2012-01-01

Comparative Investigation Of Cognitive And Pragmatic Thinking Modes Linear Algebra

Enayatollah Kalantarian
University of Texas at El Paso, enayat02@gmail.com

COMPARATIVE INVESTIGATION OF COGNITIVE AND PRAGMATIC THINKING MODES LINEAR ALGEBRA

ENAYATOLLAH KALANTARIAN

Department of Mathematical Sciences

APPROVED:

Hamide Dogan-Dunlap, Ph.D., Chair

Miguel Argaez, Ph.D.

Mahesh Narayan, Ph.D.

Benjamin C. Flores, Ph.D.
Dean of the Graduate School

Dedication

To my son Asadollah, my daughters Maryam and Narges, and my son Hamid who have been so patient with me in all phases of my life.

COMPARATIVE INVESTIGATION OF COGNITIVE AND PRAGMATIC THINKING MODES LINEAR ALGEBRA

by

ENAYATOLLAH KALANTARIAN

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF ARTS IN TEACHING

Department of Mathematical Science

THE UNIVERSITY OF TEXAS AT EL PASO

May 2012

Acknowledgements

I would like to express my gratitude to my dear advisor Dr. Hamide Dogan for her support, patience, and encouragement throughout my graduate studies. Her technical and editorial advisor was essential to the completion of this thesis and she has taught me innumerable lessons and insights on the working of academic research in general. She taught me the value of hard work by her own example.

My thanks also go to my members of my major committee, Dr. Miguel Argaez and Dr. Mahesh Narayan for providing many valuable comments that improved the presentation and contents of this thesis.

I also want to remember my dear deceased wife Sandra Bouche Kalantarian for her memorable and beautiful educational and social statements in our lives. Her valuable advises will stay in our memory for good.

I am grateful to my kids (Asadollah, Maryam, Narges Khatun and Hamid Kalantarian) for the inspiration and moral support they provided throughout my research work.

Abstract

A careful study of research done on mathematical thinking modes indicates the uncertainty level of understanding by students who have difficulties with linking the high level of linear algebra problems to their own mathematical backgrounds. Thus, the tangibility approach to linear algebra becomes irritating and difficult for this type of students. The topics furthermore don't seem remarkable and pleasant to them.

In this work, I have investigated the conceptual contrast of the cognitive experiences of two non-math major students' mathematical thinking modes using two categorized class assignments compared to two videotaped interviews relevant to basic linear algebra concepts such as linear independence, span and spanning sets.

The subjects of this thesis came from a Matrix Algebra course offered mainly to non-math major students. The composed data in this thesis were extracted from the transcripts of two videotaped interviews with two Matrix Algebra students alphanumerically referred to as A24 and A25. Interviews were designed to investigate students' understanding level through mathematical thinking modes. To analyze the outcomes, I categorized student responses using the framework of thinking modes by Sierpinska (2000) based on the valuable collected phrases from the transcripts and class assignments. Then, I used the students' responses to evaluate, and compare the results obtained from transcripts and assignments.

The key element of this thesis was to document and compare student thinking modes from interviews and class assignments on linear algebra concepts. To resemble the tunes emerged in the transcripts with the interviews, I categorized and examined the conceptual indicators (obtained phrases from transcripts) and derived codes from the student responses.

In short, I sat out to answer the following research question:

What are the similarities and differences between student thinking modes displayed on their interviews and class assignments?

Based on the results analysis, it was concluded that the level of exposure of students A24 and A25 to all three different thinking modes types (Analytic-Arithmetic, Analytic-Structural, and Analytic-Geometrical) during their interviews are about the same, but responding to the questions of the assignment are not following the thinking mode types that Sierpinska established in 2000.

Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction

1.1 Philosophy and Problem Statement

Many mathematical investigators are concerned with the improvement and validity of learning skills including mathematical thinking modes of non-math majors who have fundamental difficulties with abstract mathematics topics at the college level.

As students' progress in their theoretical ability, the subject of linear algebra also morphs into a more challenging and captivating state. Students might believe that linear algebra was shaped to depress them with abstract concepts, but the reality is that linear algebra can be applied to our daily situations, and therefore it can be respected as a subject of interest.

The focal point of this thesis concentrates on the assessments and evaluation of the validity of two different assessment measures via mathematical thinking modes provided by two students using a couple of assignments and videotapes of interviews regarding basic linear algebra topics. I will measure and evaluate the fallacy and mental mathematical visions by comparing the results of the assignments and videotapes with each other. Bloom (1990, p. 560) defines the misconceptions as a significant element in our lives. Bloom continues that this acts as a qualitative evaluation that can uncover much about the assumed processes that lead a learner along a particular path of understanding.

Classification of modes of thinking while learning linear algebra seems to be promising in determining obstacles students face in linear algebra (Dogan-Dunlap, 2010). Hence, the goal of this thesis is to address the following issues:

(a) To qualitatively interpret the experiences of two non-math major students relevant to their level of understanding from mathematical thinking modes extracted from a set of displayed assignments and their corresponding videotape interviews regarding linear algebra concepts.

(b) To analyze the possible similarities and differences between the documented modes from their interviews and assignments.

(c) To qualitatively analyze and thus inject my opinions in developing the notion of mathematical understanding through thinking modes.

To fulfill the above objectives, I will collect and categorize data from many different mathematical expressions achieved by students due to their own understanding from the first year linear algebra class at the university level. For the sake of simplicity, I will accordingly classify the thinking modes by sorting the students' self-conceptual responses to the questions concerning homework and interviews into "explanation, indicators and codes format".

What Is Mathematical Thinking?

To obtain mathematical thinking modes from the linear algebra concepts, it requires thinking mathematically, which usually needs to follow a "process" required to accomplish mathematical problem solving. To perform the process, learners need to get the mathematical language and terminologies of the context. The performance of this process also requires reaching a higher level of understanding mathematical problems and thus development of the individuals who are pursuing the process. Of course, there could be different set of definitions and explanations of the thinking modes by the individuals based on their level of mathematical knowledge and absorption from the subject. In Australia, Stacey (2005) furthermore states that "working mathematically" is one of the important goals in mathematics. She verifies that mathematics learners should directly or indirectly shadow a process counting a set of required steps to cognitively tolerate the contents.

This is to be expected because a well-defined meaning or explanation of mathematical thinking has yet to be developed (Lutfiyya, 1998; Cai, 2002). As a result, there is no detailed description of the words "mathematical thinking" in most national mathematics curriculum documents (Isoda, 2006). As such, different perspectives on mathematical thinking are evoked. Mason, Burton and Stacey (1982) for example, defined mathematical thinking as a dynamic process enabling one to increase the complexity of ideas that are able to be handled, and

consequently expand understanding. Katagiri (2004) defined mathematical thinking as the ability to think, and to make judgments independently while solving mathematics problems.

Accordingly, the mathematical thinking modes could also be considered as:

a) Active mental skills pursued through obtained information by the individuals from the content.
b) It also depends on the level of mindfulness and control of learners' thinking modes from the content.

1.2 Background and Foundation

Linear algebra produces great deals of notable challenges for students who need to improve and raise their understanding level of linear algebra and advance their mathematical learning skills. In fact, these challenges and lack of clear knowledge, and uncertainty about mathematics become more obvious once learners participate and get more involved in mathematical conferences, workshops or seminars.

My background search consists of two efforts:

1- It covers several guides from many mathematics investigators that contributes a lot in this thesis and then authenticate my study regarding mathematical thinking modes in linear algebra.
2- It also contains qualitative positions concerning the method comparison technique that clarify the presentation and the context of the method.

1.2.1 What is Method Comparison?

Engineers usually use method comparison to measure and calibrate the relative groundwork between two analytical methods that measure the same quantity and may produce the same errors in measurement. As an example the Bland-Altman plot (see Fig. 1) or difference plot is a graphical method to compare two measurement techniques with each other. In this graphical method the differences (or alternatively the ratios) between the two techniques are

plotted against the averages of the two techniques (Bland & Altman, 1986 and 1999). The plot may also be used to assess the repeatability of a method by comparing recurrent measurements using one single method on a series of subjects. An example of a Bland & Altman (Bellomo R. 2007) is a plot as follow:

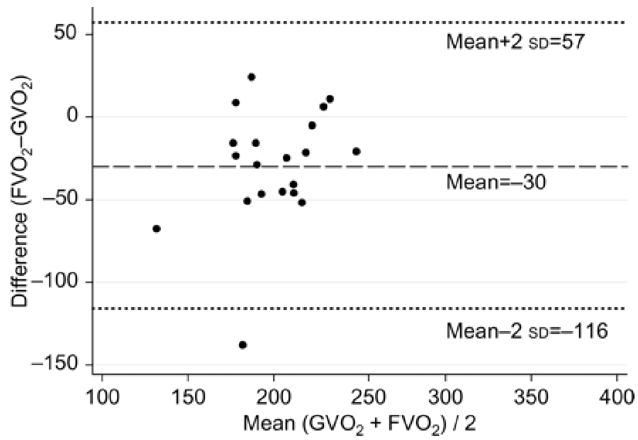


Figure 1.1: Bland–Altman plot of the difference between inspired/expired gas analysis-derived oxygen consumption (GVO_2) and Fick-derived oxygen consumption (FVO_2) against the mean of GVO_2 and FVO_2 in the 20 patients in the study

In medicine, they calibrate problems (comparing two different methods together) of measuring some quantity, such as blood pressure, gestational age, or cardiac stroke volume. Sometimes they compare an approximate or simple method with a very precise one. Frequently, however, they cannot regard either method as giving the true value of the quantity being measured. In this case they want to know whether the methods give answers which are, in some sense, comparable. Altman and Bland (2010) have used methods of analysis to compare two methods of measurements to obtain correlation, regression and the difference between means based on analysis of variance, and simple graphical methods for a simple parametric approach.

In clinical measurement comparison of a new measurement technique with an established one is often needed to see whether they agree sufficiently for the new to replace the old. Such investigations are often analyzed inappropriately, notably by using correlation coefficients. Bland a b and Altman a b believe that many studies provide the correlation coefficient (*r*)

between the results of the two measurement methods as an indicator of agreement. In the scientific measurement the correlation depends on the range of the true quantity in the sample. If this is wide, the correlation will be greater than if it is narrow. An alternative approach, based on graphical techniques and simple calculations, is described, together with the relation between this analysis and the assessment of repeatability (Bland and Altman, 2007).

Shoukri and Hashim (1998) used method comparison to examine the relative agreement between two rational methods that measure the same quantity. Their main goal was to see whether two methods produce the same errors in measurement.

For the resolution of demonstration, for example, data on bacterial counts from nasal swabs taken from cattle and measured by two clinicians were analyzed to evaluate the inter-clinicians agreement.

1.2.2 Linear Algebra Cognition

Harel (2005) argues that understanding an algebraic system without any genuine mental connection or visual representation will create many cognitive obstacles for students. In other words, it would not be expedient for algebra learners to comprehend algebra courses without being mentally visualized and prepared.

Dorier and Sierpinska [2004] approve Harel's idea by presenting a case review about the needs of cognitive agility for a deeper understanding of linear algebra concepts. They also believe that the derived thinking modes would be directly proportional to the mental preparation and abilities of algebra learners.

Grossman (2010) in her master thesis determined that if her students use multiple representations of mathematical thinking, they can solve their math problems faster with less miss-perception. In her research study, she utilized two classrooms of 8th grade mathematics students to investigate how students can solve open ended math problems and what representations will used as tools to figure out the solution. Notice that the two classrooms involved in the study were students that couldn't pass their state instructed assessment. She

realized from data taken from the school's standard assessment that overall in the area of algebra, that these students achieved better than the 8^{th} grade students who were branded as passing or exceeding the state standard of mathematics. One of the representations that the students struggled was the representation of writing an explanation of the solution or patterns involved in the given mathematics problems. Finally, she suggested to the participating teacher and other readers to continue with the multiple representations by having the students work in a more open-ended environment without any haste to choose the representations at hand to help them figure out the solution of the problem.

Shoemaker and Tucker (1993) believe that presenting linear algebra concepts using advanced digital computers in engineering schools will encourage and attract more and more students from other disciplines. Yue (2008) in her thesis investigated the validity of concept map assessments comparing to the level of understanding and learning skills of mathematics exposed on clinical interviews. Hong Yue's data included transcriptions of videotaped interviews and concept maps drawn by students which were designed to investigate students' understanding of the mathematics concepts of function and slope.

Dogan-Dunlap (2010) has examined different types of thinking modes using a group of matrix algebra students by responding to some questions about linear independence from two assignments conducted via arithmetic computational devices and through the graphical representations supported by an interactive web-based module. All the responses from the two assignments were qualitatively analyzed followed by the framework of Sierpinska (2000) on thinking modes.

Dogan (2003) also has followed the framework of Sierpinska's (2000) to interpret the Synthetic–Geometric, Analytic–Arithmetic and Analytic–Structural thinking modes. The research indicated that all three thinking modes have different representations in practice. For instance, Synthetic–Geometric mode utilizes geometric representations. This representation geometrically describes a line and a plane, but it doesn't define the nature of the modes (Dogan-Dunlap, 2010).

Analytic thinking mode uses numerical and algebraic representations, and can evidently define the modes and also identify two additional thinking modes; Analytic-Structural and Analytic-Arithmetic (Sierpinska, 2000; Dogan-Dunlap, 2010).

1.3 Insinuation and Hypothesis Using Method Comparison

Method comparison allows scientists to use the linear correlation coefficient to indicate the accuracy level of their data judgment using linear regression analyses. Applying method comparison as an assessment technique can generate many benefits and advantages to help evidently detect and understand the misconceptions and clarify uncertainty fluctuations in the process (James, 2004; Westgard 2010). Additionally, the contrast between two approaches (the student's class assignments and interview transcripts) will be prudently estimated using the percentage distribution differences (percentage distribution of one student effort minus the second one) to achieve a better explanation for the most frequent student thinking modes appeared within the categories. Thus, I will sum the percentage distributions of responses among the categories for the class assignments and the videotaped interviews obtained for each individual student separately, and then calculate the difference between the sums of two students. I will also make a comparison between the assignments and the interviews for each student such as:

Table 1.1: The count of one response for student A24

Code	Category	Interview	Percentage	Assignment	Percentage
NVC	Number of vectors compared to R^n	26	0.292	4	0.138

The percentage difference = 0.292 - 0.138= 0.153

The small amount of percentage difference shows that the conceptual thinking modes of two students are about comparable and reliable. Table 1 depicts a preliminary measurement of the percentage distribution for student A24. To apply the idea to this work, I will also build a table for the second student and measure the total percentage values of both interview and assignment for further calculation and argument.

1.4 Methodology

This study will analyze the responses of two undergraduate students, enrolled in a basic linear algebra course, to a set of questions asked during their one-on-one interviews scheduled, and their responses on a class assignment. Each of these students was selected (they were selected to make sure there is substantial data that can be obtained from class assignment) from a list of 22 volunteers interviewed at the end of a semester. My primary objective is to qualitatively analyze the different aspects of learning skills obtained through thinking modes expressed by each student first individually, and then compare the modes of the two students overall using percentage distribution discussion.

1.5 Analysis

A qualitative strategy, specifically the method comparison technique will be implemented in an effort to analyze the student thinking modes from responses to the interviews and assignments. The focal point will be the presence and categorization of thinking modes in students' interviews and assignments regarding the questions about a fundamental linear algebra course.

Qualitative Analysis

The interviews of two non-math major students will be transcribed and abbreviated. I will cognitively excavate through the transcripts and summarized class assignment responses, and construct the dependable indicators that match the corresponding categories of thinking modes in this thesis. Conversation regarding the appropriate modes with my thesis advisor will be done continuously to deliberate the different categories identified in each interview and assignment. Once all possible categories are listed, sample student responses and category descriptions will be formed in each entity as identifiers in a table form. Thus, the frequency, types of thinking modes identified, systematic error calculated from the difference of two percentage values (two non-math major students) will be recorded for further analysis and comparison.

Chapter 2: Literature Review

Removing difficulties confronted in linear algebra mostly requires consideration of instruction techniques and methods of gaining knowledge of linear algebra, since linear algebra courses are not taken seriously by students or taught correctly by instructors. However, the concept of linear algebra naturally is a difficult subject for non-math majors. Recently, many mathematics education investigators have been concerned with students' difficulties related to the undergraduate linear algebra courses. There is an agreement that teaching and learning algebra is a frustrating experience for both teachers and students, and despite all the efforts to improve the prospectus, the learning of linear algebra remains challenging for many students (Dorier & Sierpinska, 2001). Dorier (2000) also stated that the question in mathematics should be essentially set on using techniques and follow certain procedures rather than understanding the concepts.

In Australia, Stacey (2005) states that "working mathematically" is one of the most important goals in mathematics that student should aim for. She verifies that mathematics learners should directly or indirectly follow a process of counting a set of required steps to cognitively tolerate the contents. Isoda (2006) in National Mathematics Curriculum documents different perspectives on mathematical thinking. The Australian Curriculum *(2010)* states that mathematics provides students with the essential skills and knowledge in *Numbers*, *Algebra*, *Measurement and Geometry, Statistics, and Probability*. It develops the proficiency capabilities that all students need in their personal, work, and civic life, and provides the fundamentals on which mathematical specialties and professional applications of mathematics are built.

Mason, Burton, and Stacey (1982) defined mathematical thinking as a dynamic process enabling one to increase the complexity of ideas that they are able to handle, and consequently

expand their understanding level. Katagiri (2004) defined mathematical thinking as the ability to think, and to make judgments independently while solving mathematics problems. Dorier and Sierpinska [2004] approve Harel's idea by presenting a case review about the needs of cognitive agility for a deeper understanding of linear algebra concepts. They also believe that the derived thinking modes would be directly proportional to the mental preparation and abilities of linear algebra learners.

Caj (2008) has performed a cognitive comparative study about developing algebraic thinking modes among earlier grade students in the United States and some international countries such as Russia, Singapore, Korea, and China in particular. In this article, Caj shares and discuses two major visions from international studies. The first vision relates to the transition between arithmetic and algebra. The second stresses on the simplification of actual representations and strategies. These two insights are based on the belief that students in earlier grades are capable of, and should be expected to think algebraically. Caj (2008) concludes that U.S. students persist in using concrete, arithmetic-based problem solving strategies, even on problems where concrete strategies are inefficient. He also showed that Chinese students effectively use abstract, algebra-based strategies to solve the same problems that U.S. students approach concretely, using arithmetic-based strategies. Research shows that a possible reason that K-8 students in the U.S. use algebraic problem-solving strategies less frequently than their counterparts in China is that U.S. teachers may not expect their K-8 students to do so.

Despite apparent differences in U.S. and Chinese cultures, knowledge of Chinese students' success with the use of generalized strategies may prompt United States teachers to explore possible ways to encourage students to move toward the use of abstract strategies in Grades K-8, and thus ease the transition from arithmetic to algebra.

2.1 Mathematical Thinking Modes

Our teaching experience designates that the non-math major students have difficulties to comprehend advance abstract mathematical concepts. The main part of the student's problem is that they are not able to think mathematically and thus, follow the process of solving mathematical problems due to the absence in understanding of the terminology and language of the field. However, Herscovics and Linchevski (1994) specified, from a study where students solved fifty algebraic equations of different types, that a cognitive gap exists between mathematics and algebra which is characterized by the students' inability to operate spontaneously on the unknown. Arithmetic is algebraic to the extent that it provides opportunities for making and expressing generalizations [Carraher (TERC 2004)].

Thomas & Kota (2001) stated that when doing algebra, we see that even after two or three years, if the relation between the variables in a problem is given in the verbal form, the students show a tendency to use an arithmetic mode of thinking. They continue that a failure to solve the given problem using any arithmetic method leads them to follow intuitive methods such as guess and check or trial and error rather than algebraic ones even though these methods are known to them and they were able to use algebra on the final question.

Certainly, there are great mathematical researchers investigating the novel methods and devices to improve the student's level of algebraic knowledge and understanding the contents in math and science. An example of such a method is the framework of mathematical thinking modes used in this study. The examination on the thinking modes obtained in this study followed from Sierpinska's framework (2000) on student thinking modes, where three types of thinking modes were recognized: Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural. Sierpinska (2000) in "On some aspects on students' thinking in linear algebra," identified these cognitive modes in linear algebra based on their interrelating language: the pictorial geometric,

the arithmetic, and the structural language. These three modes co-exist in linear algebra, and the use of one of them does not imply the elimination of the other (Sierpinska, 2000). In the next sections, I introduce these modes and some of these investigators that consider the framework of thinking modes in their research.

2.1.1 Analytical Arithmetic Modes

Foundations of mathematics are the study of the most basic concepts and analytical structure of mathematics, with an eye to the unity of human knowledge (Simpson 1999). Thomas and Kota (2008) believe that many students have difficulties with algebraic problem solving because of the lack of association between their logical and mathematical exercises in their educational backgrounds. There have been a number of studies shown by arithmetic education researchers trying to escalate the nature of these difficulties. Kieran (1992) has identified how to create equations from words as the major area of difficulty for high school algebra students. Numerous researches (e.g. Lochhead, 1980; Clement, Narode and Rosnick, 1981; Kaput & Sims Knight, 1983) have analyzed the difficulties faced by students while solving algebra word problems and have offered proposals to overcome these difficulties. Other researchers like Macgregor & Stacey (1993), Thomas (1994) have tried to associate arithmetic and algebraic modes of thinking in the context of problem solving. Filloy (1985) stated that algebra is the perpetuation of arithmetic. Kieran (1992) analyzed arithmetic and algebraic thinking modes by referring to arithmetic processes carried out on numbers to yield numbers as practical and a set of operations carried out on algebraic expressions as mechanical characteristics. A failure to solve problems using these arithmetical methods leads students to follow distinctive approaches such as guess and check or trial and error rather than algebraic ones even though these approaches are known to them and they were able to use algebra on the final question, Kieran (1992).

2.1.2 Analytical Structural Modes

The key alteration between the ‘synthetic’ and ‘analytic’ modes is that in the first, objects are given directly to the students mind, and their mental processes then try to describe, connect, and form meanings from them; while in the second mode objects are given indirectly to the student, so the student tries to make sense of them by the definition of properties of their elements (Sierpinska, 2000). Dogan-Dunlap, and her students (2010) used the Sierpinska’s framework on student thinking modes analyses as the starting point for their study. Sierpinska’s framework includes three kinds of thinking modes resulting from her linear algebra students’ responses, which are: *Synthetic Geometric*, *Analytic–Arithmetic and Analytic–Structural*. Table 2.1 illustrates the outline of the modes. Dogan and her students (2010) believe that the structural modes may involve geometric means in their applications. She states an example that if a student considers the features of an object in the setting of a system with geometric features then students may be applying both the structural and geometric modes at the same time. For instance, using a dimension argument in determining linear independence may either be considered having geometric or/and algebraic underpinning depending on the context in which the argument is made. Sierpinska (2000) seems to consider the structural modes having strictly algebraic associations

The modes are shown below in Table 2.1 including examples of student responses that may represent the responses that fall into each of the modes.

Table 2.1: Sierpinska's thinking modes. (Taken from Dogan-Dunlap, 2010)

Mode of Thinking	Topic	Student Competency or Lack of
Synthetic Geometric	Graphical representation	Student will be able determinewhether vectors whose graphs
		are provided in R^2 or R^3 are linearly independent or dependent.
Analytic Arithmetic	Matrix representation	Student is able to constructmatrix from vectors, compute row-reduced echelon form and relate reduced matrix to linear dependence and independence
	Linear Combination	Student is able to use definition of linear independence as a linear combination of vectors. Students attempt to solve linear combination, not just state or provide it.
Analytic Structural		Dimension of Vector Space Student is able to makeconjectures on the cardinality-of linear independent sets in R^n for any natural number n.

The above table interprets the framework of Sierpinska in three different thinking modes based on the utilized type of representations. For example, Synthetic–Geometric mode covers geometric representations and only explains the provided objects, but not defined (e.g. drawing of a line or a plane). On the contrary, analytic modes use arithmetical and algebraic representations and it defines the provided objects in the mode. For instance, the formal meaning of linear independence uses analytic modes.

Table 2.1 also illustrates that the analytic modes have two subdivision modes that are Analytic–Arithmetic and the Analytic–Structural or algebraic modes. Dogan (20101) enlightens that Analytic–Arithmetic mode reflects objects with respect to their process and procedures. Analytic–Structural mode on the other hand considers objects in systems, and disregards the process and procedures.

2.1.3 Synthetic-Geometric Modes

Synthetic geometry is a subdivision of geometry which deals purely with geometric elements directly endowed with geometrical possessions by abstract axioms. Synthetic geometry is the kind of geometry for which Euclid is famous and that we all learned in high school.

Modern synthetic geometry, yet, has a more logically complete and reliable substance. However, in this thesis, I have not encountered any geometric representation in either the students' assignments or their videotaped interviews. Dogan-Dunlap and her students (2010) have documented the differences on the types of modes linear algebra students displayed in their responses to the questions of linear independence from two different assignments. Dogan (2010) has eloquently elaborated on the second assignment since the content of the assignment was controlled with the support of graphical reasoning through an interactive web-module. Additionally, she believes that the structural modes may involve geometric resources. If a student considers the characteristics of an object in the context of a system with geometric features then students may be applying both the structural and geometric modes. She also states that they observed many responses, in geometric responses, where the students interpreted the arithmetic mode of solutions (using the rref of matrices) in connection with their geometric and algebraic modes. She continues that some of these participants linked the geometric mode of being able to trace vectors back to another vector to the algebraic mode of having a linear combination resulting in the vector.

Dogan (2010) concludes that 17 different categories of modes of geometric representations were revealed from that assignment. She suggested that the geometric representation specifically be in presence of algebraic and arithmetic modes since it can greatly help to ease the thinking modes clarification throughout the content.

It may be that the reluctance to use algebra can be lessened by using geometric figures to express the relationship between unknowns simultaneously with verbal forms. That mathematical thinking may be encouraged by the use of visualization encouraged by multiple

representations is well known. Thomas (1995) has suggested a theory of cognitive integration which may begin to explain the mechanisms by which this may take place.

2.2 Thinking Modes in Linear Algebra Education

Dogan (2003; 2011) has followed the framework of Sierpinska's (2000) to interpret the Synthetic–Geometric, Analytic–Arithmetic, and Analytic–Structural thinking modes. The research indicated that all three thinking modes have different representations in practice. For instance, the Synthetic–Geometric mode utilizes geometric representations. This representation geometrically describes a line and a plane, but it doesn't define the nature of these objects.

Dogan-Dunlap (2010) has also examined different types of thinking modes using a group of matrix algebra students by responding to some questions about linear independence from two assignments conducted via arithmetic computational devices and through the graphical representations supported by an interactive web-based module. As reported by Dogan-Dunlap (2010), the use of geometric representations helps students consider the different representations of a concept flexibly, and allows them to move from one thinking mode to another.

Hurst (2008) describes a strategy to develop the mathematical thinking modes for higher level of students using task-based interviews as a data gathering tool. He attempts to identify different modes or levels of thinking used by eight students in a multiple case study following the implementation of an interference program. The modes of thinking were based on the mathematical background and strategic knowledge by the students.

Harel (1998 and 2000) defines a clinical task-based interview as a situation where the interviewer– interviewee interaction on a task is regulated by a system of explicit and implicit

norms, values, and rules. Task-based interviews can be useful tools for helping teachers assess the mathematical thinking modes of their students, particularly when mathematical concepts are embedded in everyday or ‘real life’ contexts [Hurst 2006]. The work was created from the ideas that there exists a continuous relationship between the three modes of thinking that a student may approach in a situation to mainly comprehend the validity of a situation. Else, the situation could be approached with contextual thinking where a student might have experienced with the particular context and use some of the embedded mathematics information to connect his/her understanding related to the aspects of the context. Figure 2.1 reflects this cyclical relationship.

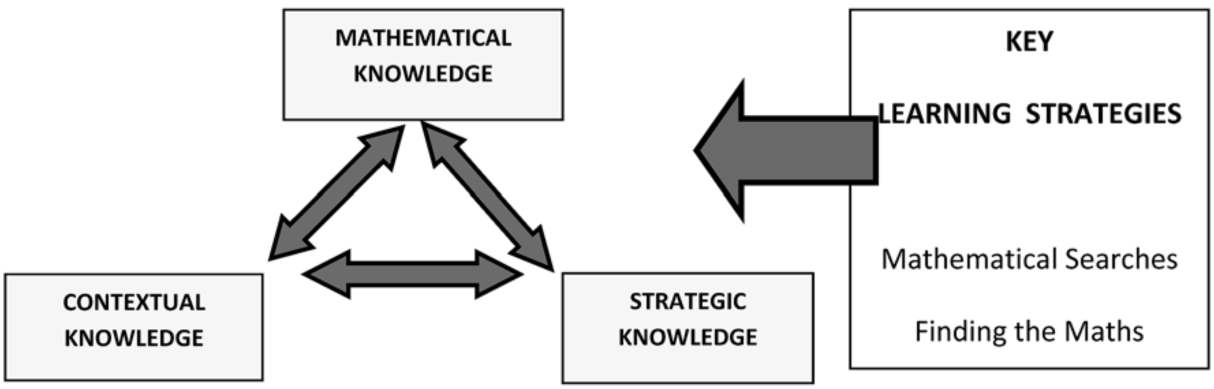


Figure 2.1: Cyclical relationship between modes of thinking (Chris Hurst)

“Curtin University of Technology”

To understand the aspects of the text, in Hurst’s study (2008), students were asked to search through pieces of a text and identify and describe the embedded mathematical information to understand features of the text, and then to develop questions that could be asked using the information.

Some of the relevant questions used in the particular work (Hurst, 2008) are as follows:

a) If we are shopping, coming to school, or watching TV, we see numbers around us. Do you take notice of them or wonder what they are telling you about?

b) If someone gave you a page out of a book or newspaper, and asked you to describe the mathematics that you saw, what sort of things would you look for?

c) Show the student the sample of the shopping docket/receipt and ask 'What mathematical ideas can you see and what do they tell you?'

d) Show the student the map samples. What mathematical ideas can you see in the maps? How could you use those ideas to help you work out or learn something?

e) Show the student the furniture advertisement. Ask 'What mathematical ideas can you see in that?' Ask 'How could you use that to help you work out something?' or 'What could you work out from that?' or 'How might that be useful to you?'

2.3 Method Comparison

In this thesis, I have considered the student's interviews and assignments as two different methods of data collectors to determine the student's thinking modes. To confirm the accuracy of the composed data sets obtained from the student's responses through the interviews and assignments, I have categorized and statistically calculated the probabilities of the occurrence of the thinking modes using relevant codes within the interview and assignment for each student and checked for the similarity and the accuracy using the linear regression approach as a subsection of method comparison. Magari (2011) stated that linear regression is probably the most popular slant in method comparison studies. Based on that approach, the regression of test method to reference method should yield a straight line non-significantly different from the equality line.

2.3.1 Linear Regression Method

The main purpose of using the linear regression method is to determine if two methods are statistically identical with 95% confidence intervals. And if they are not identical, to determine

what is the statistical relationship of these methods. Ludbrook (2010) presented that there are many reasons to justify the utilization of linear regression method in the statistical cases. Two of those reasons are:

1- To detect bias and calibrate one method or measurer against another. Fixed bias is present when one method gives higher (or lower) values across the whole range of measurement. Proportional bias is present when one method gives values that diverge progressively from those of the other.
2- Linear regression analysis is a popular method for comparing methods of measurement, when the familiar ordinary least squares (OLS) method is rarely acceptable. The OLS method requires that the x values are fixed by the design of the study, whereas it is usual that both y and x values are free to vary and are subject to error. In this case, special regression techniques must be used.

The concept of the comparative method includes a number of other methods as well, and also applies to experimental and statistical methods. It is not narrowed to only what in recent years has come to be identified with a fine definition of the comparative method—namely, comparison based on Mill's methods (Sekhon 2002). This is used in our everyday, common-sense reasoning about connection; they are also used by scientists in scheming experiments. Formerly, this leads to a question of, if the comparative method shares its analytical charisma with other methods—and the term "comparison" has indeed been applied to experimental and statistical methods—*is there a difference between the comparative method and other methods*? What is special about the comparative method? The following investigators illuminate and describe it as such:

Habib, Kaplan, Margalit, and Friedman (2010) have considered the DNA-binding specificities of transcription factors by dispersal of several algorithms simultaneously to

determine whether two DNA motifs represent the same binding preferences. They demonstrated the use of the comparison method as a basis for theme clustering and retrieval procedures, and compare it to several commonly used alternatives. Their results showed that the new method outperformed other available methods in accuracy and sensitivity.

Some authors maintain that without comparison all scientific thought is unimaginable (Swanson, 1971, p. 145) and that research, in one form or another, is unavoidably comparative (Lasswell, 1968, p. 3; Lieberson, 1985, p. 44). Almond notes (1966), "[it] makes no sense to speak of a comparative method in political science since if it is a science, it goes without saying that it is comparative" (pp. 877–878). Because comparison constitutes the core of all scientific explanation (Armer, 1973; Bailey, 1982; Blalock, 1961; Nagel, 1961), some authors object to the logical and epistemological distinctiveness of the comparative method (Grimshaw, 1973, p. 18). As Klingman (1980, p. 124) notes, many of the debates forget that all science is inherently comparative.

Altman and Bland (1983), Bland and Altman (1986) used the comparison method to statistically compare and assess degree of agreement between two methods of clinical measurement. Medical researchers often need to compare two methods of measurement, or a new method with an established one, to determine whether these two methods can be used interchangeably or the new method can replace the established one. In most of these situations, the 'true' value of the measured quantity is unknown. The Bland–Altman method calculates the mean difference between two methods of measurement (the 'bias'), and 95% limits of agreement as the mean difference (2 sd) [or more precisely (1.96 sd)]. It is expected that the 95% limits include 95% of differences between the two measurement methods. The plot is commonly called

a Bland–Altman plot and the associated method is usually called the Bland–Altman method. The Bland–Altman method can even include estimation of confidence intervals for the bias and limits of agreement, but these are often omitted in research papers.

Chapter 3: Methodology

The primary goal of this thesis is to generate and examine the cognitive thinking modes utilized by two out of 46 randomly selected first year college students. The content for this study was extracted through videotaped interviews and class assignments in a linear algebra course offered as a junior level class at a four year university. Students taking this class must meet the prerequisites for the class. The prerequisite for the class is a calculus course that covers differential and integral calculus. The students were taught during a regular college semester and were required to submit assignments throughout the semester. Assignments were graded and included in class grades. Two of these assignments were utilized for the data used in this thesis. The purpose of the assignments was to provide an opportunity for the students to experiment their own thoughts using basic linear algebra concepts, to record how these participants were mentally thinking about the algebraic concepts, and how they could in turn simplify their findings to more abstract levels of understanding. In addition to the two homework activities, a set of warm-up activities were assigned in order to provide opportunities for students to become familiar with the type of questions, and with the interactive website Trejo (2007). The students' responses used in this thesis, were studied and re-examined looking for patterns among responses which may reveal common fallacies, correct/incorrect estimations, and student difficulties with linear algebra concepts. Additionally, their responses relating to the homework questions were associated to the types of thinking modes utilized by students while they were solving linear algebra problems. The additional detailed explanation of the analysis, description of participants, and procedure applied to the data collection methods will be illustrated in the rest of this chapter.

The inscription for this investigation was derived from Sierpinska's efforts (2000) on thinking mode elaboration while learners attempt solving linear algebra problems. The full discussion was

illustrated on the previous chapter (Literature Research).. The main inscription for this thesis was originated as a method to categorize and perceive participants thinking modes while solving linear algebra problems, and to clarify the possible changes on their modes of thinking as they were introduced to numerous different cognitive approaches. Thus, to do so, the following three distinguished goals were pursued as key elements.

1- To allow students to reach their own rational level of interpretation and self-arithmetical analysis, and consequently have a much deeper sympathy of the algebraic content.
2- To draw a standard pattern of students' vocal and visual expressions of their own relevant knowledge about the subjects.
3- To provide a tangible and significant model of student reasoning mathematical ideas and non-concrete algebraic problems in particular.

I have referred and illustrated the statistical documented analyses achieved by Zamora (2010). To be able to carefully analyze his own effort, Zamora (2010) needed to select a particular algebraic class and do the arithmetical background analyses on all the students. The following table, Table 3.1, is a breakdown of student demographics in that particular class chosen for his research. Notice that the IRB procedure has been obtained for the collected data used in his work [IRB code is 84840-4]. Additionally, interactive online modules and take-home assignments for inquiry-learning were also considered in order to provide first-hand experience in the matrix algebra course.

3.1 Participants

The students who participated in this work were selected from three different sections of the first linear algebra course offered at a four-year southwestern university during the Spring

2009 semester. A large percentage of the students from each section are of Hispanic origin, and a considerable percentage has English as a second language.

The two sections which were implemented online, considered as modular course (non-traditional course), and the third one is referred to as non-modular course (traditional course). In the modular matrix algebra courses, the use of the online modules was enforced and the integration of these was encouraged through the constructivist homework assignments. The non-modular course was more traditional. The instructor assigned homework strictly from the book, and the computerized modules were not even introduced to the students registered in this section.

Some of the main characteristics of the students belonging to each section of the matrix algebra course in the spring 2009 semester are summarized in the following tables.

Table 3.1: Demographics of Group A; modular section (obtained from Zamora, 2010)

Section A	
Question	
Gender	*-Males:25 -Females: 9*
Ethnicity	*-Hispanic/Hispanic American: 79.4% -White/Caucasian/American: 17.64%* *-American-Asian/Asian: 2.94%*
Classification	*-Freshman: 0% -Sophomore: 35.29% -Junior: 41.18% -Senior: 23.53%*
Major	*-Mathematics: 20.59% -Computer Science: 44.12% -Electrical Engineering: 23.53%* *-Industrial Engineering: 8.82% -Computer Engineering: 2.94%*
Courses this semester	*-Mean: 4.4 -Standard Deviation: 0.86 -Mode: 4*
Have a job? ***For how long?*** ***Hours/week?***	*-No: 20.59% -Yes: 79.41%* *-Less than a year: 62.96% -1 to 3 years: 25.93 -No answer: 11.11%* *-Less than 20: 51.85% -Exactly 20: 18.52% -More than 20: 29.63%*
English first language	*-No: 44.12% -Yes: 55.88% 100% of the students that answered no to this question, reported to have Spanish as their first language (15 students)*
Fluency	*-10: 33.33% -9: 6.67% -8.5: 6.67% -8: 46.66% -5: 6.67%* *-Standard Deviation: 1.35 -Mean: 8.60*

There were 35 students registered in Section A at the beginning of the spring 2009 semester. Of these 35 students, only 34 students attended class the day the pre-survey was administered during one of the first class meetings (Zamora, 2010).

Below Table 3.2 summarizes the demographics of Section B (modular course):

Table 3.2: Demographics of Group B; modular section (obtained from Zamora, 2010)

Section B	
Question	
Gender	*-Males: 22 -Females: 6*
Ethnicity	*-Hispanic/Hispanic American: 75% -White/Caucasian/American: 17.86%* *-Mexican/Chicano: 3.57% -American-Asian/Asian: 3.57%*
Classification	*-Freshman: 0% -Sophomore: 7.14% -Junior: 50% -Senior: 42.86%*
Major	*-Mathematics: 3.57% -Computer Science: 35.7% -Electrical Engineering: 37.5%* *-Industrial Engineering: 7.14% -Mechanical Engineering: 7.14% -Philosophy: 3.57%* *-Physics: 3.57% -Multidisciplinary Studies: 3.57%*
Courses this semester	*-Mean: 4.04 -Standard Deviation: 1.04 -Mode: 5*
Have a job? ***For how long?*** ***Hours/week?***	*-No: 32.14% -Yes: 67.86%* *-Less than a year: 26.32% -1 to 3 years: 52.63% -More than 3 years: 21.05%* *-Less than 20: 5.26% -Exactly 20: 42.11% -More than 20: 52.63%*
English first language	*-No: 64.29% -Yes: 35.71% 77.78% of the students who answered no to this question, reported to have Spanish as their first language; 5.56% had Thai as their first language, and 16.66% gave no answer.*
Fluency	*-10: 16.67% -9: 55.567% -8: 22.22% -7: 5.55%* *-Standard Deviation: 1.35 -Mean: 8.60*

There were 35 students registered in Section B at the beginning of the spring 2009 semester. Of these 35 students, only 28 students attended class the day the pre-survey was administered during one of the first class meetings (Zamora, 2010).

Below Table 3.3 summarizes the demographics of the non-modular section, Section C. All 35 students registered for this section attended class and were able to answer the pre-survey administered at the beginning of the spring 2009 semester (Zamora, 2010).

Table 3.3: Demographics of Group C; Non-modular section (obtained from Zamora, 2010)

Section C	
Question	
Gender	*-Males:26 -Females: 9*
Ethnicity	*-Hispanic/Hispanic American: 80% -White/Caucasian/American: 11.43%* *-Mexican/Chicano: 2.86% -American-Asian/Asian: 5.71%*
Classification	*-Freshman: 2.86% -Sophomore: 17.14% -Junior: 62.86% -Senior: 17.14%*
Major	*-Mathematics: 17.14% -Computer Science: 22.86% -Electrical Engineering: 51.43%* *-Industrial Engineering: 5.71% Physics: 2.86%*
Courses this semester	*-Mean: 4.54 -Standard Deviation: 1.20 -Mode: 4*
Have a job? ***For how long?*** ***Hours/week?***	*-No: 37.14% -Yes: 62.86%* *-Less than a year: 31.82% -1 to 3 years: 50% -No answer: 18.18%* *-Less than 20: 50% -Exactly 20: 27.27% -More than 20: 22.73%*
English first language	*-No: 37.14% -Yes: 62.86% 100% of the students that answered no to this question, reported to have Spanish as their first language (13 students)*
Fluency	*-10: 30.77% -9: 30.77% -8: 23.08% -7: 50.37%* *-Standard Deviation: 1.09 -Mean: 8.77*

The students attending all three sections were pursuing degrees from a wide variety of disciplines, including electrical engineering, industrial engineering, mechanical engineering, mathematics, computer science, computer engineering, physics, philosophy, and multidisciplinary studies. The nature of this matrix algebra class introduced students from all three sections to proofs; in some cases students expected to have the same format as previous courses with less theoretical framework, such as pre-calculus and calculus. The percentage of students who had experienced proofs in previous courses from Section A was 35.29%, from Section B 60.71%, and from Section C was 40% (Zamora, 2010).

Students from sections A and C attended class twice per week for an hour and twenty minutes, while students from Section B attended class three times per week for 50 minutes. The average age among the university population was reported to be 26 years old (University of Texas at El Paso- UTEP, 2009).

In order to avoid bias in our analysis, the name of students who participated in the surveys and interviews was replaced by a code composed of a letter corresponding to the section they belonged to and a number. The codes from Section A ranged from A1 through A35, for Section B from B1 through B34, and finally for Section C from C1 through C36. Students enrolled in this course were asked to sign a consent form at the beginning of the spring 2009 semester that allowed the researchers to use their information for the purpose of this investigation.

Students interviewed were volunteers from each section of the course. The interviews were videotaped for informational purposes, so they could be transcribed by the author of this thesis. One interview from each section was randomly chosen for the purpose of this thesis from the group of students who volunteered to be interviewed. No preconceptions were placed on the base of race, gender, age, or socioeconomic status of any of the students while interviewing and analyzing the interview transcripts. The interview transcripts are available upon request.

3.2 Instruments: Assignments

Assignments of two students A24 and A25 were classified and developed as the data sets in this thesis. The following conditions are elaborated on to clarify the efforts and the utility of the assignments in this thesis.

1- They would function as sets of data for the future study and analysis as well. Notice that students were uninformed which specific set of assignments will be used as data in the future.

2- This allowed students to freely respond to the queries and remain satisfied with their responses as stated in the little or as much detail as they felt required.

3- The study of homework would cognitively help students to express their views and opinions based on their conceptual level of understanding from the content with regard to the question.

3.3 Procedure:

Participants of the study were given a take-home assignment and a group of volunteers were interviewed. Both interviews and the take-home assignment include questions relevant to linear independence. In this thesis, we report qualitative and quantitative analysis of take-home and interview responses.

The take-home assignments (each with six questions) were selected from two different students (referred to with the pseudo names; A24 and A25) to be analyzed in this research (See Appendix C). The responses to the questions were analyzed and then dissimilar thinking modes were considered in for further interpretation in this research. To advance a better knowledge and understanding from the students' thinking modes embedded within their homework; I have arranged and interpreted questions, 3, 4 and 6 from both students A24 and A25 in this thesis. The arranged student's questions 3, 4 and 6 are as follow:

Question Number 3:

Conjecture on the necessary and sufficient condition (s) for two vectors in R3 to be linearly independent vectors. Explain your reasoning

Question Number 4:

Conjecture on the necessary and sufficient condition(s) for three vectors in R3 to be linearly independent vectors. Explain your reasoning.

Question Number 6:

Answer the following questions and explain your answers:

a. *True or False: There exist three linearly independent vectors in R2.*
b. *True or False: there exist four linearly independent vectors in R3.*
c. *True or False: There exist three linearly independent vectors in R3.*
d. *True or false: The set v= ------------ is linearly independent.*
e. *On each graph below, shown is a set of vectors originated from (0,0). Circle the one(s) that are linearly independent. Explain and justify your selections(s).*

However, questions 1 has provided a great opportunity to all students (including A24 and A25) to spontaneously express their own ideas whether the set of provided vectors to them are dependent or independent. Nevertheless questions 3-6 allowed students to perform experiments using their own set of vectors to be examined. This cognitively allowed students to think and then construct the type of linear dependency or independency of vectors.

Once the responses to the questions were transcribed, classification was done independently based on their categories and type of reasoning embedded within the statements.

3.4 Interviews

One of the main purposes of the videotaped interviews with each of non-math major student was to observe their mathematical problem-solving behavior in part and draw implications from the observations about the students' understanding of mathematical thinking modes. It is important to note that the uncertainty level of students who have difficulties with linking the high level of linear algebra problems to their own mathematical backgrounds varies based on their math connected and self-interpretation from the content. Each interview was videotaped and free problem solving was encouraged for about an hour for each student. All the hints or suggestions were made only after the students had the chance to respond impulsively. The questions were intended to disclose kinds of theoretical knowledge and technical knowledge of mathematical concepts within the student's minds. The students' actual names were replaced with pseudonyms as A24 and A25. The responses given by Student A24 and A25 for each interview revealing their knowledge were recorded, and transcribed. Transcripts were analyzed, categorized and thus created a representative knowledge that conveyed in the responses. Once the categorization of the responses had taken place, the data were arranged into tables to better demonstrate the information.

Table 3.4: The outline of tables used to document the responses of students A24 and A25

No	Indicators	responses by Students	categories	Codes

Table 3.4 above show the outline of tables used to document the responses of students A24 and A25. The quality of interviews was well considered since it would affect the presentation of the interviewees and more the qualitative examination. Therefore, the interviewer had a full knowledge about the interview strategy and of course, recognized the individuality of the interviewees as well. As Ginsburg (1997) argues, an evaluation of the interview should consider five aspects such as adequacy of implementation, adequacy of response, plausibility of interpretation, replication, effects on the interviewer and the subject. The interview procedure and experiment definitely followed the scheme of "checklist for the successful interviewer" invented by Ginsburg (1997) in this thesis. The entire transcription of these interviews can be found in Appendix D and Appendix E. Chapter 4 also shows and clarifies more about the conceptual and ablated responses in detail.

Each interview included a set of 7 (check the number) pre-set questions on linear independence. These questions were similar for both student A24 and A25. During the interviews, added questions were asked whenever interviewer felt necessarily to better understand student Reponses to the pre-set questions. Interviewer did not correct student responses to reduce the potential influence of external inputs. To have a better understanding of the process and procedure used in our work, let's illustrate the theme in the following example.

3.4.1 Example:

Let's consider two students A and B being questioned and interviewed based on certain comparable questions. The interviews and questions including their responses were transcribed and translated into analytical formats for further analysis. This is exactly what you thesis is set

out to document (consistency between responses on take-home and interview; we cannot claim at this point of an existence of consistency). Again this is what thesis is set out to document; (We cannot claim any at this point). Constant comparison method was implemented to document categories of thinking modes from both assignments for each student, and furthermore, linear regression analysis was implemented, using percentages of responses on each category, to both entities for each individual student distinctly. The correlation lines were drawn in two separate x and y coordinate graphs (x= percentage of interview responses on each category, y= percentage of assignment responses on each category). Regression analysis was also implemented to document correlation between student A and B's responses.

Chapter 4: Results and Discussion

4.1 Qualitative Analysis

Two interviews and two assignments were selected, with the criterion of ensuring substantial input from homework assignments, for the purpose of this thesis; each interview contained the responses of one student from each section of the Matrix Algebra class during the spring 2009 semester. Each interview and assignment was individually conducted and analyzed. The analysis focused on the identification and classification of cognitive constructs, specifically thinking modes, found in the students' responses.

Results reported in this chapter were found by applying the Grounded theory (Glaser and Strauss, 1967). The idea of classifying the student's responses, focusing on thinking modes, was applied by using this theory. Codes found to be similar were merged into a single category. After codes were obtained, they were grouped for further analysis following the Glaser and Strauss's theory. The interview transcripts and assignments were first checked independently for similarities and differences present in the student responses in order to establish the terminology to be used for the naming of categories. Categories and codes were individually created for each student.

After all thinking modes were identified; a table of category of descriptions was created for each student. The frequency (counts) of types of thinking modes was recorded separately to address the research questions

This research shadows two main goals as follow:

1. To evidently reveal the differences and similarities between the mathematical thinking modes embedded within the student's mathematical ideas and the kinds of mental and cognitive interpretation during scientific interviews using comparison method.

2. To indicate if homework assessments can be a valid assessment tool to forecast students’ empathetic mathematical knowledge.

To address the themes above, the implemented regression analyses (sub-division) of the comparison method and the transcripts of the interviews and assignments were qualitatively and quantitatively studied and characterized. The emerging themes in the transcripts were analyzed and compared with the types of knowledge experiential on students’ thinking modes. The section below emphases on the knowledge derived from interviews and assignments of each student.

4.2 Classification of Responses

After the individual categories were formed, representative quotations were obtained from the transcripts and assignments to stand for each category. Once the students’ responses were categorized by the author of this thesis, his advisor performed the same analysis independently. The independent analyses were then compared to the findings of the author of this thesis in order to establish credibility.

The following subsections present the result found while analyzing the data obtained from the interview transcript of student A24. The following example of a student’s argument explains what aspects were taken into account and how it was categorized:

Student A24: “***They can be linearly independent if …they are...um…they form the whole plane for the whole space….because you …well, you at least need 3 to form the space, but those 3 need to be linearly independent”***

It can be seen that student A24 was focusing on the geometrical aspects of a set of vectors, he believes that 3 vectors are linearly independent if they form the whole space, since at least 3 linearly independent vectors are needed to cover the 3D space. He chose to focus on the fact that all the vectors are on the same space as well as the same plane which came from the fact

that it was a set of vectors in Rn. The category created for this argument was [NVC and PS] "Number of Vectors Compared to Rn and Planes and Space" respectively.

4.2.1 Modes of Thinking

The description of thinking modes identified by Sierpinska (2000) was followed in the analysis of the interview transcripts and assignments; this category was explained in section 2.1.2 of chapter 2. See chapter 2 of the thesis for detailed discussion on the particular framework. The thinking modes found in the responses of students A24, A25 are presented in the subsections below.

Students A24 and A25 belonged to Section A of three sections of a matrix algebra course considered in the NSF project. They were both interviewed separately at two different times by their own professor in the spring 2009 semester and answered to a similar set of questions. The following section orderly interprets the responses corresponding the interviews and assignments for student's A24 and A25.

4.2.2 Student A24

Tables 4.1 and 4.2 summarize the collected thinking modes for the interview and assignment performed by student A24. These tables show that each category of thinking mode has its own code, a response section and the indicator extracted from the transcripts belonged to student A24.

Table 4.1: Categories for Responses of Assignment Corresponded to Student A24.

No	Indicator	Assignment Analysis for Student A24	Category	Code
1	Mention/Statement/computation of Scalar Multiplication	The set W is linearly independent, because there is no scalar multiplication on a vector that can be obtained as a result of the other vector	Vector Operation	VOP

2	1-Mention/Use of Operation 2-Specific Computation of Linear combination 3- Mention/computation of infinite by many solutions	The set V is linearly dependent since the addition of the first 2 vectors gives you the third vector. And there are infinitely many results.	1-Vector Operation 2-Computed Linear Combination 3-Infinity Many Solutions	VOP CLC IMS
3	Mention/Statements/computation of Scalar Multiplication	Any scalar multiple of one vector can't be obtained as a result of the other vectors	Verbal Linear Combination	VeLC
4	Mention/Statements/computation of Scalar Multiplication	So, the accessary and sufficient condition for two vectors to be linearly independent is that, there should be no scalar multiple of a vector that will give as a result the other vector.	Verbal Linear Combination	VeLC
5	Number of Vectors Combined to the dimension of Rn	Being the same, the two vectors should not be on the same line, because they would be linearly dependent.	Geometry of Line	GL
6	1- Mention/Statement /computation of Scalar Multiplication 2-Scalar Multiplication 3-Specific Linear of Linear Combination	In order for three vectors to be lineally independent, the sum and scalar multiplication of a pair of vectors should not be equal to the third vector.	1- Vector Operation 3-Verbal linear Combination	VOP VeLC
7	1-Number of Vectors Compared to the dimension of Rn 2-Mention/Statements/of Linear combinations	If there are more than three vectors in R3, the vectors will be always linear independent, because there will be always	1-Number of vectors Compared to Rn 2-Verbal Linear combination	NVC VeLC
8	1- Number of Vectors Combined to the dimension of Rn 2-Mention/Statements/of Linear Combination	False, when there are more than two vectors in R2 the set is always linearly dependent, because there will always be a combination of any 2 or more vectors resulting in another vector.	1-Number of Vectors Compared to Rn 2-Verbal Linear Combination	NVC VeLC
9	1- Number of Vectors Combined to the dimension of Rn 2-Mention/Statements/of Linear Combination	False, when there are more than three vectors in R3 the set is always linearly dependent, because there will always be a combination of any 3 or more vectors resulting in another vector.	1-Number of Vectors Compared to Rn 2- Verbal Linear Combination	NVC VeLC
10	1-Number of Vectors Combined to the dimension of Rn 2-Mention/Used Operations	True, because none of the vectors can be gotten using the other vectors by neither scalar multiplication nor addition.	1-Verbal Linear Combination 2-Vector Operation	VeLC VOP
11	1-Indication of Application of Gauss Jordan Elimination Process 2- Mention/Statements of trivial solution	True, because when I input the matrix in the calculator it only showed that the only solution was a = 0, b = 0 and c = 0, this meaning that it is linearly independent	1-Row Echelon Form 2-Trivial Solution	REcF TS
12	1- Number of Vectors Combined to the dimension of Rn 2- Mention/ Use of Line	A.C.D and G since are only two vectors, it's in R2 and they are not on the same line, the set is linearly independent	1-Number of Vectors Combined to Rn 2-Geometry of Line	NVC GL

Table 4.2: Categories for Responses of Interview Corresponded to Student A24.

No.	Indictors	Interview's Responses by Student A24	Category and Analysis	Code
1	1- Number of vectors compared to the dimension of Rn	..depending on the space.. it's um…if you have an example R2 I take that ..if you have 2 vectors, they can be either linearly independent or not, when you have more than 2 they are always independent…no dependent…they are always dependent	Number of vectors Compared to Rn	NVC
2	1- Number of vectors compared to the dimension of Rn	if you have 3 vectors, it will always be dependent because you can get it from the combination of the initial 2 vectors you can get the third one	Number of vectors compared to Rn	NVC
3	1- Mention/Statement/Computation of Scalar Multiplication 2- Number of Vectors combined with the dimension of Rn	…yeah that they are a scalar multiple of each other, I would say that's a line because…um they can only …well there's …that's the only thing that would be formed from the combination of those 3.	1-Vector Operation 2- Geometry of Line	VOP GL
4	1- Number of vectors compared to the dimension of Rn 2- Mention/use of plane/Space 3- Dimension 4- Comparison	They can be linearly independent if …they are...um...they form the whole plane for the whole space….because you …well, you at least need 3 to form the space, but those 3 need to be linearly independent…	1-Number of Vectors Compared to Rn 2-Planes and Space	NVC PS
5	1- Specific Computation of Linear combination 2- Number of Vectors combined to the dimension of Rn	that would be linearly dependent, to form a line. because, with this vector.. you can get this one…multiplying by 2…and this one multiplying it by 3	1- Computational Linear Combination 2- Geometry of Line	CLC GL
6	1- Specific Computation of Linear combination 2- Mention/Statement/computation of Scalar Multiplication	well.. in order for it to be linearly dependent, you have to like… um either multiply this one …well you have to like.. add it up and if you get the third one…that would be the linearly dependent one.	1- Computational Linear Combination 2- Vector Operation	CLC VOP
7	Number of vectors compared to the dimension of Rn	it's because I…uh.. this is an R2 right? And R2 you have more than… two vectors that was linearly independent, so ..	Number of vectors compared to n in Rn	NVC
8	Number of vectors compared to the dimension of Rn	so when you have more than 2 vectors in R2, I have…this is where I understand	Number of Vectors Compared to n in Rn	NVC

		..okay…there are more 2 vectors in R2, there is always linearly independency		
9	Number of vectors compared to the dimension of Rn	Linearly dependent.. because this is what I know that R2 when there is more than 2 vectors in R2 that's what I..	Number of vectors compared to n in Rn	NVC
10	Number of vectors compared to the dimension of Rn	since you just added the ones, I'm just trying to figure out if that makes a difference and because this is now in R3 and there is three vectors that could be linearly dependent or independent. So, but I don't see the linearly dependency so…	Number of vectors compared to n in Rn	NVC
11	1- Number of vectors compared to the dimension of Rn 2-Mention/Statement /Computation of Scalar Multiplication	I'm trying to like subtract one vector, well just the number like one minus one, three minus two, one minus one, but if this were zero one zero, that's the way I would search for it.	1-Computational linear combination 2-Vector Operation	CLC VOP
12	1- Indication of Application of Gauss Jordan Elimination Process 2- Mention/Focus/Statement /use of identity	Well I know, like I would like try to get the RREF, row reduced. …and if this ends up in, in the identity, this would be linearly independent.	1-Row Echelon Form 2-Identity form	REcF IF
13	1- Indication of Application of Gauss Jordan Elimination Process 2- Mention/Focus/Statement /use of identity	when this, you apply Jordan relation…if you get the identity matrix it would be linearly independent.	1-Row Echelon Form 2-Identity form	REcF IF
14	1- Number of vectors compared to the dimension of Rn 2- Mention/Focus/Statement /use of identity form	um, well since I said that if It didn't like um, it's not the matrix, it's not the identity, I said that if that would be linearly dependent.	1-Identity form 2- Number of vectors compared to Rn	IF NVC
15	1- Number of vectors compared to the dimension of Rn 2- Mention/Focus/Statement /use of identity form	it's because I was trying to get, if this one is one, like I remember seeing zeros here, that would be like, I'm sure that's the linearly dependent but I'm not so sure	1-Identity Form 2- Number of vectors compared to Rn	IF NVC
16	Mention/Statement of Rows with all zeros	I would say it's linearly dependent because that last row equals to zero, but	Zero Row	ZR
17	Mention/Statement of Rows with all zeros	well yeah if I see the zeros here, I would be more sure, that it is linearly dependent but I can't figure out what's…	Zero Row	ZR
18	Mention/Statement/computation of Scalar Multiplication	um like use this vector once and multiply it twice	Vector Operation	VOP
19	Mention/Statement/computati	um, I'm trying to see that	Vector Operation	VOP

	on of Scalar Multiplication	would be linearly dependent… if the only result is zero, multiplying by zero		
20	Number of vectors compared to the dimension of Rn	I'd say that that's linearly dependent since it has more than 3 vectors in R3.	Number of vectors compared to n in Rn	NVC
21	Number of vectors compared to the dimension of Rn	Because…you only need the, in order to form this base, you only need, you only need um 3 vectors, but those three vectors need to be linearly independent.	Number of vectors compared to n in Rn	NVC
22	Mention/Statement/computation of Scalar Multiplication	and if you have a fourth one you can always get the fourth one, the fourth vectors with the combination of the first three or two	Verbal Linear Combination	VLC
23	Number of vectors compared to the dimension of Rn	I would say that it could be linearly dependent, if you use only those three vectors, and also could be if you use these two vectors	Number of vectors compared to n in Rn	NVC
24	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement /computation of Scalar Multiplication	Because that's like, the way I'm seeing it, if you have two vectors you have to only multiply one by the scalar in order to get the other one	1-Vector Operation 2-Computational Linear Combination	VOP CLC
25	Mention/Statement/computation of Scalar Multiplication	well…there are combinations that I could get 4 vector or… you could get any vector, with a combination of two or three vectors.	Verbal linear Combination	VLC
26	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement /computation of Scalar Multiplication	ok, like this one, this one, two times the first one. I am thinking, I mean I am seeing it this way, you use the first vector	1- Vector Operation 2- Computational Linear Combination	VOP CLC
27	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement /computation of Scalar Multiplication	yes, since it has only one, I would consider it the first one. Use this vector two times and just subtract the second one, in order to get the third one.	1- Vector Operation 2- Computational Linear Combination	VOP CLC
28	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement /computation of Scalar Multiplication	and also, how to get the fourth just use the product vector and subtract two line of second vector one from the second vector	1- Vector Operation 2- Computational Linear Combination	VOP CLC
29	1- Mention/Statement /computation of Scalar Multiplication 2- Mention/Use Focus on a	I wouldn't, not sure, be able to tell how to get. I am pretty sure it's linear dependent here because it has the last one	1- Verbal linear combination 2- Zero Row	VLC ZR

	row of a Matrix	equal to zero but I would not be able to tell how to get the for example forth vector, there is two		
30	Mention/Statement/computation of Scalar Multiplication	because, I am not seeing the scalar or if we get some, because if I would add it would be 1 plus two ,one plus 2 , 3 three plus one 4, but there is no one here.	Vector Operation	VOP
31	Mention/focus/Statement/use of identity	I am seeing, the first leading zero and the first lending ones. I would say that because of the ones…. would be independent b yes because of the ones	Identity Form	IF
32	Mention/focus/Statement/use of identity	since it's not a…because since it has already has one, it is independent vectors you don't't' need, you don't need these values,	Identity Form	IF
33	1- Focus on the size of a matrix compare to square matrix 2- Mention/Focus/Statement / use of Identity	. and just because the leading one and, that is very easy since it's not a square matrix I see it complete, because ends up here	1- Square Matrix 2- Identity Form	SM IF
34	Mention/focus/Statement/use of identity	if there was another vector that would like that one all the ones lined up, that would be linearly independent too	Identity form	IF
35	Mention\Use Focus on a row pf a Matrix	but if there was zero at the end that would linearly dependent …because of …oh I am not sure of those 0 0 's	Zero row	ZR
36	Focus on vectors on the same plane or not	I'm a thinking of ….I say that because the first three vectors are the same plane and the fourth one is not in the same plane, and the…b plane.	Plane Vectors	PV
37	1- Focus on vectors on the same plane or not 2- Number of vectors compared to the dimension of Rn	if they are saying that 3 vectors in the same plane. In order for it to be linearly dependent, you have two vectors in first two just in R2 vectors the first two, that the 3rd one is just …	1- Plane vectors 2- Number of vectors compared to n in Rn	PV NVC
38	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	because there is no way to get to the other vectors adding them up, so it would be linearly independent	1-Vector Operation 2- Computational Linear Combination	VOP CLC
39	1- Number of vectors combined to the dimension of Rn 2- Mention/Statement/comp	if they are on the same line, just multiply -1 by or just like shorter, just multiply by 2, got little bit more	1-Vector Operation 2-Geometric of line	VOP GL

	utation of Scalar Multiplication			
40	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication 3- Use of Linear Independence	because if there are linearly independent, they will be able to get any vector , or one of the vectors, you will get one of the vectors that are linearly dependent just adding them up or subtracting them, so just ….using first three , I say linearly dependent	1- Vector Operation 2- Computational Linear Combination 3- Linearly Independent	LI VOP CLC
41	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	in order to get any vector, 1 2 3 you will have to multiply this one by zero,	1- Vector Operation 2- Computational Linear Combination	VOP CLC
42	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	you have the four vectors, so in order to get this from you have to multiply this one by, there is one by zero and these two are	1- Vector Operation 2- Computational Linear Combination	VOP CLC
43	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	just seeing that, because you can get independent set, so the second vector here is going to be vector 5 times the first one	1- Vector Operation 2- Computational Linear Combination	VOP CLC
44	Use of Example to verify to compare the results	yes, I am seeing… I am trying to solve examples	By Example	BEx
45	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	And first, the first is going to be multiplying 5 times and the other two 1 2 3 t, so I am trying to see 6 , 7 and 8 that would be the result of the second vector 5 times	1-Vector Operation 2- Computational Linear Combination	VOP CLC
46	Number of Vectors Combined to the dimension of Rn	I can't think, but it is going to be well, is not going to be independent in R5 or more, it's not going to be.	Number of vectors compared to n in Rn	NVC
47	1- Number of Vectors Combined to the dimension of Rn 2- Mention/Statements/of Linear Combination	because they are more than 4 elements and two space the vector spaces, you can always have a linear combination of any vector using the first 4.	1-Number of vectors compared to n in Rn 2- Verbal Linear Combination	NVC VLC
48	Number of Vectors Combined to the dimension of Rn	if you have more than 3 vectors in R3, you always be linearly independent		
49	Number of Vectors Combined to the dimension of Rn	because it is in R3, there is 4 vectors	Number of vectors compared to n in Rn	NVC
50	Number of Vectors Combined to the dimension of Rn	if you use more than three vectors, in this case we will, we are going to get linearly	Number of vectors compared to n in Rn	NVC

		dependent set.		
51	Number of Vectors Combined to the dimension of Rn	there is 4 vectors, .. in this case the vector would be resulting would be linearly independent	Number of vectors compared to n in Rn	NVC
52	Number of Vectors Combined to the dimension of Rn	ok , I am going to be sure that if is in Rm, well m is equal to .. for it is less than 4 … this is going to be always linearly dependent	Number of vectors compared to n in Rn	NVC
53	1- Number of Vectors Combined to the dimension of Rn 2- Focus on number of rows compared to columns	so we have 4 sets, 4 vectors, sorry, .. and if they are 4 vectors, you started at R1 or R0?	1- Number of vectors compared to n in Rn 2-Number of rows versus number of columns	NVC
54	Number of Vectors Combined to the dimension of Rn	so if is in R0, it will be linearly dependent, and if is in one, it also going to be linearly dependent, if is in R2, it is going to be linearly dependent,	Number of vectors compared to n in Rn	NVC
55	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	because.. I mean if you have 5 times U1, which would be linearly independent, because you are using….the second set because you are adding the second vector… the results still will be independent	1-Vector Operation 2- Computational linear combination	VOP CLC
56	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	. because of U2, if you were going to use U1, that would be no problem saying that it is linearly dependent , you use the scalar in order to get the second.	1-Vector Operation 2- Computational linear combination	VOP CLC
57	Mention\Focus on the type of vector in the context of zero vector	…. of course if U2 is none zero, if its zero, the resulting will be dependent,	Zero vector	ZV
58	1- Number of Vectors Combined to the dimension of Rn 2- Use of Linear Independence	if linearly independent, it cannot get any combination of each of the vectors, using a set	1-Linear Independent 2- verbal linear combination	LI VeLC
59	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/ computation of Scalar Multiplication	because if you add 5 tines U1 to U2 , is independent since is not combination of each of the vectors , so if you multiply the first vector by 5, because you add the second vector, and is linearly dependent, it say that still is	1-Vector Operation 2- Computational Linear Combination	VOP CLC
60	Use of Example to verify to compare the results	um I'm trying to, well as I did , I was trying to think of examples	By Example	Bex
61	Number of Vectors Combined to the dimension of Rn	okay, I'm going to be sure that if it's in Rn, n is equal to , well	Number of vectors compared to n in Rn	NVC

		it's less than four…I would say that this is going to always be linearly….dependent		
62	Number of Vectors Combined to the dimension of Rn	if it's three it is also going to be linearly dependent . if it's four, that could be resulting independent or dependent, but since it's given to us that is its independent	Number of vectors compared to n in Rn	NVC
63	Number of Vectors Combined to the dimension of Rn	it's linearly independent if you cannot get any more combination of each of the vectors well using the set	Verbal linear combination	VeLC
64	Number of Vectors Combined to the dimension of Rn	Linearly combinations resulting in another vectors.	Verbal Linear Combination	
65	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	it's because I'm using both so. Since it's given to us that it's linearly independent each of the vectors is not a linear combination of each other um using the linear vectors…so even if you multiply it, the first vector, by 5…is that five? Because you add the second vector and it is linearly independent …you can say that it's still linearly independent	1-Computational Linear Vector Combination 2-Vector Operation	CLVC VOP
66	1- Number of vectors compared to the dimension of Rn 2- Mention/Statement/computation of Scalar Multiplication	uh…well I'm thinking of the matrix dividing it into vectors, so I would say that this is a vector and this is….so we're trying to see that the vector number 2 is equal to the vector 4 plus 3 times the vector 5 so…like the…like this question is	-Computational Linear Vector Combination 2-Vector Operation	CLVC VOP
67	Mention/Statement/computation of Scalar	I'm seeing it like …let's see 1 A3, 4 plus 3 times 5…A3 ,4 ….so this is the set.	Vector Operation	VOP

Based on Sierpinska's ideas, shown in section 2.1.2 Table 2.1 the categories reported in Table 4.1 can be separated into three different thinking modes recognized by Sierpinska (2000) as the ways linear algebra students think while solving problems, and answering questions. The thinking modes identified by Sierpinska (2000) are Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural.

Revisiting the information provided in Table 2.1, we can conclude that the categories omitting exact descriptions that based on their explanation on graphical representations can be classified into the Synthetic-Geometric thinking mode. Categories based on numerical and algebraic representations of objects requiring manipulation of data to arrive to conclusions, are classified into the Analytic-Arithmetic thinking mode, and the categories in which objects are analyzed with the use of theorems and definitions are classified as part of the Analytic-Structural thinking modes. Some of the categories found -contain ideas that would allow each category to be classified into more than one thinking mode. That is, categories are not mutually exclusive since the two students tended to use a combination of tools to arrive to conclusions and provide an answer. Table 4.3 illustrates the follow up technique of Sierpinska, (2000) by student A24. This table has categorized the responses and matched the student's thinking modes with the theoretical thinking modes described by Sierpinska as follow:

Table 4.3: Frequency of used categories for student A24

Frequency of Used Categories for Student A24				
		Interview	Assignment	Thinking Modes Type
Code	Category	Count	Count	
NVC	Number of vectors compared to Rn	26	4	Analytic- Structural
VOP	Vector Operation	22	4	Analytic-Arithmetic
GL	Geometry of Line	3	2	Synthetic-Geometric
PS	Planes and Space	1	0	Synthetic-Geometric
CLC	Computational Linear Combination	15	10	Analytic-Arithmetic
RREF	Row-Reduced Echelon Form	2	1	Analytic-Arithmetic
IM	Identity Matrix	8	0	Analytic- Structural
IMS	Infinitely Many Solutions	0	1	Analytic-Structural
ZR	Zero Row	4	0	Analytic- Structural
SM	Square Matrix	1	0	Analytic- Structural
PV	Plane Vectors	1	0	Synthetic-Geometric
LI	Linearly Independent	2	0	Analytic-Structural
BEX	By Example	2	0	Analytic- Arithmetic
VLC	Verbal linear combination	2	7	Analytic-Structural

The categories belonging to student A24 –summarized in table 4.2- can be classified as follows: the least amount (combined number of responses is 7) of thinking modes are categorized as Synthetic–Geometric for both the assignment and interview; GL, PS, and PV. There were a total of combined 55 responses that fit into the Analytic-Structural thinking modes: NVC, IM, ZR, SM, LI, and VLC. With these categories student A24 referred to numerical representations and manipulated numerical computations to determine linear dependence or independence. Lastly, the categories that can be classified into the Analytic-Arithmetic thinking mode are VOP, RREF, IMS, BEX, and CLC. The number of responses in the Analytic-Arithmetic categories is 56.

4.2.3 Statistical Analysis of Responses for Student A24

The following section will discuss student A24's interview and assignment results considering the conceptual and procedural knowledge of thinking modes using regression line analysis (the Comparison Method). Figure 4.1 depicts the application of the comparative method (regression line) depicting the interview response percentage on the x-axis and the assignment response percentage on the y-axis. Each (x, y) pair is constructed to reflect the interview percentage (x point) and assignment percentage (y point) for each thinking mode category present in the student's responses. Plotting all the pairs for the categories present will allow us to obtain information on the correlation of the interview and assignment responses. The relatively high value for the correlation coefficient (r=0.52) shows a relatively decent consistency level between the assignment and interview in this case, indicating that thinking mode categories with high response percentages in the student's interview is associated with high percentages in the assignment responses.

It is noted here that the ideal correlation between the x-axis and y-axis would be when x and y points equal to each other in every pair. We assume this to be the perfect case of x and y

correlation and use it to obtain the predicted value in this section. This ideal predicted value is plotted in Figure 4.1 as the red line with X=Y. The best fit line fitting the plotted (x, y) pairs is plotted as a black line and its deviation from the ideal X=Y case can be noted and even quantified in the residual error, as explained below. It is seen that at some points both predicted value (red line) and observe value (black line) are not too close to each other. We can see from the plot that the residual value is minimal in the region from 0% to about 10% on the x-axis. Thus, there is some consistency in this portion of the graph. It is important to note that the residual error percentage is calculated from the following formula:

$$Residual\ Error\ Percentage = \frac{Observed\ Value - Predicted\ Value}{Predicted\ Value}$$

Thus the residual error percentage is a measure of how far the (x, y) pair is from the x=y ideal. We can conclude that the wordings or dialogues for student A24, although not quite stable throughout his responses within the assignment and the interview, they do exhibit a rational pattern to be followed.

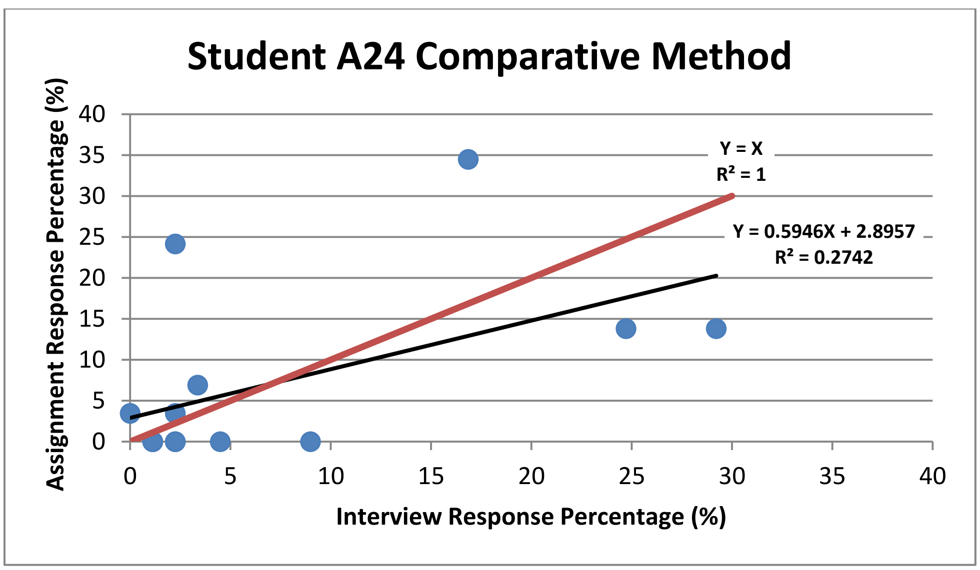


Figure 4.1: Comparative Method Analysis for Student A24

To quantify the findings in the above plot, we have calculated and categorized the results of the percent error including percent differences for both assignment and interviews in the Table 4.4 below illustrates fourteen different categories including the corresponding counts and the percent of occurrence of thinking modes throughout the student's responses for the assignment and interview. As was mentioned in Chapter 3, the statements and relevant dialogues were categorized by different themes. Each category contains more than one statement and relevant dialogue that all reveal the same theme throughout an interview and assignment.

Table 4.4: Percent Error Calculation Using Percent Responses of the Assignment and Interview of Student A24

Count of Responses for Student A24							
		Interview		Assignment		Percent Error	
Code	**Category**	**Count**	**Percentage (%)**	**Count**	**Percentage (%)**	**Pr(I)- Pr(s)**	**[Pr(I)- Pr(s)]/Pr(I)**
NVC	Number of vectors compared to Rn	26	29.21	4	13.79	15.42	0.53
VOP	Vector Operation	22	24.72	4	13.79	10.93	0.44
GL	Geometry of Line	3	3.37	2	6.90	-3.53	-0.51
PS	Planes and Space	1	1.12	0	0.00	1.12	1.00
CLC	Computational Linear Combination	15	16.85	10	34.48	-17.63	-0.51
RREF	Row-Reduced Echelon Form	2	2.25	1	3.45	-1.20	-0.35
IM	Identity Matrix	8	8.99	0	0.00	8.99	1.00
IMS	Infinity Many Solutions	0	0.00	1	3.45	-3.45	-1.00
ZR	Zero Row	4	4.49	0	0.00	4.49	1.00
SM	Square Matrix	1	1.12	0	0.00	1.12	1.00
PV	Plane Vectors	1	1.12	0	0.00	1.12	1.00
LI	Linearly Independent	2	2.25	0	0.00	2.25	1.00
Bex	By Example	2	2.25	0	0.00	2.25	1.00
VLC	Verbal linear combination	2	2.25	7	24.14	-21.89	-0.91
Sum		**89**	**100.00**	**29**	**100.00**		

Table 4.4 shows that the top 3 most frequent thinking modes for the interview and the student's questions are the categories of NVC, VOP and CLC. It is interesting to see that responses more frequently fell into the above categories. This might be because of the phraseology of the questions for both interview and assignment as well. Notice that NVC, VOP and CLC stand for the "Number of Vectors Compare to R^n", "Vector Operation" and "computational Linear Combination" respectively.

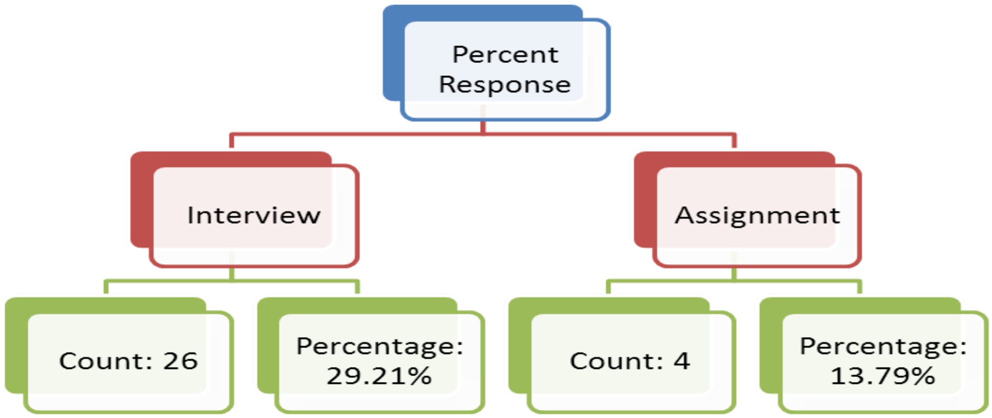
To clarify the above correlation, I have statistically measured the percentages for both the interview and assignment performed by student's A24. As an example, the highest percentage within the interview's category (29.21%) signifies the percent value of the thinking modes corresponding to student's A24 responses that fell under the NVC category. Concurrently, the highest percentage (34.48%) of the thinking mode responses falling within the assignment of the student's A24 signifies the most frequent use of the corresponding category of CLC.

Table 4.4 displays the calculated percent error between both interview and assignment in the category of NVC for the student A24 to be about 0.53. The percent error indicates the measure of closeness of the thinking modes from both interview and assessment. It is important to note that the smaller the error percentage is, the more accurate and closer the thinking modes for both the assignment and interview are. We can see from the table that no interview response fell into the category of "Infinitely Many Solution" (IMS), while the assignment did have such responses. This means that there does not appear to be any consistency or agreement between the categories of the two assessment tools. The least number of responses fell into the "planes and Spaces" category for both assignment and interview. A total of 14 categories were revealed during the interview and assignment responses.

4.3 Percent Error Calculation for Student A24's Responses

This example illustrates the calculation of the percent error for one of the responses in Table 4.1 for the student's responses on the interview and assignment considering their measured percentage values. To calculate the error, first we need to calculate the percentage values for both interview and assignment using the number of their occurrence (counts) of thinking modes within each of the category where 89 and 29 are the total number of counts (frequencies) appeared in both interview and assignment respectively for each student's responses separately.

Given Information (Refer to Table 4.1):
Category: ***Number of vectors compared to Rn (NVC)***



Computation:

$$\% \text{ Error} = \frac{Percent(I) - Percent(A)}{Percent(I)} = \frac{29.21 - 13.79}{29.21} = 0.53$$

$$\text{Percentage (A) } \% = \frac{4}{29} = 0.1379 = 13.79\%$$

$$\text{Percentage (I) } \% = \frac{26}{89} = 0.2921 = 29.21\%$$

4.3.1 Statistical Analysis of Responses for Student A25

The argument about the interview and assignment for the student's A25, is very similar to the discussion and analogy of student's A24. The theoretical analogy of thinking modes clarification using regression line (comparison method), has been similarly performed and the results were categorized and tabulated as table 4.5 and 4.6 depicting the categories of the assignment and interviews for student's A25. This section eloquently discusses and analyzes the dialogues formed from the mentioned transcripts within the categories given the tables.

Table 4.5: Student A25 Assignment Categories

No	Indicator	Assignment's Responses By Student A25	Category	Code
1	Verbally….the use of linear combination	linearly independent as one vector is not a multiple of the other	Verbal Linear Combination	VLC
2	Verbally….the use of linear combination	dependent as one vector is a linear combination of other two	Verbal Linear combination	VLC
3	1- Mention\use of Rn 2- Verbally….the use of linear combination 3-Mention/Use of vectors and Scopes in a Matrix	one vector needs not be a multiple of a second vector to be linearly independent vectors in R3. These vectors have to create a plane and will do so if they are two different vectors.	1-Rn 2- Verbal Linear combination 3-Number of vectors and dimensions	1-Rn 2-VLC 3-Nvc&d
4	Mention\use of Rn	one vector must not lie in plane formed by other two vectors	1-Rn	Rn
5	1- Mention\use of Rn 2- Mention/Use of vectors and Scopes in a Matrix 3- Verbally….the use of linear combination	In R3 there can be a maximum of three independent vectors if they create three different planes. Additional vectors can be expressed as linear combination of other vectors	1-Rn 2- Number of vectors and dimensions 3- Verbal Linear Combination	1-Rn 2-Nvc&d 3-VLC
6	1- Mention\use of Rn 2- Mention/Use of vectors and Scopes in a Matrix 3- Verbally….the use of linear combination	False. In R3 a maximum of three linearly independent vectors can exist	1-Rn 2- Number of vectors and dimensions 3- Verbal Linear Combination	1-Rn 2-Nvc&d 3-VLC
7	1- Mention\use of Rn 2- Mention\use of Vector Component	False. In R3 a maximum of three linearly independent vectors can exist	1-Rn. 2-Vector Component	1-Rn 2-VcCo
8	1- Mention\use of Rn 2- Mention\use of Vector Component	False. Not always will there be three as one could be linear combination of others	1-Rn. 2-Vector Component	1-Rn 2-VcCo
9	Verbally….the use of linear combination		Verbal Linear combination	VLC
10		True. It is linearly independent.	No explanation	N\A
11	Verbally….the use of linear combination	One is a linear combination of the other two	Verbal Linear Combination	VLC

Table 4.6: Student A25 Interview Categories

No	Indicator	Responses By Student A25	Category	Code
1	1-Mention\use of computation Solution type 2- Mention\use of Identification of singular or non-singular Matrix	It would be that um, so the vectors you have that you for a linear Independence's you have none trivial solutions so that it is...singular.	1-Solution type 2- Singular	1-ST 2-Sing
2	Mention\use of Identification of singular or non-singular Matrix	Singularity it would be that, if it's linear independent, and it means that it is … is it singular?	Singular	Sing
3	1-Mention\use of	Singularity would mean um, its	1-Solution type	1-ST

	computation Solution type 2- Mention\use of Identification of singular or non-singular Matrix	going by, or going with a none trivial solutions that it has, so it is singular it means he only has none trivial solutions…	2- Singular	2-Sing
4	1-Mention\use of system of linear equation 2- Mention\use of Identification of singular or non-singular Matrix	so if it's, so if giving a system that is singular, meaning that the set represents a system it's a linear independent or dependent?	1-System of linear Equation 2-Singular	1-SLE 2-Sing
5	1-Mention\use of application of verbal operation 2-Verbally….the use of linear combination 3- Mention\use of computation Solution type 4- Mention\use of Identification of singular or non-singular Matrix 5- Mention\use of system of linear equation	so if it's a linear independent it means that it cannot be other multiples of the other vectors so it means it only has the none trivial solutions here, so a singular system	1-Vector Operation 2-Verbal Linear Combination 3-Solution Type 4-Singular 5- System of Linear Equation	1-VOP 2-VLC 3-ST 4-Sing 5-SLE
6	Mention\use of Identification of singular or non-singular Matrix	the set is none singular, and it would be also linear independent	Singular	Sing
7	1-Mention\use of Rn 2- Mention\use of application of verbal operation 3-Mention of computation of Linear Combination	Say in R3, would be a set given by um, R3, 1 ,1, 0, another set 2, 2, 1, so that there in a multiples of each other and then the third one would gonna have to be multiples of the other two or a linear combinations with it so it would be 4, 0, 1, um, that should work	1-Rn 2-Vector Operation 3-Computed Linear Combination	1-Rn 2-VOP 3-CLC
8	1- Mention\use of application of verbal operation 2- Mention of computation of Linear Combination	Well try to see if this one is a multiple of the other two so it is not, and also if adding these two and multiplying by any constant it would yield , so we .., by three	1-Vector operation 2-Computed Linear Combination	1-VOP 2-CLC
9	1- Mention\use of application of verbal operation 2- Verbally….the use of linear combination	Just saying if this one times the constant plus this one times any other constant would yield the third one.	1-Vector operation 2-Verbal Linear Combination	1-VOP 2-VLC
10	1- Mention\use of system of linear equation 2-Mention\use of computation of row reduced form	so,, it could be a system of equations and then you could solve,, just trying to see mentally if it could work, and it would be a linear independent system	1-system of Linear Equation 2-RREF	1-SLE 2-RREF
11	1- Mention\use of application of verbal operation 2- Mention of computation	actually no, it is a linear dependent system. Because this one um, multiply by 2 and added with this vector yields that their	1-Vector Operation 2-Computed Linear Combination	1-VOP 2-CLC

	of Linear Combination	vectors is a Linear dependent system		
12	1- Mention\use of system of linear equation 2- Mention\use of computation Solution type	So I'm trying to ,, pretty much do the systems of equitation's so it be that 1x plus 1 y plus 0z equals to 0 vector and then 2x plus 2y, I guess this would be equal, yeah, plus 2 y plus 1z is 0 and then 4x plus 0y plus 1z equals to 0, so we find any solutions of x y and z and see if the solutions would be the.. um	1-System of Linear Equation 2-Solution Type	1-SLE 2-ST
13	1- Mention\use of Rn 2-Mention\use of Vector Component	well, we know that.. this is one vector and just equals to 0 so have them all each of the vectors, the components so if it's in R3, this would be the x component for all of them, this would be the y component for all of them and this would be the z component for all of them so set that to um,, another 0 vector	1-Rn 2-Vector Component	1-Rn 2-VCm
14	1- Mention\use of Rn 2-Mention\use of Vector Component	Just the... components of any vectors in the R3, so it can be any other variables. it's just	1-Rn 2-Vector Component	1-Rn 2-VCm
15	Mention\use of Vector Component	Right, just that this one,, this components are a different component from second row and then from the third row	Vector Component	VCm
16	Mention\use of computation Solution type	If a solution is given by each variable, has a given value, then it would be,, or a 0 value, then it would be a independent set, independent set, if otherwise it gives that x y or z are depending on the other variable then it would be a dependent set if it's going to be in terms of the other..	Solution Type	ST
17	Mention\use of computation of row reduced form	So that this one has this value there, so we just divide it by 4, the first row,, so it would be... 1, 0 and 1/4 and the rest would stay the same.. and again.	RREF	RREF
18	Mention\use of identification of identity matrix	If it would be linear independent it would just be umm, like a …identity.. matrix... 0 0 0	Identity Matrix	IM
19	1-Mention\Application of Matrix Equation 2- Mention\use of computation Solution type	Linear independence?.. It would mean that given the vector equation of Ax equaling to 0, that the only solution x would be a .. it would have only the nun-	1-Matrix Equation 2-Solution Type	1-ME 2-ST

		trial solution		
22	Mention\use of computation Solution type	If a solution is given by each variable, has a given value, then it would be,, or a 0 value,, then it would be a independent set , independent set, if otherwise it gives that x y or z are depending on the other variable then it would be a dependent set if its going to be in terms of the other..	Solution Type	ST
23	Mention\use of computation of row reduced form	With the matrices. so i guess its easier to just put it as a matrix, so it would be 1,1,0,2,2,1, 4,0,1, so I'm just applying Gauss Jordan It would be umm, it would be changing this one, just dividing it by half and adding it to the first row so it would give you.. umm.. 401,221, and 110.	RREF	RREF
24	Mention\use of computation of row reduced form	So that this one has this value there, so we just divide it by 4, the first row,, so it would be... 1, 0 and 1/4 and the rest would stay the same.. and again.	RREF	RREF
25	Mention\use of identification of identity matrix	If it would be linear independent it would just be umm, like a …identity.. matrix... 0 0 0	Identity Matrix	IM
26	1- Mention\Application of Matrix Equation 2- Mention\use of computation Solution type	Linear independence?.. It would mean that given the vector equation of Ax equalling to 0, that the only solution x would be a .. it would have only the nun-trial solution	1- Matrix Equation 2- Solution Type	1-ME 2-ST
27	Mention\use of Vector Component	This is the own dimensions. X Y and Z.. So it would have to be a 3 by 1.	Vector Component	VCm
28	1- Mention\use of computation of row reduced form 2- Mention\use of identification of identity matrix	so you could just write a 3 by 3 matrix using that set, so 2,4,8, and 3,5, 11, so carrying custodian elimination.. it would yield umm, and then the last row being equal to 0.	1-RREF 2- Identity Matrix	1-RREF 2-IM
29	1- Mention\use of 2-computation Solution type column Matrix Mention of Column of a Matrix	yeah for it to be linear independent it could also be constant the last column, so that means that x depends on the 3rd variable and , or y depends on the other variable.	1- Solution Type 2-Computed Matrix	1-ST 2-CM
30	Mention\use of Identification of singular or non-singular Matrix	It would be that x plus 2y plus 3c equals to 0, and 3x plus 4y equals 5z, and then 5x plus 8 y plus 11z equals 0. and then we	System of linear Equation	SLE

		are setting it equal to 0 because we are saying that the vector equation, having the vector 0 so		
31	Mention\use of Identification of singular or non-singular Matrix	It would be x plus y,, hmm, 2x plus 2y equals z, and then 4x plus c,, it would be this vector and then it would have to equal the 0 matrix.	System of linear Equation	SLE
32	1- Mention\use of computation of row reduced form 2- Mention\use of identification of identity matrix 3- Mention\use of Identification of singular or non-singular Matrix	the matrix, 1, 2, 3, 4, 5, divide it by 11,and i wanna plug in the x y z matrix and then that should give you the 0 matrix, so then this can be re-written as just x plus 3y plus 5z, 2x plus … It should equal to 0 with zero matrix	1-RREF 2- Identity Matrix 3- System of linear Equation	1-RREF 2-IM 3-SLE
33	Mention use of Compute Application of Matrix Operation	because its carrying the matrix multiplication, so just by definition it would be this row times that one and then adding them together and then this row times that one and adding the components and so on.	Matrix Operation	MOP
35	1- Mention\use of Rn 2- Mention\use of Vector Component	well, since, here we are dealing with vectors with in R3 so it could be said that this is special dimensions and in coordinates, x y z coordinates so that’s why.	1-Rn 2-Vector Component	1-Rn 2-VCm
36	1- Mention\use of Rn 2 Mention\use of Vector Component 3-Mention Use Computed of the Size of Matrix	Yeah cuz that’s the way i was thinking that it should be since given one vector so its R3 so this should be the x y and z but then one thing it has a whole 3 by 3 matrix cannot be said that this or the x y and z that rows are .. because it doesn’t work when writing a system of equations	1-Rn 2- Vector Component 3-Matrix Size	1-Rn 2-VCm 3-MSz
37	1- C 2- Mention\use of identification of identity matrix	mhm, yeah so the,, um, gas Jordian approach could be done to them to this matrix so as to get weather or not its result in identity matrix or just with the one or the last two rows equal to 0.	1-RREF\ 2- Identity Matrix	1-RREF 2-IM
38	Mention\use of identification of identity matrix	Well you stop and then once you see that if the last row or the last two rows are 0 have all 0 entries, then it means that its linear dependent.	Identity Matrix	IM

39	1- Mention\use of identification of identity matrix 2- Mention\use of computation Solution type	Since the last rows is all 0s it means that that row where it should be a 1 here that means, so that the z component depends on the, it has a given value, if its all zero that means that the z is um, arbitrary so that means that x and y can have multiple values, given any z since its arbitrary so that means, that's many solutions then its linearly dependent.	1-Identity Matrix 2- Solution Type	1-IM 2-ST
40	1- Mention\use of identification of identity matrix 2- Mention\use of computation Solution type 3- Mention\use of Identification of singular or non-singular Matrix	well, like i said its just if the row is,,it has zero entries for the row, then that means that the z or whichever other variable considering it has no real given value, no given value,, so then, it has many values it can be assigned, infinite values , which means that the set has multiple solutions. So going back with uh tying linear independent with the singularity that means it means it has multiple solutions, so it's a singular system and its a linearly dependent set	1- Identity Matrix 2- Solution Type 3-Singular	1-IM 2-ST 3-Sing
41	1- Mention\use of computation Solution type 2- Mention\use of computation Solution type	the set…the sys.. uh…the system. given by this matrix has multiple solutions so the set that is representing here with the vectors, it will be singular. So it means it has, its a linearly dependent	1- Solution Type 2- Singular	1-ST 2-Sing
42	Mention\use of computation Solution type	well since we are saying that that Z's is arbitrary, so that I can have any value, it means it can be any multiple of whatever value x or y has. Is that what you..	Solution Type	ST
43	Mention\use of computation Solution type	yeah the z can have a multiple of x and y same same could be any value.	Solution Type	ST
44	1- Mention\use of Rn 2- Mention\use of Vector Component 2- Mention\use of Identification of singular or non-singular Matrix 3-Mention\Use of Comparison of Number of Equations to Number of Unknowns	Again since we have, i guess, also dealing with the dimensions say since we said that the columns represent um the compound of that coordinate which means that....the vertical so since we have three unknown variables and four equations, that means that one equation is the	1-Rn 2- System of linear Equation 3-Number of Equations compared to Unknowns	1-Rn 2-SLE 3-NECU

		multiple and linear combination of other equations of say .. so since we have three vectors, and then each has..uh i don't know how to explain it.		
45	1- Mention\Use of Comparison of Number of Equations to Number of Unknowns 2- Mention\use of system of linear equation 3- Verbally….the use of linear combination	well the dimensions, since we have, this could be seen as a system with 4 equations and only 3 variables, so since there is more equations than variables that means that one equation would have to be a linear combination of the other	1- Number of Equations Compared to Unknowns 2- System of linear Equation 3- Verbal Linear Combination	1-NECU 2-SLE 3-VLC
46	Verbally….the use of linear combination	since we are saying that one equation is the combination of the other, then it will be an linearly dependence, this matrix represents linearly dependent set.	Verbal Linear Combination	VLC
47				VecAnal
48	Mention\Use of Comparison of Number of Equations to Number of Unknowns	well there is, there would have to be another variable, but we don't since we have half the columns. So the number of variables that are in the system...I'm trying to figure out where the extra row, if it gives extra equations if it is just be that...	Number of Equations Compared to Unknowns	NECU
49	Mention\use of Vector Component	well because I was trying to see if this row was the x component and this was the y and z and see in the same trouble I´m having here	Vector Component	VCm
50	1- Mention\use of Rn 2- Verbally….the use of linear combination	because a plane can be formed with just two vectors, and it's the linearly combination of the two vectors that can form a plane. So having an additional vector means that that set will be linearly dependent.	1-Rn 2- Verbal Linear Combination	1-Rn 2-VLC
52	Verbally….the use of linear combination	a set just another set assigned by U1 U2 and U3. So if we add U4 vector to the set, it will still be linearly deponent since we already said that U3 will be a linear combination.	Verbal Linear Combination	VLC
53	Verbally….the use of linear combination	yes. So it's saying that replacing one of the vectors in the set, to be an addition of that and the	Verbal Linear Combination	VLC

		other one, so that means that this vector already is the combination of another vector and thus the set will be linearly dependent.		
54	Verbally….the use of linear combination	so if we have the given set, U1 U2 and U3 which is linearly independent, it means for having another set to be linearly dependent it means that we have another vector which is just another linear combination, which is U1 and U2 or it can be U3 or U4. This could be linearly dependent since we have this vector is already a combination of previous vectors which are in the set. But since this one U2 is not there anymore, this vector is not a real combination of previous vectors, it's just a new assigned vector, so this set will be linearly independent	Verbal Linear Combination	VLC
57	Verbally….the use of linear combination	since we have U2 and it does not belong to the set anymore, it means that U2 plus U1 is not the combination of the other vectors, since U2 does not belong to the set anymore, but it is only the orientation of…	Verbal Linear Combination	VLC
58	Verbally….the use of linear combination	well if it is just given those vectors we will have to define that at least that U1 is not a multiple of U3 or U4 and so on that the others…is that what you?	Verbal Linear Combination	VLC
59	Verbally….the use of linear combination	okay well you need to define already define a set containing those vectors but we still need to define the vectors so that U1 is not any constant times U3 or U1 is not the constant times U4.	Verbal Linear Combination	VLC
60	Verbally….the use of linear combination	so just say…so umm … okay so we just have to define U1 cannot be equal to any constant time U3 or U4 or so on. For c is just a constant.	Verbal Linear Combination	VLC
61	1- Mention\use of application of verbal operation 2- Verbally….the use of linear combination	because if one other vector is a scale multiple of the others, it mean that one is a linear combination of the other once	1- Vector Operation 2- Verbal Linear Combination	1-VOP 2-VLC

		since it’s the multiple of it.		
63	Verbally….the use of linear combination	yeah I would also have to check, that U1 plus U2 is not any constant times scalar multiple of U1 or U3 or U4	Verbal Linear Combination	VLC
64	1-Mention\use of application of verbal operation 2-Verbally….the use of linear combination	uh also that… just U1 is not equal to another, the scalar multiple of U3 plus any other scalar multiple of U4, so that this defines U1 is not a linear combination of the others, which is the the definition of linear independence	1 Vector Operation 2-Verbal Linear Combination	1-VOP 2-VLC
65				
66	1-Mention\use of application of verbal operation 2-Verbally….the use of linear combination	yeah so for this one it will be linearly independent if and only if U2 plus U1 does not equal CUn. So that the summation of this fold does not add up to the scalar multiple of U1.	1 Vector Operation 2-Verbal Linear Combination	1-VOP 2-VLC
67	1- Mention\use of computation of row reduced form 2- Mention\use of Rn 2- Mention\use of Vector Component	well given that, you know the values and dimensions of U1 and the other vectors, and you can carry on the operations and check.	1-RREF 2-Rn 3- Vector Component	1-RREF 2-Rn 3-VCm
69	1-Mention\use of application of verbal operation 2-Verbally….the use of linear combination	well since we already said that U1 is now combination of U3 and U4, by the Independence of this set, but here we’re adding these two vectors, but since neither of them are the combinations of the other two that means that it’s just an addition, will not be just be a combination of the other two as well, so it will not really , we don’t really have to check if this one is the scalar multiple of this one or that one. You only have to check that adding these two vectors is not a scalar multiple of the vectors that It contains.	1 Vector Operation 2-Verbal Linear Combination	1-VOp 2-VLC

71	1- Mention\use of identification of identity matrix 2- Mention of computation of Linear Combination	well we're saying, not the second column, but since we're dealing with all I's then we're saying already all rows, so the second entry of any row we can just generalize it to the second column will be equal to the fourth column plus the third, three times the 5th column.	1-Identity Matrix 2- Computed Linear Combination	1-IM 2-CLC
72	1- Mention\use of identification of identity matrix 2- Mention\use of identification of identity matrix	okay so we can write the set as the matrix A, and then rewrite it, carry a Gas Jordan elimination process then we yield that the process is not an identity matrix, first of all since it's not a square matrix and also the last row, or any other rows, could have zero entries.	1- Identity Matrix 2-RREF	1-IM 2-RREF
73	Mention\use of identification of identity matrix	well since like with this example we are saying that, if it's linearly dependent it would have zero entries and so on, so it's the same thing, with just any given matrix	Identity Matrix	IM
74	Mention\use of identification of identity matrix	well since it's not square, we already know that at least you cannot yield identity matrix, and since identity matrix are around the square matrices	Identity Matrix	IM
75	Mention\use of computation Solution type	hmmm okay, so we have infinitely many solutions for this set, which is saying in any given N, so since it has infinitely many solutions, that it's a dependent set,	Solution Type	ST

4.3.2 Counts of Responses for Student A25

The counts and residual values analogy of responses by student's A25 are exactly similar to the analogy performed by student A24 in section 4.2.3 with different results. To save time redoing the process, I have just shown the measurements of the percent error calculation and thinking modes discussion through Tables 4.6, 4.7 and Figure 4.2 below.

Table 4.6: Percent Error Calculation Using Percentages of the Assignment and Interview of Student A25

Count of Responses for Student A25								
		Interview		Assignment		Percent Error	
Codes	Category	Count	Percentage (%)	Count	Percentage (%)	Pr(I)- Pr(s)	[Pr(I)- Pr(s)]/Pr(I)
ST	Solution type	12	12.50	0	0.00	12.50	1.00
Sing	Singular	8	8.33	0	0.00	8.33	1.00
SLE	System of linear Equation	7	7.29	0	0.00	7.29	1.00
VOP	Vector Operation	9	9.38	0	0.00	9.38	1.00
VLC	Verbal Linear Combination	16	16.67	7	36.84	20.18	0.55
Rn	Rn	6	6.25	7	36.84	30.59	0.83
CLC	Computed Linear Combination	4	4.17	0	0.00	4.17	1.00
RREF	Row-Reduced Echelon Form	9	9.38	0	0.00	9.38	1.00
VCm	Vector Component	7	7.29	0	0.00	7.29	1.00
ME	Matrix Equation	2	2.08	0	0.00	2.08	1.00
IM	Identity Matrix	10	10.42	0	0.00	10.42	1.00
CM	Computed Matrix	1	1.04	0	0.00	1.04	1.00
MOP	Matrix Operation	1	1.04	0	0.00	1.04	1.00
MSz	Matrix Size	1	1.04	0	0.00	1.04	1.00
NECU	Number of Equations compared to Unknowns	3	3.13	0	0.00	3.13	1.00
Nvc&d	Number of vectors and dimensions	0	0.00	3	15.79	15.79	1.00
VcCo	Vector Component	0	0.00	2	10.53	10.53	1.00
Sum		**96**	**100.00**	**19**	**100.00**		

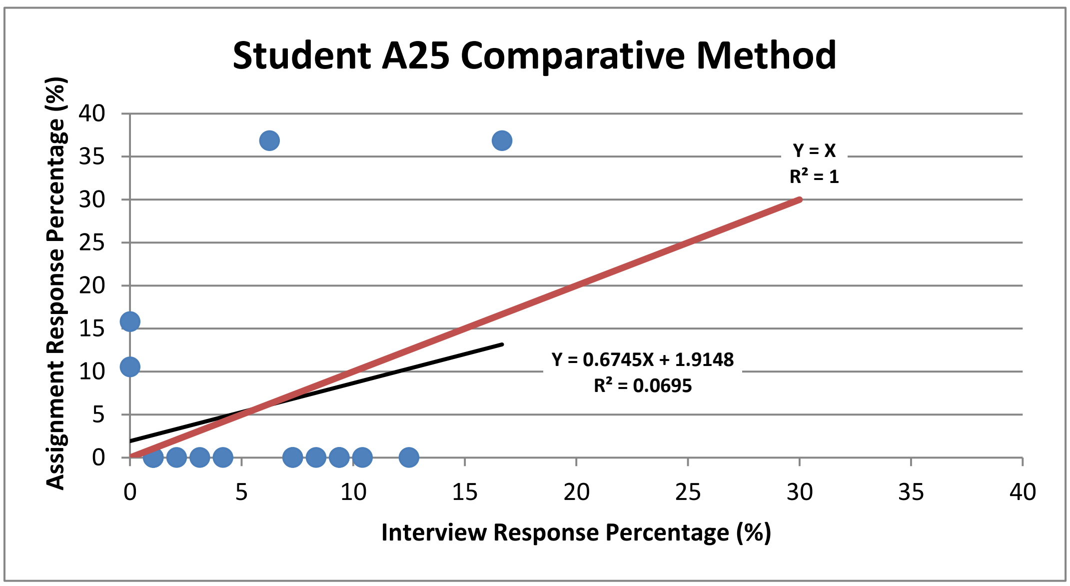


Figure 4.2: Comparative Method Analyses for Student A25

The following section will discuss the student A25's interview and assignment results belonged to the thinking modes using regression line analyses. Figure 4.2 portrays the application of the comparative method (regression line) including the percentages for the

interview and assignment questions for the student A25. We can see from the Figure 4.2 that the observed and the predicted values don't accord well. So, we can conclude that the thinking modes provided by assignment and interview are not consistent at any portion of the experiment.

Table 4.7 shows the frequency of the thinking modes adaptation using three thinking modes derived by Sierpinska (2000). As we said in the above section, since the thinking modes of assignment and interview are not consistent, I cannot have a consistency argument using the thinking modes categories in Table 4.8.

Table 4.7: Frequency of used Categories for Student A25

Frequency of Used Categories for Student A25				
		Interview	Assignment	Thinking Modes Type
Codes	Category	Count	Count	
ST	Solution type	12	0	Analytic-Structural-
Sing	Singular	8	0	Analytic-Arithmetic
SLE	System of linear Equation	7	0	Analytic-Structural-
VOP	Vector Operation	9	0	Analytic-Arithmetic
VLC	Verbal Linear Combination	16	7	Analytic-Structural-
Rn	Rn	6	7	Analytic-Arithmetic
CLC	Computed Linear Combination	4	0	Analytic-Structural-
RREF	Row-Reduced Echelon Form	9	0	Analytic-Structural-
VCm	Vector Component	7	0	Analytic-Arithmetic
ME	Matrix Equation	2	0	Analytic-Structural-
IM	Identity Matrix	10	0	Analytic-Arithmetic
CM	Computed Matrix	1	0	Analytic-Structural-
MOP	Matrix Operation	1	0	Analytic-Arithmetic
MSz	Matrix Size	1	0	Analytic-Structural-
NECU	Number of Equations compared to Unknowns	3	0	Analytic-Arithmetic
Nvc&d	Number of vectors and dimensions	0	3	Analytic-Arithmetic
VcCo	Vector Component	0	2	Analytic-Structural-

4.3.3 Examination of Thinking Mode Frequency in Student A25's Responses

The counts and residual error analysis of responses from student A25 are similar to the analysis performed on student A24 in section 4.2.3 but with different results as follows:

This section discusses the results of the interview and assignment obtained from the responses collected from student's A25 work in this thesis. The procedure of calculation of percent error

and percent responses including the extracted thinking modes using regression line analyses are portrayed in example of section 4.3, Tables 4.7, 4.8, and Figure 4.2. We can clearly detect the trend of the responses and thinking modes in Figure 4.2. We can conclude that the thinking modes provided by the analysis of assignment and interview are not relatable at any portion of this experimental plot. For the sake of clear plot analysis, we can see from the calculation of percentage error values in Table 4.6, the lack of correlation between the assignment and interview's categories. This table shows the only two categories with percent values below 100% as "Verbal Linear Combination" (VLC) and "Rn" with error percentages of 55% and 83**%,** respectively. The rest of the categories demonstrate a 100% error percentage. So, we conclude that there is no consistency or any relation between the dialogues made in assignment and interview by student A25 in this thesis.

Table 4.7 shows the frequency analysis of the categories for student's A25. This table indicates only 9 counts for Analytic-Structural and 10 counts for Analytic- Arithmetic corresponding to the assignment category. It also depicts zero value for the frequency of other categories used to evaluate the types of thinking modes for the assignment. We can only rely on the interview counts to adopt the thinking modes type examination in our case with, 52 counts for Analytic-Structural, 42 counts for Analytic-Arithmetical and zero count for Analytic-Geometrical thinking modes.

It is important to note that each category contained more than one statement and relevant dialogue that all revealed the same theme throughout an interview and assignment in student A25's task.

4.4. Thinking Modes Comparison between the Interviews and Assignments of Students A24 and A25

Going back to Tables 4.3, 4.7, and Figures 4.1, 4.2, we can conclude that the level of exposure of student A24 to all three different thinking modes types (Analytic-Arithmetic, Analytic-Structural, and Analytic-Geometrical) in his dialogues during his interview and responding to the questions of the assignment, is more than that of student A25. To elaborate more on this statement, let's look at Table 4.8; This table shows more than 76% of the responses regarding the thinking modes types is zero value compare to the interview's responses performed by student A24 and student A25 as well. Moreover, the total amount of counts in the categories of Analytic-Arithmetical (55 counts) and Analytic-Structural 56 counts) for student A24 are less than the total amount of counts of Analytic-Arithmetical (61 counts) and Analytic-Structural (54 counts) used in the student A25's responses. This analysis also shows that the students used different conceptual thinking modes briefly based on their cognitive way of understanding the algebraic problems. Finally, Figures 4.1 and 4.2 also illustrate this analysis since there isn't any similarity within their observed values and their corresponding error percentage in their categories.

One of the similarities between students A24 and A25 is that they all used arguments that were classified into the Analytic-Arithmetic mode with a higher frequency; even though, they didn't use all three thinking modes evenly with a slight change in frequency of the Analytic-Structural mode.

Most of the students' responses were independent of each other's opinions dealing with linear independence of a set of vectors and the corresponding thinking modes. The most common categories of thinking modes used by both students A24 and A25 were "Vector linear Combination (VLC)", "Computed Linear Combination (CLC)" and "Vector Operation (VOP)."

The results obtained from the analysis of the interview transcripts and the responses from the assignments reported in this chapter will further be discussed in the following chapter – Discussion and Conclusions- by comparing each student's thinking modes analysis and their corresponding category response frequency.

4.5 Aggregate Model Created from the Combination of Interviews and Class Assignments Category Responses for both Students A24 and A25

It is theorized that one of the main factors contributing to lowering of the correlation between the interview and assignments for the two students being examined is the size of the response sample size. Although relatively lengthy assignments and interviews were performed for each student, the accuracy of the correlation would only improve with larger numbers of response inputs. One way to improve this limitation is by comparing the combined works of both or many students together and examining the correlation between the thinking modes present in their assignments versus their interviews. The presence of a clear correlation would establish the concept of consistency between the thinking modes in the two different assessment methods used.

We examined the data from both students A24 and A25 together in Figure 4.3 where the interview and assignment response percentage pair for every category in each student's works is plotted separately. Performing the comparative method analysis, as explained previously, results a correlation coefficient of (r=0.39). We consider this as a control case where the data is not yet combined and similar thinking modes for each student is plotted separately. To assess the impact of the aggregation of responses from the two students, we combine the response percentage for the thinking mode categories in common between both students. In other words we have added the frequencies of responses for assignments and interviews for both A24 and A25 for the

common categories of VLC, CLC, VOP, IM, and RREF also shown in Table 4.8. The result of this analysis is shown in Figure 4.4 is which the correlation coefficient is (r=0.43).

As expected the correlation between the interview and assignments has improved as the data set is increased by using the responses of both students in the analysis. It is predicted that including more students to the analysis would further improve this correlation coefficient (approaching the value of r=1), establishing the thinking modes as general indicators of students' thinking process and understanding level.

Table 4.8: Combined Count of Responses for Students A24 and A25

Combined Count of Responses for Students A24 and A25			
Codes	**Category**	**Interview Percentage (%)**	**Assignment Percentage (%)**
NVC	Number of vectors compared to Rn	29.21	13.79
VOP	***Vector Operation***	***34.09***	***13.79***
GL	Geometry of Line	3.37	6.90
PS	Planes and Space	1.12	0.00
CLC	***Computational Linear Combination***	***21.02***	***34.48***
RREF	***Row-Reduced Echelon Form***	***11.62***	***3.45***
IM	***Identity Matrix***	***19.41***	***0.00***
IMS	Infinity Many Solutions	0.00	3.45
ZR	Zero Row	4.49	0.00
SM	Square Matrix	1.12	0.00
PV	Plane Vectors	1.12	0.00
LI	Linearly Independent	2.25	0.00
Bex	By Example	2.25	0.00
VLC	***Verbal linear combination***	***18.91***	***60.98***
ST	Solution type	12.50	0.00
Sing	Singular	8.33	0.00
SLE	System of linear Equation	7.29	0.00
Rn	Rn	6.25	36.84
VCm	Vector Component	7.29	0.00
ME	Matrix Equation	2.08	0.00
CM	Computed Matrix	1.04	0.00
MOP	Matrix Operation	1.04	0.00
MSz	Matrix Size	1.04	0.00
NECU	Number of Equations compared to Unknowns	3.13	0.00
Nvc&d	Number of vectors and dimensions	0.00	15.79
VcCo	Vector Component	0.00	10.53

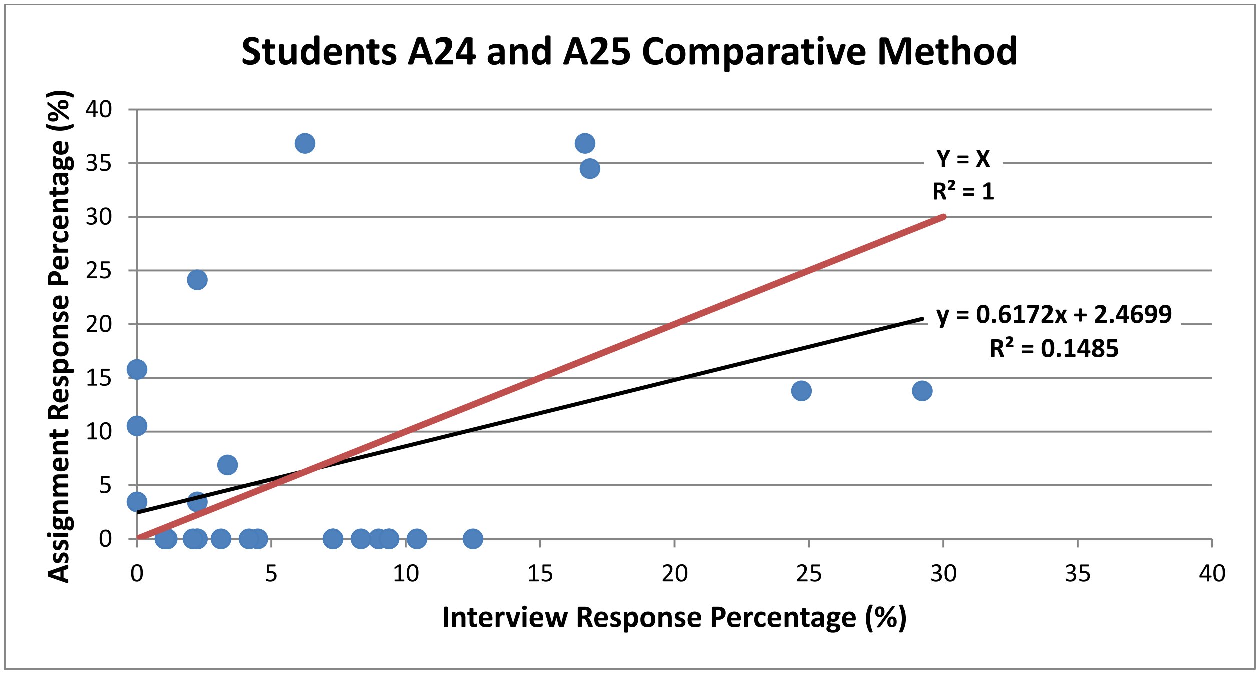


Figure 4.3: Comparative Method Analyses for Students A24 and A25

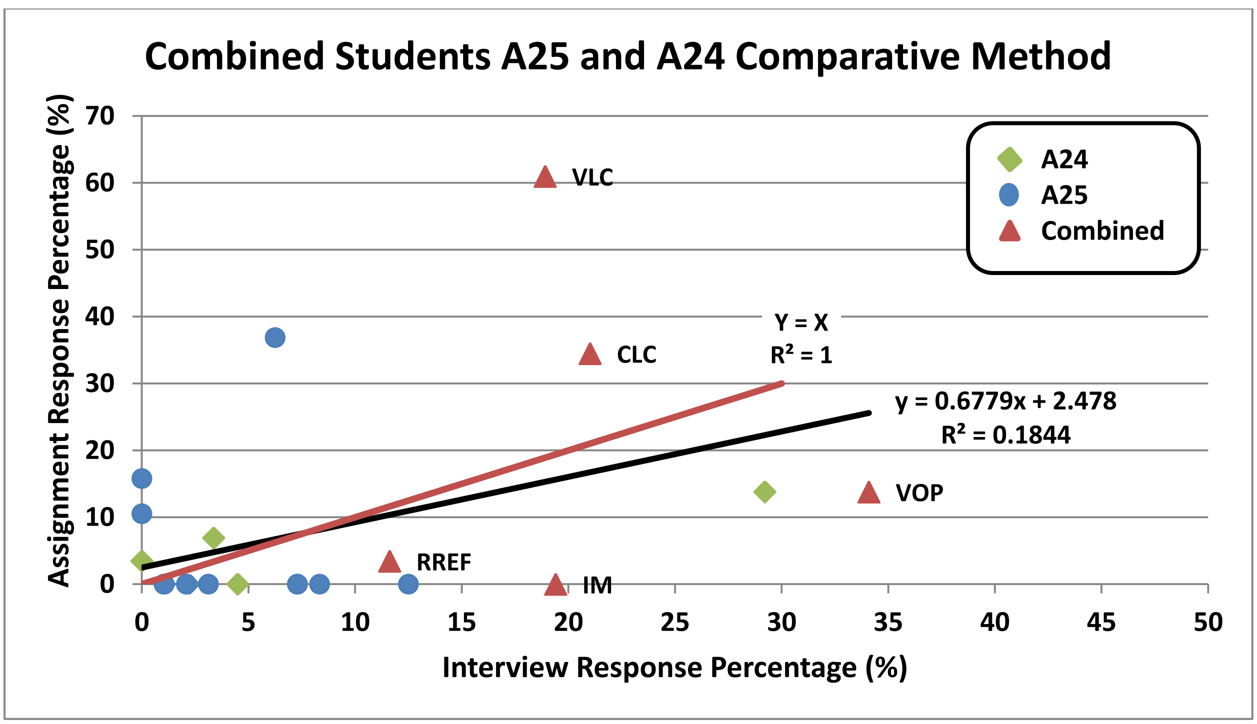


Figure 4.4: Combined Comparative Method Analyses for Students A24 and A25

Chapter 5: Discussion and Conclusion

5.1 Discussion and Conclusion

Once the analysis of interview transcripts and the assignments for students A24 and A25 were completed, the collected data indicated that there are dissimilar thinking modes present in each student's cognitive tasks. These thinking modes help students construct their own cognitive understanding of notions presented in their first linear algebra course at the university level. Both students have a propensity to think in algebraic and mathematics modes while they were asked to response to linear independence related questions The data obtained from the analysis of the transcripts shows clear signs that students were able to create their own influences by touching from one thinking mode to the other –counting the Synthetic-Geometric thinking mode- to relate numerical and algebraic features in the matrix algebra course.

This chapter comprises possible clarifications for each student's reasoning to use a specific thinking mode as a portion of their understanding. We will deliberate some of the issues that might have led to the student's favorite for an explicit mode. We will also discuss some of the resemblances and alterations throughout the data for each student as well as the aspects moving the results and the research limitations for the upcoming insinuations. This chapter adds efforts and sufficient information to the field about the possible amplifications of reasons to why students may choose to use the thinking modes-reported in solving mathematics problems. Furthermore, this chapter will discuss factors that may have caused these thinking modes.

5.2 Comparison of the used Thinking Modes by Sierpinska (2000)

The following part of this study discusses the adaptation and modification of the obtained thinking modes compared to the thinking modes established by Sierpinska (2000).

5.2.1 Reviews and Discussions

Signifying the data reported in Chapter 4 on the classification of the categorized arguments representing the thinking modes used by students during their interviews, Table 4.4 shows that student A24 used 14 different types of categories that can be classified into the three thinking modes presented by Sierpinska (2000). There were 3 categories fitting to the Synthetic-Geometric mode of thinking, 6 categories classified into the Analytic-Arithmetic mode, and finally there were 5 categories classified into the Analytic-Structural mode (Table 4.3). Compared to studentA24, Table 4.5 depicts that student A25 used seventeen different types of categories in his category analyses. There was a zero category appropriated the Synthetic-Geometric mode of thinking, 8 categories classified into the Analytic-Arithmetic mode and 9 categories classified into the Analytic-Structural mode (Table 4.7).

Going back to the information provided in Tables 4.3 and 4.7 for students A24 and A25, we can observe that the Synthetic-Geometric mode had a frequency of 5 and zero respectively, the Analytic-Arithmetic mode had a frequency of 56 and 51, and lastly the Analytic-Structural mode had a frequency of 54 and 52 for student's A24 and A2, respectively.

Considering the above frequency analysis on interview responses, we can say that both students A24 and A25 appear to use dominantly the Analytic-Arithmetic mode in their reasoning and seem to be able to go from this particular mode t to Analytic-Structural and vice versa quite often. To implement the thinking mode analogy to this study, let's refer to the following passage obtained from the original transcripts of both students exemplifies the use of the two analytic modes interchangeably in order to try to make sense of their arguments.

5.2.2 A Passage from Student A24's Original Transcript

I. uhu uhu, so um initially you said when you looked at a set the first two don't sum up to this one you usually tend to identify this as ***linearly independent*** *but when you went to the R2, you said you know what, that perspective you have is saying this is linearly dependent . Am I correct? Yeah?*

I. how about this set?

SA24. I would say that would be linearly…..

I. think out loud, remember, I'm here, I'm not interested in you telling me just this set that is linearly dependent or not, I would like to know your perspectives on this. Just like this one, that was very helpful for me, you did an excellent job, you told me the perspectives. So when you're thinking, tell me what you're thinking, if you say "hmms I'm focusing on the ones, or I'm focusing on two…" you know, think out loud.

SA24. since you just added the ones, I'm just trying to figure out if that makes a difference and because this is now in R3 and there is three vectors that could be linearly dependent or independent. So, but I don't see the linearly dependency so…

I. so how are you searching for the dependency?

SA24. I'm trying to like subtract one vector, well just the number like one minus one, three minus two, one minus one, but if this were zero one zero, that's the way I would search for it.

I. oh okay. Are you doing just the two or all of them?

SA4. I'm just doing this one, like adding up this one and getting three and getting this one

I. so you just told me both are in R3, so that both are possible in this case?

SA24. there could be two possibilities, it could either be either linearly independent or dependent.

I. so do you think maybe we can have any other way we can try to eliminate one of the possibilities? Like this one, you nicely eliminated the first one right?

SA24. uhum

I. first approach. And then you said are you sure about the second one?

SA24. yeah. Well I know, like I would like try to get the RREF, row reduced?

I. Row reduction form?

SA24. …and if this ends up in, in the identity, this would be linearly independent.

Student A24 is suggesting that the set can be either linearly independent or linearly dependent because of the fact that there can only be 3 linearly independent vectors in R3 (use of Analytic-Structural mode; this could also be a geometric mode if student is considering this fact in context of visualization of R3 and x, y and z coordinates), then decides to verify by reducing the matrix via the Gauss-Jordan elimination method (use of Analytic-Arithmetic mode), then he rephrases the statement that since "*R2 exists in the matrix, the identity is present*" (use of Analytic-Structural mode), and lastly decides to confirm his view by converting the entries of the matrix representing the set of vectors in reduced form into a system of equations (use of Analytic-Structural mode).

The following section discusses the three thinking modes adaptation to student A25. I used a passage as an example to discuss the obtained modes to the student A25's task. The passage obtained from the original transcript of student A25 demonstrates how his ability to move from one thinking mode to another was helpful in connecting ideas.

5.2.3 A Passage from Student A25's Original Transcript

I: Okay, what is singular?

SA25: Singularity would mean um, it's going by, or going with a none trivial solutions that it has, so it is singular it means it only has none trivial solutions..

I: okay.

SA25: so if it's, so if giving a system that is singular, meaning that the set represents a system it's a linear independent or dependent?

d: If it has none singular solutions?

SA25: yeah.

d: Okay, so which one do you think it is?.. yeah go ahead.

SA25: So...

d: What are you thinking?

SA25: ummm, I'm trying to think the definitions

d: aha,

SA25: so if it's a linear independent it means that it cannot be other multiples of the other vectors so it means it only has the none trivial solutions here, so a singular system

I: Mhm,

SA25: A non-singular would have to be linear independent as well.

I: Mhm, Okay, now, your using the term singular and none singular, you're saying it is singular. What do you mean by the "It"?

SA25: umm, the set.

:: The set? okay.

SA25: the set is none singular, and it would be also linear independent.

I: Okay. Okay, Can you give me an example of it, of a linear dependent set of vectors?

SA25: Um, Linear dependent?

I: mhm, you can write it too, yeah

SA25: Say in R3, would be a set given by um, R3, 1 ,1, 0, another set 2, 2, 1, so that there in a multiples of each other and then the third one would going to have to be multiples of the other two or a linear combinations with it so it would be 4, 0, 1, um, that should work.

I: okay. So this is an example of?

SA25: Of a Linear independent set vectors

I: Okay, so how did you make sure that this is a linear independent?

Student A25 concluded that the set of vectors was linearly independent using a reasoning of existence of 3 vectors –which he assumed were in R3 (using Analytic-Structural mode), then he analyzes the first two rows of matrix via the scalar multiple of vectors and then he mentions that the third one would be a linear combination of vectors in R3 (using Analytic-Structural mode). In the above passage student's A25 only uses the Analytic-Structural mode to justify his idea considering the row's matrix elements.

5.3 Assignments and Discussions

Data composed of thinking modes used to response the questions of the assignments. Going back to Chapter 4, Tables 4.3 and 4.7 illustrate the categorized arguments representing the types of thinking modes obtained throughout the 6 questions given on a class assignment by A24 and A25 works congruently (See Appendix C). Table 4.3 shows there was only one category appropriating to the Synthetic-Geometric mode of thinking. Only three of the categories indicated both student's responses regarding the assignment that are discrete thinking modes used by students to response the questions connected to linear independence of vectors . The results from Chapter 4 show that apparently students in some cases are unable to comprehend the concept of linearly independent sets of vectors, despite the interference of lectures, and content

discovery based assignments. Finally there were 3 categories classified into the Analytic-Structural mode (Table 4.3) for both students. Likened to student A24, Table 4.5 portrays that student A25 used seventeen different types of categories in his category analyses. There was a zero category appropriated to the Synthetic-Geometric mode of thinking, two categories classified into the Analytic-Arithmetic mode and two categories classified into the Analytic-Structural mode (Table 4.7). According to the analysis of Tables 4.3 and 4.7, there is no consistency among the types of obtained modes among the responses of both students A24 and A25, we can fairly judge that students responses for both interviews and assignments don't follow a similar pattern. For example, student A24 doesn't utilize any geometrical mode in his responses, but student A25 has used that mode only once in his responses to the questions in which data interview or class assignment

5.3.1 Thinking Modes Comparison

Based on the results, we can conclude that the level of exposure of students A24 and A25 to all three different thinking modes (Analytic-Arithmetic, Analytic-Structural, and Analytic-Geometrical) during their interviews are about the same, but responding to the questions of the assignment are not following the thinking mode types that Sierpinska (2000) has established. As an example the amount of the responses regarding the thinking modes types obtained from the interview responses are more than the responses obtained from the questions of the assignments for student A25. On the other hand, the responses to the interview and the assignments for student A24 are more correlated and relevant.

Finally, we may be able to conclude that:

1- The schemes and the conceptual algebraic contents have been differently utilized by both students A24 and A25. Thus, the selection of the thinking modes essentially was based on their cognitive way of understanding the algebraic problems.

2- Finally, there isn't any similarity (does it mean they are significantly different????) within their observed values (categorized responses) and their calculated error percentage in their categories.

The significant similarity between students A24 and A25 is that they all used arguments that were typically classified into the Analytic-Arithmetic and Arithmetic-Structural modes with a relative higher frequency than geometric mode.

5.4. Comparative Analysis between Combined Interview Categories and Class Assignment Response Categories for Two Students

The examination of the data from both students A24 and A25 to assess the impact of the aggregation of responses from the two students showed improved correlation between the interview and assignment response percentages. The common categories among both students included VLC, CLC, VOP, IM, and RREF as was shown in Table 4.8.

It is predicted that including more students to the analysis would further improve this correlation coefficient (approaching the value of $r = 1$), establishing the thinking modes as general indicators of students' thinking process and understanding level.

5.5 Research Limitations

Students were only taking their first year of linear algebra, and we can strongly confess that if they imply their experiences earned during this research for the second linear algebraic course, the results might be different. An additional limitation is the objectivity of the categorization obtained from the analysis of the author of this thesis. Another limitation for this thesis is that the interview transcripts and the assignment were analyzed by independently only by the author of this thesis and his advisor. The analysis performed on these interview transcripts

was strictly qualitative; therefore the students' responses were subject to individual's potential subjective interpretation.

Finally, students who participated in the interviews volunteered and very minuscule credit in the form of a participation point for their matrix course was offered to them, so there exists the possibility of bias involving the reasons to why some students did and some did not volunteer.

5.6 Implications

Concerning future implications for this type of research at the university level, a similar study can be conducted with a similar process in which students are exposed to different technological and visual aspects of learning, but are being taught by the same instructor. If the same research was to be conducted again, students could benefit from tutors and/or a computer-math lab available to students enrolled in the matrix algebra course to enhance students understanding.

The cognitive constructs analyzed for the purpose of this research –thinking modes- provide an insight into the students' reasoning while taking their first course in linear algebra at the university level, but it is important to mention that some students had previous knowledge (depending on their backgrounds) of certain concepts, such as vectors and matrices. The analysis of students with the same backgrounds can provide a better understanding of the significance of the students' responses while reasoning and understanding linear algebra concepts.

5.7 Final Remarks

The analysis presented in the thesis has the purpose of documenting the cognitive structures and thinking modes- present in the two students' responses while enrolled in their first linear algebra course in the effort to make sense of the cognition of the abstract concepts. The number of students, whose interviews were analyzed, does not reflect a significant sample of the

students registered in the matrix algebra course during the spring 2009, and therefore generalizations cannot be made from this research. The sole purpose of this thesis is to document those cognitive constructs and not to make any generalizations.

References

- Altman, D. G. and Bland, J. M. (1983). *Measurement in Medicine: the Analysis of Method Comparison Studies.* The Statistician 32 (1983) 307-317.

- Beaven, P. S. (2011). *Metaphors, metonymies, modes and linear algebra. M.A.T., The University of Texas at El Paso, Thesis.*

- Cai, J., and Moyer, J. (2008). *Developing Algebraic Thinking in Earlier Grades: Some Insights from International Comparative Studies*. National Council of Teachers of Mathematics Yearbook.

- Dogan-Dunlap, H. (2003). *Technology-Supported Inquiry Based Learning in Collegiate Mathematics*. The Electronic Proceedings of the 16th Annual International Conference on Technology in Collegiate Mathematics (ICTCM).

- Dogan-Dunlap, H. (2006). *Lack of Set Theory Relevant Prerequisite Knowledge*. International Journal of Mathematical Education in Science & Technology, Vol. 37 n4 401-410.

- Dogan-Dunlap, H. (2007). *Reasoning with Metaphors and Constructing an Understanding of the Mathematical Function Concept*. In Woo, J.H., Lew, H.C., Park, K.S., & Seo, D.Y. (Eds.). Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, 209-216.

- Dogan-Dunlap, H. (2010). *Linear Algebra Students' Modes of Reasoning: Geometric Representations*. Linear Algebra and Its Applications (LAA), 432. pp. 2141-2159.

- Dogan-Dunlap, H. (2011). *Metonymies and Object Formation, Vector space Theory*. Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education.

- Dorier, J. L (Ed.) (2000). *On the Teaching of Linear Algebra*. Dordrecht: Kluwer Academic Publishers.

- Dorier, J.L. and Sierpinska, A. (2001). *Research Into the Teaching and Learning of Linear Algebra*. In Holton, D. (Ed.) The Teaching and Learning of Mathematics at University Level: An ICMI Study, Kluwer Academic Publishers, 255-273.

- Harel, G. (1998). *Two Dual Assertions: The First on Learning and the Second on Teaching (or Vice Versa)*. The American Mathematical Monthly, Vol. 105, No. 6, 497-507.

- Harel, G. (2000). *Three Principles of Learning and Teaching Mathematics*. The Teaching of Linear Algebra in Question. Klamar Academic Publishers, 177-189.

- Hurst, C. (2008). *Using Task-Based Interviews to Assess Mathematical Thinking of Primary School Students.* Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia.

- MacGregor, M., and Stacey, K. (1993). *Cognitive models underlying students formation of linear equations.* Journal for Research in Mathematics Education, 24(3), 217-232.

- Sierpinska, A. (2000). *On Some Aspects of Students' Thinking in Linear Algebra*. In Dorier, J-L (Ed.) The Teaching of Linear Algebra in Question, Kluwer Academic Publishers, 209-246.

- Stewart*, S. and* Thomas, M. O.J. (2009). *A framework for mathematical thinking: the case of linear algebra.* International Journal of Mathematical Education in Science and Technology, Vol. 40, Issue 7.

- Trejo, R. C. (2007). *Student thinking modes expressed while determining linear independence / dependence of sets of vectors.* M.A.T., The University of Texas at El Paso, Thesis.

- Yue, H. (2008). *Concept maps as assessment tools in mathematics: Comparison with clinical interviews.* M.A.T., The University of Texas at El Paso, Thesis.

- Zamora, A. (2010). *Uses of Cognitive Constructs in Linear Algebra*.

Appendix A

INFORMED CONSENT FORM

Interactive Online Modules and Take-Home Assignments for Inquiry-Learning to Provide First-Hand Experience in Matrix Algebra Course

You are invited to be part of research activities conducted at The University of Texas at El Paso.

The purpose of this work is to identify what role the online interactive modules and inquiry assignments play in improving student achievement. The evaluation of the impact of the activities will be done through the assessments of student performance, their responses on pre- and post-surveys as well as in clinical interviews. Furthermore, we will document student conceptualizations of basic abstract concepts through student responses on take-home assignments and class tests.

Your permission will make possible for the researcher to document the effectiveness of the proposed activities in addressing obstacles in learning basic matrix algebra concepts.

You must be 18 years of age or older to participate. Your participation is completely voluntary and you may end your participation at any time with no consequences. There are no known risks involved in your participation in this study. You are given the opportunity to ask questions concerning the procedure, and any questions will be answered to your satisfaction.

Every effort will be made to keep your data confidential. No name will be released to anyone and in any published results; to keep the identity of the participating students confidential, a random numerical/letter code will be assigned to each of the respondents. Each participant will be referred to by this numerical/letter code only in presentations and publications of qualitative or descriptive data. Neither the faculty of UTEP nor the subjects' supervisors or colleagues will be provided with the names referring to the codes.

This project, (IRB protocol number: 84840-1), has been reviewed by The University of Texas at El Paso Institutional Review Board. Any questions regarding the conduct of this research or your rights as a research participant may be directed to Lola Norton, IRB Administrator, at (915) 747-8841 or irb.orsp@utep.edu at UTEP.

If you agree to participate, you are invited to sign this consent form and receive a copy of it after thoroughly reading it and asking the researcher any questions until you understand the proposed research activities.

____ Date__________
Student's name and signature

____ Date__________
Lola Norton, IRB Administrator

____ Date __________
Researcher' name and signature

Appendix B

Interview Questions

INTERVIEW QUESTIONS

NSF/CCLI PROJECT

SPRING 2009

1. Define the linear independence of a set of vectors.

2. Given an example of a linearly dependent set of vectors.

3. Given the set {u1, u2, u3, u4}where the vectors u1, u2, u3 are on the same plane and u4 is not. Determine if the set {u1, u2, u3, u4}is linearly independent. Explain your answer.

4. Given a linearly independent set {u1, u2, u3, u4} in R^n. Determine the linear independence of the set {u1, u2 +5u1, u3, u4}.

5. Given an nxm matrix a where ai2=ai4+3ai5 $\forall 1 \leq i \leq n$. Determine if the set {A1, A2, A3,…, Am} (Here Aj is the jth column of A) is linearly independent. Explain your answer.

6. Given a singular 3x3 matrix a. determine if the vectors of the set {A1, A2, A3}, where Aj is the jth column of A, are on the same plane. Explain your answer.

7. Given that the vector equation xu+yv+zw=0 has infinitely many solutions. Determine if the vectors u, v, w are on the same plane. Explain your answer.

8. Given the vector equation a1u1+a2u2+a3u3=0 with the solution a1=1, a2=-2, and a3=0. determine the linear independence of the set {u1, u2, u3}.

9. Given that dim(Span{u,v,w})=1. Determine the linear independence of the set {u,v,w}.

Appendix C

Take Home Assignment's Questions

Math 3323

Spring 2009

Name: Students A24 and A25

1. Determine the linear independence of each of the sets, and compare the two sets in light of your findings. Explain and justify your answer.
2. To answer the questions 3-6 accurately, you may need to experiment on more sets of vectors of your own.
3. Conjecture on the necessary and sufficient condition (s) for two vectors in R3 to be linearly independent vectors. Explain your reasoning.

   One vector needs not be a multiple of a second vector to be linearly independent vectors in R3. These vectors have to create a plane and will do so if they are two different vectors.
4. Conjecture on the necessary and sufficient condition(s) for three vectors in R3 to be linearly independent vectors. Explain your reasoning. One vector must not lie in the plane formed by other two vectors.
5. State your conjecture on the linear independence of set of any number of vectors in R3. Explain your reasoning. In R3, there can be a maximum of three linear independent vectors if they create three different planes. Additional vectors can be expressed as linear combinations of other vectors.
6. Answer the following questions and explain your answers:

   f. True or False: There exist three linearly independent vectors in R2.

False. In R2 a maximum of two linearly independent vectors can exist.

g. True or False: there exist four linearly independent vectors in R3.

   False. In R3 a maximum of three linearly independent vectors can exist.

h. True or False: There exist three linearly independent vectors in R3.

   False. Not always will there be there as one could be linear combination of others

i. True or false: The set v= ------------- is linearly independent. True

j. On each graph below, shown a set of vectors originated from (0,0). Circle the one(s) that are linearly independent. Explain and justify your selections(s). (Note that the vectors are in R2).

Appendix D

Transcription:
Interview of Student A24 (90 Minutes)

I: uhum
S: First I didn't get them at all; I'm like, what is this? But (uhum) I started like, on the one module I didn't finish, like what is, the last one, (uhum). I tried and tried and tried and that's when I learned um what happened and what's um, and what happened when we put like a number??
I: oh okay so on the last one you're saying…but on the earlier ones you did not?
S: well I did but I was like struggling a bit
I: but later you were able to
I: so what do you think the role in…. you understanding some of the concepts we cover in class?
S: I really think that (uhum)…it helps a lot (yeah) because I mean you don't always see it in paper, but you always see it in like a graph or… what happens when you multiply by two or…
I: I see, you get to see that, that's how you figure…it's helpful when you see it like, what happens with it when it's multiplied or added.
I: oh I see okay
I: would it be as helpful if modules were given after I cover the topics? If you noticed the modules were given before I cover the topic? Right?
S: uhu
I: um yeah
S: uh..I'd say like,(uhum) just move little parts, like you can divide linear dependencies, start with it, and then just focus more on what happens on the planes and that kind of stuff, because I that way I couldn't get it at all.
I: I see, it felt overwhelming…yeah?
S: Yes
I: okay, so you're suggesting dividing it into sections maybe?
S: yes divide it into little sections.
I: is it still a good idea to um have it before I cover or
S: well…I mean I think it's a good idea to get like…for you to understand what's happening. What's going to happen…but I think it's good
I: it's good? it's ok to give it before?
S: yeah
I: okay, do you think you get about the same if I were to give it after I cover, say linear independence module, the vectors module and I cover that in class and then I give it afterwards
S: well…
I: what would be the difference in your understanding? Would you have the same understanding?
S: no, more understanding than before. I would do better if...if you gave the class first then then the module
I: oh I see, so you would do better on the module.
I: okay. Sounds great!
I: so um here's what we're going to do, as I've mentioned before… I'm really interested… as a learner here, try to understand your perspective. What are the things that you're focusing on

like, when you read a question I want you to read out loud and then while you're thinking, I want you think out loud too, for instance if you read it and you say "given this set for the vectors" that's the reading right you might be thinking uhum They are giving a set with 4 vectors and they are saying this is the thinking part where they usually don't say it out loud while we think it…for the interview if you could think it out loud then well be... ….it will be helpful in seeing the perspective you are coming from and the aspects you are focusing on.
S: okay?
I: okay, and whenever you need to write to try to explain or do work, you're welcome to write on and here are my little things, recording you. Those are for you since you didn't have lunch
S: thanks
I: um let's start with the first questions, let me give you this one, if you could read the first question and maybe tell me what you think.
S: okay
I: and try to be vocal as much as you can
S: "define the linear dependence of a set of vectors"
I: uhum
S: well what I understand from this is like, if …what were the cases when...When the vectors are linearly dependent or not, how can you tell if they are linearly independent or not. That would be it.
I: that would be it? So can you elaborate on what are the cases you think of um
S: well…when they're… depending on R what it's called, when you say R2, it's just R2
I: yeah
S: depending on the R,
I: the space you mean?
S: yeah
I: okay
S: depending on the space.. it's um…if you have an example R2 I take that ..if you have 2 vectors, they can be either linearly independent or not, when you have more than 2 they are always independent…no dependent…they are always dependent
I: why do you say that
S: umm…because… there are more than, for example, there's R2 is a plane right?
I: yea
S: and to form a plane you only need 2, and if you have a third one, you can get the third one from the combination of those initial two vectors
I: um okay, what else do you say you think of when you look at the first question when you said " when …
S: when the…
I: um... you said some other thing…when and where and how is that what you said?
S: no haha…
I: no? what did you say? can you repeat what you when you said define linearly dependence, you think of …
S: when they are linearly independent or not, and well they are linearly independent or dependent
I: okay
S: that's what I said

I: okay and then you said when uh you think of say R2.. and you think of when they are linearly dependent, meaning when there is 2 they make…is that what you said?
S: that there is two vectors there can only be…well there may be either independent or linearly dependent, or independent and that's because …if they are the…if they are on the same line, they can only be dependent because they are just…they are scalar to each other, and if they are independent they will form a plane and that's how you get to
I: oh I see, so if you have 3 vectors?
S: if you have 3 vectors, it will always be dependent because you can get it from the combination of the initial 2 vectors you can get the third one
I: oh I see, what if you have 4?
S: dependent too
I: why?
S: because you can get…um...with the combination of 3 or 2 you can get the other ones
I: oh I see, what if we have 3 in R3 …??
S: they can be either linearly dependent or linearly independent because um like the like the um…well, if they that's how they see it, you can make a formal plane or from the whole R3 space because when you have…if they're…how would you say linearly or scalar? How would you say?
I: scalar multiple?
S: but if they are on the same line?
I: yeah scalar multiple of one another. Yeah so one is the scalar multiple of another, is that what' you're trying to say?
S: yeah
I: ok
S: yeah that they are a scalar multiple of each other, I would say that's a line because ...um they can only …well there's …that's the only thing that would be formed from the combination of those 3. They can be linearly independent if …they are...um...they form the whole plane for the whole space….because you …well, you at least need 3 to form the space, but those 3 need to be linearly independent… because if they are linearly dependent there could be a …uh…a form with a plane…
I: oh okay, um…do you remember the formal definition of linear independence? Can you think of it in that sense or do you always think of it in a sense of planes and spaces?
S: yeah I always think of it as like uh…if it's either a line, space or a plane..
I: oh ok, can you then give me an example of linearly dependence set of vectors?
S: would you want it in R3 or
I: it doesn't matter, you give me one and tell me what it is...haha… you need another one?
S: no that's fine
I: oh I see I see, got you got you
S: okay,
S: I will give an example of R2, a linearly dependent vectors.
I: uhum
S: so I say like, this would be linearly dependent,
I: because...
S: because, you well there's scalar, you can get two times this one, make this one
I: oh okay
S: so if, well that's linearly dependent
I: are you sure?

S: yes
I: so you're reason for that is, you said that is linearly dependent because?
S: because the scalar …um
I: scalar yeah
S: scalar? Uh...well the scalar you need to put…well
I: if you know, Asesuna speaks Spanish, so any time you feel like you're not able to express, just switch to Spanish
S: no that's fine
I: that's fine? okay, just giving you another option here. Okay. So this means scalar multiple, one scalar multiple of this vector, you're saying resulting this one. Okay, can you give me one in R3?
S: sure …R3...in order for it to be…well.. hmm..
I: uhum
S: dependent right?
I: yeah linearly dependent set of vectors
silence…
S: that would be linearly dependent, to form a line
I: okay tell me why you think that's linearly dependent
S: because, with this vector.. you can get this one…multiplying by 2…and this one multiplying it by 3
I: okay
S: by four? Yeah by four.
I: four? Okay okay
I: can you also identify this as better, linearly independent or dependent
S: that would be…I would identify it as linearly independent
I: and reasoning?
S: well... in order for it to be linearly dependent, you have to like… um either multiply this one …well you have to like... add it up and if you get the third one…that would be the linearly dependent one.
I: okay
S: but if this were a 2 it would be linearly dependent
I: oh I see, I see, so you're sure it's linearly independent right?
S: yes
I: are there other ways we could check to see that the one you told me is adding these two right…and then on these you give me the scalar multiples… okay…um... so do you think other perspectives, do you think of when you do homework, or when we talk about sets are there other ways you think about this and try to identify it as linearly independent ? so you already told me based on that, you think this is linearly dependent, but if you ..it looks like that is more of...you know… I tried …I wonder if there is another way we can think of it…try to verify that this is linearly independent…that's what you said right?
S: okay hold on.
I: I mean you don't have to force it to come out with different ways, I wonder if you have other ways that you think of
S: it's because I…uh.. this is an R2 right? And R2 you have more than… two vectors that was linearly independent, so ..
I: you used that one too? Is that why you were like…"hmm I'm not sure if these two are…"

I: so the first way you said this is linearly independent because these two don't sum up to this one right?
S: uhu.
I: if in a second way you say this is an R2,
S: so when you have more than 2 vectors in R2, I have…this is where I understand...okay…there are more 2 vectors in R2, there is always linearly dependency.

I: okay so what do you think might be happening here? Which one do you think is..
S: the correct one?
I: yea
S: linearly dependent
I: why is that?
S: because this is what I know that R2 when there is more than 2 vectors in R2 that's what I..
I: so you're sure about that one?
S: yeah
I: how about the first one?

Appendix E

Transcription:
Interview of Student A25

S: and then for physics I'm taking optics, um
D: mhm.
S: um, modern physics. Physics
D: Ah, Ok!
S: and uh, analytical mechanics so it's all heavy.
D: Ah, hehe, it looks like you did end up having a lot of stuff heavy. Um,, Did you say you're taking applied analysis?
S: Two.
D: Who are you taking it with?
S: With uum, Mendez.
D: Mendez? How's it going? Good?
S: Yeah.
D: What made you decide that you want to switch to Math, or well both
S: Well I already had in mind, but I just hadn't done it.
D: oh,, oh ok.
S: for I already had it in mind at the beginning of the previous semester, just have time to go to office and actually change it.
D: oh, I see. I see. Um,, so which one of the topics you are more interested in? Physics or Math? Or are they equal?
S: um,, physics, I should say.
D: Aha,, so what are you planning on doing with it?
S: Uh,, research
D: Research?
S: in nuclear simulations,
D: Oh I see.
S: so theoretical so that's why I need more Math.
D: a lot of math?
S: yeah.
D: Oh I see, I see. So what are you guys learning in applied math?
S: Uhh, right now we are doing uh, with uh,, well, the last two weeks we were doing … expansions, just for uh,, analysis of PD's
D: Oh ok. I see. How many cour.. um,, what math courses have you taken so far?
S: umm, last semester I took differential equations and Cal 3.
D: Oh ok.. So you've taken some of the series and related topics your familiar with.
S: Yeah.
D: I see, gotcha. Ok. Um, what I would like you to do um, If you, if it is possible I'm going to give you a few questions.
S: sure.
D: and you again, this is not for a class.
S: mhm.

D: It is strictly for me to try to understand some um, some aspects of the modules I'm running and the way we are running the classes. So I don't want you to be thinking... Oh she is asking maybe I'm in trouble, no, that's not the case at all. um, if I ask you or tell you I don't understand what you're doing it's not because again I'm thinking you're wrong. It's because I may not have grasped what you are saying fully. Um so please think it that way and um, if you could think out loud not think and then tell me the answer, but try thinking out loud so that I can try to see the thought processes that you're going through . I would really appreciate it. And if you have questions at any time don't hesitate to ask , feel free to ask .
S: Ok,
D: Um, and um, and if , while you're trying to explain or thinking if not explaining, thinking, if you feel you want to write, feel free to write on this pad. Um, and I have all this stuff ... so you can use any of those. And I really Am appreciative that you're allowing us to get our …... On this one.
S:
D: so , um, what we're going to do is start out with some questions and then I'll, if I have questions I'll ask you and I may remind you to think out loud. so that we can hear what you are thinking.
S: ok,, mhm,,
D: um, lets, can we start with your opinion on matrix algebra in general um,, aha,,
S: hehe,, how so?
D: mhm, like what do you think of Matrix Algebra
S: Well I know it's fundamental
D: mhm.
S: for a lot of electromagnetism courses in physics.
I: mhm,
S: It's a lot of vector calculators which matrix algebra is useful for
I: Aha, ok.
S: so ,
D: So you're saying you've seeing some of the applications already.
S: Yes,
D: mhm, ok . Um, How about the modules that we have been um, getting in the class. What do you think about them?
S: Um, I think they're kind of dry,
D: They are?
S: It's just so much work and suddenly getting, well it kind of gets you to think but it doesn't help much. It's just really redundant, and it's just work, busy work I would say.
D: Oh ok. Um,, When you start seeing, say that we have a module, when you start seeing me covering a topic that was related to what you did before, um do you at some point, feel recalling it or completely, when I start talking about say linear Independence,, was it completely new to you?
S: well, A little.
D: mhm,
S: But I had a little background missing of Linear Independence and um, and it goes from umm, applied analysis to dealing with ….so it's all linear independence sets.... So I was kind of familiar with that just didn't know fully into definitions.

D: Ah, ok,, ok, so are you saying without the modules it would still be ok. the way you're going with the class?
S: yeah,
D: Ok, um, um, what would you suggest maybe it's not your case but what would you, what would you suggest that we do with the modules and how we cover them …. cover them we uses them at all in the class?
S: Um, I guess not mandatory to do them, just an aside, I would say.
D: aha, ok,, ok so optional you mean.
S: yeah, optional.
D: Oh ok,, ok,, got you, got you. um, any other things that you want to share with me about the class? And again, this is really is going to stay in this class room, I'm not going to take it with me to the class or..
S: um, over all the class is Okay and it's not too fast, not too slow, so it's okay pace.
D: Okay, good. How about the abstraction in the class. Like the theorems, and proofs, what do you think about them?
S: I'm getting used to it cuz I'm not fully familiarized with the fully mathematical language, but I'm getting used to it.
D: Okay.
S: so this is the class that I'm actually learning more the... how to write mathematically properly,
D: Ah, okay. That's good. I'm glad that's happening. That's excellent. Okay, let's start with the first question then. Um, can you define the linear independence of a set of vectors?
S: Umm, Independence would be having vectors that are, a set of vectors which each vector is not a linear combination of the others.
D: okay,
S: and it would be also dealing with um, a set that is also a system that is represented by a um, i forgot how to say um,,,
D: can you speak out, while you're thinking if you could just try to speak out. um. Not just telling me, but say umm, thinking this,, as if your, your, studying this at home and talking to yourself.
S: It would be that um, so the vectors you have that you for a linear Independence's you have none trivial solutions so that it is.. singular.
D: ok so you said none trivial solutions to what?
S: to a system again by that set of vectors
D: and the singular.. what are you referring?
S: Singularity it would be that, if it's linear independent, and it means that it is .. is it singular?
D: Okay, what is singular?
S: Singularity would mean um, it's going by, or going with a none trivial solutions that it has , so it is singular it means he only has none trivial solutions..
D: okay.
S: so if it's, so if giving a system that is singular, meaning that the set represents a system it's a linear independent or dependent?
D: If it has none singular solutions?
S: yeah.
D: Okay, so which one do you think it is?.. yeah go ahead.
S: So...
D: What are you thinking?
S: ummm, I'm trying to think the definitions

D: aha,
S: so if it's a linear independent it means that it cannot be other multiples of the other vectors so it means it only has the none trivial solutions here, so a singular system
D: Mhm,
S: A none singular would have to be linear independent as well.
D: Mhm, Okay, now, you're using the term singular and none singular, you're saying it is singular. What do you mean by the "It"?
S: umm, the set.
D: The set? okay.
S: the set is none singular, and it would be also linear independent.
D: Okay. Okay, Can you give me an example of it, of a linear dependent set of vectors?
S: Um,, Linear dependent?
D: mhm, you can write it too, yeah
S: Say in R3, would be a set given by um, R3, 1 ,1, 0, another set 2, 2, 1, so that there in a multiples of each other and then the third one would gonna have to be multiples of the other two or a linear combinations with it so it would be 4, 0, 1, um, that should work.
D: okay. So this is an example of?
S: Of a.. Linear independent set vectors
D: Okay, so how did you make sure that this is a linear independent?
S: Well try to see if this one is a multiple of the other two so it is not, and also if adding these two and multiplying by any constant it would yield , so we .., by three
D: Okay… can you um, can I give you an example of,, I had something here, then I guess I, um , Oh... Can you tell me if this is a linear independent or dependent set?
S: Hmm, I'm doing mental math, so I'm trying to figure out
D: Okay,
S: so it would..
D: So, What kind of mental math are you doing?
S: Just saying if this one times the constant plus this one times any other constant would yield the third one.
D: Okay
S: so,, it could be a system of equations and then you could solve,, just trying to see mentally if it could work, and it would be a linear independent system.
D: Because?
S: actually no, it is a linear dependent system. Because this one um, multiply by 2 and added with this vector yields that their vectors is a Linear dependent system.
D: Oh Ok, okay, So would it be possible for your example that some multiple of this plus that or the other combinations result in this?
S: It could be possible, we just have to work it out.
D: So how do you work it out?
S:Just .. a system of . …. as vectors it would be, x, vectors x y z times.
D: can you tell me what your trying to
S: So I'm trying to ,, pretty much do the systems of equitation's so it be that 1x plus 1 y plus 0z equals to 0 vector and then 2x plus 2y, I guess this would be equal, yeah, plus 2 y plus 1z is 0 and then 4x plus 0y plus 1z equals to 0, so we find any solutions of x y and z and see if the solutions would be the.. um
D: Okay, so where did the 4 come from?

S: From this Vector.
D: Oh, right there, okay... um how did you write these equations?
S: well, we know that.. this is one vector and just equals to 0 so have them all each of the vectors, the components so if it's in R3, this would be the x component for all of them, this would be the y component for all of them and this would be the z component for all of them so set that to um,, another 0 vector
D: Okay.
S: trying to see if you have a system of linear equations to solve for the x y and z values.
D: So what are X Y and Z's?
S: Just the... components of any vectors in the R3, so it can be any other variables. it's just.
D: Um,, Okay so X could be any other variable?
S:Right, just that this one,, this components are a different component from second row and then from the third row.
D: So,, are you ,, let me just see,, Are you saying 1, 2, 4, is one component and 1,2,0 is another and 0,1,1 is another component?
S: Yes,
D: and then X is for the first component. um So when you solve the system of linear equations ,, what kind of,, what do you get and what is that telling you?
S: If a solution is given by each variable, has a given value, then it would be,, or a 0 value,, then it would be an independent set, independent set, if otherwise it gives that x y or z are depending on the other variable then it would be a dependent set if it's going to be in terms of the other..
D: Okay, um,, do you think you can solve it for me?
S: Hmm, Sure.
D: Okay, go ahead.
S: hmm, we solve it algebraically or with the vectors
D: Do you want me to...
S: With the matrices. so I guess it's easier to just put it as a matrix, so it would be 1,1,0,2,2,1, 4,0,1, so I'm just applying Gauss Jordan It would be umm,, it would be changing this one , just dividing it by half and adding it to the first row so it would give you.. umm.. 401,221, and 110.
D: What did you do there?
S: Just switched the first and the.
D: Aha, ok.
S: So that this one has this value there, so we just divide it by 4, the first row,, so it would be... 1, 0 and 1/4 and the rest would stay the same.. and again.
D: Okay, so say that you got umm,, what would you expect to get if this was an linear … dependent.
S: If it would be linear independent it would just be umm, like a …identity.. matrix... 0 0 0
D: Okay. Do you remember the formal definition of linear independence?
S:Linear independence?.. It would mean that given the vector equation of Ax equalling to 0, that the only solution x would be a .. it would have only the nun-trial solution
D:For the linear independence?
S: Yes.
D: So x has to have only the ..
S: umm 0 .
D: solution for linear independence?
S: yes.

D: Yes. umm Can you tell me what a should be in the context of your set here?
S: Umm, it would be the intermetrix, 1, this matrix.
D: umm, Ok.. How about the vector equation in the definition, can you write the vector equation, do you remember have you been writing the vector equation?
S: um,, no.
D: No? umm, so A should be this and this would be what?
S: The... just the... so it would be .. A would be this one, 1401, and the x would be …….
D: Ok, umm, can I, is this the Yeah, go ahead.
S: This is the own dimensions. X Y and Z.. So it would have to be a 3 by 1.
D: So,, What do you mean it has to be 3 by 1?
S: So it has to be compatible so that the multiplication could be carried on yielding this vector.
D: Okay, not your example but you already told me that this thing is linearly dependent, right?
S: Right.
D: Umm, can you actually carry this out and tell me,, show me that it is linearly dependent, because you already told me that this is really 2 times this plus that, right,, so can you carry out what you suggested here for this set?
S: so you could just write a 3 by 3 matrix using that set, so 2,4,8, and 3,5, 11, so carrying custodian elimination.. it would yield umm, and then the last row being equal to 0.
D:Ok, do you think it would always yield this?
S: it could also be that.. either that or yields 0 and then the last 2 rows equal to 0.
D: Okay, umm,, is it possible to have a 1 over here?
S: well in this case..
D: could it be? so are you telling me
S: yeah for it to be linear independent it could also be constant the last column, so that means that x depends on the 3rd variable and , or y depends on the other variable.
D: Okay,, um,, now in this one you wrote for this particular set you wrote 1 ,2, 3 as the first column, here for this set though you wrote the row as the first vector. which one is correct?
S:it's this way.
D: so, What were you thinking when you wrote this one?
S: Oh, because when I looked at just this,, I thought I read it wrong and just assuming it was the matrix but it should be 1, 2, 4, ..cuz when I, I was trying to write the matrix from this one.
D: but can you tell me one more time, how did you write this one? or umm.. can you write a similar system of linear equations for this set?
S: It would be that x plus 2y plus 3c equals to 0, and 3x plus 4y equals 5z, and then 5x plus 8 y plus 11z equals 0. and then we are setting it equal to 0 because we are saying that the vector equation, having the vector 0 so
D:Ok, so can you tell me what this matrix multiplication gives us if you were to carry it out?
S: It would be x plus y,, hmm, 2x plus 2y equals z, and then 4x plus c,, it would be this vector and then it would have to equal the 0 matrix.
D: Okay, um .. can you compare these two? so um,, earlier you said solving this is the same as taking that matrix …. elimination process looking at it ….. so that means solving this for this particular set of vectors is gonna be solving this matrix equation or this system of linear equations right?
S: Mhm,
D: So, um out of this you get this one right there, and you wrote this initially. so are they the same?

S: well this one is for this vector, for this, I mean..
D: Did you,, oh..
S: Yeah that's why they are not the same because this one i wrote it for this set.
D: Oh, Oh, I see, that's my fault, I wanted to have it for this set of vectors.. So if you had it for this set of vectors what would it be?
S: It would be.. using this matrix.. Yeah so it would be,, haha, I am confused now.
D: So you want to restart?
S: Yes,
D: Maybe um right here, use this set and write the.. Yeah.
S: the matrix, 1, 2, 3, 4, 5, divide it by 11,and I wanna plug in the x y z matrix and then that should give you the 0 matrix, so then this can be re-written as just x plus 3y plus 5z, 2x plus … It should equal to 0 with zero matrix.
D: so you had this earlier, can we check to see if they are the same. It's tough on me to.. umm So you had this for that right, before,, and this is what you got. Are they the same?
S: hmm, no.
D: So the same thing with these two. What do you think might be happening. Because um in one rout you get this, and the other way just looking at the vectors you get this. Which one do you think is the correct form?
S: This one.
D: Why do you think it's the correct form?
S: because its carrying the matrix multiplication, so just by definition it would be this row times that one and then adding them together and then this row times that one and adding the components and so on.
D: ok, so what might be happening here if you think this is the correct one?
S: Yeah, umm, so here was,, since we have the vectors i was saying that this one times the x this one times the y this one times the z should equal to 0 and then solve them for the other two vectors so i was just doing 1x times 2y plus 3 z equaling to 0 and so on.
D: so ,, on this one,, what are we doing then? so here we are doing this one times x y z and here what are we doing?
S: its saying this one times that one that one times that one times that one
D:so are you saying this one times x, this one times y , this one times z.
S: mhm, yeah, cuz what is as trying to do here is that each row was the components but each column has the components.
D: Components, what do you mean by the components?
S: Just the variable, it would be.. the components of the x component of this system and y component and z component, so it would be, this con would be the constants for the x variable and this would be for the y variable and this for the z variable.
D:So when you say the x is, this is the component for the system. Can you elaborate on that? what exactly do you mean. like give me an example because I'm not sure what you mean by them?
S: well, since, here we are dealing with vectors with in R3 so it could be said that this is special dimensions and in coordinates, x y z coordinates so that's why.
D: Oh, I see, so do you think this obeys that rule,, or which one?
S: This one is obeying that rule this one is not because it should be that the row the columns are..
D: mhm, but if you think about it this is the x component if you think about each vector and this x this y this z component, yes,, yet this one is saying this whole thing is associated with x, and this

whole thing is associated with y. So that means,, um are we really considering entries, vector entries and saying this is associating with x, this is y this is z? Let's,, lets.
S:mmm.
D: yeah go ahead. What are you thinking. tell me what you are thinking.
S: Yeah cuz that's the way I was thinking that it should be since given one vector so its R3 so this should be the x y and z but then one thing it has a whole 3 by 3 matrix cannot be said that this or the x y and z that rows are .. because it doesn't work when writing a system of equations.
D: mhm, ok,, so say that we have this form as a result of producing this matrix.. can you use this form to tell me where you go from there to be able to um, identify this set as a nearly dependent?
S: what do you mean?
D: um,, like um.. You already told me this is nearly dependent right? and you gave me a linear combination of the two that was the same as this one. right.. so can you show me the same thin, using this approach. where you got to the real recessional form and you showed me that that is …. the case?
S: mhm, yeah so the,, um, gas Jordan approach could be done to them to this matrix so as to get weather or not its result in identity matrix or just with the one or the last two rows equal to 0.
D: okay, so after that what do you do? Once you have the real recessional form of the metrics and you found whatever form it is what do you do after that?
S: If,..
I: mhm, do you stop there or do you continue?
S: Well you stop and then once you see that if the last row or the last two rows are 0 have all 0 entries, then it means that its linear dependent.
D: Okay, um, so if, last row or some rows are all zeros it means linearly dependent. um, what if you,, somebody asks you um well you told me this is linearly dependent can you tell me um,, how the solution to the system that you wrote relates to linear dependency? here. what would you say? SO instead of stopping right here and somebody says I don't see why your saying all 0 are 0 is implying this is linearly dependent.
S: Well it means that,
D: mhm.
S: Since the last rows is all 0s it means that that row where it should be a 1 here that means, so that the z component depends on the, it has a given value, if it's all zero that means that the z is um, arbitrary so that means that x and y can have multiple values, given any z since its arbitrary so that means, that's many solutions then its linearly dependent.
D: so if,, yeah if somebody says how is that solution is telling me or implying that the set is linearly dependent? what would you say?
S: well because, Oh, if,, like I say if one of them is zero
D: Why did you say "OH"?
S: Well.
D: aha, You're thinking something.
S: yes,
D: Yeah what is it?
S: so if its, I don't know how to,,
D: Think it out loud
S: Say it in words
D: Yeah, its ok, think it out loud. Don't worry about how it comes out.

S: well, like I said it's just if the row is ,it has zero entries for the row, then that means that the z or whichever other variable considering it has no real given value, no given value,, so then, it has many values it can be assigned, infinite values , which means that the set has multiple solutions. So going back with uh tying linear independent with the singularity that means it means it has multiple solutions, so it's a singular system and it's a linearly dependent set
D: what has multiple solutions?
S: the set…the sys.. uh…the system. given by this matrix has multiple solutions so the set that is representing here with the vectors, it will be singular. So it means it has, it's linearly dependent
D: okay. Say that I, consider me not knowing this topic and you're teaching me, say that I'm still not sure why you're telling me that this solution, having a lot of solutions tells me this is linearly dependent. How would you explain this to me. So you're saying Z has a lot of values, how could you tell me that this set is linearly dependent. Can you show me an example maybe, using this form , this matrix right there? So how is this matrix, lead you or anybody to say this is linearly dependent? Other than the correlation that you're telling me. If it has zeros its gonna implies that I this is linearly dependent. I don't understand the gap between the two. I have a matrix with zeros, and I have a set, but I'm no sure how this goes to the set and say you're linearly dependent.
S: hmmm
I: what are you thinking?
S: I'm blank, I don't know how to explain it really, it's uh
I: what makes this, to force, or … get back to the set? Let me put it this way, what is the relationship between this matrix and the vectors of this set? So that this type it's saying the vectors are linearly dependent?
S: hmmm
I: what are you thinking?
S: I'm stuck hehe
I: heeh you're stuck, okay – try to explain, so other than saying that matrix has zero's are there other things that you know of 31:41
S: um
I: you already told me the relations between z's and X's
S: uhum
I: so how is that relation, do you think is going to come back to this linearly dependence of this set? How did you define linearly dependence? You did before. The way you defined it before, what did you say?
S: linearly dependence would be, that given a set of vectors that each of the vectors are not multiples of one another, or linear combinations
I: okay so how is the Z's or the x's and y's here gets to what you just describe to me?
S: well since we are saying that that Z's is arbitrary, so that I can have any value, it means it can be any multiple of whatever value x or y has. Is that what you..
I: what do you mean 'it'? the z you mean?
S: yeah the z can have a multiple of x and y same same could be any value.
I: uhum, can you write this for me, for this matrix, what you just described, for this particular matrix i mean.
S: so for this one, since we're saying X's , the zeros's….zero equal zero, Z is not implied in the equation so that means Z can be any value.
I: okay, so how would you write that? Z can be any value?

S: so z, is just equal to Z itself
I: oh okay. Um now what if I write this matrix and I say you know this matrix can form …..for another set of vectors. Just…I'm just writing it…um ….let's say we have this matrix, sorry let me write it again… what if this is the one for a matrix, that it's coming from the vectors we had, a set of vectors, but as a result we ended up having this row in that matrix?? What would that..
S: Again since we have, I guess, also dealing with the dimensions say since we said that the columns represent um the compound of that coordinate which means that....the vertical so since we have three unknown variables and four equations, that means that one equation is the multiple and linear combination of other equations of say .. so since we have three vectors, and then each has...uh I don't know how to explain it.

I: what are you looking at_ what are you focusing on here?
S: well the dimensions, since we have, this could be seen as a system with 4 equations and only 3 variables, so since there is more equations than variables that means that one equation would have to be a linear combination of the other
I: okay so how does that imply in the linearly dependence or independence of this set?
S: since we are saying that one equation is the combination of the other, then it will be a linearly dependent, this matrix represents linearly dependent set.
I: Okay, linearly dependence, um okay. um on the earlier one, on this one when you said all zeros is that what you were thinking? when you had the all zeros on this one? were you thinking it's a multiple of the other equations?
S: yeah
I: how is that related to this solution you gave me? like you said, just now you said this equation related to this row is the multiple of the others. and then you said, when I have a solution of this type that implies linearly dependent set. Right? Am I correct? Is that what you said?
S: mhm
I: so and you used the same argument, saying this is the multiple of the other equations, and um you´re saying this implies linearly dependence. So from what you´re saying, what will this solution give us? for the system, represented by the solutions.
S: infinite solutions?
I: okay can you write it? 36:44
S: okay so x equals zero...and then..
I: what happened what are you thinking? why did you say that?
S: since we said that the columns represent the variables, then we have another row, which has no other variables.
I: uhum
S: we see how that ties in with the others.
I: what are you thinking? think out loud, try to give … you are thinking something don`t be quiet.
S: chuckles
I: tell me what you´re thinking
S: well there is, there would have to be another variable, but we dont since we have half the columns...so the number of variables that are in the system...i´m trying to figure out where the extra row, if it gives extra equations if it is just be that...
I: so why are you saying these three are another variable?
S: well I'm trying to think whether there would have to be one or not
I: okay

S: and it´s related to the last row
I: okay, so you´re thinking the first is first the second is second varible the third is third and the last is the last variable?
S: uhum
I: okay, but yet you´re saying this doesn´t match what I’m thinking there so what might be happening there?
I: what are you thinking? are you focusing on something?
S: no no no
I: not really?
S: I´m trying to see relate what I’m trying to see here and then I messed up writing this
I: what do you mean you messed up writing this?
S: well because I was trying to see if this row was the x component and this was the y and z and see in the same trouble I’m having here
I: uhum, okay. um and here compared to this, i think this is it, this or this…one of those, well we said one of them is using x for the columns and the other is using x for the rows, is that what you´re thinking? on this one for instance?
S: yes
I: what do you think x and y and z´s are representing in the context of linearly dependence of vectors, the formal definitions...if you consider the formal definitions of linear vectors, what are x y z´s representing in that formal definition ?
S: the components in the x vector, x matrix,
I: okay, so in terms of the vectors, what are they representing? so what are...so in this case you´re saying x y z are the components and in the context of the vectors you are writing this with respect to the vectors, what are you thinking those are representing in the components of x y z´s with respect to these vectors. are they representing anything or?
S: they should be because the set, since we have vectors in the set and each vector gives the special since here we are in <r3…..coordinates of the vectors, so that this vector represents the vectors in the x, the y coordinate the z coordinate, so for the other vectors
I: okay. can I then ask you to go to...any questions for me? remember I’m asking these to understand something, I’m not asking to judge you and say ooh he´s right or wrong...seriously I’m asking to understand um. push back that you are thinking these and umm every question i ask does not have any implications of

~ 41
I: seriously I’m asking to understand, um push back if you’re thinking this…every question I ask does not have any implication to whether you know it or not, it’s just I want to know what you’re thinking. Um can you take a look at number 4 and tell me what you think.
S: reads something
I: so number four says, given this set which has U1 U2 U3 U4 where the vectors U1 U2 U3 are on the same plane and U4 is not. And the question is determine if the set that U1 U2 U3 U4 is there is linearly independent or not, and it says explain your answer. What do you think?
S: well it first

I: can you tell me what the question is asking.
S: it's asking if there is three vectors which are on the same plane that means since we have, um, 3 vectors, on one plan that means that the set formed by, formed by U1 U2 and U3 that would be linearly dependent set.
I: why is that?
S: because a plane can be formed with just two vectors, and it's the linearly combination of the two vectors that can form a plane. So having an additional vector means that that set will be linearly dependent.
I: okay
S: so regardless of whether U4 is on the plane or not, doesn't get U1 U2 U3 already says that the set is linearly dependent.
I: which set?
S: a set just another set assigned by U1 U2 and U3. So if we add U4 vector to the set, it will still be linearly dependent since we already said that U3 will be a linear combination.
I: okay okay, so you're saying you didn't include U4 here, this is just the new set not the set that is given here.
S: right.
I: okay. Um can you then take a look at …um I'm trying to find one...okay can you take a look at um…what was I thinking…uh number three. 44.01
I: so number three says given a linearly independent set, prove or disprove that the set is linearly, so that the new set is linearly independent. Are you thinking?
S: yes. So it's saying that replacing one of the vectors in the set, to be an addition of that and the other one, so that means that this vector already is the combination of another vector and thus the set will be linearly dependent.
I: okay, so um this vector is the linearly combination of which vectors? 44:47
S: of the U2 and U1 since they are adding the 2 vectors. So it's the combination of both.
I: okay um so U2, or U3, or U2 is not there by itself would that make a difference? Can you show me that linear combination between…cuz you defined linear combination as ..or linearly dependence as there are linear combinations in these vectors resulting in other vectors right? Is that how you defined it?
S: yes.
I: okay can you show me this one? You said this is linearly dependent. Maybe give me one linear combination?
S: so if we have the given set, U1 U2 and U3 which is linearly independent, it means for having another set to be linearly dependent it means that we have another vector which is just another linear combination, which is U1 and U2 or it can be U3 or U4. This could be linearly dependent since we have this vector is already a combination of previous vectors which are in the set. But since this one U2 is not there anymore, this vector is not a real combination of previous vectors, it's just a new assigned vector, so this set will be linearly independent.
I: okay um so is there a way you can verify it, you can show it, what you are claiming right now? Are you sure 100 % it's linearly independent?
S: yes
I: yeah, and can you make your argument convince me and everybody else?
S: okay since we have the vectors and U1 can be expressed in matrix to see if we're having entries for U1 or U2 or …try to set the limitations so it doesn't mix with the rest

I: okay uhm say that you just set it up so you're gonna have the vectors, where will you go from there?
S: so having the vectors solve them to Un entries it has, then this vector 2 to so on, so that for it to be linearly dependent you have to have U1 plus U2 given that they are already , U1 and U2, are already exist …or belong to the set.
I: okay what if you have this one? The one that is given us on number three
S: since we have U2 and it does not belong to the set anymore, it means that U2 plus U1 is not the combination of the other vectors, since U2 does not belong to the set anymore, but it is only the orientation of…
I: how can we make sure this is not a linear combination of the others anymore? Is there any way we can? Or can we show in fact that it is not? Not just saying that U2 is not here..
S: yeah I'm trying to think.
I: is there another way? Is it possible that this sum is going to be the linear combination of U2 U3 U4? What are you thinking?
S: well if it is just given those vectors we will have to define that at least that U1 is not a multiple of U3 or U4 and so on that the others…is that what you?
I: okay well tell me why you think you need to make sure ath U1 is not a linear combination of the others?
S: okay well you need to define already define a set containing those vectors but we still need to define the vectors so that U1 is not any constant times U3 or U1 is not the constant times U4.
I: okay so what od you mean by defining the vectors? can you give me an example
S: so just say…so umm … okay so we just have to define U1 cannot be equal to any constant time U3 or U4 or so on. For c is just a constant.
I: okay, so you just want to make sure that U1 is not some scalar multiple of the others?
S: uhum
I: so how is that gonna help us that this set is linearly independent? Why do you think this is the condition for it?
S: because if one other vector is a scale multiple of the others, it mean that one is a linear combination of the other once since it's the multiple of it.

50

[50:00]
S: because if one of the vector is a scale multiple of the others, it mean that one is a linear combination of the other once since it's the multiple of it
D: okay
S: so it would be just dependent set
D: so you're saying if I show, or make sure U1 is not some scalar multiple of U3 and U4, You're ok, well you ignored the one in the middle. Is it possible this could also result in that?
S: yeah I would also have to check, that U1 plus U2 is not any constant times scalar multiple of U1 or U3 or U4
D: mhm are there other kinds of linear combinations we can consider or we just need to focus on the scalar multiplication?
S: uh also that… just U1 is not equal to another, the scalar multiple of U3 plus any other scalar multiple of U4, so that this defines U1 is not a linear combination of the others, which is the definition of linear independence

D: so why did you just focus on U1?
S: well it could also be that U3 is not the same as the others
D: okay okay , now um …did we… is the fact that this set is given as linearly independent, will that have any implications to what you just told me?
S: since the previous answers gave us linearly independence that means that all this hold, so and then you said since we don't have U2 written, U2 does not belong to the set anymore as it self, so that means that it is linearly independent
D: okay, so U2 does not depend on the others, and how about U1? You still have U1 in there
S: yeah so for this one it will be linearly independent if and only if U2 plus U1 does not equal CUn. So that the summation of this fold does not add up to the scalar multiple of U1.
D: so you think there may be a possibility of that happening?
S: it could, that's why this would have to hold for that set to be linearly independent
D: now, is there a way we can verify? Whether it's happening or not? Can we um, actually try to see if this is happening in this set ? Analyze
S: it could,
D: say that one tells you " you know what, you need to show me that this is happening for this set or not" what would you do?
S: well given that, you know the values and dimensions of U1 and the other vectors, and you can carry on the operations and check.
D: which operations?
S: oh kinda like what we were doing with the other examples, just uh
D: which examples?
S: uh, like those,
D: these?
S: so we have U1 and all the component of that vector into Un , and then so we are just writing this equation as in matrix so that won't then this one is with U2, and this one will be with U1 and U2, so that it is not equal to just, the it is not equal to the scalar time the matrix of U1 …all the answers
D:okay, is it possible that this would also be summoning a combination of U3 or U4? would that be possible?
S: umm, all could be possible, but I'm just trying to see if this conditions already puts of in the question in that these are not…
D: how are you trying to see that? What are you thinking?
S: well since we already said that U1 is now combination of U3 and U4, by the Independence of this set, but here we're adding these two vectors, but since neither of them are the combinations of the other two that means that it's just an addition, will not be just be a combination of the other two as well, so it will not really , we don't really have to check if this one is the scalar multiple of this one or that one. You only have to check that adding these two vectors is not a scalar multiple of the vectors that It contains.
D: okay, umm now. Can you look at number five. It says given an m by m matrix a where that is satisfied…well tell me what the question is asking first and then tell me what you're thinking
S: umm, so it's saying that for any row that the second column entry is equal to the fourth entry plus 3 times the 5th entry ,
D: so when you say fourth and fifth entry, are they rows or columns?
S: um, the columns, the column entries for any given row
D: okay

S: so it has to determine whether the columns all the way to aM
D: what are you thinking?
S: I'm trying to understand the question
D: oh okay go ahead
S: so if it's saying that if the second column is the combination of the 4th and 5th columns, it's asking to determine the linearly dependence of the set given by the columns of the matrix as vectors . so since we have already one vector, and it's the combination of the other two, then we say it's a linearly dependent set.
D: so what do you say vector?
S: well because we were saying that the set A1,that vector A1, is just the first column of matrix A, A2 would be the second column of matrix a and so on, all the way to the end column
D: Okay, so you point at this one, why did you say the second column for this one?
S: well we're saying, not the second column, but since we're dealing with all I's then we're saying already all rows, so the second entry of any row we can just generalize it to the second column will be equal to the fourth column plus the third, three times the 5th column.
D: okay so you're saying because of that …the set is linearly dependent
S: yes
D: okay so how would that tie to this approach you had? When you had a matrix and then you applied Gauss Jordan in here. You told me just now that there is a matrix right?, m by m matrix, and the second column is that same as the forth plus 3 times the 5th column and your telling that because of that this set is linearly deponent. If you were to use this approach, and use the matrix, what would you arrive at as a result of Gas Jordan's process, apply to this particular matrix?
S: okay so we can write the set as the matrix A, and then rewrite it, carry a Gas Jordan elimination process then we yield that the process is not an identity matrix, first of all since it's not a square matrix and also the last row, or any other rows, could have zero entries.
D: okay, so why did you say rows? Because we started out with columns, but you're saying rows
S: well since like with this example we are saying that, if it's linearly dependent it would have zero entries and so on, so it's the same thing, with just any given matrix
D: okay, why did you say since it's not square?
S: well since it's not square, we already know that at least you cannot yield identity matrix, and since identity matrix are around the square matrices
D: okay, so if we ever have a non-square matrix, is it ever possible that we make it a linearly Independent set of vectors? Um like the one I gave you before, like this one right there. So we have three vectors in this set, and the matrix lead to this row special form that has three rows but four columns, it's not a square matrix right, is that why you said this one the set is linearly dependent ?
S: mhm.
D: Okay, can you take a look at number 7? And that's gonna be the last one for us
S: the vector
D: can you think out loud?
S: I'm just reading the questions right now. Figure out what it's asking so if you have a vector equation with infinitely many solutions it means that x y and z are unknowns and then UV are vectors, so it's asking whether or not if it's on the same plane, so if we have the vector equation has infinitely many solutions, that means that it's a linearly dependent set, so if it's a linearly dependent set...
D: what is a linearly dependent set? What is the set?

S: well, a set? since the vector equation represents, is used to represent the systems so,, the set it represents
D: what is the set? can you tell me the set?
S: well the set in that case, since we have U V W are the vectors, so we just have
D: oh I see, okay so U V W are are the vectors, so the set will be? Well in fact there's no W there. So you said this has infinitely many solutions.
S: mhm.
D: okay what are you thinking.
S: okay it's because I'm getting confused, at
D: what are you getting confused at? Now tell me
S: the X and the U's just trying to say that the x are the plane….seeing them are the vectors
D: oh so you're seeing X Y Z as vectors ? okay
S: yeah, ok so I'm just trying to set this at …since we have this being the unknowns, and these are the vectors,
D: mhm, you're shaking your head, tell me what you're thinking.
S: hmmm okay, so we have infinitely many solutions for this set, which is saying in any given N, so since it has infinitely many solutions, that it's a dependent set,
D: what is the set? tell met the set again
S: the set is the set that contains by the multiply the matrix,
D: can you write the set
S: so the set will be…

Vita

Enayatollah Kalantarian was born in Dezful, Iran on August 6, 1955, the eldest son of Farajollah Kalantarian and Maryam Delijani. After completing his studies at Qotb High School in Dezful, he came to the United States in January 1979. Enayatollah attended East Arkansas Community College in Forrest City for two semesters, and then transferred to the University of Arkansas at Fayetteville where he studied for two years. He ended up finishing his bachelor in mechanical engineering in 1984 at The University of Texas at El Paso (UTEP). He worked for about three years and then returned for the master program in 1988. He finished his master degree in mechanical engineering in about a year in 1989. After working about eight years he came back and pursued a Ph.D. degree in Environmental Science of Engineering with a minor in mathematics. He was teaching math and science at EPCC and UTEP while he was working on his Ph.D. program. From his many years of experience he found that he is most needed by the early college level students in math and science, rather than at the university level. He decided to improve his teaching skill in mathematics and stay at EPCC in El Paso, TX.

PERMANENT ADDRESS:

6732 Dakota Ridge Dr.
El Paso, TX 79912

This thesis was typed by the author, Enayatollah Kalantarian.