the third… ah… that will fit inside of one of those holes…"	third vector in R2	fit inside one of the two holes
21		holes	entries in R2
22	"…where the two are already occupied, so fitting three in two means…plane and dependent…"	two occupied holes	two vectors in R2
23		three vectors in R2	plane
24		plane	linearly dependent set
25	"…and then to determine if this set…uhm… u4, u4 is not… so u4 is not in R2, which means it's in R…then there is {inaudible} then it should be R3…"	u4 not being on the plane	u4 not being in R2
26	"… this one made another plane {student draws a fourth vector outside the plane, but inside another plane}…if R4 made another plane…like a cube or something…those three will still be part of a R2…and it will still be dependent…"	plane	any dimensional shape (cube)
27	"…you said that these two make up the plane in R2…and then we add u4…which is not in R2, so all of them will be in R3 now…	plane	R2
28		three dimensional space	R3

29	I2: …how do you know it is in R3?... and not in …SA33: … because we said that u4 is not in the same… space as R2…so…adding up one more…	u4 not being on the plane	u4 not being in R2
30		vector not being on a plane	vector being in R3
31	"...if say you have a plane, right…and then you add another dimension… oh, it does not matter what that dimension is…you are still gonna have three dimensions…"	three dimensional space in Rn	R3
32	"…cause we said they form a plane…if they… if these two are in R2 were dependent, then they will form a line…"	two dependent vectors in R2	line
33	"…then they will only be in R1 and if they were dependent then they will be a plane…or they will be a line…"	two dependent vectors	vectors in R1
Q5. Given an nxm matrix a where $a_{i2}=a_{i4}+3a_{i5}$. Determine if the set {A_1, A_2, A_3,…, A_m} (Here A_j is the jth column of A) is linearly independent. Explain your answer.			
34	"...which in this case means dependence…{student writes the word dependence after the a2=a4+3a5} … so each …uhm…this column {referring to column A2…} it's dependent on these {referring to columns A4 and A5…} two columns…"	linear combination of columns	linear dependence among vectors
35	SA33: …because we called each of these columns vectors and this vector {referring to column A2…} will be a combination of these two {referring to columns A4 and A5 }… I2: …okay, then the set would be… SA33: ..dependent…because…	dependence among columns	dependence among vectors
Q7. Given that the vector equation xu+yv+zw=0 has infinitely many solutions. Determine if the vectors u, v, w are on the same plane. Explain your answer.			
36	"…and this has infinitely many solutions, so there is a case where…non trivial…that's…has infinitely solutions, so it includes the trivial solution and other solutions…"	infinitely many solutions	nontrivial solution
37		infinitely many solutions	includes trivial solution
38	"I2: …okay…so here the solutions…if where its…whenever it says "has infinitely many solutions"…what does it mean?...do you know?... SA33: …it means that other than the trivial solution, there can be another solution…"	infinitely many solutions	other solution other than trivial
39	"…then if they are infinitely many that means it's	infinitely many solutions	singular matrix

40	singular…uhm…which means they are dependent…	singular matrix	dependent
41	"...dependent can also mean that these {referring to u, v, w…} three are a line…"	dependent vectors	scalar multiples
42	"…well we do know is that since they are dependent…this is not in R3…{student writes S does not belong to R3 }…S is at least is R2, {student writes S Є R2 ‖ S Є R1…} or S is in R1…"	three dependent vectors	vectors do not exist in R3
43		set S is dependent	vectors are in R2 or R1
44	I2: …okay, so if they are in R3, I mean in R2…they are uhm…SA33: …then at least…then all three of them can be on the same plane…I2: …uhm, uhm…SA33: …and if it is in R1, then they are all in one plane…	vectors in R2	vectors on the same plane
45		vectors in R1	vectors on a line
Q9. Given that dim(Span{u,v,w})=1. Determine the linear independence of the set {u,v,w}.			
46	"…like in the meaning of a spanning set, it could be that it requires all three to be a span, or it could be that it just takes one of them to be the span, then each of them is dependent…"	dim(Span {u, v, w})=1	linear dependence
47	I2: …uhm, uhm…so what…what does it mean, the span?...SA33: …like looking for the basis…I2: …what does the span of u, v, and w represent?...SA33: …uhm…the vectors which makes this unique…	Span	basis
48		Span	uniqueness
49	"…or remember when we were talking about…like discrete…like we are looking at graphs and trees…well it's like looking at the skeleton that you need…"	Span	skeleton needed
50	"…if it is one, then of course there exist one…and that would be… that all three of these {referring to vectors u, v, w…} are dependent…but if it is more than one dimension…then it will be R2…determine if the span…okay, so we get rid of that span…it could be that instead of having like two dimensions, there is just one…{student writes vectors [1] and [2]}…"	dim(Span {u, v, w})=1	linear dependence
51		dim(Span {u, v, w}) > 1	vectors are in R2
52	"…so now that we know that each of them…each of them has just… one row…so we are talking that the dimension is one…"	vectors in R1	dim(Span {u, v, w})=1

53	SA33: …cause if this was 2, then this would be two dimensional…and a three {student writes vector [1; 1; 1]}…a three dimensional… I2: …oh, okay…so if…if the dim of the span u, v, w were 2…we would have… SA33: …something in R2…	dimension of span	change of the space (R1, R2, R3,…)
54	"…one dimension and three vectors…by the pigeon hole, then it's dependent…"	pigeon hole	number of vectors exceeds space
55	I2: …okay…so if we had the dimension of the span of u, v, w in…it's equal to 2…would you know if it is dependent or independent?...I mean dependent or independent… SA33: …if it was two, we still have three vectors…trying to go in two dimensions…so it will still be dependent…	dimension of span	space
56		number of vectors exceeds space	linearly dependent

By looking at table 4.10, we can conclude that a high percentage of metonymies used by student A33 dealt with his reasoning of linear independence and linear dependence. One of the metonymies frequently used by student A33 was the word identity to represent a linearly independent set of vectors. The following passage (obtained from the original transcript of the interview) indicates the use of the metonymy as part of his reasoning.

SA33: … I still have my calculator… but then to do reduced row and then if it is identity, then that means each of those probably {are} independent…

I2: if what is the identity?…

SA33: …uhm…I'm kind of worry now…

I2: …so what do you do with this…but let's say this is a set of vectors {referring to vectors previously written, but they are not visible to the camera} …like this

SA33: Then you just… uhm… consider them as a matrix

I2: … uhm, uhm…so you can do it there…if you want

SA33: … yeah, I just want a matrix …{student writes a matrix $\begin{bmatrix}1 & 3 & 2\\0 & 2 & 0\\0 & 4 & 1\end{bmatrix}$ *…}, and then do the rref on this…*

I2: …uhm, uhm…

SA33: …and if it's identity or if you get {inaudible…}, then we probably know to add these two together {referring to first and second columns of the matrix previously written}… to form some sort of combination…

Student A33 also appears to believe that vectors on a plane (regardless of the space vectors are located in) automatically means that the vectors are in R^2, due to this association student A33 repeatedly uses the word plane to stand for R^2.

SA33: …oh right… Given the set {u_1, u_2, u_3, u_4} where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not. Determine if the set {u_1, u_2, u_3, u_4} is linearly independent and explain your answer. So…u1, u2, and u3 are on one plane…that means they are on R2, right?...that's…at least that's what I assume…so if…

I2: …you said…uhm…which ones are in R2?...

SA33: …the u1, u2, and u3… are elements of the plane which is in R2…{student writes u1, u2, u3 Є R2}…

I2: …okay…how do you determine that?...

SA33: …that planes are in R2?…

I2: …oh, okay…

SA33: …uhm…well… because…it would be a…more of a …. space or 3D, but… uhm… the model of those are in R3, so R2 means it's a plane…

Another tendency shown by student A33 was the interchangeable use of dimension of the span of a set of vectors with the space R in which the vectors are located.

SA33: …we don't know which ones will make up the span…they don't tell us what the answer to the span is… we don't know how many of the vectors are gonna be left in dim…

I2: …okay…

SA33: …it can be anywhere from one to three {student writes dim(1 -> 3)}…depending on the span…

I2: …uhm, uhm…

SA33: …if it is one, then of course there exist one…and that would be… that all three of these {referring to vectors u, v, w…} are dependent…but if it is more than one dimension…then it will be R2…determine if the span…okay, so we get rid of that span…it could be that instead of having like two dimensions, there is just one…{student writes vectors [1] and [2]}…

I2: …uhm, uhm…

SA33: …in this case wouldn't matter which…how many vectors we have…

I2: …uhm, uhm…

SA33: …then the dimension is still be one, but if the span has like one and one {student writes vector [1; 1]…} then it's two dimensional…so now that we know that each of them…each of them has just… one row…so we are talking that the dimension is one…

I2: …okay…

SA33: …cause if this was 2, then this would be two dimensional…and a three {student writes vector [1; 1; 1]}…a three dimensional…

I2: …oh, okay…so if…if the dim of the span u, v, w were 2…we would have…

SA33: …something in R2…

In the passage reported above, student A33 attempted to make sense of the concept of dimension of span by relying in his own metonymies and in some cases chaining them to arrive to a conclusion.

4.2.2.3 Student B6

The transcript analysis done by the author of this thesis yielded a total of 30 metonymies and 1 metaphor (see number 4) used by student B6. Table 4.11 summarizes the metonymies and metaphor used and the part of the transcript where they were found.

Table 4.11. Metonymies and Metaphors displayed by student B6 (in order of appearance).

Sample Response		***Metonymy***	***used to represent***
Q1:Define the linear independence of a set of vectors; Q2. Given an example of a linearly dependent set of vectors.			
1	…well, I know that when a vector…well, not the vector...when a matrix...I mean a matrix is to be defined as a set of vectors, right?...and then when…when the matrix is invertible, that means it is linearly independent…	invertible matrix	independent set of vectors

2	…when you have a…a matrix and it has a bunch of numbers and you can reduce it, and if at the end you have any free variable…that means it is not a unique solution because that free variable can be whatever…	free variable	not a unique solution
3	SB6: …I don't know, one…{Student writes down matrix [1, 2, 3, 4; 3, 3, 3, 3; 2, 2, 2, 2]}… actually this is going to be dependent anyway…I1: …why is that?...SB6: …because you can do this and it will give you a row of zeros…I1: …when you did this, what do you mean by that?...zeros?...are these…is that what you meant?...SB6: …yeah, like this…{referring to the second row}…because you can…oh, no…well, you can eliminate…like when you solve it…I1: …can you think out loud?...SB6: …when you solve it, this is going to give you…{student writes down matrix [1, 2, 3, 4; 1, 1, 1, 1; 2, 2, 2, 2]…}…and then you can eliminate this one {referring to third row}…with this one {referring to second row}…I1: …I see…SB6: …and then this one is going to be zeros…{student writes matrix [1, 2, 3, 4; 1, 1, 1, 1; 0, 0, 0, 0]}…and this is going to be a bunch of ones…there is more than…at least this is going to be…at least these two {referring to entries 3 and 4}… are going to be free variables because there is not {inaudible} to get all the…the…how do you call them?...	row of zeros	linear independent set
4	…that the matrix is indepen…I mean linearly dependent and it has…it doesn't have a unique solution and it's not invertible…	(metaphor) matrix	set of vectors
5		not a unique solution	linear dependence
6	I1: …you don't know?...okay…but when you said it doesn't have a unique solution…that's why it is…it is linearly…SB6: …dependent…I1: …dependent?...okay. If it has a unique solution, what do you say?...SB6: …that it is independent…	not a unique solution	linear dependence
7		unique solution	linear independence

8	SB6: …like if you write the set, it is going to be something like…{student writes down matrix {1, 0; 0, 1; 0, 0; 0, 0]}…I1: …okay, I see…SB6: …this is…it's x…equals to zero?...{student writes x=0, 0=0, and 0=0, next to the first, third, and fourth rows}…I1: …okay…SB6: …and then 0 equals zero…something like that…I1: …how about his one?...{referring to the entry in the second row, second column}…SB6: …oh, yeah…{student writes x2 = 0}…	unique solution (trivial solution)	linearly independent set
9	…and zero, one, two, three…if there is a connection from zero…from one to…to like two, then you can put a one here…and if there is a connection from two to three, then there is a one here…{student writes a number one in the cells that corresponds to row 1 and column 2 and row 2 and column 3}…	connections	entries of 1
10	SB6: …and since this…if there is a directed graph, then there is no connection from four to one…I1: …oh, I see…so you put in…SB6: …zero…{student writes a number zero on the cell corresponding to row 4 and column 1 on the table}…	not a connection	entries of zero
Q3. Given the set {u_1, u_2, u_3, u_4} where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not. Determine if the set {u_1, u_2, u_3, u_4} is linearly independent. Explain your answer.			
11	…yeah, but I don't know what it means…plane…like I don't know if it is talking about like if they are in the same…dimension?...or I don't know what it is by plane…	plane	dimension
Q4. Given a linearly independent set {u_1, u_2, u_3, u_4} in R^n. Determine the linear independence of the set {u_1, u_2+5u_1, u_3, u_4}.			
12	SB6: …yeah, then you can add them and then solve…and know if they are linearly independent or not…I1: …oh, okay…so you are saying you would put them in the matrix and solve it, meaning what?...SB6: …like if it can be reduced…	solve the matrix	perform row operations to reduce the matrix
13	…to…if it can be reduced to the identity then it's linearly independent, and if not then its dependent…	identity matrix	linearly independent set
14		not identity matrix after reducing the matrix	linearly dependent set

15	…yeah, I said something like this…this were…what I thought about it is, this is a set of vectors and these two…I mean this vector was a combination of another two vectors…then I think since it was a combination I said it was dependent…	linear combination among vectors of a set	linearly dependent set
	Q7. Given that the vector equation xu+yv+zw=0 has infinitely many solutions. Determine if the vectors u, v, w are on the same plane. Explain your answer.		
16	…and since it has infinitely many solutions…then…well usually the infinitely many solution means that it is dependent…	infinitely many solutions	linear dependence
17	…and since it has a lot of solutions, that means the matrix couldn't go to the…the identity…	lots of solutions	infinitely many solutions
18		infinitely many solutions	does not reduce to identity
	Q8. Given the vector equation $a_1u_1+a_2u_2+a_3u_3=0$ with the solution $a_1=1$, $a_2=-2$, and $a_3=0$. determine the linear independence of the set $\{u_1, u_2, u_3\}$.		
19	SB6: …okay, so…since these are vectors also…right?...there is solution one, two, and zero, but that doesn't mean that it is independent…but it has a unique solution…I1: …okay…SB6: …but there is only one…I1: …what are you thinking?...SB6: …like this…yeah, I think it has a unique…since it has a unique solution I guess it is…independent…	unique solution (even if it is not trivial)	linear independent set
	Q9. Given that dim(Span{u,v,w})=1. Determine the linear independence of the set {u,v,w}.		
20	…uhmmm…okay…so, span means like…like it belongs to that graph… like the other one doesn't…are not really needed?...	span	something that belongs to the graph (in the hamiltonian cycle diagram)
21	I1: …okay, so…so if it says dimension of that is one, what does that say to you?...what would you think about it?...when you say, when we say dimension of the space is one…SB6: …I don't know…that…I don't know, the first thing that comes to my mind is that there only is one vector and is not needed…	dimension of the span is one	one vector not needed
22	I1: …yeah, okay…okay… so if that is the case, what would	one vector not needed to create the span	one vector was a combination of the other two

23	you say about the set?...SB6: …uhmmm…I1: …think out loud…SB6: …I guess…dependent…I1: …oh, okay…because?...SB6: …because if there is one…vector that is not needed, then one of the vectors was like a combination of the other ones or it was the same vector or something like that…	linear combination among vectors of a set	linearly dependent set
24	…like…well, not the…not the combination, but like one of the vectors was like two times v…{student writes 2v}…like if it was like one like this…and then the other one was something like this or…{student writes [1; 1; 1] and [2; 2; 2]}…	2v (vector v times 2)	scalar multiple of a vector (not a combination)
25	…well, I don't know how to…like…like if this is…because what I am saying here is like in a matrix…one of the columns is equal to these two…so…but here it's saying that…a matrix, which is this one, this…this doesn't mean that this is equal to…another column with in the same matrix…it just means it is a combination of this set…of one of this set of vectors…	one column is a combination of other two	linear combination among the vectors of the set
26	SB6: …okay…so since…one of the sets is a combination of the other one…I am guessing if you change one of…any number of this set, it depends on this one…I1: …uhm…SB6: …it is not the same dependency, but I would say that this is…is dependent…	one of the sets	one of the vectors
27		linear combination among vectors of a set	linearly dependent set
28	SB6: …because there is no numbers I can solve for…I1: …okay…are you talking about reducing the matrix?...SB6: …yeah, reducing the matrix…	solve the matrix	reducing the matrix to ref
29	I1: ..oh, okay…I see…okay…how about this set?...{interviewer writes set {[1; 2], [2; 3], [1; 0]}…}…can we identify whether it's linear independent or not?...SB6: …uhmmm…dependent…I1: …because?...SB6: …because if you build a matrix with that, they won't be able to get to the identity…	linearly dependent set	not being able to reduce down to identity

30	I1: …oh, I see…I see, but now you are sure this is linearly dependent, because?...(50:02.5)SB6: …because there is more…like it is in two dimension and there is two…two…I1: …vectors?...SB6: …yeah, like there is two dimension and three vectors…	three vectors in R2	linearly dependent set
31	…no, not at the first…but after trying to solve it…I couldn't go to…identity, and then therefore it {inaudible} dependent…	matrix did not reduce to identity	linearly dependent set

By looking at table 4.11, we can conclude that a high percentage of metonymies used by student B6 dealt with her reasoning of linear independence and linear dependence. One of the metonymies frequently used by student B6 was the existence of a unique solution to stand for linear independence. -The following passage (obtained from the original transcript of the interview) indicates the use of the metonymy -unique solution- as part of her reasoning and shows the only metaphor used by student B6 during the interview in which an independent set of vectors was mentioned as a linearly independent matrix.

I1: ...you don't know?...okay, so you had this matrix and you did all this process…you came up with this one…is this the end though?...what we are supposed to be doing or…

SB6: ...I think so…

I1: ...you think so?...okay, having this is telling us what?...

SB6: ...that the matrix is indepen…I mean linearly dependent and it has…it doesn't have a unique solution and it's not invertible…

I1: ...okay…what do you mean by its…its…it's the matrix unique solution…what do you mean by unique solution?...

SB6: ...uhmmm…

I1: ...like you said, it doesn't have a unique solution, right?...and I am assuming you were referring to the matrix…you said the matrix doesn't have a unique solution…uhmmm...that means that you have that unique solution structure in your knowledge, so I am trying to understand what you mean by that…

SB6: ...uhmmm…actually I don't know…

I1: …you don't know?...okay…but when you said it doesn't have a unique solution…that's why it is…it is linearly…

SB6: ...dependent…

I1: ...dependent?...okay. If it has a unique solution, what do you say?...

SB6: ...that it is independent…

Student B6 also showed signs of being able to create a chain of signifiers (invertible matrix→ independent set of vectors → unique solution→ no free variable → trivial solution), in an Analytic-Structural thinking mode, to connect ideas at the beginning of the interview, this was the first time the -unique solution- metonymy was used.

SB6: ...well, I know that when a vector…well, not the vector...when a matrix...I mean a matrix is to be defined as a set of vectors, right?...and then when…when the matrix is invertible, that means it is linearly independent…

I1: ...so what do you mean by it?...you said that means it is linearly independent…what are you referring to?...

SB6: ...uhmmm…it doesn't have…I get confused with the words, but…

I1: ...that is okay, whatever you are thinking…remember…

SB6: ...I know this has a unique solution…and it doesn't have a free variable…

I1: ...uhm, uhm…okay…

SB6: ...and that...I don't remember if it is trivial or nontrivial…

The passage included above shows signs of student B6 not being sure about the definition of trivial solution, this problem persists during the interview and manifests again when asked to answer question number 8.

SB6: ...okay, so…since these are vectors also…right?...there is solution one, two, and zero, but that doesn't mean that it is independent…but it has a unique solution…

I1: ...okay…

SB6: ...but there is only one…

I1: ...what are you thinking?...

SB6: …like this…yeah, I think it has a unique…since it has a unique solution I guess it is…independent…

I1: …oh, okay…so since this one has a unique solution, you are saying this is linearly independent…

SB6: …uhm, uhm…

I1: …okay…uhmmm…so earlier I asked you what you meant by unique solution…and this is what you meant then…

SB6: …yeah, there is only one solution…

I1: …oh, okay…what if we say the solution…with a solution a1 is zero, a2 is zero and a3 is zero...would that be a unique solution ?...so if you replace the one and two with zeros…

SB6: …with what?...zeros?...

I1: …yeah…

SB6: …I guess…

I1: …yeah?...okay, so you are saying that would be a unique solution…as long as you are saying that our values for a1, a2, a3 then this is a unique solution that implies…the vectors to be linearly…

SB6: …independent.

The passage reported above clearly shows how the use of the metonymy unique solution led to a misunderstanding and made student B6 provide an incorrect response for question 8.

4.2.2.4 Student C7

The transcript analysis done by the author of this thesis yielded a total of 49 metonymies and 6 metaphors (see numbers 7, 21, 27, 35, 38, 40) used by student C7. Table 4.12 summarizes the metonymies and metaphor used and the part of the transcript where they were found.

Table 4.12. Metonymies and Metaphors displayed by student C7 (in order of appearance).

	Sample Response	***Metonymy***	***used to represent***
Q1:Define the linear independence of a set of vectors; Q2. Given an example of a linearly dependent set of vectors.			
1	...and then all the way to vn…and…uhm…that is…pretty much covers all sets of vectors…	n number of vectors	represent all sets of vectors
2	...when you put these vectors into let's say a matrice…v1…and when I do that I mean that that's is vector one…the column is vector one…	vectors	columns of matrix
3	SC7: …and once reducing it…if…let's say we get the identity matrix…which is the one, zero, all the zeros…zero, one, zero, all the way down, etcetera…until you get over here, for where the bottom is one…I1: …okay…SC7: …if you get that then it is linearly independent…	identity matrix	linear independence
4	SC7: …however if you get some matrice where you have what is known as free variables…I1: …uhm, uhm…SC7: …so if you have like a bunch of zeroes here…one here…but over here you had something like a two or a…and then a three…a bunch of zeroes…and then once again you come down here…then it is not linearly independent…	free variables	linear dependence
5	…it is linearly dependent because of this…because this vector right here can be expressed in terms of the other vectors…it depends…	linear dependence	vector expressed in terms of the others
6	…so by free variable I mean that it…this row…this column right here doesn't only have a leading one, it also has numbers above it or below it…	free variables	column not on row reduced echelon form
7	…the free variable can be expressed in terms of the other variables…	free variables	expressed in terms of the others
8	…and only if you get the identity matrix it would be linearly independent…	identity matrix	linear dependence

9	SC7: …I believe it is, as long as x1 is equal to x2 is equal to x3 etcetera …is equal to xn… as long as that is the only…the only way for this system to be equal to zero…I1: …oh, okay…SC7: …for the only way for this to be equal to zero vector is this…the trivial solution…	trivial solution	system equals the zero vector
10	…yes…so the only way they can equal to zero is if x1 plus x2 equals zero…all of them equal zero…then it will be linearly independent…	trivial solution	linear independence
11	SC7: …okay, so you are saying that…like if I had a full zero column right here…I1: …uhm, uhm…SC7: …then this whatever the…the variable is right here wouldn't be present in any of these…	column of zeros	variable not present
12	…and the reason that I know automatically that they are linearly independent is the fact that when you just write them into that matrix, it's already the identity matrix…	linear independence	identity matrix
13	… this can be a scalar multiple because it obviously depends on the other one already…and it becomes a row of zeroes…	scalar multiple	dependence
14		linear dependence	row of zeros
15	…uhm…well…because of the fact that x6 is zero, it kind of I guess disappears in the…in a way that, these two, right here…show that it is linearly dependent…	x6 equaling zero	variable disappears
16		variable disappears	linear dependence
17	I1: …great!...what if I put your vectors?...instead of having [1; 2; 0], I have [1; 2; 0; 0], [2; 4; 0; 0], [0; 0; 1; 0]…SC7: …same thing…I1: …the same thing?...so…SC7: …because of the fact that it is just a zero row at the bottom…in the end…so if you add a zero at the bottom, this doesn't affect it…it is just a bunch of zeroes, pretty much…	row of zeros at the bottom	linear dependence
18	…yeah, those are just the augmented…matrix…with the zero at the end, because of the zero vector that is equaling to…that the set of equations is equaling to…	augmented matrix	set of equations equaling the zero vector
19	…well, it's obviously already linearly dependent because of	row not containing the leading one	matrix not equaling the identity matrix

20	the fact…this right here…this row does not contain the leading one…	matrix not equaling the identity matrix	linear dependence
Q3. Given the set {u_1, u_2, u_3, u_4} where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not. Determine if the set {u_1, u_2, u_3, u_4} is linearly independent. Explain your answer.			
21	…given the set S…u1, u2, u3, and u4…where the vectors u1, u2, u3 are on the same plane…so they are coplanar…	coplanar vectors	vectors on the same plane
22	…just from knowing that these three are coplanar…then they are automatically linearly dependent…because of the fact that if they are coplanar, then you can express any of them in terms of the others…	three coplanar vectors	linear dependence
23		coplanar vectors	expressed in terms of the others
Q4. Given a linearly independent set {u_1, u_2, u_3, u_4} in R^n. Determine the linear independence of the set {u_1, $u_2 + 5u_1$, u_3, u_4}.			
24	SC7: …when I answered I was thinking because of the fact that this is a linear combination of two of the other ones, that it depends already…it depends on those vectors…I1: …oh, okay…SC7: …so saying that…that's probably why would be linearly dependent…	linear combination	linear dependence
25	…because if you have four vectors, and you only have three in R…you have R3…then it is automatically dependent…	four vectors in R3	linear dependence
26	…you will get a free variable; they will be linearly dependent…because you cannot have more vectors than there is space…	free variable	linear dependence
27		space	dimension
28	…I guess unless the last row is a bunch of zeros, that will be the only way that this will be linearly independent…if the last row was a row of zeros…	row of zeros	linear independence
29	…and it reduces into…the identity matrice…matrix…meaning that…that set is also linearly independent…	identity matrix	linear independence
30	…it doesn't prove it either way you have to show a vector of all…so u1 would be, like I said earlier…u1a, u1b, …all the way down to u1 whatever it is…n…where u n is whatever number you choose to be…	u1a, u1b, … all the way down to u1n (a b c order)	u11, u12, u13, …u1n (n as a number)
31	…oh, and that is equal to the zero vector…that is the only way that this could be linearly independent is…if x1, x2, x3, and x4 is equal to zero…	trivial solution (x1, x2, x3, and x4 equal to zero)	linear independence

32	…I am just going to see if I do like this…u2x2 plus 5u1x2 plus u3x3 plus u4x4 is equal to zero…but the only thing that I see here is the fact that this is still a…a scalar of that…{ }…but you are adding something to it…like another vector to it…	scalar multiple	dependent set of vectors
Q5. Given an nxm matrix a where $a_{i2}=a_{i4}+3a_{i5}$. Determine if the set {A_1, A_2, A_3,…, A_m} (Here A_j is the jth column of A) is linearly independent. Explain your answer.			
Q6. Given a singular 3x3 matrix A, determine if the vectors of the set {A_1, A_2, A_3}, where A_j is the jth column of A, are on the same plane. Explain your answer.			
33	…well, right away you can't say if they can or can't…I guess, they way to say that is they are or they aren't …they only way they could…if this set right here is linearly independent, then they automatically do not lie on the same plane…	linearly independent set	vectors not on the same plane
34	SC7: …they could be linearly independent…if they are scalars of each other , but then it would just be something like that…I1: …oh, you mean they could be linearly dependent...SC7: …yeah, sorry…dependent, sorry…	scalar multiple	linearly dependence
35	…unless you change the vector to be that one…that is the only way that you can create that one, so because they don't depend on each other to be created by themselves…to be created…then they would be linearly independent…	vector created by itself	not a linear combination or scalar multiple
36		not a linear combination or scalar multiple	linerly independent set
37	…they are not…because coplanar, three coplanar vectors cannot be linearly independent because of the fact that…like I said earlier, if we had another one like that you could use these two to create this one…by some scalar multiple plus addition I guess…	three coplanar vectors	linearly dependence
38	…yeah, that one of these can be expressed in terms of the others meaning that…excuse me…it is on the same plane, I guess…as the others…	one vector expressed in terms of the others	linear combination
39		one vector is the a linear combination of the other two	vectors on the same plane
40	SC7: …the equation of the vectors?...well, there is a…or A1…that is what it is…A1 is equal to, whatever point, some point…which is the vector…or that is a line, sorry…uhm…I1: …what is the line?...SC7: …oh, I was just writing out the equation of a line…it is just a…	vector	a line
41		vector equation	equation of a line

42	…well, I mean…same plane…the equation of a plane would be A times… {student writes a(x-0)+b(y-y0)+c(z-z0)=0}…	equation of a plane	vector equation
43	SC7: …I remember that…but…sorry I didn't even see that…uh…okay, so a singular matrix is not invertible…which means that…when you augment it, you can't put it into the identity matrix…I1: …okay…SC7: …so that probably means that they are dependent…I1: …oh, because the matrix cannot be…SC7: …put into the identity matrix…	singular matrix	not invertible matrix
44		not invertible matrix	can't get identity matrix after reducing the augmented matrix
45		no identity matrix	linear dependence
46	…dependent…so there exists some sort of… scalar multiple, meaning that there are on the same plane…	linear dependence among vectors	one vector is a scalar multiple of other
47		scalar multiple	vectors on the same plane
48	…a line right there, that is just showing that is not actually on the plane…is just the projection of it, I guess…	line outside the plane	projection of a vector
49	…yeah, I am trying to figure out if…my mind is getting confused here…hold on…and that is a scalar multiple of that one…so that is why that one is on…	scalar multiple	vectors on the same plane
50	…so…if it is linearly dependent, then it can be expressed in terms of the others…meaning that it can't be on a different plane, it has to be on the same plane…	linear dependent vectors	linear combination among the vectors of the set
51		linear combination among vectors of the set	vectors on the same plane
Q7. Given that the vector equation xu+yv+zw=0 has infinitely many solutions. Determine if the vectors u, v, w are on the same plane. Explain your answer.			
52	…so…infinitely many solutions, meaning that is not only zero…x doesn't always…meaning that you don't only get the trivial solution…	infinitely many solutions	not only the trivial solution
53	…and these are on the me plane so… because of the fact that you have infinitely many solutions…means that they are linearly dependent…	infinitely many solutions	linear dependence among vectors
Q8. Given the vector equation $a_1u_1+a_2u_2+a_3u_3=0$ with the solution $a_1=1$, $a_2=-2$, and $a_3=0$. determine the linear independence of the set $\{u_1, u_2, u_3\}$.			
54	…so, so a1…oh, I can tell you right now because of the fact that the solution for it… is not the trivial solution, then they are linearly dependent…	not the trivial solution	linearly dependent

55	I1: …like, why do we say they are linearly dependent?...SC7: …because of the fact that you can create one of the other ones by a combination of the other two…	linear dependence	linear combination among vectors

By looking at table 4.12, we can conclude that a high percentage of metonymies used by student C7 dealt with her reasoning of linear independence and linear dependence. Two of the metonymies frequently used by student C7 were –identity and free variables- to imply linear independence or linear dependence. The following passage (obtained from the original transcript of the interview) indicates the interchangeably use of the metonymies –identity and free variables- as part of her reasoning.

SC7: …and once reducing it…if…let's say we get the identity matrix…which is the one, zero, all the zeros…zero, one, zero, all the way down, etcetera…until you get over here, for where the bottom is one…

I1: …okay…

SC7: …if you get that then it is linearly independent…

I1: …what is linearly independent?...

SC7: …linearly independent means that…I have to remember…

I1: …no, what I mean is yeah…go ahead you can explain it…when you say it…you say it is…what do you mean by "it"…that pronoun…

SC7: …the set of vectors…

I1: …okay…

SC7: …okay, sorry…the set of vectors is linearly independent…

I1: …oh, okay…

SC7: …okay, so if you get this {referring to the identity matrix} then they are linearly independent…however if you get some matrice where you have what is known as free variables…

I1: …uhm, uhm…

SC7: ...so if you have like a bunch of zeroes here…one here…but over here you had something like a two or a…and then a three…a bunch of zeroes…and then once again you come down here…then it is not linearly independent…

We can see the repeated use of the metonymies –identity and free variables- again in the following passage, and it also shows how student C7 uses them as opposites.

I1: ...okay, so just the fact that this can be expressed in terms of the others…so when you say this, are you talking about the free variable here?...

SC7: ...yes…

I1: ...okay, free variable can be expressed in terms of the others…okay, say it again…

SC7: ...the free variable can be expressed in terms of the other variables…

I1: ...okay, and here you use free variable as t…

SC7: ...t, which is just a parameter…so you can set t to be…it's an arbitrary number as long as it is a real number…

I1: ...oh, okay…

SC7: ...so for this…it would be…as long as t belong to all real numbers…

I1: ...oh, okay great...so which ones are you expressing here?...

SC7: ...oh…what do you mean by that?...

I1: ...a like you say this is referring to this parameter, right?...

SC7: ...right…

I1: ...and this can be expressed in terms of the others you say…

SC7: ...right…so…if I remember correctly…I have to think about this, hold on a second…

I1: ...oh, I guess what I am saying is…are you expressing are you expressing this in terms of this?...

SC7: ...well this can also be expressed in terms of this…

I1: ...oh, okay…okay…

SC7: ...so that what is being expressed right here in this right here…

I1: …oh, I got you…okay, so…so you are saying because of that fac…fact…

SC7: …uhm, uhm…

I1: …that implies the set being linearly dependent?...

SC7: …right…

I1: …okay…

SC7: …and only if you get the identity matrix it would be linearly independent…

The results obtained from the analysis of the interview transcripts and reported in this chapter will further be discussed in the following chapter –Discussion and Conclusions- by comparing each students' thinking modes and their respective use of metonymies and metaphors

Chapter 5: Discussion and Conclusion

After analyzing the interview transcripts, the data obtained indicates there are different thinking modes present in every student's reasoning. These thinking modes are important because they help students construct their understanding of concepts introduced in their first linear algebra course at the university level. Students have a tendency to think in algebraic and arithmetic modes while prompted with linear independence related questions because of the easiness of their representations and computational methods to arrive at conclusions. The data obtained from the analysis of the transcripts shows clear signs that students were able to create their own arguments by moving from one thinking mode to the other –including the Synthetic-Geometric thinking mode- to relate numerical, algebraic, and graphical representations seen in the matrix algebra course.

This chapter includes possible explanations for each student inclination to use a particular thinking mode and the use of metonymies and metaphors as part of their understanding. We will discuss some of the factors that might have led to the student's preference for a specific mode and the repeated use of certain metonymy or metaphor. We will also discuss some of the similarities and differences among the data belonging to each student as well as the factors affecting the results, the research limitations, and the future implications.

5.1 Discussion

In this section a comparison of the use of thinking modes by Sierpinska (2000) and the metaphors and metonymies defined by Presmeg (1998) for students A18, A33, B6, and C7 belonging to the three sections –modular or non-modular- of the matrix algebra course will be discussed.

5.1.1 Student A18

Going back to the data reported in chapter 4 on the classification of the categorized arguments representing the thinking modes used by students during their interview, it can be said

that student A18 utilized 8 different types of categories that can be classified into the three thinking modes presented by Sierpinska (2000). There was only one category belonging to the Synthetic-Geometric mode of thinking. There were 6 categories classified into the Analytic-Arithmetic mode, and finally there were 3 categories classified into the Analytic-Structural mode, while 2 categories fell into multiple modes of thinking.

Going back to the information provided in table 4.2, we can conclude that the Synthetic-Geometric mode had a frequency of 1 (used 5% of the time), the Analytic-Arithmetic mode had a frequency of 17 (used 85% of the time), and finally the Analytic-Structural mode had a frequency of 8 (used 40% of the time). Taking this percentages into consideration, it is safe to say that student A18 appears to use dominantly the Analytic-Arithmetic mode in his reasoning and seems to be able to go from that mode to Analytic-Structural and vice versa quite often. The following passage obtained from the original transcript exemplifies the use of both analytic modes interchangeably in order to try to make sense of his argument:

I2: ...oh, okay…uhm…what if you have...uhm… some other set of vectors… let's say in…let me see an example… {interviewer flips through pages looking for an example and writes something that is not visible to the camera}…

SA18: {student writes down matrix [1, 2, 6; 4, 3, 10; 5, 6, 16; 2, 1, 6]}…a… I think… in this case… this is a high probability that it is independent because…there is…this is R4 and we have three vectors, so there is a good chance that it is independent… but…just to be sure….I'll do Gauss Jordan… do you wanna do this one?...{referring to student A}..

SA18: ...yeah…

I2: how do you know if it is in R4?...what do you look at in the set of vectors or…in the vectors…to know that it is in R4?... you say this is in R4, right?...

SA18: I just….just…the number of rows…

I2: …oh, okay…

SA18: …and…let's see…{ student waits for other student to be done with the calculator}…well again the calculator is…{ student writes matrix [1, 0, 0; 0, 1, 0; 0, 0, 1; 0, 0, 0]}…which means it found…uhm…the calculator found that the R3 exist in this… which is pretty true because we always have that…but it couldn't explain this last one{referring to the third column of the matrix}…so…uhm…this set will be linearly independent because… no matter what the outc…this

set will be dependent… I should said…uhm… because this… uhm…last system…this last equation doesn't really matter…so… I can have, have a random…let's see…ah, I am trying to think here…

I2: …what are you thinking?...

SA18: …uhm…normally I just… if I had a… thinking of this system of equations… I can say a, b, and c...{student labels columns a, b, c}…

I2: …uhm, uhm…which…what are those a, b, and c?...are scalars or the vectors, or…

SA18: …ah…these will be scalars…

I2…okay…

SA18: …and with {} thinking of x, y, and z….so this first one we will have {} an a…plus zero, plus zero…and a b plus zero, plus zero….and a c plus zero, plus zero…and the last one is 0a, plus 0b, plus 0c student writes equations a+0+0, 0+b+0, 0+0+c, and 0a+0b+0c}…okay…and this means I can put in any values for a, b, and c and still get the…. I still get the same exact results from here…because…. it makes them irrelevant in all of those… you know what I mean?...

Student A18 starts by implying that the set may be independent because of the fact that there are only 3 vectors in R^4 (use of Analytic-Structural mode), then decides to verify by reducing the matrix via the Gauss-Jordan elimination method (use of Analytic-Arithmetic mode), then he expresses that R^3 exists in the matrix –implying the identity is present-(use of Analytic-Structural mode), and finally decides to verify by converting the entries of the matrix representing the set of vector in reduced form into a system of equations (use of Analytic-Structural mode).

Looking at the data obtained with respect to the use of metonymies and metaphors, we can conclude that a high percentage of the metonymies used were directly related to linear dependence and independence arguments. Out of the 47 metonymies found, 22 of those represented linear independence. It is obvious that the metonymy preferred and extensively used by student A18 was –linear combination associated to linear dependence (linear combination → linear independence). This metonymy is closely related to the Analytic-Arithmetic thinking mode, among the categories classified into this mode were LCST (Linear combination stated),

LCALG (Linear combination algebraically), and LCNUM (Linear combination numerically). The following passage obtained from the original transcript demonstrates how the extensive use of the metonymy (linear combination → linear dependence) became a problem while trying to answer a less obvious question.

I2: ...so what are you thinking?...what do you have?...what would you do?...or how do you determine if ...

SA18: ...ah…if there is…if there is a linear combination… the two vectors to make a third… to make a third one… in this set or a scalar multiple of one makes another vector in the set, then the set is linearly dependent…but…in this case we are just subtracting one of the vectors from another…we are not necessarily…it doesn't actually necessarily say that we are making any of the other vectors in the set…you know what I mean? it's like…uhm…u2 plus 5u1 doesn't necessarily equal a combination of u1 plus u3 plus u4…

I2: ...okay…

SA18: ...uhm…and actually I don't think it would either because…u2 plus 5u1 exist in the…in the set… in the original set…but it's independent, so this doesn't make any of the other vectors… but that's given..so I'm gonna say this set it's still linearly independent…

I2: ...okay…uhm…could you do anything…like uhm… algebraically to determine if it is dependent or independent?...or are you 100% sure that it is independent?...

SA18: ...I don't think that… that I could really use Gauss Jordan here… uhm…I would imagine seeing this graphically, but I am not sure how I can understand any dimension…maybe three dimensions…yeah, I can understand, but n dimension?...maybe not… uhm…I am really just using…just trying to use some logic here…to say this…

I2: ...okay…what if I tell you that it is in R4… would it help?...

SA18: ...uhmmm…

I2: ...the…the original set is in R4…would you be able to do something?...like uhm…algebraically?...to really show that it is independent…because you are saying that it is independent, right?...

SA18: ...uhm, uhm… they are independent…even adding one of these vectors to the other…uhm…if they were in R4…I can say that this set is equal to the standard basis of R4…{student writes the 4x4 identity matrix} but…it kind of helps me out a little…visually…and it's saying that… the problem here is this other one…is this strange back here, so… if I look at u2 and then add u5…

5.1.2 Student A33

Student A33 utilized 10 different types of categories that can be classified into the three thinking modes reported by Sierpinska (2000). There were three categories belonging to the Synthetic-Geometric mode of thinking. There were 6 categories classified into the Analytic-Arithmetic mode, and finally there were 4 categories classified into the Analytic-Structural mode, while 3 categories fell into multiple modes of thinking.

Going back to the information provided in table 4.4, we can conclude that the Synthetic-Geometric mode had a frequency of 9 (used 47.37% of the time), the Analytic-Arithmetic mode had a frequency of 9 (used 47.37% of the time), and finally the Analytic-Structural mode had a frequency of 11 (used 57.89% of the time). The use of all thinking modes was evident by student A33. He was the only student whose interview (out of the four students' interviews analyzed by the author of this thesis) demonstrated a pretty close frequency for all three thinking modes. Even though the frequency of the Analytic-structural mode is slightly higher than the others, we cannot make any conclusions about his preferred thinking mode. Student A33 clearly shows signs of being able to move from one thinking mode to another and in some occasions, of being able to combine them all at once. The following passage obtained from the original transcript illustrates the use of the Synthetic-Geometric mode and the Analytic-Structural mode interchangeably in order to try to make sense of his argument:

SA33: …oh right… Given the set {u_1, u_2, u_3, u_4} where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not. Determine if the set {u_1, u_2, u_3, u_4} is linearly independent and explain your answer. So…u1, u2, and u3 are on one plane…that means they are on R2, right?...that's…at least that's what I assume…so if…

I2: ...you said…uhm…which ones are in R2?...

SA33: …the u1, u2, and u3… are elements of the plane which is in R2…{student writes u1, u2, u3 Є R2}…

I2: ...okay…how do you determine that?...

SA33: …that planes are in R2?…

I2: …oh, okay…

SA33: …uhm…well… because…it would be a…more of a …. space or 3D, but… uhm… the model of those are in R3, so R2 means it's a plane…

I2: …oh, okay…

SA33: …and that would be a line… uhm… so … three vectors in … the dimension is…where the dimension is two means it's indepe…a…dependent…already that these three {referring to u1, u2, u3} are somehow dependent…

I2: …how do you know that?...

SA33: …three vectors in…three vectors in two…uhm…I guess…what was that there?...it was in discrete …pigeon hole…

I2: …okay…

SA33: …that… where…one of these will fit as… at…or two of these will be independent and the third one, will just be the third… ah… that will fit inside of one of those holes…

I2: …okay…

SA33: …where the two are already occupied, so fitting three in two means…plane and dependent…

This shows how student A33 arrived to the conclusion that the set of vectors was linearly independent by connecting the notion of having 3 vectors in a plane –which he assumed were in R2- to the fact that the number of vectors in the set was four, and therefore the number of vectors in the set exceeded the dimension of the space.

Looking at the data obtained with respect to the use of metonymies and metaphors, we can conclude that a high percentage of the metonymies used were directly related to linear dependence and independence arguments. Out of the 56 metonymies found, 19 of those represented linear independence or linear dependence of a set of vectors. Student A33 had more metonymies frequently used than student A18, which is not a surprise due to the fact that student A33 was able to use all three modes around the same amount of times. Among his more frequently used metonymies are: planes associated with linear combination, dimension to imply linear independence, infinitely many solutions to stand for a singular matrix, and the identity matrix to stand for a linearly independent set of vectors. The following passage obtained from

the original transcript demonstrates how his ability to move from one thinking mode to another was helpful in connecting ideas, but his tendency to use the word plane to stand for R^2 and the number of vectors (dependent or not) to determine the dimension created some confusion before providing an answer.

SA33: …then if they are infinitely many that means it's singular…uhm…which means they are dependent…

I2: …uhm, uhm…

SA33: …but…dependent can also mean that these {referring to u, v, w…} three are a line…

I2: …uhm ,uhm…

SA33: …in R2…and that…well we do know is that since they are dependent…this is not in R3…{student writes S does not belong to R3 }…S is at least is R2, {student writes S Є R2 || S Є R1…} or S is in R1…

I2: …okay…

SA33: …and, I guess kind of like the previous question…if we can't say that…at least two of the vectors are on the same plane…cause all three can be on the same line…then we don't know…yeah…

I2: …so you were saying that because this…this set is…

SA33: …is not in R3….cause it's uhm…dependent…

I2: …so it has to be…

SA33: …in either R2 or R1…

I2: …okay, so if they are in R3, I mean in R2…they are uhm…

SA33: …then at least…then all three of them can be on the same plane…

I2: …uhm, uhm…

SA33: …and if it is in R1, then they are all in one plane…

Students A18 and A33 are twin brothers who were registered in the same modular section of the matrix algebra course and were interviewed together, and even though they were taught in the same way and completed the same assignments their ability to move from one thinking mode to another was different. Student A33 demonstrated the use of the Synthetic-Geometry, Analytic-Arithmetic, and the Analytic-Structural thinking modes evenly, while student A18 used the Analytic-Arithmetic mode at a higher percentage. It is my opinion that the excessive use of the metonymy –linear combination → linear dependence- by student A18 and his desire to find a linear combination numerically and/or algebraically impeded him from using the Synthetic-Geometric thinking mode more often. Student A18 also expressed his preference to think in the analytic modes to prove arguments and no preference for visuals.

I2: ...okay…uhm…do you think… uhm…like a…uhm…can you imagine how it would be graphically?, or…or you don't really see it graphically, at all?...when you do this…when you do…like for example if it is…if we are in R3, for example…and you… they tell that…uhm… they are three vectors in R3 and that they are independent…do you imagine them?...in…on the plane, or a line, or…

SA18: ...uhm…I don't really look at them…

I2: ...no?...

SA18: ...graphically…a…

I2: ...you don't imagine them at all?...so you just do this and see…oh…if it is u1…combinations...and stuff like that?...

SA18: ...uhm…just…uhm…when I…when I first started…uhm…I did use to look at things graphically…uhm…now I kind of look…just looking at the algebra for the most part…but th..those modules we had…

I2: ...uhm, uhm…

SA18: ...uhm…they were kind of nice…to usually get the concept…

I2: ...uhm, uhm…

SA18: ...and then once I had…I started looking at… okay…how I apply this concept easy?...prove it using math…not just proving using visuals…

5.1.3 Student B6

Going back to the data reported in chapter 4 on the classification of the categorized arguments representing the thinking modes used by students during their interview, it can be said that student B6 used 9 different types of categories that can be classified into the three thinking modes presented by Sierpinska (2000). There was only one category belonging to the Synthetic-Geometric mode of thinking. There were 6 categories classified into the Analytic-Arithmetic mode, and finally there were 6 categories classified into the Analytic-Structural mode, while 4 categories fell into multiple modes of thinking.

Going back to the information provided in table 4.6, we can conclude that the Synthetic-Geometric mode had a frequency of 2 (used 9% of the time), the Analytic-Arithmetic mode had a frequency of 19 (used 86.36% of the time), and finally the Analytic-Structural mode had a frequency of 16 (used 72.7% of the time). Taking this percentages into consideration and the amount of categories classified into both analytic modes, it is my opinion that student B6 has the ability to move from the Analytic-Arithmetic mode to the Analytic-Structural mode with no problem. -The category MINSPT (Minimal spanning tree) was the only category classified into the Synthetic-Geometric and the Analytic-Arithmetic; moreover MINSPT was the only category belonging to Synthetic-Geometric. -The arguments provided by student B6 showed some superficial knowledge of the minimal spanning tree and Hamiltonian cycles, so it is my opinion that the student was not able to fully understand the geometrical aspects of sets of vectors and/or matrix algebra concepts. -The following passage obtained from the original transcript reflects some of the obstacles that kept student B6 from answering one of the questions during her one-on-one interview due to her inability to think in a Synthetic-Geometric way.

SB6: …okay… given the set of…vectors…{u_1, u_2, u_3, u_4} where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not…

I1: …are you thinking?...

SB6: …it should be…this is a vector right?... so it's like…something…it is representing something like this…from…let's say…all of them are in a plane…

I1: …okay…what are you thinking?...

SB6: …that I don't know what it means by plane…

I1: …oh I see…so the three vectors are on the same plane and the fourth one is not…that is what it is saying…is that what you were…

SB6: …yeah, but I don't know what it means…plane…like I don't know if it is talking about like if they are in the same… dimension?...or I don't know what it is by plane…

I1: …okay, so you are thinking maybe there is something about they are in the same dimension…are there other options?...when they say on the same plane?...

SB6: …uhmmm…

I1: …no, you can't?...okay…uhmm…I see…by on the same plane, what they are saying is…yes, they are on the same plane…like this is a plane in space…and the three of them are on this one…

SB6: …okay…

I1: …that is what they mean…okay?...(20:07.6)…oh, okay, so if the three are on this one…on this plane, and the fourth is not…not on this plane…

SB6: …okay…

I1: …that is what the question is saying…okay…and the question says, can you then determine based on that information, whether the set with the four vectors is linearly independent or not?...

SB6: …this one I have no idea…

I1: …any, any ideas?...like what it might be or…

SB6: …no, I got stock in the concept of…plane…I don't understand…

Looking at the data obtained with respect to the use of metonymies and metaphors, we can conclude that a high percentage of the metonymies used were directly related to linear dependence and independence arguments. Out of the 31 metonymies found, 16 of those represented linear independence or linear dependence of a set of vectors. The most commonly used metonymies by student B6 were –unique solution → trivial solution, free variables → linear dependence, trivial solution → linear independence, and the identity matrix → linear independence. The following passage obtained from the original transcript demonstrates how student B6 use of unique solution to stand for the trivial solution created some misunderstanding.

SB6: …okay, so…since these are vectors also…right?...there is solution one, two, and zero, but that doesn't mean that it is independent…but it has a unique solution…

I1: …okay…

SB6: …but there is only one…

I1: …what are you thinking?...

SB6: …like this…yeah, I think it has a unique…since it has a unique solution I guess it is…independent…

I1: …oh, okay…so since this one has a unique solution, you are saying this is linearly independent…

SB6: …uhm, uhm…

I1: …okay…uhmmm…so earlier I asked you what you meant by unique solution…and this is what you meant then…

SB6: …yeah, there is only one solution…

I1: …oh, okay…what if we say the solution…with a solution a1 is zero, a2 is zero and a3 is zero...would that be a unique solution ?...so if you replace the one and two with zeros…

SB6: …with what?...zeros?...

I1: …yeah…

SB6: …I guess…

I1: …yeah?...okay, so you are saying that would be a unique solution…as long as you are saying that our values for a1, a2, a3 then this is a unique solution that implies…the vectors to be linearly…

SB6: …independent…

5.1.4 Student C7

Going back to the data reported in chapter 4 on the classification of the categorized arguments representing the thinking modes used by students during their interview, it can be said that student C7 used 15 different types of categories that can be classified into the three thinking modes presented by Sierpinska (2000). There were three categories belonging to the Synthetic-

Geometric mode of thinking. There were 9 categories classified into the Analytic-Arithmetic mode, and finally there were 5 categories classified into the Analytic-Structural mode, while only 2 categories fell into both Analytic-Arithmetic and Analytic-Structural thinking modes.

Going back to the information provided in table 4.8, we can conclude that the Synthetic-Geometric mode had a frequency of 6 (used 13.95% of the time), the Analytic-Arithmetic mode had a frequency of 31 (used 72.09% of the time), and finally the Analytic-Structural mode had a frequency of 21 (used 48.84% of the time). Taking this percentages into consideration and the amount of categories classified into both analytic modes, it is my opinion that student C7 has the ability to move from the Analytic-Arithmetic mode to the Analytic-Structural mode with no problem and vice versa. Also, there were some instances where student C7 showed signs of being able to relate the analytic modes to the Synthetic-Geometric mode. It should be noted that even though section had technology activities implemented, the classroom observations indicated the use of geometric modes (by the instructor) more in section C than section B (Zamora, 2010), many of which were static geometric modes. The following passage obtained from the original transcript reveals some of the arguments in which student C7 connected all three modes to provide an answer.

SC7: …uhmmm…I am trying to figure out if there is any way that something like this, where the third one isn't coplanar with it…if there is any way that…

I1: …uhm, uhm…

SC7: …oh, actually it is a scalar multiple, so it wouldn't be like that…it would be like that…

I1: …okay…

SC7: …or shorter, or something…

I1: …yeah, very good…

SC7: …so then, is there any way that…oh… hold on…then they would be on the same plane…

I1: …why is that?...

SC7: …because…well…would they be on the same plane?...

I1: …are you thinking of a plane?...what are you thinking right now?...

SC7: …yeah, I am trying to figure out if…my mind is getting confused here…hold on…and that is a scalar multiple of that one…so that is why that one is on…

I1: …is this one an extension to this one?...

SC7: …no, that one is another…I'll do it this way…

I1: …okay, okay…

SC7: …that is another vector in space…I am trying to see if its…

I1: …you can also use pens or things like that, if you want to…to demonstrate…

SC7: …okay…uhmmm…so…well then yeah, they are coplanar…

I1: …because?...

SC7: …because of the fact that when you take two vectors, so you have A1 and you cross it with A2, it gives you…it gives you the norm of it…the norm of it. You take cross multiplication of it, you get the norm of it, and then you can create a plane…with that…

I1: …okay, so which factors here are you taking?...

SC7: …I am taking…well because of the fact that this is…these are scalar multiples, even if you multiply this by a scalar…let's say A1…let's say this is A2, this is A1, and then A3 is long…so A3 would be equal to…{student writes $A3 = \alpha A1$}…

Looking at the data obtained with respect to the use of metonymies and metaphors, we can conclude that a high percentage of the metonymies used were directly related to linear dependence and independence arguments. Out of the 55 metonymies found, 29 of those represented linear independence or linear dependence of a set of vectors. The most commonly used metonymies by student C7 were –trivial solution → linear independence, free variables → linear independence, linear combinations → linearly dependent set, scalar multiples → linear dependence, and the identity matrix → linearly independent set. The following passage extracted from the interview transcript is a clear example of how the use of metonymies helped student C7 arrive to a conclusion based on her previous knowledge.

SC7: …I believe it is, as long as x1 is equal to x2 is equal to x3 etcetera …is equal to xn… as long as that is the only…the only way for this system to be equal to zero…

I1: …oh, okay…

SC7: …for the only way for this to be equal to zero vector is this…the trivial solution…

I1: …okay…

SC7: …and then that makes it linearly independent…

I1: …okay, and again is this set?... I'm assuming…

SC7: …I'm sorry…the set of vectors linearly independent…

I1: …okay, good…very good, so how is that tight to…having it…

SC7: …oh, okay…well because over here you have something like…x1, x2,etcetera is equal to zero…the zero vector, that's what I mean by that…{student writes [x1; x2; …] = 0}…

I1: …uhm, uhm…

SC7: …and the only way to do this…you multiply…rows by columns, I believe…

I1: …okay, are you sure about that?...

SC7: …not really, I have to think…I am sorry…

I1: …so you say rows by columns…okay…

SC7: …yeah, so here it would be x1 so you here…obviously if it is the identity matrix, all the rest of them are zeros…so all of these when you multiply by zero would be zero…so you have x1 and then you have plus, and then the next one would be plus x2 because all the rest of them…and etcetera all the way to plus xn…and you want this to equal the zero vector…well the only way…

I1: …okay, so did you so the first row times this?...or all of it?...

SC7: …well, that's what I…I did all the rows… I did first row times that which will give you x1…

I1: …okay…

SC7: …because the rest are zero…then you do the next one and because that's zero then it will be zero…and then one times x2 is that…and then all the way the down…

I1: …okay, great and then you add them up…

SC7: …yes…so the only way they can equal to zero is if x1 plus x2 equals zero…all of them equal zero…then it will be linearly independent…

5.1.5 Comparison

Based on the observations obtained from Zamora (2010) and reported in chapter 3 of this thesis, we can conclude the level of exposure of students belonging to section B and C to geometrical representations was lower (although more in section C) than the level of exposure of students belonging to section A. Even though the computer modules were part of the course for students belonging to sections A and B, the use of the modules was more extensive during lectures in section A where the instructor used the computer modules in an attempt to connect different representations and make sense of new concepts (Zamora, 2010).

One of the similarities among students A18, B6, and C7 is that they all used arguments that were classified into the Analytic-Arithmetic mode with a higher frequency, while student A33 used all three thinking modes evenly with a slightly higher frequency of the Analytic-Structural mode. Students A18, A33, and C7 demonstrated considerable ability to move from one thinking mode to another, while student B6 had a hard time moving from the analytical to the Synthetic-Geometric mode.

Among all four students, most of their metonymies used dealt with linear independence and/or linear dependence of a set of vectors. The most common metonymies used were –linear combination → linearly dependent set of vectors, unique solution → trivial solution, free variable → linear dependence, scalar multiples → linearly dependent set, and trivial solution → linear independence. The metonymy used by all four students was –identity matrix- to stand for linearly independence, which was sometimes used in an Analytical-Arithmetic mode (if a student had to carry out computations) and others mostly in an Analytical-Structural mode (if a student did not use any numerical computations and only used it as an argument).

5.2 Factors Affecting Results

There are some factors that may have influenced the results reported in this thesis. The exposure duration to the computer modules in sections A and B in which students had the opportunity to look at visual representation of some linear algebra concepts such as linear combinations, linear dependence, linear independence, span, spanning set, and vectors. Additional factors to consider the instructors' teaching style (constructivist vs. traditional) and structure of the homework assignments.

5.3 Research Limitations

One of the limitations to this study includes the amount of time students were followed during this research. All students were only taking their first year of linear algebra, and it is safe to say that if we had followed them in their second course of linear algebra, the results might had been different. An additional limitation is the objectivity of the categorization obtained from the analysis of the author of this thesis. The interview transcripts were independently analyzed by the author of this thesis and two additional raters, the process was explained in detailed in chapters 4 and 3. The analysis performed on these interview transcripts was strictly qualitative, therefore the students' responses were subject to individual subjective interpretations with a potential objectivity that may have occured. A measure of reliability among raters was reported and explained in section 3.3 of chapter 3.

Finally, students who participated in the interviews volunteered and extra credit was offered to them, so there exists the possibility of bias involving the reasons to why some students did and some did not volunteer.

5.4 Implications

Concerning future implications for this type of research at the university level, a similar study can be conducted with a similar process in which students are exposed to different

technological and visual aspects of learning, but are being taught by the same instructor. If the same research was to be conducted again, students could benefit from tutors and/or a computer-math lab available to students enrolled in the matrix algebra course to enhance students understanding.

The cognitive constructs analyzed for the purpose of this research –metonymies, metaphors, and thinking modes- provide an insight into the students' reasoning while taking their first course in linear algebra at the university level, but it is important to mention that some students had previous knowledge (depending on their backgrounds) of certain concepts, such as vectors and matrices. The analysis of students with the same backgrounds can provide a better understanding of the significance of the students' responses while reasoning and understanding linear algebra concepts.

5.5 Final Remarks

The analysis presented in the thesis has the purpose of documenting the cognitive structures –metonymies, metaphors, and thinking modes- present in the 4 students' responses while enrolled in their first linear algebra course in the effort to make sense of the cognition of the abstract concepts covered. The number of students, whose interviews were analyzed, does not reflect a significant sample of the students registered in the matrix algebra course during the Spring 2009, and therefore generalizations cannot be made from this research. The sole purpose of this thesis is to document those cognitive constructs and not to make any generalizations.

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Appendix A

Pre-Survey Administered at the Beginning of the Semester

Survey

Math3323—Matrix Algebra Spring 2009

Instructor: ________________________ Date: _______________

Class: __________________________

The National Science Foundation (NSF) agency has granted funds to the Department of Mathematics of the University of Texas at El Paso (UTEP) to conduct research to identify the kind of problems and difficulties students face in learning Matrix Algebra concepts and to develop instructional tools to address these issues.

This study will help researchers to better understand the effect of technological learning devices on the learning of difficult math concepts. Our project is also interested in the effect of interventions on the learning of matrix algebra concepts among groups with various backgrounds. For this purpose, we ask your input- via this survey- to better represent the demographics of students who are taking a matrix algebra course at UTEP.

Please respond to the survey questions to the best of your knowledge.

---

1. Please circle your answer:
   Gender: Male Female

2. Please circle your answer:
   Ethnicity: American African-American Hispanic
   Asian Native- American
   Other:_________________

3. Please circle your answer:
   Classification: Freshman Sophomore Junior Senior

4. Your Major: _________________________

5. Please provide your overall GPA at the start of the semester: ___________

6. To the best of your knowledge, please list the College Mathematics courses you have taken before attending this class and the grade you earned in each course. _____________________
   ____
   ____
   ____

7. How many courses were you enrolled in at the start of the semester? _____________________

8. Did you drop any courses this semester? (Circle your answer) Yes No
   If yes, how many courses did you drop? ____

9. Have you had a job this semester? (Circle your answer) Yes No

10. If yes, how long have you been working/worked on the job? ________________

How many hours per week are (or were) you working (on average)? Circle one.
Less than 20 hrs. 20 Hours More than 20 hrs.

11. Is English your first language? (Circle your answer) Yes No
    If not, what is your first language? ________________________

12. If English is not your first language, what level of fluency in English would you say you have in a rating 1-10 (10 being the highest)? ____________

13. Do you agree that language played a significant role on your learning and understanding of the topics of this course? (Circle your answer) Yes No.
    If yes, please explain how: ____
    ____
    ____

14. In a rating of 1-10 (10 being the highest level of difficulty) what level of difficulty did this class present to you? ____

15. Before this course, had you taken any classes that involved proving theorems? (Circle your answer) Yes No
    If yes, please provide a list of the classes you attended. ___________________________
    ____
    ____

16. Assign a rating from 1-10 (10 being the highest difficulty) to each of the topics below according to the difficulty you experienced while learning, studying, and/or practicing it. Note: If a topic in the list hasn't been covered in your class yet, please indicate it by writing "NC."

    Linear systems ______ Matrices ______ Subspaces ______
    Linear Independence ______ Span & Spanning sets ______
    Linear transformations ______ Eigenvalues & eigenvectors ______
    Inner product spaces ______ Others: ________________________________

17. Was there a time, while taking the matrix algebra course, you wished a topic (s) was covered differently to help you understand better? (Circle your answer) Yes No.
    If yes, please explain.____
    ____

18. Do you agree that you needed some additional explanations of the topics from a different perspective while learning them –through visualization, through real life applications, etc. (Please circle your answer). Yes No
    If yes, please explain ____
    ____
    ____

19. How would you recommend the topics that were difficult for you to learn to be covered?
____
____

20. Any suggestions on how to improve the teaching and learning of matrix algebra topics?
____
____

21. Would you like to add anything else regarding the matrix algebra course? ________________
____
____

Thank you for your collaboration on responding to this survey!

Appendix B

INFORMED CONSENT FORM

Interactive Online Modules and Take-Home Assignments for
Inquiry-Learning to Provide First-Hand Experience in Matrix Algebra Course

You are invited to be part of research activities conducted at The University of Texas at El Paso.

The purpose of this work is to identify what role the online interactive modules and inquiry assignments play in improving student achievement. The evaluation of the impact of the activities will be done through the assessments of student performance, their responses on pre- and post-surveys as well as in clinical interviews. Furthermore, we will document student conceptualizations of basic abstract concepts through student responses on take-home assignments and class tests.

Your permission will make possible for the researcher to document the effectiveness of the proposed activities in addressing obstacles in learning basic matrix algebra concepts.

You must be 18 years of age or older to participate. Your participation is completely voluntary and you may end your participation at any time with no consequences. There are no known risks involved in your participation in this study. You are given the opportunity to ask questions concerning the procedure, and any questions will be answered to your satisfaction.

Every effort will be made to keep your data confidential. No name will be released to anyone and in any published results; to keep the identity of the participating students confidential, a random numerical/letter code will be assigned to each of the respondents. Each participant will be referred to by this numerical/letter code only in presentations and publications of qualitative or descriptive data. Neither the faculty of UTEP nor the subjects' supervisors or colleagues will be provided with the names referring to the codes.

This project, (IRB protocol number: 84840-1), has been reviewed by The University of Texas at El Paso Institutional Review Board. Any questions regarding the conduct of this research or your rights as a research participant may be directed to Lola Norton, IRB Administrator, at (915) 747-8841 or irb.orsp@utep.edu at UTEP.

If you agree to participate, you are invited to sign this consent form and receive a copy of it after thoroughly reading it and asking the researcher any questions until you understand the proposed research activities.

____ Date___________
Student's name and signature

____ Date___________
Lola Norton, IRB Administrator

____ Date ___________
Researcher' name and signature

Appendix C

Interview Questions

INTERVIEW QUESTIONS
NSF/CCLI PROJECT
SPRING 2009

1. Define the linear independence of a set of vectors.

2. Given an example of a linearly dependent set of vectors.

3. Given the set $\{u_1, u_2, u_3, u_4\}$where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not. Determine if the set $\{u_1, u_2, u_3, u_4\}$is linearly independent. Explain your answer.

4. Given a linearly independent set $\{u_1, u_2, u_3, u_4\}$ in R^n. Determine the linear independence of the set $\{u_1, u_2 +5u_1, u_3, u_4\}$.

5. Given an nxm matrix a where ai2=ai4+3ai5 $\forall 1 \leq i \leq n$. Determine if the set $\{A_1, A_2, A_3,\ldots, A_m\}$ (Here A_j is the jth column of A) is linearly independent. Explain your answer.

6. Given a singular 3x3 matrix a. determine if the vectors of the set $\{A_1, A_2, A_3\}$, where A_j is the jth column of A, are on the same plane. Explain your answer.

7. Given that the vector equation xu+yv+zw=0 has infinitely many solutions. Determine if the vectors u, v, w are on the same plane. Explain your answer.

8. Given the vector equation $a_1u_1+a_2u_2+a_3u_3=0$ with the solution $a_1=1$, $a_2=-2$, and $a_3=0$. determine the linear independence of the set $\{u_1, u_2, u_3\}$.

9. Given that dim(Span{u,v,w})=1. Determine the linear independence of the set {u,v,w}.

Vita

Persis Beaven was born in El Paso, Texas and was raised in Ciudad Juarez, Mexico. The only daughter of Guadalupe Sandoval, enlisted in the military in 2001 and served in the United States Army until March 2004. She met her husband, Chad Beaven, while stationed in South Korea, and they now live in El Paso, TX with their three children. Persis attended El Paso Community College from 2005 to 2006, where she worked as a mathematics tutor from September 2006 to March 2008. In the Spring 2007 semester, Persis transferred to the University of Texas at El Paso to fulfill her undergraduate studies. She graduated with a Bachelors of Science in Mathematics and a minor in Secondary Education in May 2009. As an undergraduate student, she worked as a precalculus peer leader and a calculus supplemental instruction leader under the supervision of Dr. Emil Schwab during 2008. She was also a scholar of the Math and Science Teachers Academy at UTEP in 2008 and 2009. Persis Beaven entered the Masters of Arts in Teaching Mathematics at UTEP in the summer of 2009, and started working on her thesis in January of 2010 under the supervision of Dr. Hamide Dogan. Since August 2009, Persis has been part of the Cross Institutional Implementation of the Supplemental Instruction UTEP-EPCC cooperative project working as a supplemental instruction leader at EPCC Northwest campus where she also teaches one of the developmental math courses.

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This thesis was typed by the author, Persis Beaven.