2012-01-01

Analysis Of Intermittence And Log-Periodicity Of Foreign Exchange Rates Near A Crash

Arturo Casillas
University of Texas at El Paso, nerano7@yahoo.com

"‘ANALYSIS OF INTERMITTENCE AND LOG-PERIODICITY
OF FOREIGN EXCHANGE RATES NEAR A CRASH"’

ARTURO CASILLAS

Department of Mathematics

APPROVED:

Maria Christina Mariani, Chair, Ph.D.

Andrzej Pownuk, Ph.D.

James Upson, Ph.D.

Benjamin C. Flores, Ph.D.
Interim Dean of the Graduate School

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by

ARTURO CASILLAS

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Department of Mathematics

THE UNIVERSITY OF TEXAS AT EL PASO

May 2012

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1.1 Introduction

The modeling and prediction of financial crashes has been a growing concern in academia. One model, developed in [16] [30], for studying financial indices near a crash. This paper applies the developed model to foreign exchange rates based on the Icelandic krona. Like common financial indices, foreign exchange rates are often seen as indicators of their respective country's economic state [33]. Unlike common indices, foreign exchange rates have a volatile nature that can quickly alter the value of a portfolio that contains assets denominated in multiple currencies. Furthermore, an incoming crash in the domestic currency should present the indicators of a crash in all foreign exchange rates involving said currency, assuming the other currencies are behaving normally. This will in turn allow us to test our model. The symmetry associated with pairs of foreign exchange rates, however, makes some issues with extrapolating the model more evident.

The exchange rates of the Icelandic krona against the U.S. dollar are analyzed in the weeks before the Icelandic financial crisis of 2007 and 2008. An R program was created to run the model on the two sets of exchange rates in the months before the onset of the crisis. The krona had been appreciating for about a year before a sudden and continual depreciation. Many believed the practice of Iceland's banks of lending money to foreign

investors interested in currency speculation left the country vulnerable to an attack by speculators [15] not unlike previous instances involving the Thai baht, the British pound, the Italian lira, and others [33] [35]. The risky position of the Icelandic banks along with the fact that the event of sudden depreciation we are studying coincides with the decline of the housing market and beginning of the 2008 global financial crisis allows us to infer that exchange rates prior to the descent will show characteristics of critical phenomena [8] [46]. However in addition to certain precursory indicators in the data indicative of an upcoming crash, such as log-periodicity and the data following an overall power law, we assume the data contains log-stationary intervals. We follow the methods described in [30] and extend the model in [16] to foreign exchange rates.

Beginning with chapter one, we present the context behind our problem. Specifically, we will present some background over financial crashes and attempts to mathematically characterize financial indices in general. First, we cover some of the economic theories efficient markets and financial bubbles, followed by a review of foreign exchange rates and Iceland's economic troubles up to 2008. The final sections cover some statistical models of the overall volatility of financial indices, and their relation to financial bubbles and crashes. Chapter two will serve to review of mathematical developments in modeling financial crashes. This includes scale invariance, being log-periodic, and what it means to be a critical point. Furthermore, chapter two contains a thorough explanation of the model we are testing and an outline of the previously developed, Johansen-Ledoit-Sornette (JLS) model. The third and final chapter draws out the data to be modeled and the numerical methods, from the authors of [30], to be replicated and performed. It also contains the results and conclusions of the analysis. There is also an extensive appendix that illustrates the results of the analysis.

1.2 Preliminary Economic Theory

Economics is the study of human behavior in response to constraints or incentives. Although often associated with the study of financial matters, the field itself covers a variety of aspects of human behavior. It may focus on the motivation or rationale of the individual performing the behavior, or microeconomics [39]. Alternatively, it may focus on aspects of the greater national or global economy, or macroeconomics. Here we summarize some important concepts that will recur throughout our analysis. However, we will not discuss the individual concepts rigorously. This is meant to be an introduction for those unfamiliar with some basic tenets important to the analysis.

We will sometimes rely on the concept of efficient markets. This is not to say that we expect market prices to be 'correct'. Rather, we say efficient to imply that market participants act with a reasonable amount of competency and expediency. Therefore, we also expect prices to reflect this tendency to some extent. For our purposes, assuming that markets are efficient is akin to assuming that prices accurately reflect all available information that can be used to value assets being traded. Asset prices must react instantaneously, or at least very rapidly, to new information pertaining to the asset. Investors would also be unable to obtain an abnormally high return on their portfolios[38]. Moreover, the efficient market hypothesis heavily relies on two principles, rational investors and absence of arbitrage. There is no precise definition of 'rational' that fully encompasses the first mentioned principle. Simply put, investors in efficient markets make decisions based on reason and logic, not emotion or some prior prejudice. Investors also base their decisions on unbiased and generally accurate expectations of future prices.

That is to say, the efficient market hypothesis does not imply that irrational investors don't exist. If irrational investors do exist, they are independent, possibly few in number and unlikely to dominate the market. Even furthermore, the actions of irrational investors will likely cancel each other out. That is, an irrational investor is just as likely to believe that an asset overvalued as they are to believe it is undervalued. So, the number overly optimistic

investors will likely be offset by an equal number of overly pessimistic investors[38]. Rational investors will also react as quickly as possible to any new developments that could affect prices, as it would make no sense to delay or ignore an opportunity for profit. That is to say, investors will rush towards any underpriced asset until the price is driven to its correct value and unload an overpriced asset in a similar manner. The concept of rational investors will often be referred to as rational expectations. For our purposes, an efficient market with rational expectations implies that prices reflect all available and relevant information over an asset. Implicitly, all investors have the same access to the same information at the same time.

The condition of no arbitrage is also sufficient for efficient markets, though it can be thought of a natural consequence of rational and fast acting investors. Arbitrage is any opportunity or investment that will give a riskless or instantaneous profit [38]. Usually, arbitrage opportunities occur when two assets or financial instruments that should have exactly the same value have different prices. Clearly then, the presence of such profit opportunities imply that investors are either not being thorough or not being rational. Akin to rational investors reacting to incorrect prices, the process of investors eliminating arbitrage can lead to efficiency. Any investor that realizes an opportunity for arbitrage, a riskless profit, would be willing to commit a portfolio of any size to taking advantage of the opportunity until the mispriced asset or assets correct themselves [38] [39]. So in a market dominated by rational investors, all riskless opportunities for profit have been eliminated. Rather than assume that there exists no arbitrage, we use the presence of arbitrage to eliminate specific assumptions or scenarios in our analysis, as any model that allows for ubiquitous or ever-present opportunities for arbitrage will likely be an inaccurate description of investor behavior.

1.2.1 Bubbles and Crashes

Two empirical observations that cast doubts over market efficiency are bubbles and crashes. Crashes are sudden and unexpected plummet in the price of an asset. Two of such crashes

that have caught the attention of academia are the 1929 crash and the October 1987 crash. It is currently believed that stock prices up to those points rose wildly beyond their original values [38]. Although there is nothing within the efficient market hypothesis that rules out large fluctuations in stock prices, many economists and critics of the efficient market hypothesis don't believe that such rises and crashes could have resulted from a truly efficient market [33]. The term 'bubble' was created specifically to describe such situations. A bubble is, therefore, when asset prices increase, or wildly differ, from their true fundamental values. The presence of bubbles and crashes has led many to advocate a weaker form of the efficient market hypothesis.

To clarify, there exists a variation of the efficient market hypothesis called 'weak-form market efficiency'. However, this variation of the efficient market hypothesis makes a modification based the information available to investors. In the weak-form hypothesis, a rational investor can make a profit only through qualitative economic analysis [38]. He would be unable to make a profit simply by analyzing past patterns in asset prices. The 'strong form' and 'semi-strong' form entail that an investor is unable to make a profit off private and publicly available information, respectively. The efficient market hypothesis and rational expectations are associated with the semi-strong form, in that all public information is assumed to be contained in prices. In contrast, by 'weaker form' we actually mean a relaxed or less stringent form of the hypothesis.

The prediction and characterization of market crashes has been of growing interest in the world of academia. The challenge in characterizing such phenomena lies in finding a theoretical framework that is also empirically testable. Many debate whether or not investors within a bubble are acting rationally or irrationally, given the circumstances. Even if investors behave more or less rationally, the fact that crashes are both unexpected and undesired suggests that investors' behavior does not fully conform to what is implied by the rational expectations portion of the efficient market hypothesis [33]. The question becomes: to what extent must rational expectations be modified to account for the presence of financial bubbles and crashes.

In the framework we are choosing to adhere to, the investor herd behavior that generates a bubble manifests itself as long-correlations that persist until the bubble pops. Furthermore, the tendency of noise and periodic fluctuations in the data to diminish near the critical point suggests that some deterministic factor in the data takes over as the bubble grows, [16] [19]. The overall behavior of the bubble resembles a critical state found in phase transitions in two dimensional fluid mechanics, and served as the inspiration for the above framework. The Ising phase transition model states that adjacent atoms ferromagnetically 'line up' just before reaching the critical temperature for a phase change. Moreover by line up, we mean that the spins of adjacent particles coincide, which corresponds to cooperation or imitation among traders. For our purposes, the two possible states of the particles, $\{-1, 1\}$, also correspond to *buy* and *sell*, two possible dispositions for traders. The benefit in using the Ising model as a mathematical analogy lies in that the model explains a deterministic natural phenomena from the nondeterministic or random behavior of the individual elements of the system.

1.2.2 Brief Summary of Exchange Rates

Simply put, an exchange rate is the rate at which one country's currency will be exchanged for another's. The rate will be quoted or abbreviated using both a base currency and a quoted currency. The base currency corresponds to the home currency or the currency to be exchanged and the quoted currency thus corresponds to the currency to be received. For example, an exchange rate of 100 USD/ISK implies that you can exchange 1 dollar for 100 Icelandic krona. By convention, the base currency is always placed first when abbreviating a rate. Also note the forward slash '/' is not always used as an abbreviations but will be used for this paper. The corresponding rate for exchanging krona to dollars must be .01 if no arbitrage is possible. As you can see, for a rate r the rate corresponding to the reverse exchange, r' which we will refer to the corresponding rate, must be approximately equal to

$\frac{1}{r}$ [38]. That is

$$r' \approx \frac{1}{r} \tag{1.1}$$

If two particular exchange rates deviate too far from the above relationship, then arbitrage may possible.

As information, a single quoted exchange rate is not directly indicative of anything. This is due to the fact that a single quoted rate is not associated with any amount of money, just the rate at which it is exchanged. In that sense, an exchange rate can be considered as the relative price of one currency in terms of another. Therefore, they indirectly signify the relative prices of goods between countries [39] and have broad implications for international trade. Movements in exchange rates are, thus, predominantly controlled by changes in the values of the underlying currencies, which is in turn determined by relative supply and demand of each country's goods. As convention, changes in value of a particular currency are referred to as 'appreciation' and 'depreciation'.

Exchange rates are quoted by 'bid' and 'ask' prices. The bid price would be the price that the broker or bank would be willing to pay the base currency in exchange for the counter currency. The ask price is, thus, the price the bank would be willing to receive for the base currency in exchange for the counter currency. As can be expected, the ask price must always be higher than the bid price to prevent any opportunities for arbitrage. This difference in bid and ask price is referred to as the bid-ask spread. In fact, some opportunities for arbitrage may become unprofitable if the difference between the ask and the bid price is too high, which may result in minor discrepancies between corresponding pairs of exchange rates [51]. Moreover, this is an example of a transaction cost that, besides limiting arbitrage, also affects liquidity.

Liquidity is a term that roughly means how easily an asset can be sold. This includes how quickly it can be converted to cash or how well it retains its value if it must be sold in a short amount of time [33] [39]. Some also refer to an asset's ability to be traded in large volumes without affecting market prices as a type of liquidity, as this implies that there is an established market for the asset where there exist many traders who cumulatively

trade in large volumes. In addition to being easier to sell, currencies that are regularly traded in large volumes tend to be more liquid and have narrower bid-ask spreads than currencies that are not as widely traded. Equivalently, exchange rates that have widely traded currencies as base currencies enjoy narrower bid-ask spreads than other exchange rates.

Although exchange rates have historically been fixed or determined by governments, many countries now allow their exchange to be flexible. That is, they allow the relative price of their currency to be partially or entirely determined by markets. This brought forth the ability to speculate and profit over exchange rates. Since then, an international, decentralized, and over the counter market has developed devoted to trading currencies. It consists of dealers, mostly banks, willing to buy and sell deposits denominated in various currencies [33]. The foreign exchange market also trades in foreign exchange derivatives, such as forwards, swaps, futures, and options [51].

1.2.3 What Determines Exchange Rates

There exist multiple theories on what determines the value of currencies and exchange rates. First, we should also introduce the concept of purchasing power parity, often abbreviated as PPP. Say the same commodities are traded within two different countries. Both countries may decide to export some of these commodities and will demand a price in their respective currencies. If the value of the same commodities is the same regardless of what country they came from, then the exchange rate between the two currencies involved is simply the ratio of the prices demanded by each country [10]. Then, Purchasing power parity arises from the notion that the same amount of money should have the same amount of purchasing power in different countries, barring arbitrage [51]. Therefore, there should be some equilibrium of exchange rate that appropriately equates the prices of these commodities in the two currencies. Moreover, changes in exchange rates result from one country's change in inflation or deflation relative to another's. Specifically, for $\Delta E_{i,j}$ denoting the change in exchange rate from country i to j then

$$\Delta E_{i,j} = \frac{\text{change in inflation in country } j}{\text{change in inflation in country} i}$$

In reality taxes, transportation costs, trade barriers, differences in local demand, and the fact that two countries will rarely produce commodities that are exactly the same make it difficult to determine the exact relative prices between two countries' goods. The relative prices between different commodities are often not the same even within a country, which gives rise to the concept of real exchange rates. Furthermore, the existence of multiple countries trading in the international market complicates the model. Nonetheless, statistical evidence does indicate a causal link between relative inflation rates and exchange rates between two countries [10] [51].

Since the relative prices of commodities may not be the same between countries, a particular currency may actually be able to purchase more of the same goods. The real exchange rate is the exchange rate adjusted for the price levels in two countries. If two countries produce the same commodity but the actual commodity is priced P_i in country i and P_j in country j, then we may adjust the exchange rate for i to j, call it r, via

$$R = r\frac{P_i}{P_j}$$

where R would be the real exchange rate quoted from countries i/j. To get an aggregate measure of the real exchange rate, the prices of commodities may be replaced by the Consumer Price Index (CPI), or some other overall measure of the prices of consumer goods.

Alternatively, you can treat the currencies themselves as commodities and attempt to characterize the relative supply and demand between two currencies. The *balance of payments* method attempts to gauge the overall flow of exchange rates. Essentially, it is a statistical record of a country's international transactions over some period of time [51]. In measuring the total flow of goods and capital entering and leaving country you are indirectly measuring the total supply and demand of everything denominated in that country's currency.

Like bookkeeping, a debit entry in the account indicates a transaction akin to purchasing some foreign good or service. Likewise, a credit is akin to a sale of domestic goods and services. Entries are recorded within one of two major categories, the 'current account' and the 'capital and financial account'. The current account tracks commodities, services, cash transfers, and foreign made income. Essentially, it tracks all trade and manufacturing denominated in that country's currency, as well as any paid services purchased abroad but paid in that country's currency. It also includes any foreign earned income from either wages or investments that are sent back via remittances. The capital and financial account tracks capital transfers, the buying or selling of nonfinancial assets, and basic financial assets and liabilities. Among other things, the capital and financial account counts how much of overall, international capital and financial investment left or entered the country.

As one can infer, the balance of payments is actually an extension to the theory that inflation and economic performance are what drive exchange rates. The various categories and forms of classification attempt to add precision to measuring these factors. As a general rule, countries with a deficit (too many debits or too much importing) in their balance of payments will see their currency depreciate and countries with a surplus (an excess of credits) should see their currency appreciate. However this theory also contains complications. For example, countries sometimes engage in direct trade with each other, without exchanging any currency. Also, changes in prices don't have the same effect on all goods. Sometimes, a country can experience a net balance of payments deficit relative to other countries but still see an appreciation in exchange rates because some transactions have a greater effect on demand than others. Such was the case in the 1980s where capital and financial transactions were causing an appreciation in the U.S. dollar despite the balance of payment account deficits recorded by the United States. We must also mention that the International Monetary Fund (IMF) has devised a guideline for writing and managing a proper balance of payments account, though the exact guidelines will not be mentioned here.

As mentioned, most governments exercise some control over their currency. Most gov-

ernments aim to keep the market value of their currency moderated and prevent sudden fluctuations [51], although some governments do attempt to completely control specific exchange rates. The usual mechanisms involve the country's central bank changing the 'supply of money' in some way. Depending on their goals, a central bank can directly alter a particular exchange rate by buying or selling foreign currencies or by changing the interest rate, usually by borrowing or lending to the private sector, relative to the opposing currency [39]. Governments and central banks actually have many tools for setting monetary policy and controlling the overall economy.

As we've seen, multiple theories attempt to explain movements in exchange rates. However, the rates themselves carry a type of ambiguity. Take for example, a sudden appreciation of the ISK/USD rate could signify that the Icelandic krona increased in purchasing power, or that the dollar has decreased in purchasing power. All that can be concluded is that the krona is now more expensive relative to the dollar. If a third currency is introduced, say the euro, and the ISK/EUR rate has appreciated but the EUR/USD and USD/EUR rates have stayed the same, then it becomes more likely that the Icelandic krona has appreciated. Though it may still be the case that both the dollar and the euro have depreciated and the krona remained idle. As mentioned, an exchange rate is, by itself, a relative measure. For a more precise indicator of a currency's behavior, we ought to use effective exchange rates or rates against a neutral element, such as gold. An effective exchange rate is a weighted geometric average of one currency's exchange rates to various others. Essentially

$$E_i = \prod_{j=1}^{m} (S_{i,j})^{w_{i,j}}$$

Where E_i denotes the effective exchange rate of country i and $S_{i,j}$ is the traditional exchange rate between the currencies of countries i and j. $w_{i,j}$ is the weight assigned to the particular exchange rate $S_{i,j}$ assigned via some measure of country j's importance or involvement in the domestic market of country i.

1.3 Economic Developments in Iceland (1991-2008)

Iceland's economy was subject to tight regulations from the 1950s up to the 1980s [50]. Beginning in 1991, then Prime Minister David Oddsson implemented extensive deregulation, privatization, and unbalanced tax cuts. Prior to privatization, the government owned and regulated Iceland's three largest banks and decisions to borrow or lend were highly influenced by the political affiliation of the bank chairman. Similarly, the government controlled the exchange rate but was willing to adjust it frequently. By October of 2008, Iceland's three largest banks were in control of 85 percent of the country's total bank assets and had gotten too big for the government to effectively bail them out[8], which many attribute to the lack of oversight that followed liberalization of the financial sector [15].

Politicians and economists have voiced multiple explanations for the collapse of Iceland's economy in 2007-2008. Some believe that the lack of oversight allowed banks to greatly leverage risky strategies, and others believe that its economy was simply unfortunate to have been involved in an extremely rare but disastrous event in the financial world [50]. This paper will favor the former explanation based on gathered evidence and the analysis in [18] suggesting that it is highly unlikely that financial crashes can be explained by statistical variation, which will be elaborated upon later. Moreover, the IMF as well as some Icelandic economists warned in the summer of 2006 that the banking sector of Iceland was exposed to vulnerabilities that put its long term stability at risk. [15] and [8] also note that starting in 2006 Icelandic banks faced difficulty borrowing, at least through traditional methods, and resorted to selling bonds to foreign investors interested in currency speculation as a source of financing. Loan officers at these banks also encouraged households to borrow in cheap foreign currency. It is our belief that Iceland's high dependence on foreign debt, and hence foreign willingness to lend, along with their banks' heavy exposure to risk to be the primary factors behind the high volatility in Iceland's currency in 2007-2008. The first signs of this volatility coincided with the onset of the financial crises in Britain and the United States. By October 2008 the currency had plummetted too far and Iceland's banks

had officially collapsed [50] [15].

1.4 The Statistical Behavior of Financial Indices

1.4.1 The Random Walk Hypothesis

The question as to whether the market prices are truly random has been debated for some time. Although many concede that small recurring trends do exist [38], a common theory is that financial indices move in a 'random walk' [48] [25]. To be a random walk, a financial index or process, call it Z_t, must have consecutive differences that follow a normal distribution with a zero mean, which can be generalized to

$$Z_t = Z_{t-T} + \epsilon_t \tag{1.2}$$

where the ϵ_t are independent with zero mean and variance $\sigma^2(T)$ proportional to the distance T between points [49]. Like a stationary process, we typically assume that a random walk has a constant mean and variance. However, a random walk actually corresponds to a unit root. Note, both stationary processes and unit roots are explained further in chapter two. The system is characterized by an unconditional variance of Z_t that does not exist.

One variant of the random walk goes as follows. Say an agent, or 'walker', begins at position $x(t)$ and takes a step of length $l(\tau)$ between time t and $t+\tau$ [42]. The position at time $t+\tau$ is simply

$$x(t+\tau) = x(t) + l(\tau) \tag{1.3}$$

Similarly, the position at time $x(t+n\tau)$ would be after the position of the walker after n steps. By 1.3 the position at time $t+n\tau$ should be

$$\begin{aligned} x(t+n\tau) &= x(t+(n-1)\tau) + l(\tau) \\ &= x(t+(n-2)\tau) + 2l(\tau) \\ &\ \vdots \\ x(t+n\tau) &= x(t) + nl(\tau) \end{aligned}$$

Here, the walker has taken n steps of length $l(t)$ where each step is independent. The above implies that the expected value can be calculated via $E[x(t+n\tau)] = E[x(t)] + nE[l(\tau)]$, which can also be done for $Var[x(t+n\tau)]$. The variables $l(\tau)$ can be either discrete steps that depend on the temporal difference τ or they can follow a normal distribution $\sim N(0, \sigma^2)$ with mean 0 and a variance independent of t.

From these properties, we have $E[x(t+n\tau)] = E[x(t)]$. Since the variable $x(t)$ can be recursively brought down to zero via the same process if $t = N\tau$, we have

$$x(t) = \sum^{N} l(\tau) + x(0) = Nl(\tau) + x(0) \tag{1.4}$$

and, thus, the variance of $x(t)$ is $N\sigma^2$

The continuous limit of the above process, or letting $\tau \to 0$ and $N \to \infty$ in the 1.4 process while still having a finite t, with $x(0) = 0$ is often known as a Wiener process, or standard Brownian motion [26]. Continuity is a convenient mathematical property and is required for solutions to differential equations to be well defined.

The model was initially developed by Louis Bachelier in 1900 and consequently put through various empirical tests and modified [25]. The model has been criticized by both academics and market practitioners, each for disparate reasons. Academics point out that empirical evidence suggests that the actual distribution of successive differences is much more peaked in the center and 'fatter' in the tails of the distribution, often referred to as leptokurtosis [29] [25]. Also, there persist some small and recurring trends, but none that are capable of consistently generating significant profits [12]. On the other hand, some market practitioners have claimed that prices do not follow a truly random walk and that a skilled money manager can use certain indicators to predict whether or not prices will rise or fall. These indicators are often one time opportunities that statistical tests fail to catch in historical data [24]. However, it is also important to point out the long-run performance of most money managers does not outperform the market [38]. In fact, the short run performance of the portfolios of most stockbrokers does not do better than picking stocks at random [39] [33]. Furthermore, a random walk model doesn't account for the presence

of risk premiums, or how some financial indices have a large expected return, but that also exhibit a large amount of risk.

The random walk hypothesis, then, is related to the efficient market hypothesis [48] and the concept of a fair game. In the efficient market hypothesis, investors are incapable of achieving abnormally large returns, beyond the expected market return. Similarly, the random walk hypothesis states that successive differences are both independent and identically distributed. Independent movements imply that historical data cannot be used to predict future price movements. In a random walk model, any investor that does achieve an abnormally high return does so out of luck, not skill. Therefore, the random walk model can accurately describe stock prices under the efficient market hypothesis. However, it would have to be modified to fully account for such price behavior as the efficient market hypothesis allows for growth and trends depending on what information is publicly available. One way to modify it would be to add an expected return component, μ_t, based on available information

$$Z_t = Z_{t-1} + \mu_t + \epsilon_t$$

so the price Z_t can incorporate all the information that has become available up to time t [38].

Empirically, the general consensus is that financial indices do follow random walks in the very long run [49] for stock prices. Also, random walks tend to accurately describe foreign exchange rates [32]. However for first differences in logarithm and short time intervals, exchange rates are uncorrelated but also tend to show volatility clustering [4] [5]. For longer scales, a random walk model does not account for the leptokurtosis, in addition to the issues with nonconstant variance. However, the distribution of error terms ε_t lose their leptokurtosis as the measured time interval increases from daily to weekly and monthly data. We must point out that even though multiple studies have detected unit roots in exchange rates [3], as well as the presence of a common trend even in seemingly independent daily exchange rates, a stationary model is also capable of describing exchange rates, just not as well.

1.4.2 ARCH(q) and GARCH(p,q) Models

The term 'ARCH' stands for auto-regressive conditional heteroskedasticity. It refers to an econometric model where the variance of each term depends on the squared errors from the previous q periods [49]. Symbolically, this is demonstrated below

$$\sigma_t^2 \equiv E[\varepsilon_t^2 | W_t] = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots \alpha_q \varepsilon_{t-q}^2 \tag{1.5}$$

where W_t is some set representing the information up to time t. In general, the error terms, ε_t^2, are considered to be normally distributed and uncorrelated [6]. However, the model could easily be extended where the error terms follow a student's t-distribution[7]. An ARCH(q) is often written in terms of lag polynomials

$$\sigma_t^2 = \alpha_0 + \alpha(L)\varepsilon_{t-1}^2$$

where $\alpha(L)$ is considered a lag polynomial of order $p-1$. Note that in order for the variance, σ_t^2, to remain nonnegative, we need the terms α_0 and $\sum_{j=1}^{q} \alpha_j$ to also be nonnegative. As a reminder, a lag operator, call it L is defined as

$$Lx_t = x_{t-1} \tag{1.6}$$

Algebraically, the lag operator can be treated like a constant

$$L^2 x_t = L(Lx_t) = Lx_{t-1} = x_{t-2}$$

or more generally

$$L^p x_t = x_{t-p} \tag{1.7}$$

Hence, a lag polynomial for some factor or coefficient, α, of order p

$$\alpha(L) = 1 - L\alpha_1 - L^2\alpha_2 - \cdots L^p \alpha_p \tag{1.8}$$

Often, lag polynomials serve to turn one time series into a different time series. We will use them to simplify sums.

Unlike the random walk theory where the differences in financial indices are independent and identically distributed, the ARCH(q) model allows volatility to vary depending on how volatile the previous q terms have been. This then allows the ARCH(q) model to describe sets characterized by alternating periods of volatility. Some claim this to be the case for time series of financial indices, [25] [7], in that markets can have quiet periods followed by highly volatile periods. Note, that the above does not imply that the data can't be stationary. In fact, the unconditional variance of σ_t^2 is as follows

$$\sigma_t^2 = E[\varepsilon_t^2] = \alpha_0 + \alpha_1 E[\varepsilon_{t-1}^2] + \cdots \alpha_q E[\varepsilon_{t-q}^2]$$

If the value $E[\varepsilon_t^2]$ is constant for all t, then the process is allowed to be stationary and

$$\sigma_t^2 = \frac{\alpha_0}{1 - \alpha_1 - \cdots \alpha_q}$$

as long as $\sum_1^q \alpha_i \in [0, 1)$. Note, the subscript t in the symbol σ_t^2 is included for the sake of convention, as the variance of a stationary process is independent of t.

As a side note, the presence of volatility clustering or similar behavior indicative of and ARCH(q) process does not necessarily undo the effectiveness of the OLS estimators. Rather, it implies that other, nonlinear estimators will be better able to describe the data [49].

Such models can be generalized in various different ways. One such model, initially proposed in [6], is known as GARCH or generalized autoregressive conditional hereoskedasticity [49] and allows for different behavior of lag patterns. Specified with two parameters p and q, the GARCH(p,q) model consists of

$$\sigma_t^2 \equiv E[\varepsilon_t^2 | W_t] = \alpha_0 + \sum_{j=1}^{p} \alpha_j \varepsilon_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{1.9}$$

A GARCH(p,q) process can also be written in terms of lag polynomials

$$\sigma_t^2 = \alpha_0 + \alpha(L)\varepsilon_{t-1}^2 + \beta(L)\sigma_{t-1}^2 \tag{1.10}$$

Where the terms $\alpha(L)$ and $\beta(L)$ are also referred to as lag polynomials. Often, the simple GARCH(1,1) is used where we only have the parameters, α_0, α_1, and β

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{1.11}$$

For σ_t^2 to be nonnegative, we generally require the three terms α_0, α_1, and β to be nonnegative as well. A GARCH(p, q) process also has a stationary solution when you consider the unconditional variance

$$\sigma^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j E[\varepsilon_{t-j}^2] + \sum_{j=1}^{q} \beta_j E[\sigma_{t-j}^2]$$

and under the condition that the variance is independent of time, t

$$\sigma^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j \sigma^2 + \sum_{j=1}^{q} \beta_j \sigma^2$$

$$\sigma^2 = \frac{\alpha_0}{1 - \sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j}$$

Although it may appear obvious that the GARCH(p, q) model includes the ARCH(q) model, a GARCH(p, q) model may actually be approximated to within an arbitrarily defined error by an ARCH(q) process under the right circumstances [6]. As long as all possible roots to the equation $1 - \beta(z) = 0$, complex numbers included, lie within the unit circle, then 1.10 may be recursively substituted into itself to obtain an equation of the variance in terms of past squared errors.

$$\sigma_t^2 = \frac{\alpha_0}{1 - \beta(1)} + \alpha(L) \sum_{j=1}^{\infty} \left(\beta^{i-1}(L) \varepsilon_t^2 \right)$$

which resembles an ARCH(∞) process. The $(1 - \beta(1))^{-1}$ is the simplification of the series $1 + \sum_{j=1}^{q} \beta_j + \left(\sum_{j=1}^{q} \beta_j \right)^2 + \left(\sum_{j=1}^{q} \beta_j \right)^3 + \ldots$. When $\beta(1) < 1$ or when the sum of the β_j coefficients are less than one, the coefficients per error term, ε_{t-j}^2 are geometrically decreasing. Furthermore, the squared error terms must be weakly stationary for the above equality to hold. That is, each ε_{t-j}^2 must have an expected value and variance independent of t. Then said GARCH(p, q) can be approximated to any predetermined accuracy via an ARCH(Q) process for a sufficiently large Q.

1.4.3 Stable Distributions and Lévy Flight

Another response to the discrepancies between the implications of the random walk hypothesis and historically accumulated data came in [25], and later in [29]. Analyzing the successive differences of the logarithm of price, $\log(Z_t)$, [25] suggests the differences follow a stable Paretian distribution. To define a stable distribution, consider the variable S_n, the sum or any linear combination of n random variables, comprising of individual random variables x_i. Then, a distribution, call it $P(x_i)$, is considered stable if the functional form of $P(S_n)$ is the same as the functional form of $P(x_i)$ [26].

The most general functional form that all stable distributions follow comes from the characteristic function $\varphi(q)$. Taking the natural logarithm of $\varphi(q)$, we have

$$\log(\varphi(q)) = \begin{cases} i\mu q = -\lambda|q|^\alpha \left[1 - i\beta\frac{q}{|q|}\tan\left(\alpha\frac{\pi}{2}\right)\right] & \alpha \neq 1 \\ i\mu q = -\lambda|q|^\alpha \left[1 + i\beta\frac{q}{|q|}\frac{2}{\pi}\log|q|\right] & \alpha = 1 \end{cases}$$

Where $\alpha \in (0,2]$, $\gamma > 0$ corresponds to the scale factor, μ can be any real number, and $\beta \in [-1,1]$ corresponds to an asymmetry parameter. Only a few stable distributions have a known analytical characteristic function. Allowing the mean and scale (μ and γ respectively) to vary, the values of α and β that give the three known distributions are

1. $\alpha = \frac{1}{2}$, $\beta = 1$: the Lévy-Smirnov distribution
2. $\alpha = 1$, $\beta = 0$: the Lorentz or Cauchy distribution
3. $\alpha = 2$: the Gaussian distribution

Hence, the case of stationary differences that follow a Gaussian distribution with mean 0 is a particular case of stable distribution[26]. For simplicity, let us restrict ourselves to symmetric distributions, $\beta = 0$, with 0 mean. As is custom in nomenclature and notation, the above family of stable distributions and special analytic cases are often called Lévy stable distributions.

As we'll see, stable distributions can have peculiar behavior. Recall that the characteristic function of a probability density function is equivalent to the Fourier transformation of said probability density function. We may then take an inverse Fourier transform of the characteristic function of symmetric Lévy stable distributions to obtain the corresponding pdf. Since such characteristic function is $\varphi(q) = e^{\gamma|q|^{\alpha}}$, we have $P_N(x)$, the pdf of x a sum of N stable random variables is

$$P_N(x) \equiv \frac{1}{\pi}\int_0^{\infty} e^{\gamma|q|^{\alpha}} \cos(qx)dq \tag{1.12}$$

Which is often called a Lévy flight. For x such that $|x|$ is very large and $\gamma = 1$ and from a series expansion, we can obtain

$$P_N(|x|) \sim \frac{\Gamma(1+\alpha)sin\left(\alpha\frac{\pi}{2}\right)}{\pi|x|^{1+\alpha}} \sim |x|^{-(1+\alpha)}$$

Where $\Gamma(x)$ is the Euler function. Hence, for large $|x|$, $P_N(|x|)$ is approximately a power law distribution. This will allow us to approximate the moments of the distribution of $|x|$, which correspond to $E[|x|^n]$ [26]. For $n \geq \alpha$ and $\alpha < 2$, the moment $E[|x|^n]$ will not converge, which implies that such Lévy processes do not have finite variance. Since physical systems have finite variance and only a few of the stable Lévy distributions are analytically known, [29] modified traditional Lévy flight to have finite variance.

The term truncated Lévy flight refers to a stochastic process that obeys the following probability distribution

$$T(x) = \begin{cases} 0, & x > l, \\ c_l L(x) & -l \leq x \leq l, \\ 0, & x < -l, \end{cases} \tag{1.13}$$

Where c_l is a normalizing constant meant to ensure that $T(x)$ is a proper probability distribution, and $L(x)$ obeys the distribution

$$L(x) = \frac{1}{\pi}\int_0^{\infty} e^{\gamma|q|^{\alpha}} \cos(qx)dq \tag{1.14}$$

A symmetrical Lévy distribution with infinite variance, or $\alpha \in (0, 2]$, and both $\gamma > 0$ and $l > 0$.

Recall, Lévy stable distributions encompass the most general functional form that every stable distribution must follow [26]. Therefore, a truncated Lévy flight (TLF) is itself not stable, as it is not equivalent to a Lévy stable distribution. However, the restriction of x to the interval $[-l, l]$ ensures that $T(x)$ will have finite variance.

Due to finite variance, the TLF will converge, albeit slowly, to a Gaussian process. Let S_N be the sum of N random variables that are independent but identically distributed by truncated Lévy distribution as described in 1.13. To investigate the distribution of $T(S_N)$ as $N \to \infty$, we look at the probability of $S_N = 0$, which depends on the parameters N, l, and α from 1.12 . For small values of N, the truncated process, $T(S_N)$, closely approximates the probability $P(S_n = 0)$. The function is given by

$$P(S_n = 0) \approx T(S_N) = \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\pi\alpha\left(\gamma\Delta t\right)}$$

where Γ in this equation is the gamma function. For large values of N however, $P(S_n = 0)$ resembles that of the Gaussian, normal process

$$P(S_n = 0) \approx N(S_N) = \frac{1}{\sigma_0(\alpha, l)\sqrt{2\pi n}}$$

Where $\sigma_0(\alpha, l)$ is the standard deviation of the individual variable x. Moreover, for $\alpha \in [1, 2)$ the approximate crossover point, call it n_x, from the truncated Lévy to the normal process was determined in [29].

$$n_x \approx Al^{\alpha}$$

Where A is equivalent to

$$A = \left[\frac{\pi\alpha}{2\Gamma(1/\alpha)\sqrt{\Gamma(1+\alpha)sin(\pi\alpha/2)/(2-\alpha)}}\right]^{2\alpha/(\alpha-2)}$$

Empirically however, simulations done in [29] imply that for some values of α and l convergence to the Gaussian normal requires a massive N, number of summed variables, to occur.

On one hand, truncated Lévy flight performs better than a Gaussian, random walk process. Analysis summarized in [27] and [28] suggests that a Lévy process very adequately

describes the S&P 500 index from 1984 through 1989. Using time scales ranging from one to 1000 minutes, in [28], it was initially determined that the logarithm in base 10 of the successive differences of the variables could not be described by the Gaussian distribution. As mentioned earlier, the tails were too 'fat' or, more precisely, the distribution at the ends of the distribution was too high to be accurately described by the Gaussian normal. However, modeling the distribution of the probability $P(S_N = 0)$ against the various scales of Δt, the tails were now too 'thin'. The distribution at the tail end for the empirical data was lower than that of the fitted Lévy distribution. Since the calibrated scalar, α, lay such that $\alpha = 1.40 \pm 0.05$, the fitted distribution has an infinite second moment and, thus, an infinite variance (from [26]). However, the experimental data exhibited an exponential drop-off that clearly suggested a finite variance.

Since the center of the distribution can be accurately described by a stable Lévy distribution, truncated Lévy flight comes in use. Having a finite second moment, and most of the other useful properties of traditional stable Lévy distributions, truncated Lévy flight can describe many of the most important properties of financial indices [27]. It accurately captures the general behavior, or central shape of the pdf, of financial indices in the short run while still converging to a Gaussian process for long time periods. It also captures the fat tails of the distributions of empirical data, in short time horizons, better than the Gaussian process. The authors of [27] have also argued that it performs better than an either an ARCH(1) or a GARCH(1,1) process. The article, [7], also states that ARCH(q) and GARCH(q,q) processes don't fully describe the leptokurtosis. However, it doesn't explain certain memory effects that retain in the data, such as for $\Delta t \leq 20$ minutes or trends that persist in particular months. Also in contrast, [4] [5] have found that ARCH and GARCH processes accurately describe trends in volatility in hourly and daily foreign exchange rates. They also believe that volatility clustering is an inherent property of foreign exchange data at short time measurements, which is also an inherent characteristic of conditional heteroskedasticity models. In contrast, ARCH and GARCH models fail to capture the power-law scaling witnessed in empirical data [26].

1.5 A Probabilistic Description of Crashes

Many academics and practitioners believe that bubbles and crashes need to be studied in modeled within their own context, or at least separately from the stable or standard periods of the market. Crashes, when they occur, are usually unexpected and involve price changes well outside what is predicted by models relying on Gaussian statistics [1] [46]. Contrary to random walks, financial indices in a bubble show a clear trend and are almost always unexpected. Thus, many believe that bubbles are investor driven and not simply rare events that can be explained as consequences of statistical noise.

As mentioned earlier, some evidence based on historical data and simulations suggests that financial crashes in general cannot be explained by statistical variance. If you define a 'draw down' as the percentage loss from a local maximum to a local minimum (referred to as DD), evidence presented in [18] and summarized in [20] and [46] show that the number of draw downs in the Dow Jones Industrial Average within a century that are less than 15% in magnitude is accurately fitted by an exponential law

$$N(DD) \approx N_0 e^{-\frac{DD}{DD_c}}$$

Where N_0 is the total number of draw downs larger than 1% in the previous one hundred years calibrated to be about 2360, with the actual number about 2789, and DD_c was calibrated at about 1.8%. Using such a fitting, they extrapolated the expected number of draws one would need to see draw downs comparable to the three largest crashes that occurred in recent stock market history. The second largest draw down had a magnitude of 28.8%. Upon determining the probability of finding such a draw in a century, it was calculated that the expected number of centuries of find such a draw totaled to about 3000. However using a higher bin of draw downs, a more conservative value of 2.2% was calibrated. In this analysis, one would be expected to find a draw-down of magnitude 28.8% in about 160 centuries.

Second, an analysis was performed where 10,000 data sets consisting of a century's

worth of returns and covering one million years were created based upon a GARCH(1,1) model estimated from the historical data with a t- distribution with four degrees of freedom [18]. From these generated sets, only 2 contained 3 draw downs larger than 22% in magnitude. No century contained more than 4 of such draw-downs. Also, three of these six crashes exhibited behavior that was unseen in historical data up to that point. These draw downs were preceded by similar 'draw-ups' where it was as if the data crashed upwards. Obviously, this is due to the volatility of one point variable being proportional to the volatility of the point preceding it, a characteristic of GARCH(1,1) processes. Rather than a slow buildup to the crash, the simulated data jumped upward before jumping downward, something unobserved in crashes that have occurred. Thus in one million years of simulated GARCH(1,1) data, three crashes with the characteristics of empirically observed crashes failed to occur.

Chapter 2

Review of Discrete Scale Invariance, Some Preliminary Concepts, and Previously Developed Models

2.1 Scale Invariance

For a function to be considered invariant it would essentially have to reproduce or retain certain properties on different scales of time or magnitude, or after some kind of modification. Hence, a function or observable $f(x)$ is considered scale invariant if for some scalar, λ, there exists another scalar, μ, such that

$$f(x) = \mu f(\lambda x) \tag{2.1}$$

[43]. An immediate consequence of scale invariance is the following property. Take two distinct values x_1 and x_2, then

$$\frac{f(x_2)}{f(x_1)} = \frac{f(\lambda x_1)}{f(\lambda x_2)}$$

[16]. Or similarly,

$$\frac{f(x_2) - f(x_1)}{f(x_1)} = \frac{f(\lambda x_2) - f(\lambda x_1)}{f(\lambda x_1)}$$

Which signifies that for a percentage change from a point x_2 also remains invariant for a change in scale. The solution, or the type of observable that would exhibit scale invariance, is a power law. That is, a function or observable that can be expressed as $f(x) = Cx^{\alpha}$ where $\alpha = -\frac{\log \mu}{\log \lambda}$. Furthermore, if the relationship $f(x) = \mu f(\lambda x)$ holds for λ being

any real number then we have continuous scale invariance. If it only holds for specific, countable values of λ then we have discrete scale invariance. Also note that if a system exhibits discrete scale invariance for some λ then it also exhibits scale invariance for λ^n where $n \in \mathbb{Z}$.

The concept of scale invariance is important because of its power law solution. Power laws can be used to describe 'critical' phenomena, which is essentially a system that is approaching a singularity. If we let K_c be the specific time, price, temperature, or such measurement at which the critical event is set to occur, then clearly the power law

$$f(K) \propto (K - K_c)^{-Al}$$

Implies f will reach a critical point at K_c. [19] and [42] go so far as to call power laws the hallmark of criticality.

2.2 Log-Periodicity

A discretely scale invariant system can be supplemented by a function $P(\log_\lambda x)$, where P is a periodic function with period 1. As a reminder, a periodic function, call it g, is called periodic with period λ if for the number λ we have $g(x+\lambda) = g(x)$. The function $P(\log_\lambda x)$ repeats itself as x is magnified by λ, which can be verified as follows

$$P(\log_\lambda(x+\lambda)) = P(\log_\lambda x * \log_\lambda \lambda) = P(\log_\lambda x + 1) = P(\log_\lambda x)$$

Therefore, by multiplying a power law function that is scale invariant under λ we can obtain a system that is periodic with respect to λ and retains the relationship $f(x) = \mu f(\lambda x)$. Our function $f(x)$ becomes

$$f(x) = Cx^\alpha P\left(\frac{\ln x}{\ln \lambda}\right) \tag{2.2}$$

As a periodic function, $P(x)$ can be decomposed into a Fourier Series $\sum_{n=-\infty}^{\infty} a_n e^{i2n\pi\frac{\ln x}{\ln \lambda}}$ which can transform $f(x)$ into

$$Cx^{-\log_\lambda \mu} \sum_{n=-\infty}^{\infty} a_n e^{i2n\pi\frac{\ln x}{\ln \lambda}}$$

We can see that systems exhibiting discrete scale invariance resemble power laws with complex exponents in the form $\alpha = -\log_\lambda \mu + \frac{i2\pi n}{\ln \lambda}$ where $n \in \mathbb{Z}$ [43]. That is to say for a real number x and some complex number α as we defined, we have the relationship

$$\begin{aligned} Cx^\alpha &= e^{\alpha \log Cx} = e^{-\log_\lambda(\mu) * \log Cx + \frac{i2\pi n}{\ln \lambda} \log Cx} \\ &= e^{-\log_\lambda(\mu) * logCx} * e^{\frac{i2\pi n}{\ln \lambda} logCx} \\ &= Cx^{-\log_\lambda \mu} e^{\frac{i2\pi n}{\ln \lambda} logC} * e^{\frac{i2\pi n}{\ln \lambda} logx} \end{aligned} \tag{2.3}$$

where $a_n = e^{\frac{i2\pi n}{\ln \lambda} \log C}$ [2]. Like equation 2.2, a power law with complex exponents, the system 2.3 above, consists of a standard power law exhibiting continuous scale invariance multiplied by a second function that is periodic with respect to variable x. The power law corresponding to the function $Cx^{-\log_\lambda \mu}$ and the periodic function corresponding to

$$P\left(\frac{\log x}{\log \lambda}\right) = e^{\frac{i2\pi n}{\ln \lambda} \log C} * e^{\frac{i2\pi n}{\ln \lambda} \log x}$$

which can be verified to be a constant multiplied by a periodic function. Recall the relation to the relation $e^z = e^{(z + i2\pi n)}$ for complex numbers z where $n \in \mathbb{Z}$. It can easily be verified that making the transformation $x \to x\lambda^m$ where $m \in \mathbb{Z}$ gives us

$$e^{\frac{i2\pi n}{\log \lambda} \log(x\lambda^m)} = e^{\frac{i2\pi n}{\log \lambda} \log(x) + m * i2\pi n}$$

Because of this periodic function, the entire system cannot exhibit continuous scale invariance like a standard power law. This second function will only repeat itself for powers of λ, or λ^m for $m \in \mathbb{Z}$. Therefore, we will only be able to retrieve the original system for these particular scales. For any other scales, no μ exists that would satisfy the relationship 2.1

A basic result of a discrete invariant and log-periodic systems is simply that they are invariant in the translation in the logarithm of their variables. Specifically if let $y = \log x$, and let f be a function that exhibits

$$f(y) = \mu f(y + \log \lambda)$$

which can be referred to as translational invariance with respect to $\log x$. The set $f(\log x)$ is then scale invariant with respect to x.

Empirically, log periodic structures were first noticed by [47] and [13] independently in the 1929 and 1987 stock market crashes. Furthermore, these log-periodic oscillations decrease as the data approaches the critical date. However, the question of whether or not log-periodic signatures form a strong enough basis for a predictive model continues to be debated. First mentioned in [19], [44] and [21], though concisely summarized in [46], power laws with log-periodic signatures have additionally been observed in various other financial bubbles, for example the Nasdaq bubble that burst in April 2000, and multiple bubbles that occurred in developing countries, the 1997 Hong Kong stock market crash, and the Russian stock market bubble whose collapse began in 1997. The signatures have also been found decay after the crash, or critical point. Often referred to as 'anti-bubbles', the investor behavior that drives the price of an asset down after the critical point follow log-periodic powers laws from the critical point. Examples include the Japanese stock market (Nikkei) and the Russian stock market. Within currency crashes the signatures have been observed in the Yen and Euro crash on Jan. 2000 and the U.S. dollar and Euro incident in October 2000. Moreover, log-periodicity had been observed in 'slow crashes' in the Hang Seng, Nasdaq, and S&P 500. It is because they've been observed in various different contexts and even after a critical event, as well as noting similarities in the parameters of calibrated models [46], that some believe that discrete invariance and log-periodicity characterize certain types of 'regime changes'. The term 'regime' refers to the particular underlying dynamics that shape the data.

2.3 Log-Stationary Intervals

2.3.1 A Stationary Process

For a set of time-series data points, call each point W_t, is considered stationary if the joint distribution of $W_{t+j_1}, W_{t+j_1}, \cdots W_{t+j_k}$ is independent of the variable t [34]. More specifically,

if $F(W_t)$ is the cumulative distribution function of W_t, then we have

$$F(W_{t_1+j_1}, W_{t_2+j_2}, \ldots W_{t_k+j_k}) = F(W_{t_1}, W_{t_2}, \ldots W_{t_k})$$

For $F(W_t, W_{t+1}, \ldots)$ referring to the joint cumulative distribution of points W_{t_i}. It follows that certain properties of the time series such as mean and variance do not vary within a shift in time [49]. More specifically when the joint distribution of the time-series is unaffected by an arbitrary shift in time, the series is considered strictly stationary and the invariance of mean, covariances and variances naturally follow. When only the means and variances are independent of time, or

$$E[W_t] = \mu \qquad \text{for all } t \tag{2.4}$$

$$E[(W_t - \mu)^2] = \sigma^2 \qquad \text{for all } t \tag{2.5}$$

$$E[(W_t - \mu)(W_{t+s} - \mu)] = \delta_s \tag{2.6}$$

Then the system is considered to be weakly stationary. Note, the covariance between two points, W_t and W_{t+s}, also depends only on s, not the point t.

Our set W_t can be defined as the intervals between the logarithm of the data points. For a set with N observations, we have $N-1$ intervals between adjacent points. Using the notation f_t for our data set and $d(\log(f_t))$ for the difference $\log(f_{t+1}) - \log(f_t)$, we simply say f_t is log-stationary when $d(\log(f_t))$ satisfies the properties above. Note that if we don't care which interval we use or if $d(\log(f_t))$ can also refer to $\log(f_t) - \log(f_{t+1})$, then the only way for $d(\log(f_t))$ to satisfy a constant mean is for $d(\log(f_t)) = 0$. Since the expectation remains constant, we can say that each interval approximately satisfies

$$d(\log(f_t)) = \mu$$

For some positive μ. If we let $a = e^{\mu}$ then we have

$$d(\log_a(f_t)) = 1$$

So there must exist some a for which the log-difference is approximately one.

2.3.2 Tests for Stationarity

Although there exist several tests for stationarity, we will only go over a few common tests. Consider the following process.

$$Z_t = \theta Z_{t-1} + \varepsilon_{t-1} \tag{2.7}$$

that has a constant θ and has error terms, ε_t, have a constant mean of 0 and a constant variance. Any process that follows 2.7 but with $|\theta| < 1$ is considered a stationary process [49]. Any such process can be shown to have constant variance, covariance, and an approximately constant mean. Consequently, any process that can be modeled with $\theta = 1$, called a unit root, is not stationary. This is a standard result, so it will not be proven.

Kowing this, we can test a time series for stationarity by assuming it follows the 2.7 process and subsequently testing the hypothesis, $\theta = 1$, against the alternative, $|\theta| < 1$. Ordinary least squares may be used to find a consistent estimate for the series' parameter θ. However, the ratio

$$\frac{\hat{\theta} - 1}{se(\hat{\theta})}$$

where $se(\hat{\theta})$ is the standard error for the ordinary least squares estimate of θ, does not follow a student's t distribution. So appropriate critical values for $se(\hat{\theta})$ must be sought. This test is often referred to as a unit root test [49], although the original equation 2.7 is often modified to

$$Z_t - Z_{t-1} = \Delta Z_t = \theta Z_{t-1} - Z_{t-1} + \varepsilon_{t-1} = (1 - \theta) Z_t + \varepsilon_{t-1} \tag{2.8}$$

where $\hat{\theta}$ is taken from the estimate of $(1 - \theta)$.

Another way to modify the test is to assume the series is stationary but adheres to some nonzero constant mean, μ. Equation 2.7 can thus be modified to

$$(Z_t - \mu) = \theta(Z_{t-1} - \mu) + \varepsilon_{t-1} \tag{2.9}$$

and perform the same tests mentioned above to the process

$$Y_t = \theta Y_{t-1} + \varepsilon_{t-1} \tag{2.10}$$

Where $Y_t = Z_t - \mu$. Note that in both versions of the test, the presence of a unit root implies the first differences of Z_t and Y_t, call them $\Delta Z = Z_t - Z_{t-1}$ and $\Delta Y = Y_t - Y_{t-1}$, are stationary.

2.4 Previous Models

2.4.1 The JLS Model

One well-known model used to characterize crashes is the Johansen-Ledoit-Sornette, or JLS, model. Many consider it to be a robust model, in part because has been applied to a wide range of financial indices, including foreign exchange rates, [44] [46], and emerging markets indices [21]. The model believes that asset prices follow faster than exponential growth while exhibiting log-periodic behavior, which was discovered independently by [13].

The JLS model's economic framework revolves around finding a justification for a financial bubble based on investors' rational expectations and all the current empirical evidence. Beginning with the 1987 crash, [47] and [19] proposed an explanation based on positive feedback loops and investors engaging in herd behavior. Investors have access to a limited amount of information and imitate their near neighbors in a process that resembles a complex dynamic system leading to a critical point. [19] and [46] extended this particular theory by formulating how investors forecast the asset price. At any given time, a crash may occur with a probability of $h(t)|_t$, otherwise known as the conditional hazard rate or probability that a crash will occur given that a crash has not occurred yet. Therefore, it is also rational for investors to remain in a bubble as long as they also expect the price of the asset to rise. That is they expect some reward for their undertaking risk. Adding a martingale hypothesis,

$$\text{for all} \quad t' > t \qquad E_t[p(t')] = p(t)$$

where $E_t[p(t')]$ is the conditional expectation of asset price at time t given its known price at time t' and also that holding or buying the asset price is a 'fair game', namely $E[p(t)] = 0$,

emphasizes the original assumptions that both the crash is inherently uncertain and that investors are rational for choosing to remain in the bubble.

As an economic side note, this theory of investor behavior does not fully conform to the previously mentioned rational expectations and efficient market hypothesis. In being inherently uncertain, the crash presents a variable that can only be probabilistically factored into the crash. This may contradict the notion that investors have access all publicly available information and that all decisions are rational, in that the model states that investors can only guess the time of the crash. The authors in [19] chose to use 'Adaptive Expectations', where investor's knowledge of the future is either limited or imperfect but investors adapt their expectations as new information becomes available. The authors of [17] raises a similar concern about the 'fair game' condition. He points out that it may be inconsistent for the expectation to be 0 and yet for the bubble to continue rising. The authors of [46] readdresses the concern and possible discrepancy. The expected rise in the asset price can be thought of as a type of risk premium. Investors are aware of the possibility of a crash and its implications for the volatility of the asset. The asset, therefore, increases to compensate investors for the added volatility. However, [17] also presents concerns over the ambiguity over cause and effect. Specifically, whether it is investors expectations that drive the asset price or if the asset price drives investors' expectations. As mentioned, this is a consequence of investors' dynamic and constantly changing perceptions of market risk [19].

To predict the crash, the authors, [46] [19] [20], assume that the cdf $Q(t)$, pdf $q(t) = \frac{dQ}{dt}$, and conditional hazard rate $h(t) = \frac{q(t)}{[1-Q(t)]}$. We will elaborate on why this has been assumed later. However, note that the hazard rate contains an asymptote, or critical point, when the cdf $Q(t) = 1$, which indirectly implies that the hazard rate can never be known with absolute certainty and, thus, coincides with the model's beliefs on investor behavior. Assuming the price of the asset will drop a fixed percentage $\kappa \in (0, 1)$ after some reference point, the overall dynamics of the asset price can be represented by

$$dp = \mu(t)p(t)dt - \kappa p(t)dj \tag{2.11}$$

Where $\mu(t)$ corresponds the expected drift of the asset with respect to time and the integration element j is a jump process with a value of 0 before the crash and a value of 1 after the onset of the crash. Combining our previous fair game and martingale hypotheses to $E_t[dp] = \mu(t)p(t)dt - \kappa p(t)h(t)dt = 0$, we may obtain the relationship

$$\mu(t) = \kappa h(t) \tag{2.12}$$

Using the above relationship, 2.11 can be transformed to an ordinary differential equation with the solution

$$log\left[\frac{p(t)}{p(t_0)}\right] \approx \kappa \int_{t_0}^{t} h(t')dt' \tag{2.13}$$

for all t before crash and $p(t)$ before the crash price

If we assume that in the crash the price will drop by a fixed percentage, κ, of some assumed critical price, call it p_c, the dynamics of the asset price become

$$dp = \mu(t)p(t)dt - \kappa\left[p(t) - p_c\right]dj$$

and a very similar martingale argument will allow us to conclude

$$\mu(t)p(t) = \kappa\left[p(t) - p_c\right]h(t)$$

and derive the solution

$$p(t) \approx p(t_0) + \kappa\left[p(t_0) - p_c\right]\int_{t_0}^{t} h(t')dt' \tag{2.14}$$

which is similar to equation 2.13.

In determining how investors estimate the hazard rate, $h(t)$, Johansen, Ledoit, and Sornette looked towards statistical mechanics for a plausible solution. Taking into account that each individual trader has his own perceptions of risk but also that the system as a whole must behave like a bubble and eventually crash, the JLS model borrowed some concepts from the Ising model of ferromagnetism that attempted to explain spontaneous phase transitions.

The Ising model assumes, for a sufficiently low temperature, that atoms are arranged in a geometric lattice, a lattice simply being a set of equally spaced points arranged in some particular shape [9]. Also, the atoms at each lattice sight usually take on two states, 1 and -1, that represent the atoms' magnetic state. The current theory states that a system must gain or lose some amount of energy in order to switch from its liquid phase to its solid phase, or some other phase change. The challenge was to create a theory that explains why all of the atoms in the system spontaneously switch from on macroscopic state to another. In making the assumption that the overall energy of the system is primarily determined by interactions between adjacent atoms, Ising allowed for the presence of long-range correlations where groups of adjacent atoms synchronized in a 1 or -1 state are allowed to remain locked in that state for some time. This is due to the fact that an atom's magnetic state is influenced by the overall state of the system and, in the Ising model, the magnetic state of adjacent atoms. For more on applying the Ising model to financial data and critical phenomena, please refer to [31] and the references therein.

Johansen, Ledoit, and Sornette believed that this had the potential to shed light on investor behavior during a bubble [19] [20]. Like atoms, investors may be arranged in some kind of network where each investor communicates with other investors but prioritizes communication with adjacent investors in his or her network. Investors can also make the decision to *buy*, 1, or *sell*, -1. Even though most traders make this decision on their own, each trader has the propensity to imitate the nearest neighbors in their network. Although this greatly simplifies investor behavior, the model serves to represent the basic nature of investor herding behavior in a. Such behavior is qualitatively elaborated upon in [40] and [41]. In these cases, investors behave in a semi-rational manner. We use the term 'semi-rational' because investors do not act 'rational' as in the rational expectations hypothesis described earlier. In this theory, investors may choose to imitate their investors when they lack or believe they lack information. Like in the Ising model, persistent trends or fads may spontaneously accumulate. The bubble is akin to the build-up of long range correlations where the spontaneous polarization is the critical event corresponding to the peak of the

bubble.

A solution to the Ising model exists when you assume the traders are arranged in a hierarchal diamond lattice, shown in figure A.16. Starting with two traders that are directly connected, we can break the link to form two new links with two new traders. The now four traders can be arranged to form a square, or some sort of diamond. The process can be continually repeated on the links between adjacent traders, where each link is broken in order to form a diamond with two new traders. After x repetitions of this process, we will have a network consisting of $\frac{2}{3}(2+4^x)$ links and 4^x traders. Again, the process is illustrated for two iterations of this process in figure A.16. The actual solution involves modeling the approach to the critical point through the interactions between the linked atoms. Equivalently, we can describe the hazard rate as

$$\frac{dh}{dt} \propto h(t)^{\delta}$$

where $\delta - 1$ quantifies the effective interactions felt by a particular trader. The hazard rate grows exponentially, depending on the number of adjacent traders. It also describes the possible self-fulfilling prophecy that arises when investors believe a crash may be imminent, so price increases to compensate investors for undertaking additional risk thereby further inflating the bubble. The solution to the Ising model for atoms in a diamond lattice contains a critical point. This corresponds to the popping of the bubble and the most likely time of the crash.

$$h(t) \propto (t - t_c)^{-\alpha} \tag{2.15}$$

Note, the exact solution can be found in terms of t, t_c and δ and is actually $h(t) \approx \left(\frac{h_0}{t-t_c}\right)^{\alpha}$ where α is defined as $\frac{1}{\delta-1}$. However, the exponent α of the solution will most likely be found numerically, as it is unreasonable to assume the exact number of interactions per trader. For practical purposes, it simply assumed that $(t - t_c)$ carries some negative exponent $-\alpha$.

To better capture the log-periodic nature of the data, a first order Fourier expansion is taken in lieu of the above equation.

$$h(t) \approx B_1 (t - t_c)^{-\alpha} + B_2 (t - t_c)^{-\alpha} \cos(\omega \ln(t_c - t) + \varphi) \tag{2.16}$$

Note the general solution 2.16 can also be obtained economically, by plugging in equation 2.15 into 2.14

$$p(t) \approx p(t_0) + B_0 \left(t - t_c\right)^{-\alpha}$$

and consequently performing a first order Fourier expansion on the above. However the first equation used to characterize crashes, and first presented in [47] was actually the following

$$\ln[p(t)] \approx \ln(p_c) - \frac{\kappa}{\beta} \left\{ B_1(t_c - t)^{\beta} + B_2(t_c - t)^{\beta} \times cos\left(\omega \ln(t_c - t) + \varphi\right) \right\}$$

Where, like before, $p(t)$ is the price of the asset at time t, p_c is the critical price, t_c is the critical time, but β is the critical exponent and the constant κ is meant to simulate the coupling interaction energy constant of the Ising model. Also note that the time variables t may be replaced by the varable $\tau = \frac{t}{t_c}$. The equation undergoes a renormalization and transforms to

$$ln \frac{[p_c]}{[p(t)]} \approx \frac{(t_c - t)^{\beta}}{\sqrt{1 + \frac{(t_c - t)}{\Delta t}}} \times \left\{ B_0(t_c - t)^{\beta} + B_1(t_c - t)^{\beta} * cos\left[w * ln(t_c - t) + \frac{\Delta\omega}{2\beta} ln\left(1 + (\frac{t_c - t}{\Delta t})^{2\beta}\right)\right]\right\}$$

although the original equation is more common.

2.4.2 A Model with Log-Stationary Intervals

In addition to a non-gaussian, leptokurtic distribution of returns, price indices tend to exhibit intermittency. Intermittency, in terms of fluid dynamics, is when a fluid alternates between steady, predictable behavior and chaotic, irregular behavior. In physics, there also exists the concept of turbulence in fluid dynamics. Intuitively, turbulence refers to the nature of the flow of a particular material. When the material's flow appears random or chaotic, it is considered turbulent. Although the problem of fully describing turbulence remains unsolved, foreign exchange data has been found to contain similarities to data exhibiting turbulence [14] [26].

Again, physics may not offer an analytical solution but the general theory behind turbulence can also serve as an analogy for modeling price indices. Intermittency is believed to be the result of energy being periodically pumped into the flowing system. The equivalent concept in finance relates to public information being gradually made available. A turbulent system also has a leptokurtic distribution of velocity fluctuations in short time scales that eventually converges to a Gaussian distribution, not unlike many financial indices. However, turbulent velocity fluctuations tend to exhibit different scaling properties in their pdfs than financial indices. Financial data has also been found to contain intermittences.

The authors of [16] decided to create a model that incorporates intermittent stable periods via stationary intervals. In determining the overall dynamics of the asset price or exchange rate, recall the differential equation for the change in the price of an asset

$$\frac{dS}{S} = \mu dT + \sigma dX \tag{2.17}$$

Where S is the price of the asset, μ and σ are constants known as the drift and volatility of the asset respectively, and dX is a random variable. [16] and [30] also assume that dX exhibits, more or less, a normal distribution. Focusing solely on the deterministic part of the above equation, not including the volatility, the equation above may be reduced to

$$\frac{dS}{dT} = \mu S \tag{2.18}$$

And is then further modified as follows

$$\frac{dS}{dT} = \mu \frac{(S_c - S)}{(T_c - T)} \tag{2.19}$$

Where $(S_c - S)$ and $(T_c - T)$ are the distances from the critical price, S_c, and critical time, T_c, respectively. Then, based on similar reasoning as the previously discussed model, investors drive up the price of the asset as the critical date approaches, [19] [46]. Since variations are expected to become smoother, the deterministic component, or drift, will dominate when the critical date is near. Like the JLS model, a power law with log-periodic signatures could be a good way to model the behavior of the asset price. Making the change

of variables P for the distance to critical price and t for distance to the critical time, the modified equation 2.19 becomes

$$\frac{dP}{dt} = \mu \frac{P}{t} \tag{2.20}$$

The next assumption relates to intermittency and log-stationary intervals. If we assume, or if we can empirically demonstrate, the variable t to be log-stationary, we have

$$d(\log(t)) = K$$

which can be reduced to

$$dt = \mu t$$

Therefore on the basis that the logarithm of the temporal steps is approximately discrete, we can reduce the modified equation 2.20 to

$$d(\log P) = \mu d(\log t) \tag{2.21}$$

log referring to the natural logarithm which in turn leads to the relationship

$$P \approx e^{\mu \log t} \tag{2.22}$$

Once again, investors' behavior leading up to the crash is being modeled as a type of critical point.

We can only claim an approximate relationship as the assumption is that the steps in the natural logarithm of t are discrete. However, $d(\log(t)) = K$ and equation 2.21 imply

$$d(\log P) = \mu K$$

Or just that $\log P$ also moves in discrete steps. Also, note that 2.21 and the equation above can imply the reverse relationship. So observing stationary behavior in $d(\log P)$ gives reason to believe that the temporal steps, $d(\log t)$, will also exhibit stationary behavior, given our assumptions about the deterministic behavior of the asset price.

In order to incorporate intermittency in the equation 2.22, [16] and [16] considered the following equations

$$P \approx \beta e^{\alpha F(\log_a t)} \tag{2.23a}$$

$$P \approx \beta e^{\alpha C(\log_a t)} \tag{2.23b}$$

Where F are C the floor and ceiling functions respectively. Recall the floor and ceiling functions are variations of the integer function. So for some number x, $F(x)$ returns the closest integer, $n \in \mathbb{Z}$ such that $n \leq x$, and the ceiling function, $C(x)$ returns the nearest integer such that $x \geq \mathbb{Z}$. Focusing on the floor function, it is easy to demonstrate that equation 2.23b is discrete scale invariant, and hence is akin to a power law with a critical point. Modifying t to λt, we have

$$\beta e^{\alpha F(\log_a \lambda t)} = \beta e^{\alpha F(\log_a t + \log_a \lambda)} = \beta e^{\alpha (F(\log_a t) + n)} = e^{n\alpha} * \beta e^{\alpha F(\log_a t)}$$

for $n \in \mathbb{Z}$. Letting $\mu = e^{n\alpha}$, then equation $f^F(t)$ satisfies the condition for discrete scale invariance. To show it this also contains log-stationary steps, for two points, x_2 and x_1, the quantity

$$d\left(\log\left(f^F(t)\right)\right) = \log\left(f^F(t_2)\right) - \log\left(f^F(t_1)\right)$$

may be reduced to

$$\alpha F(\log_a t_2) - \alpha F(\log_a t_1) = \alpha * n \tag{2.24}$$

where n is a nonnegative integer. For a particular value of the basis a, there will be a set of values of t for which $d\left(\log\left(f^F(t)\right)\right) = 0$. That is to say, if for some t_j we have $\alpha F(\log_a t_j) = \alpha K$, then there must be some point t_{j+l} where for t_{j+l} and all points after we will have $\alpha F(\log_a t_{j+l}) \neq \alpha K$. For all points t in between, we have $\alpha F(\log_a t) = \alpha K$ and $d\left(\log\left(f^F(t)\right)\right) = 0$ by the relationship above.

The authors of [16] and M.C.Mariani2006 have already applied the aforementioned model to the NASDAQ and S&P 500 before the crashes of April 2000 and October 1987, respectively, as well a few other international indices. However, the model has yet to be directly applied to currencies where such a model may be useful, though applying this model to exchange rates based on the krona will pose its own problems.

Chapter 3

Data, Numerical Methods and Analysis

3.1 The Chosen Data

We decided to analyze the behavior of the Icelandic krona up to a sudden fluctuation in July of 2007, which was when traders, most likely, began turning against the currency. The actual event began as a reaction to the bursting of the housing bubble and onset of the financial crisis [50]. The study of financial crashes via the previously mentioned models is dominated by the study of stock indices. The few attempts of currencies have yielded results that are comparable, but not as promising[46] [44] [16]. Furthermore, Iceland's economic crisis has mostly been studied qualitatively, not mathematically. Its growth and collapse is similar to other countries [35] [15] and can serve as further warning to other rapidly growing nations. In addition to the collapse of its banks, Iceland also had to deal with a collapsed currency.

Since the summer of 2006, the krona had been experiencing a very consistent appreciation. Prior to that summer, the krona had actually been depreciating [8]. The currency's lowest point came amidst warnings that Iceland's banks had been undertaking overly risky strategies that left them exposed to market volatility. As input, we chose to analyze daily exchange rates involving the krona. The exchange rates offer a representation of investor sentiment towards the krona.

Against the krona, we paired the Swiss franc, Japanese yen, and U.S. dollar for a total of six data sets from which to sample. As critical prices, we chose a price corresponding to

local maximum or minimum before a clear downward or upward trend per each data set. We also decided upon three hundred and sixty-five data points sampled daily between July 2007 and April 2006. This initial sample is chosen arbitrarily relying only on the assumption that the number of data points is sufficient. Since in [16] and [30], there were no specifications on the chosen data, for now we assume that the analysis does not require a minimum number of data points in the interval of interest. Nonetheless, we also chose a second sample that began on an economically significant date. Since the model assumes that new information changes price dynamics, a sample chosen based on an important economic announcement may improve the model's accuracy. The second sample was chosen beginning on July 3rd 2006, the day that the *isec*, an Icelandic small cap securities market, opened for trading [22]. This sample ended up having a total of three hundred and seven days. We will focus on these two samples to test the efficacy of the analysis.

The goal was simply to test the model's performance for the different exchange rates with different counter currencies. Each exchange rate will have slightly different underlying conditions due to the different counter currencies. Each rate can be expected to exhibit power-law growth, but will likely grow at different rates. Each rate should also exhibit log-periodic intermittences, but may vary in the time and length in which they are presented. The authors of [16] and [30] imply the reliability of the model lies mostly with its ability of the model to detect exponential growth and log-periodic intermittences, and that detecting these trends allows for the prediction of a crash date. Testing on different rates involving the krona is a way of testing the model's sensitivity to noise of each individual set. If the analysis is genuinely focusing on trends characteristic to a crash, its results should not vary much with between equal samples of exchange rates involving the krona.

Note,the analysis may not even perform the same on two corresponding pairs of exchange rates. Specifically for a pair of exchange rates, p and q where $p * q \approx 1$, the analysis may not treat the two rates the same way. We will illustrate this argument visually. First we note that for a scalar λ and the data set λp, we may choose $p_c \to \lambda p_c$ such that

$$|\lambda||p_i - p_c| = |\lambda p_i - \lambda p_c|$$

for each p_i. Both of the above sets will also be modeled by the equation

$$|\lambda|Be^{\alpha F(\log_a|t-t_c|)}$$

which is a scalar multiple of the modeled equation for the set p_i. Hence, modifying the data by a scalar will only modify the magnitude of the modeled equation but not anything else. The numerical calibration of the equation coefficients will also remain consistent with this result.

Also let $P_i = \frac{|p_i-p_c|}{max|p_i-p_c|}$ and $Q_i = \frac{|q_i-q_c|}{max|q_i-q_c|}$. Note that within the two sets, $max|p_i - p_c|$ and $max|q_i - q_c|$ are both constants. Therefore, P and Q should be modeled by scalar multiples of $|p_i - p_c|$ and $|q_i - q_c|$ respectively. Choosing p_c and q_c such that $p_c * q_c = 1$ for the sake of consistency, figures A.13 A.14 and A.15 show that the two sets have a very similar periodic structure but are not exactly the same. Therefore, they must each be governed by different power laws that differ by more than just the scale. Moreover, one set of data points seems to be consistently higher in value than the other, when adjusted for scale. This implies that the set's differences in logarithm must necessarily be different which may, in turn, affect how the model determines the number of log-stationary intervals for fitting. For example, the exchange rate that prefers to remain higher than the other, demonstrated in the figures, much decrease faster to reach the minimum distance to the crash price. The step function fitted to this set of exchange rates therefore needs to have shorter steps that allow for more jumps, at least near the end.

Note, we mentioned that exchange rates may have some discrepancies due to liquidity between currencies. Iceland, being a small nation, has a currency that is less liquid than the more widely traded dollar, franc, and yen. This liquidity difference may be the cause for these discrepancies between corresponding sets. However, we argue that this is a mathematical property characteristic of the inverse function

$$f(x) = \frac{1}{x}$$

which exchange rates must follow. This is because each corresponding set will have a different average slope between two points. For two points, x_i and x_j, the absolute value

of the average slope would be

$$\frac{|x_j - x_i|}{|t_j - t_i|}$$

For their inverses it would be

$$\frac{|\frac{1}{x_j} - \frac{1}{x_i}|}{|t_j - t_i|} = \left(\frac{1}{|x_j x_i|}\right)\frac{|x_i - x_j|}{|t_j - t_i|} \neq \frac{|x_j - x_i|}{|t_j - t_i|}$$

Since our model uses $|p_i - p_c|$, the distance to some point p_c, each corresponding set will approach their respective critical prices differently. As we argued, this could imply that the model will perform differently on two corresponding sets. Nevertheless, the issue of liquidity between two currencies may be a bigger issue when analyzing data sampled at higher frequencies. In our data, quoted to six significant digits, corresponding sets obeyed $p_i \approx \frac{1}{q_i}$ within an accuracy of seven decimal places.

In all, economic expectations say dictate that the model should give the same results on all of the data sets. The subtle differences in the data sets suggest that this may not be the case.

3.2 Numerical Methods

Our goal is to replicate the analysis done in [30] and [16] and therefore we developed a program in R to emulate the data analysis methods as much as possible. To recap, we are modeling $|p - p(t)|$ via the equation 2.23a, which is essentially $\beta e^{\alpha F(\log_a |t-t_c|)}$ where p_c is the estimated critical price, t_c is the estimated critical time, and $p(t)$ is the exchange rate at time t. We begin by estimating the general power law that characterizes the system.

$$|p(t) - p_c| \approx \beta e^{\alpha \log_a |t-t_c|} \tag{3.1}$$

by running ordinary linear regression on

$$\ln |p(t) - p_c| \approx B + \gamma \ln |t - t_c|$$

where the estimated coefficients β and γ are $\ln B$ and $\frac{\alpha}{lna}$ respectively. From this relationship, we can then estimate the base a to use in our desired equation

$$|p(t) - p_c| \approx \beta e^{\alpha F(\log_a |t-t_c|)}$$

which corresponds to equation (3.9a) and F is the floor function. Furthermore, let $G(\ln |t - t_c|)$ be as follows

$$G(\ln |t - t_c|) = \ln |t - t_c| - \ln \alpha \cdot F\left(\frac{\ln |t - t_c|}{\ln a}\right) \tag{3.2}$$

Note that the function $G(\ln |t - t_c|)$ above is equivalent to

$$G(\ln |t - t_c|) = \frac{1}{\gamma}\left(\ln(\beta e^{\alpha \log_a |t-t_c|}) - \ln(\beta e^{\alpha F(\log_a |t-t_c|)})\right) \tag{3.3}$$

and is therefore equivalent to the difference between $ln|t - t_c|$ and the step function $\ln \alpha \cdot F\left(\frac{\ln |t-t_c|}{\ln a}\right)$. We note that $G(\ln |t - t_c|)$ is periodic with period a. To prove this,

$$\begin{aligned} G(\ln |t - t_c| + \ln a) &= \ln |t - t_c| + \ln a - \ln a \cdot F\left(\frac{\ln |t - t_c| + \ln a}{\ln a}\right) \\ &= \ln |t - t_c| + \ln a - \ln a \cdot F\left(\frac{\ln |t - t_c|}{\ln a} + 1\right) \\ &= \ln |t - t_c| - \ln a \cdot F\left(\frac{\ln |t - t_c|}{\ln a}\right) = G(\ln |t - t_c|) \end{aligned}$$

Another property of the function G is that for a scalar λ we rewrite the above equation as

$$G(\ln |t - t_c|) = \frac{1}{\gamma}\left(\ln(|\lambda|\beta e^{\alpha \log_a |t-t_0|}) - \ln(|\lambda||p - p_c|)\right)$$

And we can use the product identity of logarithm to eliminate $\ln |\lambda|$. Therefore, multiplying each point in the data by a scalar doesn't change the periodic relationship between the underlying power law and the step function that makes the final approximation. To find the period of G, we take a discrete Fourier transform to find the frequency in which G is periodic, and ideally a frequency with a high power in its domain. In particular, we use

$$Q(k) = \sum_{n=1}^{\frac{N}{2}} G(x_n) e^{-i2\pi(k-1)(n-1)/(NK)}$$

where K is the period of the Fourier transform and x_n for $n = 1, 2, N$ are equidistant points between $\ln|t_1 - t_c|$ and $\ln|t_f - t_c|$, where t_1 and t_f are the initial and final dates of our observation window. Due to the natural log function, it is highly unlikely that the time points $\ln T_i$ will be equidistant. Since our discrete Fourier transform requires the points in its domain to be equally spaced, we must interpolate to obtain x_n. In our convention, we let N be the number of our data points minus one when they were odd and equal to the number of our data points when they were even. To obtain our x_n and $G(x_n)$ to be

$$G_{est}(x_n) = \frac{G_i[j] + (G_i[j+1] - G_i[j])}{(n[j+1] - n[j])} * (x_n - n[j] + n[1])$$

For each point $n[j] < x_n \leq n[i+1]$ where $n[i] = \ln|t_i - t_c|$. Since $\|Q(k)\|$ returns an approximation of the power at each freqeuncy, to pick our dominant frequency, f, we pick one with a local maximum at its frequency domain. In our case, we analyzed every frequency except the smallest frequency and half our sampling frequency, which is one day. Subsequently, we chose whichever provided the $a = e^{\frac{1}{f}}$ with the best fit.

To choose the best a, we did another linear regression. This time, $\ln|p - p_0|$ was fitted against $B + \alpha F^{\frac{\ln|t-t_c|}{\ln a}}$ for each a determined by our frequencies. The base a that produced the smallest sum of squared errors will be the one taken into consideration. Since the model cannot return explicit probabilities for the likeliness of our chosen t_c and p_c, the (t_c, p_c) that has the lowest sum of squared errors of all the candidate pairs is taken as the most likely crash date and price.

3.3 Foreseeable Questions

First, some observers may note that the data does not exhibit behavior typical of a crash. Traditionally, crashes are associated with very sharp downturns after a history of consistent growth. Historically, it has been well documented that Iceland's currency was overvalued for a large period of time after the country's liberalization. It is also well known that the currency went on a long uncontrollable plunge beginning in 2007. The actual plunge,

however, occurred during the course a year, not the traditional behavior of a financial crash. We argue that the atypical behavior alone is not enough to disqualify the data as a crash. For example, what is currently considered to be the Asian crash or Asian crisis began in 1997 and lasted for well over a year [35]. In fact, the Asian crisis has several things in common with the Icelandic economic crisis. Both countries decided to liberalize extensively in the years before the crisis. Both crises were exacerbated by excessive borrowing in foreign currencies by each region's newly empowered financial sector. More importantly, both crashes were the result of a 'speculative attack'. That is, in both cases the crash was driven largely by the market speculators betting on the currency. The two governments also refused or were unable to act appropriately when investors acted upon fears over each currency's stability [15]. Furthermore, power laws and log-periodicity are meant to describe self-organized criticality, a broad term that encompasses different kinds of events. The economic similarities to previously studied critical events along with our critical dates coinciding with economic events, that may themselves also indicate a polarizing shift in investor sentiment, all suggest that the sudden depreciation we are studying corresponds to a self-organized critical event.

Second, our data suggested that the Icelandic krona underwent sharp depreciation in July 2007 and recovered close to its initial value in December 2007 before going through a long period of depreciation. Furthermore, Iceland remained in denial about the exact nature of their economy even up to February of 2008 [50]. This raises questions with our choices of critical dates. Some may question whether the event in July, which is demonstrated in A.1 A.2 A.5 A.6 A.9 and A.10, was when investors changed their stance on the krona.

Economically, can be difficult to know exactly when investors all decided to turn against the currency. However, we believe the choice of earliest possible critical date is justified. The first sudden bout of depreciation coincided with the earliest markers of the 2008 financial crisis [8]. The housing markets in Britain and the United States were showing signs of trouble in the summer of 2007, when the krona reached its peak and began to fall. We believe that this event either represented or directly caused the change in investor senti-

ments, at least for the month of July. As to what extent that was represented in the data, we call upon the assumption of a more or less efficient market.

Furthermore, reports in 2006 by various reputable institutions also suggested that Iceland's financial sector was exposed to market volatility. The reports warned that Iceland's banks would not be able to handle a bad turn in the economy. Even after months of growth, news of upcoming financial turmoil would have been clearly bad for the krona. Even furthermore, Iceland's growth model relied on a heavy krona. Speculators believed that government and banks made would make some attempts keep the krona's value high, should it ever depreciate too much [50].

On the other hand, bondholders would not have an incentive to attack the currency, as it would devalue their returns [10]. So, a speculative attack may not have been the direct cause of the event we are studying. Still, investors tend to flee from risky and illiquid currencies should they suddenly become risk averse [23]. Since the event we are studying coincides with the onset of the housing crisis and since investors had been warned of Iceland's instability, there is also a good chance that the event we are studying is actually investors fleeing to more liquid currencies and not necessarily an 'attack' on the krona. However, this is a subject that goes a bit beyond what is needed to address the concern. The actual dynamics of speculative attacks can be complex. For further study on speculative bubbles on currency the reader may consult [11]. However, this all goes a bit beyond what is needed to address the concern. It should suffice to provide evidence that investors had reason to turn against or abandon the krona in July, in a manner akin to a self organized critical event.

Moreover, some may wonder be why we planned to analyze foreign exchange rates based in the krona and not the krona's effective exchange rate. In analyzing a crash in the krona, it seems to make more sense to analyze a direct measure of the krona and not an indirect measure of the krona. We maintain any foreign exchange rate involving the krona should mimic the behavior of a bubble and crash, as long as the other currency is behaving relatively normally. However if the counter currency is also undergoing a crash, it may also be displaying the characteristic signatures of a crash, thereby obscuring said behavior in

the exchange rate. Still, having multiple data sets that encode the same event allows us to test the model's consistency, as the different exchange rates will vary in their own way. An ideal model should be able to detect this common trend, the crash, in spite of each data sets particular, variational, behavior. Admittedly, this may not allow us to extrapolate the model to describing any exchange rate that is nearing a crash.

Also, the assumptions of log-stationarity and diminishing periodic oscillations, at first glance, don't appear to be present in the data. Although we don't require continual log-stationarity only that it be present during particular intervals, the domination of the deterministic component of equation 2.17 is required for the proposed equations 2.22 and 2.23a to be a good fit.

As the data approaches the critical price, the distance to the critical price becomes closer to zero. The logarithm of the price then becomes very negative and very large in magnitude. Similarly, small changes in the asset price, p_i, will lead to large changes in $\log|p_i - p_c|$ when the price is close to the critical price. This arises from an innate property of the log function, which is illustrated in A.3, A.4, A.7, A.8, A.11, and A.12 for a particular sample where $d\log|p - p*|$ is compared to the original data and a distant critical price $p*$. Note, the oscillations are greater in magnitude for the smaller critical price but their overall behavior remains the same even as $p*$ varies. Normally, this would not be a problem, as the assumptions allow this final interval to have its own weakly stationary properties. Moreover, this would not necessarily imply that this particular analysis is inappropriate. It could, however, imply that a different process, such as an ARCH or GARCH process, could be more accurate. Our concern is that this appears to be the opposite behavior that the model requires. Recalling equation 2.17

$$\frac{dS}{S} = \mu dT + \sigma dX$$

We eliminate the drift term based on the assumption that volatility will 'smooth' out, [16], and derive equation 2.18

$$\frac{dS}{dT} = \mu S$$

Yet the consecutive differences appear getting more volatile as the exchange rates approach their respective critical prices. This may imply that the transformation $p_i \to |p_i - p_c|$ leads to behavior that contradicts our assumptions in deriving equation 2.18 and, consequently, equation 2.23a. Choosing a large critical price appears to fix this problem, again referring to figures A.3, A.4, A.7, A.8, A.11, and A.12 where we see that oscillations are closer to the appropriate scale for large critical prices. The model should, therefore, work better for large critical prices. If this is true, then the results of the analysis may be difficult to interpret, as the model will prefer very large and unrealistic critical prices.

Some may dislike the method of prediction to be used in this analysis. In previous papers [30], the authors relied heavily on detecting an underlying power law behavior with intermittences that generate log-periodic behavior around the power law. They, thus, rely heavily on the assumption that a causal link exists between log-periodic behavior around a general power law trend and an crash. To what extent the data exhibits these signatures at reasonable critical times and critical prices further determines whether the analysis will give useful information. Understandably, the practical application of this technique is debatable.

Note that we will use the word *predict* interchangeably with *prefer*. This is because prediction involves assuming that there will be an upcoming crash and using the goodness of fit of the model to gauge whether our assumption made sense. Admittedly, the authors of [30] did not provide explicit guidelines for determining whether or not the model is performing well, or performing badly. We decided to compare the model to a basic power law fitting, to determine if the inclusion of log-periodic intermittences allows us to describe the data better than a simple exponential fitting. Failure to outperform or provide more useful information than an exponential fitting will be the primary factors in determining the usefulness in predicting a crash.

Also, goodness of fit will only allow us to answer: if a crash will occur, can the model predict it? To determine the model's practical usefulness, we must answer: if the model predicts a crash, will a crash occur? Thus to determine whether or not a crash will occur

if the model predicts a crash, we also have to assess the model's propensity to give a false positive. If the model is highly likely to give a false positive, the likelihood of a crash actually occurring on the condition that the model predicts it will occur may be low. Such examples are often used to illustrate Bayes' rule. Mathematically, Bayes' rule is simply the following

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$$

where $P(B_j)$ is the probability of event B_j [37]. For example, say there exists a rare disease that only 1% of the population has. Also say there exists a test that is 95% accurate. That is, the test will falsely read positive only 5% of the time. If the test were administered regularly, the vast majority of people coming to get tested will not have the disease and the probability of a person having said disease on the condition that the tests reads positive is about 16.1%. Without knowledge of how the analysis performs on nonsense sets where no crash or significant event is set to occur, the practical predictive applications of the analysis are limited. Short periods where financial indices may exhibit an exponential growth that resembles a power law and have periodic log-stationary intervals with respect to $\log_\lambda$ but were followed by no significant events may actually be common. The older and more tested JLS model offers a remedy for this by claiming a universality of the fitted parameters [45]. The authors of [46] have also claimed that in the eighty six years, the JLS model was only able to fit eleven, four hundred week periods that had sums of squared errors comparable to that of three major crashes. Every single one of these periods closely preceded one of these three crashes, and only six periods exhibited the universal fitted parameters. Furthermore, there exists a diagnostic test to know when to apply the JLS model [36]. The same cannot be said for this model. For one, it has much fewer parameters to fit. Even the parameter a that determines the frequency of the stationary intervals will vary with the sample and sample size. Since it is meant to correspond to investors reacting to information being injected into the system, it would make sense that different crashes have different dynamics with regard to this intermittent behavior.

This is not to say the model does not have the potential to predict an upcoming crash,

simply that it still falls short of being able to give both an accurate and precise prediction when the true critical time is unknown.

3.4 Results

First, we present results on the primary data sets. Then we present results for various iterations of the analysis on different subsets of the data to assess the robustness of the initial results. As mentioned earlier, [16] and [30] do not give explicit specifications over choice of data. Therefore we assumed the analysis would not vary much with small changes in our choice set, which proved to be the case in practice. The analysis replicated from [16] and [30], however, did not provide a measure of accuracy. The sums of squared errors resulting from the analysis will be compared against a pure power law to gauge the model's general effectiveness. As a reminder, the original analysis was performed over major stock indices, not exchange rates. The effectiveness of this model on pairs of exchange rates has yet to be confirmed. Therefore, the ability of the model to outperform a basic power law will be important in measuring its success.

Appendix B shows contour graphs of the sum of squared errors for the model on progressively increasing critical dates and critical prices. Furthermore, a value of .5 was taken to represent a critical date of the current date with progressive increments in critical date consisting of one day up to thirty days. Similarly, a critical price of .6125% past the final data point was taken to represent a critical price very near the final price and progressive increments in critical price were set at 1.25%. Critical prices were such that $p_c \times q_c = 1$, where p_c and q_c are the critical prices for the corresponding rates, for the sake of consistency. In examining the contour graphs, we are most interested in the scale and distribution of the sum of squared errors, since a previous metric for success has not been established for foreign exchange rates. First we present the specific results.

In our primary analysis on the three pairs of data sets, the analysis performed quite well. In each case, our model preferred an immediate critical date, which we dubbed to

be the final data point coinciding with a maximum or minimum before a sudden change. Furthermore, the analysis presented results that remained completely consistent between each pair. As we see from figures B.1 B.2 and B.3, the analysis here prefers the same critical date for every set as well as a similar range of critical prices, which ranged from an increase of 2.5% to 10% and their corresponding percentage decreases of 3.6% to 9.1%. The only major discrepancies come from each exchange rate's choice of critical price and from each rate's distribution of the sum of squared errors. Also, the two exchange rates based on the U.S. dollar were very consistent with each other. The yen and the franc based rates, on the other hand, exhibited similarities between each other, yet a different distribution of errors between corresponding rates. The analysis, over all of the different exchange rates, also outperformed a pure power law, both in predictive accuracy goodness of fit. Overall, the model proved to be accurate and consistent, on this first sample.

To ensure our initial results were meaningful, we shifted the interval of observation back by a few days from the peak before the crash. The analysis should, thus, predict later critical dates. One indication that our results were not meaningful would be if the analysis always predicts an immediate upcoming crash date, even when a crash is relatively distant. Then, the analysis would have very little predictive power. It would at most be able to indicate that it detected a power law with log-periodic behavior, though those are two very important symptoms of a bubble. Ideally, shifting the sampling interval backward in time by one day should shift the preferred critical day forward in time by one day. However, we do not expect the model to be this exact, as it relies on statistical properties of the data to fit the presumed critical dates. We found that for small changes in scale, the analysis performed the same for these small shifts. For larger shifts of four or more days before the actual crash date, the analysis began predicting crash dates farther into the future, though few corresponded to the actual crash date. Once again, corresponding rates were consistent in their predicted critical dates, but preferred critical prices varied more than in the initial analysis. Results are demonstrated for a sample shifted backwards five days. As seen in figures B.4 B.5 and B.6, the model still prefers current critical dates or dates in the

very near future. It also continues to outperform the power law in short time scales. This indicates that the model can pick up the symptoms of a crash, but is otherwise incapable or unlikely to provide a precise prediction for a crash date, especially if the actual crash date were to be unknown.

Our next sample begins with the opening of the isec small securities market, on July 3rd 2006 and continues up to the common critical date of July 20th 2007. Over this sample, the analysis now varies in how in predictive accuracy, consistency between corresponding exchange rates, and goodness of fit relative to a pure power law, which can be seen in figures B.7 B.7 B.8 and B.9. The model over the U.S. dollar sets continue to prefer immediate critical dates, but at critical prices 15% higher and a corresponding 9.1% lower than the presumed true price. It also only marginally outperformed a power law on short time scales and high critical prices. The model did not perform consistently on corresponding sets for the Swiss franc and Japanese yen. However, there were similarities between the distributions of sums of squared errors between the franc and the yen. The analysis over both the Swiss franc and the Japanese yen sets preferred critical dates four and six days into the future, but at massive 25% and 20% increases and decreases. Like the analysis on the U.S. dollar sets, the model only slightly outperformed the power law on short time scales and high critical prices. Though, it could be that these inconsistencies result from a sample that is insufficiently large.

3.5 Discussion

The model is capable of functioning properly, under the right circumstances. For the first and largest set, predictions were close to the true critical date and inconsistencies between corresponding exchange rates were very few and usually associated with the choice of critical price. The results, however, were not very promising on the isec set. It appears that the model is indeed capable of describing the exponential growth and general periodic characteristics of a crash.

Given that data points differ only by a day and that the model cannot consider zero distance to the critical date, the range or general temporal scale of good critical dates is probably more indicative than which particular critical date the model preferred. As mentioned, shifting the sampling interval backwards did not change the results of the analysis by much. Mathematically, this is because the model relies on statistical regularities that do not change much by small changes in the sample. Even if the true critical date is looming, the analysis may give an inaccurate prediction in terms of the exact critical date. The authors of [30] and [46] imply that the exact date of the crash must be inherently uncertain. Economically, this is because if the crash were truly predictable from publicly available information, investors would have predicted it and incorporated it into the price according to rational expectations. Similarly, some investors, who would have predicted the crash, fearing that they would be unable to switch their positions in time may begin selling the currency, or assets denominated in the currency, beforehand. If, as in the JLS model, investors generate bubbles via herd behavior, then selling their assets early may convince other investors to do the same, thereby causing the crash prematurely. Then, by definition, these investors who sold early did not accurately predict the crash. This may sound philosophical, but the essence is that it is highly unlikely that a crash can be precisely predicted from public information alone. Furthermore, the analysis has demonstrated that it is unlikely to prefer the exact critical date and critical price, even under ideal conditions. Also we don't know how the analysis will perform in noncrash data, so the model may not have the ability to give an accurate prediction when the true crash time is unknown. We thus interpret the improvement of the fittings as indicators of the presence of exponential growth and with log-periodic behavior, two important symptoms of a crash.

The results also indicate that the analysis performs relatively consistently between the various sets of exchange rates, but most for the large samples. For these larger samples, nearly all of the results indicated a preference for the same critical date. The range of preferred critical prices was also fairly narrow. For these sets, the distributions of the sums of squared errors all exhibited similarities, again indicating that the model should be

able to pick up the important characteristics of a crash despite the differences between the exchange rates. However, the second sample, that was chosen based on the opening of the isec, did not appear to be as consistent. Again, only the U.S. dollar sets predicted on the same day while the analysis over the yen and the franc preferred critical dates two days apart and with a massive price increase. The distributions of errors appeared noticeably disparate between the corresponding exchange rates for the yen and the franc. One on hand, this confirms our concern, in that the model is capable of noticeably different results on different rates that encode the same crash. However, it continued to perform adequately on the shifted interval of the large sample, which suggests that the model is sensitive to sample size and, hence, the amount of information to be considered for the analysis.

One additional thing to note is that the inital samples tended to prefer values of parameter a tend to be between 1.4 and 1.56, which is a little higher than the values calibrated in [16]. Higher values of a imply lower frequencies and fewer intermittences. Fittings at smaller samples tended to have values of a as small to 1.22, implying such samples prefer many intermittences. It appears, then, that analysis over the smaller, isec, sample has a hard time describing the overall log-periodic trend and prefers to focus on either the noise or the more subtle, higher frequency periodicities in the data. Hence, the fittings on the smaller samples may not be very meaningful, as they may be focusing on the capturing the day to day noise of the data and not the overall periodicity.

3.6 Conclusion

The model has demonstrated that is capable of performing well, but that it is also capable of giving unreliable results. The results on the largest, primary sets of exchange rates, however, are very promising. Here, it was able to detect the general symptoms of a crash and improve fittings across exchange rates. It predicted consistently despite the subtle differences in noise between rates. The isec set, however, confirmed our concerns about the model's consistency between exchange rates.

The model's proposed advantage over the JLS model was its simplicity [16]. Since we are modeling after a step function,

$$|p_i - p_c| \approx \beta e^{\alpha F(\log_a |t_i - t_c|)}$$

the exponential trend and identification of the dominant frequency via Fourier or spectral analysis is all that is necessary. Despite its simplicity however, the model appears to require more insight from the modeler in determining whether or not its results are meaningful. To some extent, this is true of all models. However, the JLS model claims that its calibrated parameters exhibit some type of universality [46]. Perhaps a third condition can be imposed on our model to improve its accuracy.

Empirically, the fluctuations of the data around the power law have been found to smooth out and increase in frequency [19] [16] as it approaches the critical price. Our model relies on capturing a general periodicity around the data. However, the data can have more than one type of periodicity. What appears to be noise may actually be a natural periodicity that exists within the intermittences. Identifying the greater periodicity would allow us to remove it from the data, or control for it in some way. Akin to

$$|p_i - p_c|^* =\approx |p_i - p_c| - \beta e^{\alpha F(\log_a |t_i - t_c|)}$$

Logically after removing the intermittences, the only periodic behavior that should remain would be a higher frequency periodic behavior. When the true crash date is unknown, the periodic behavior around the power law near the end of the sample can be analyzed and compared to the high frequency periodicities in the data. Perhaps via analysis of volatility clustering or more sophisticated spectral analysis, the smooth precursors to the true critical date may be identified and distinguished from any natural periodicity that may remain in the data. The presence of these smooth fluctuations would, thus, impose stronger conditions for predicting the critical date.

We also cannot conclude that this would be the best model to characterize exchange rates near a crash. The krona's volatility is closest to currencies in emerging markets [23],

so results like ours are most likely to be seen in such currencies. A GARCH or ARCH model may explain some of the volatility clustering we see the differences in logarithm, shown in figures A.3 A.4 A.7 A.8 A.11 and A.12. The JLS model is capable of imitating a greater range of behavior, so it may provide a more accurate fit. Nevertheless, under somewhat ideal conditions like a sufficiently large sample size and being able to test a known crash date, the model proved to be accurate and consistent among the different rates. We may at least conclude that the model describes these sets of foreign exchange rates better than a pure power law.

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Appendix A

Figures and Fundamental Properties of the Data

Table A.1: This is a brief summary of the currencies used. The non-krona currencies were color coded for easy identification

Country	**Currency**	**Letter Code**	**Color**
Iceland	krona	ISK	None
United States	dollar	USD	Blue
Switzerland	franc	CHF	Red
Japan	yen	JPY	Green

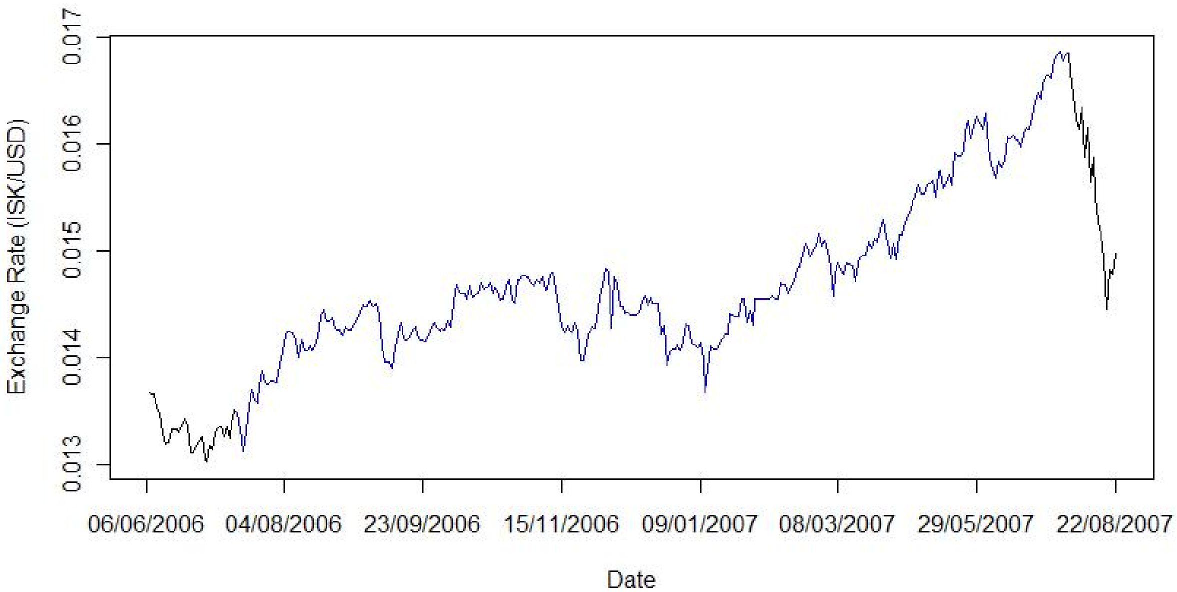

Figure A.1: A plot of the dollar to krona exchange rate (ISK/USD) from June 2006 to August 2007. The interval to be studied is highlighted in blue.

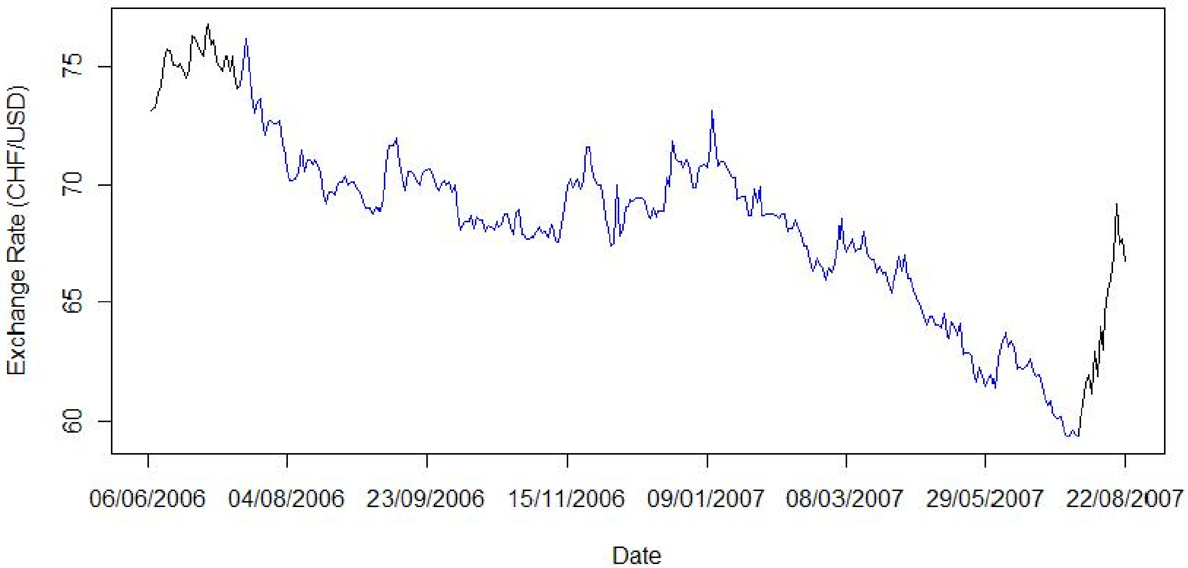

Figure A.2: A plot of the krona to dollar exchange rate (USD/ISK) from June 2006 to August 2007. The interval to be studied is highlighted in blue.

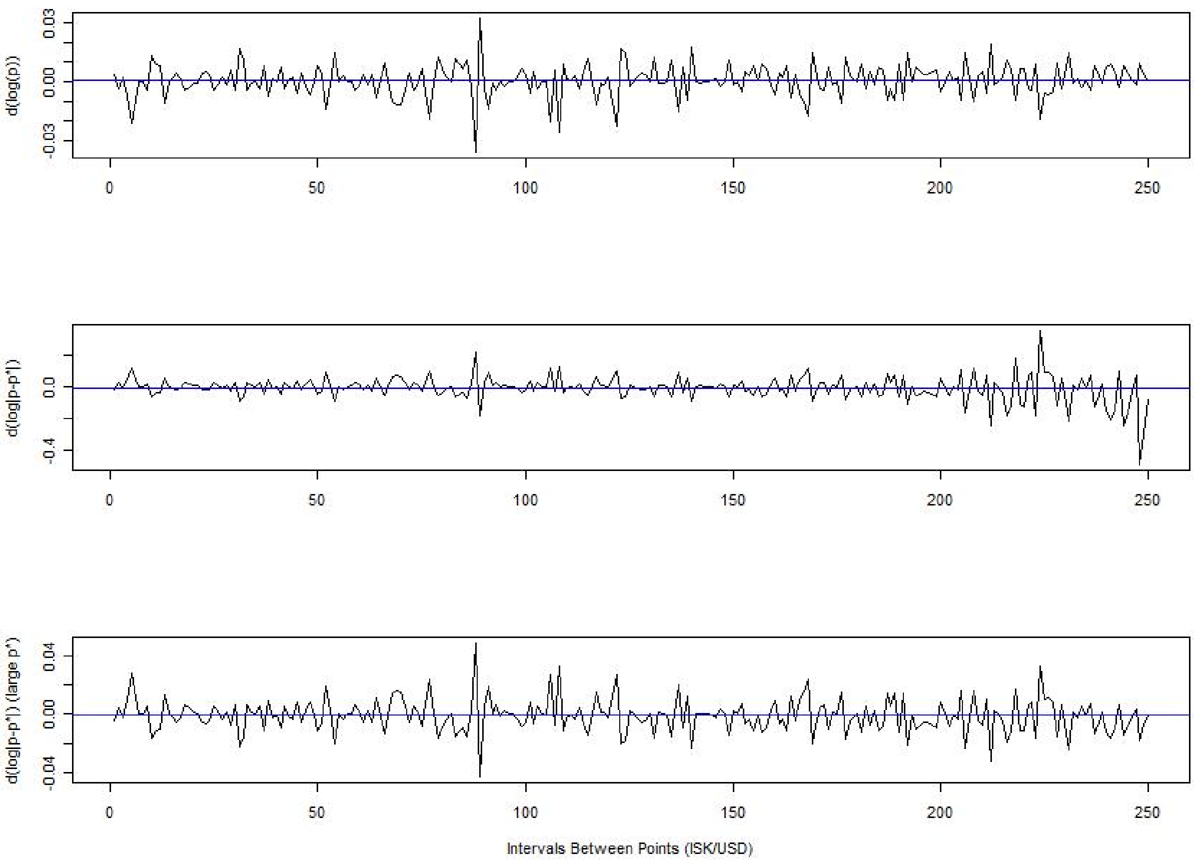

Figure A.3: A plot showing the difference in the natural logarithm of the dollar to krona exchange rate (ISK/USD) within the interval of interest.

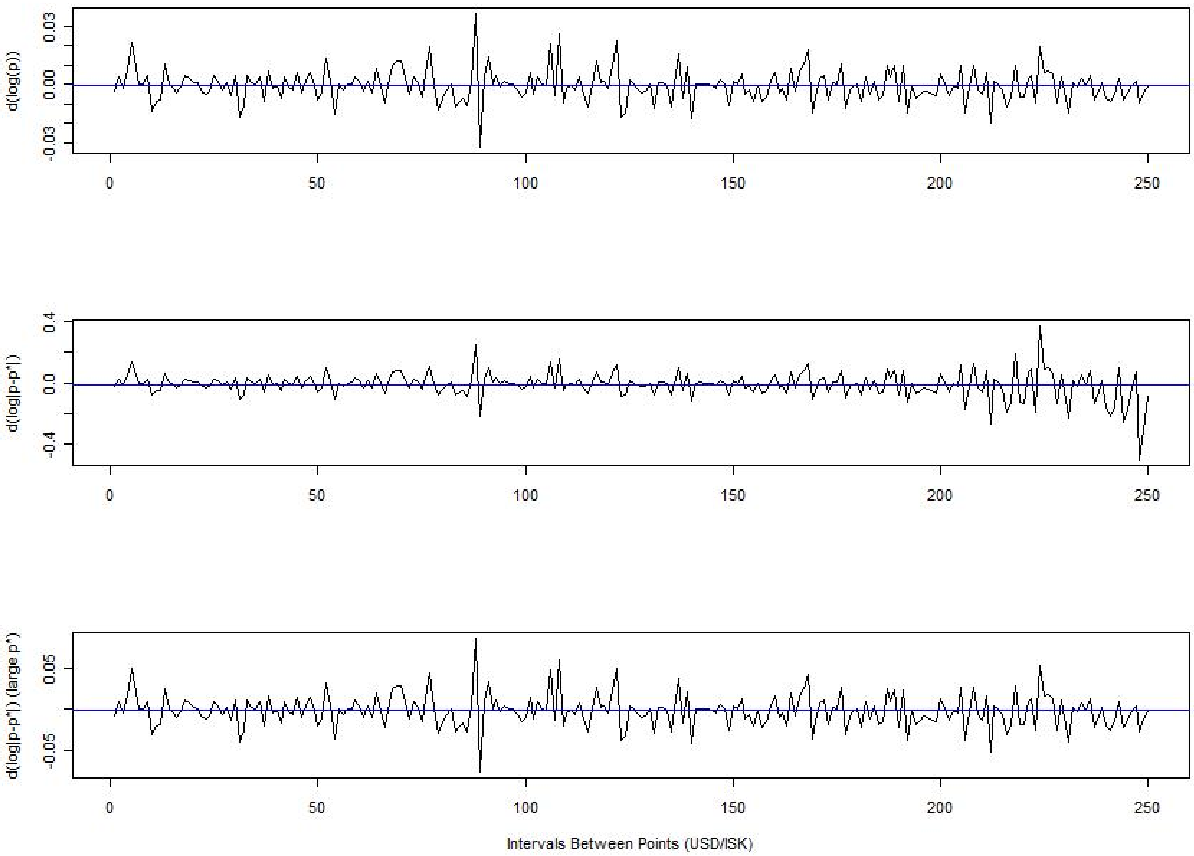

Figure A.4: A plot showing the difference in the natural logarithm of the krona to dollar exchange rate (USD/ISK) within the interval of interest.

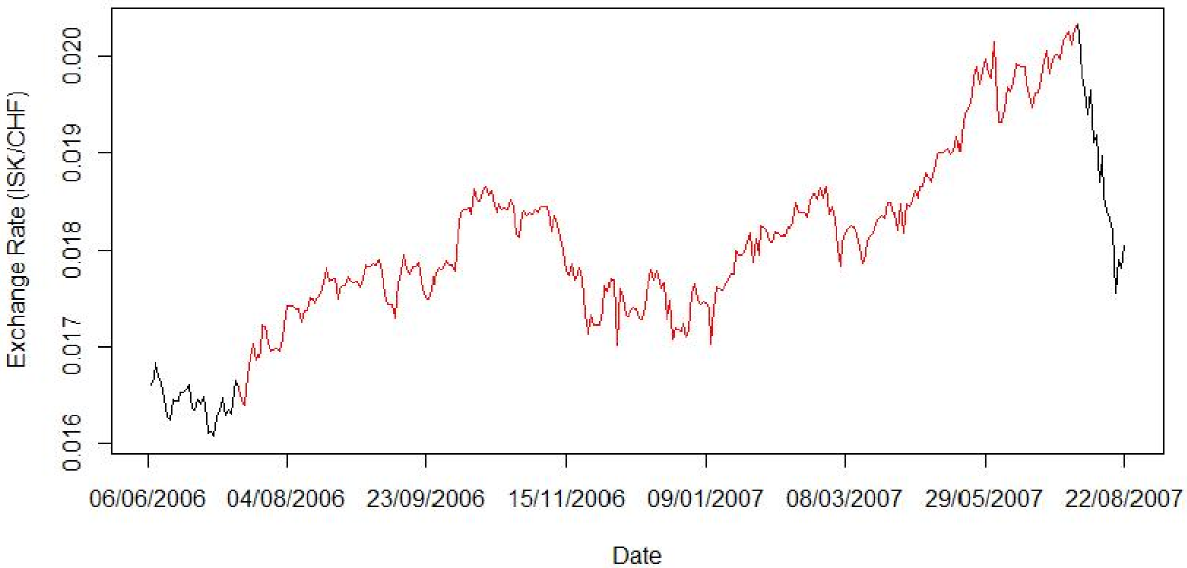

Figure A.5: A plot of the Swiss franc to krona exchange rate (ISK/CHF) from June 2006 to August 2007. The interval to be studied is highlighted in red.

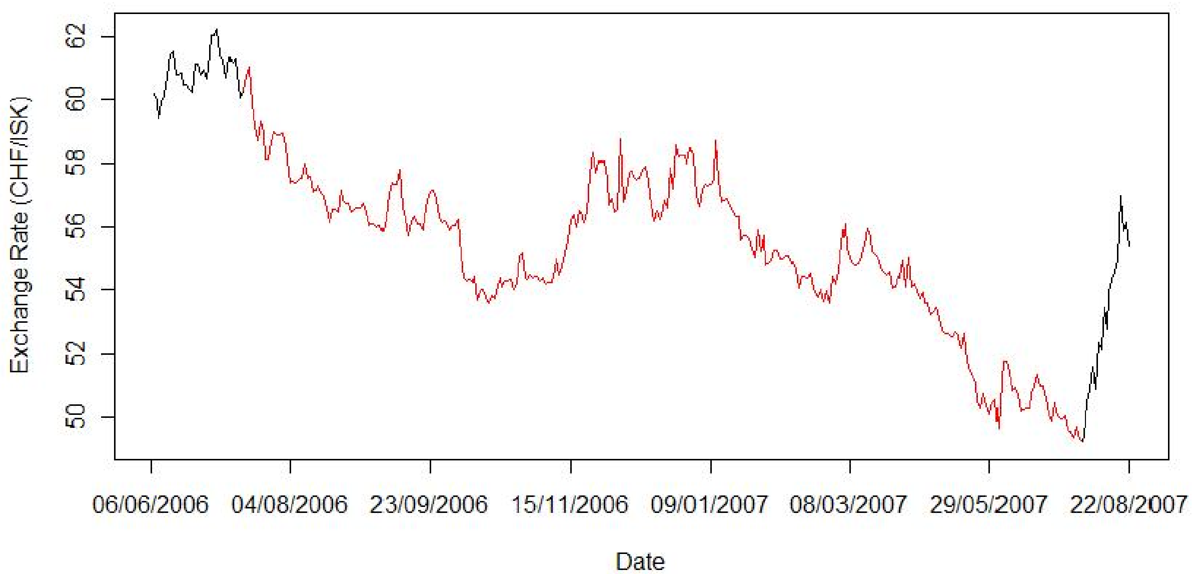

Figure A.6: A plot of the krona to Swiss franc exchange rate (CHF/ISK) from June 2006 to August 2007. The interval to be studied is highlighted in red.

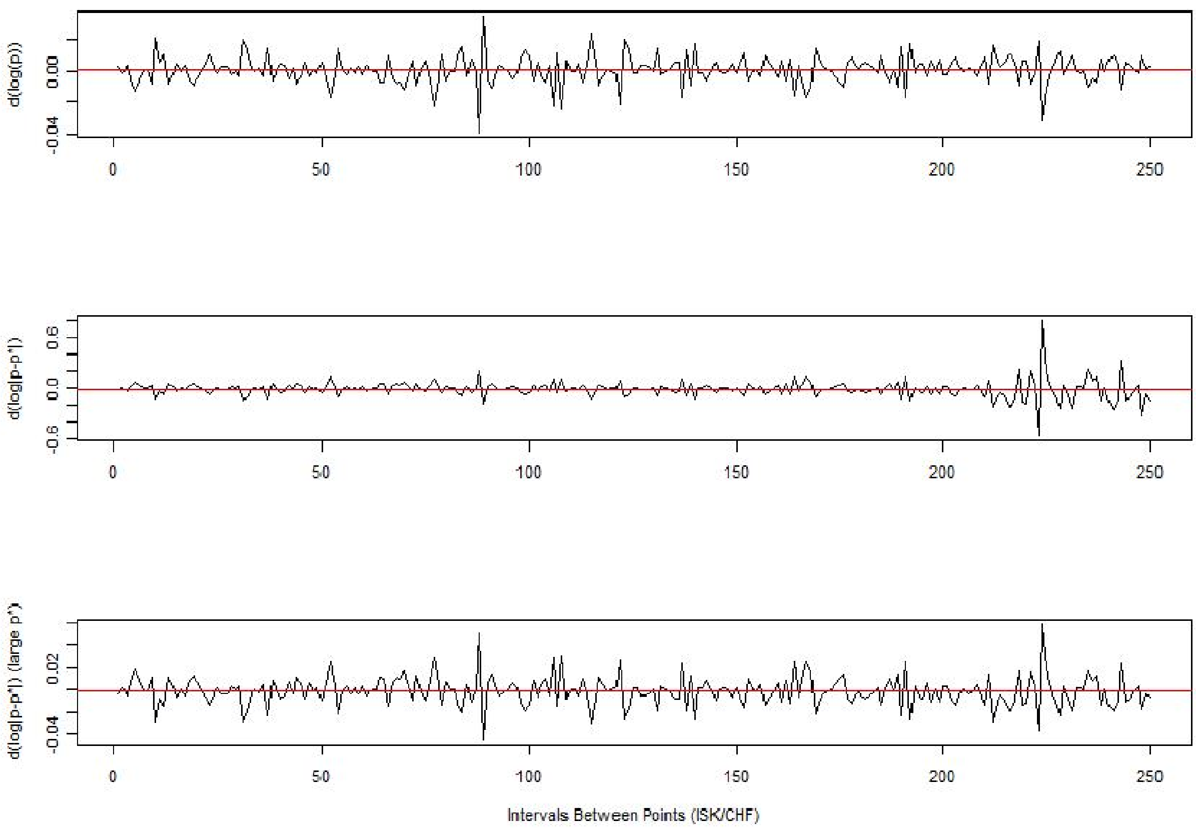

Figure A.7: A plot showing the difference in the natural logarithm of the Swiss franc to krona exchange rate (ISK/CHF) within the interval of interest.

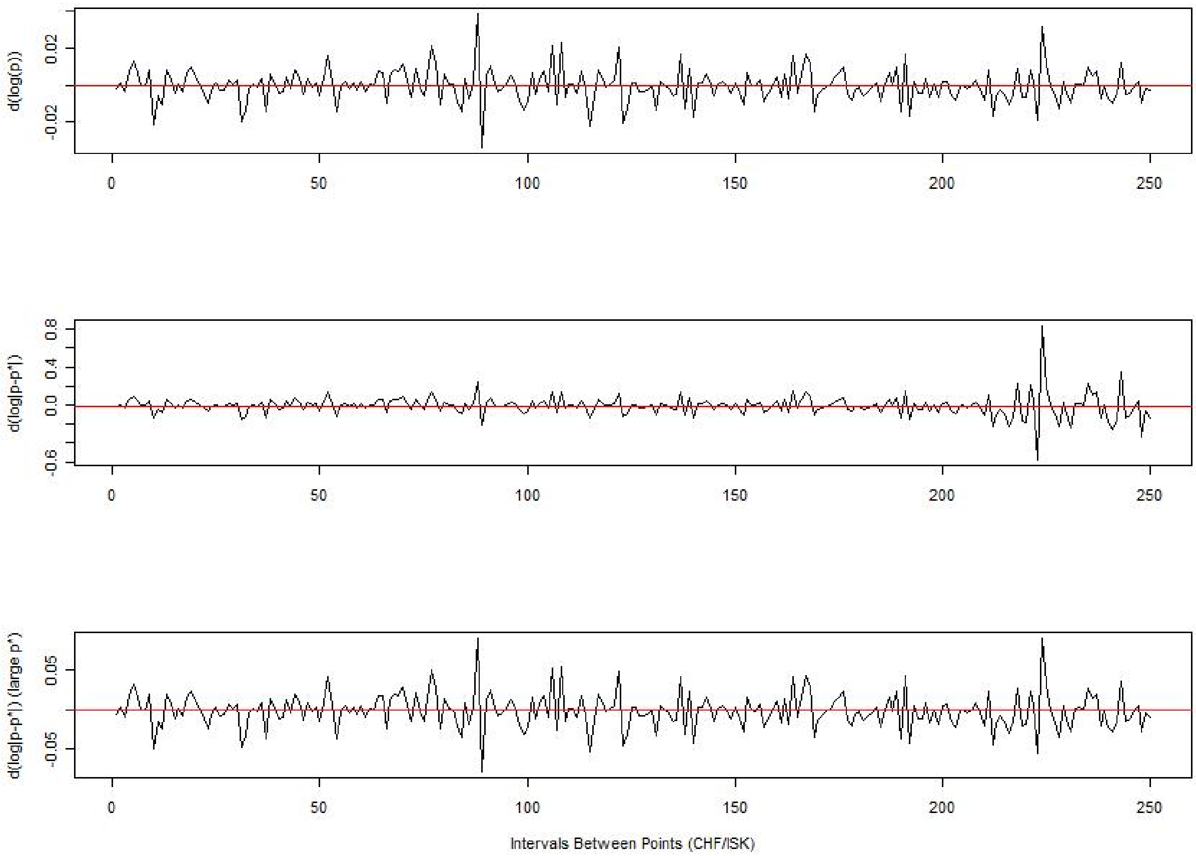

Figure A.8: A plot showing the difference in the natural logarithm of the krona to Swiss franc exchange rate (CHF/ISK) within the interval of interest.

Figure A.9: A plot of the Japanese yen to krona exchange rate (ISK/JPY) from June 2006 to August 2007. The interval to be studied is highlighted in green.

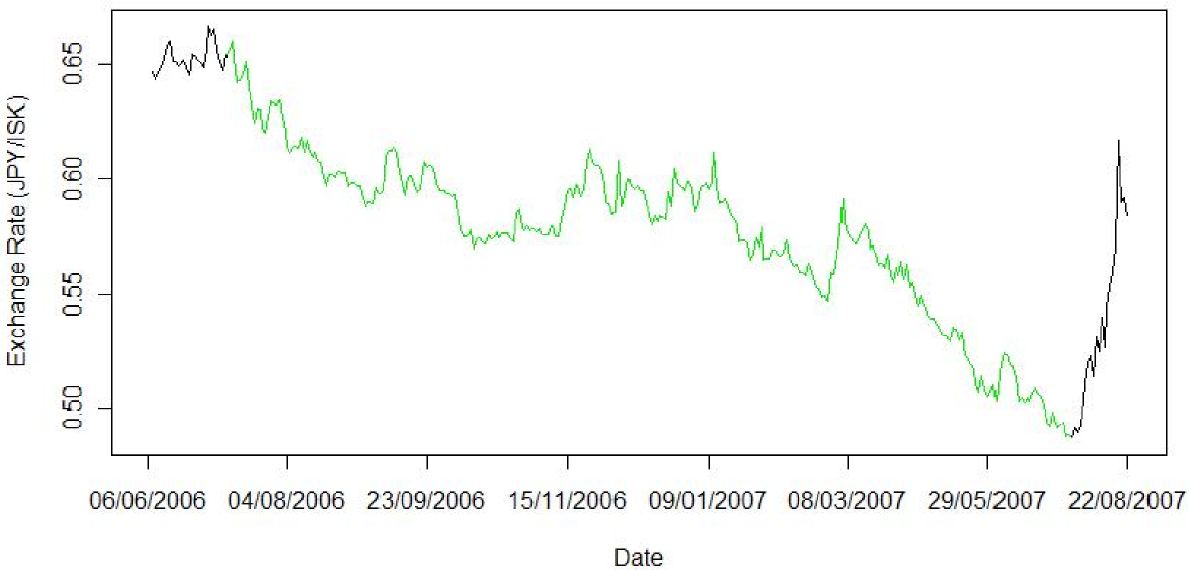

Figure A.10: A plot of the krona to Japanese yen exchange rate (JPY/ISK) from June 2006 to August 2007. The interval to be studied is highlighted in green.

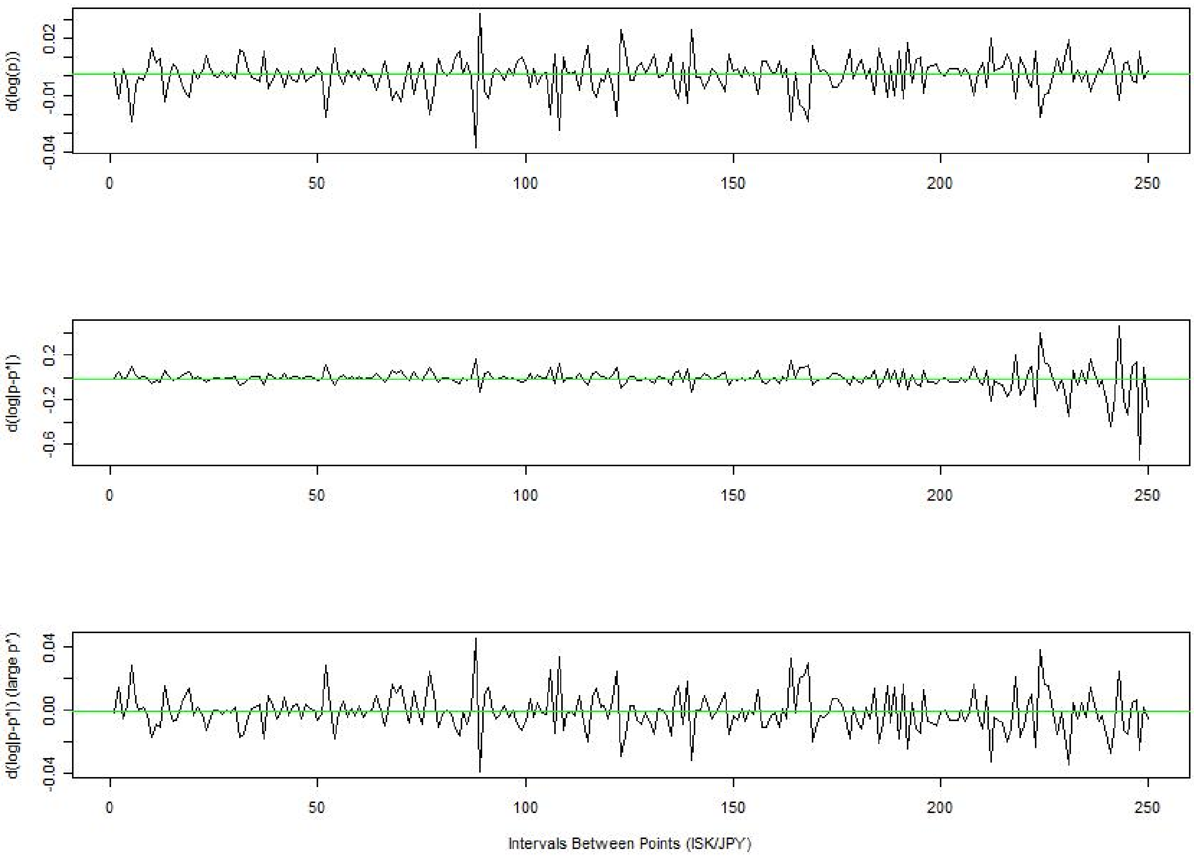

Figure A.11: A plot showing the difference in the natural logarithm of the Japanese yen to krona exchange rate (ISK/JPY) within the interval of interest.

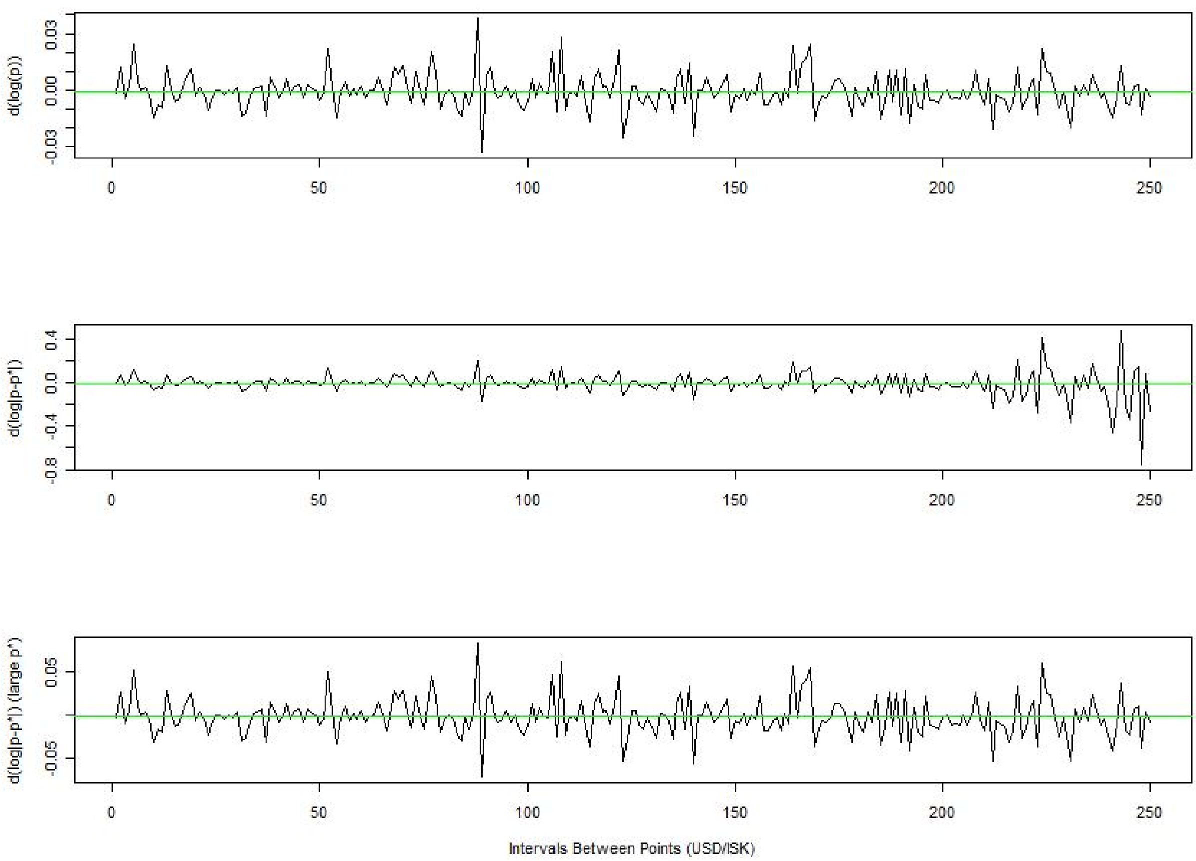

Figure A.12: A plot showing the difference in the natural logarithm of the krona to Japanese yen exchange rate (JPY/ISK) within the interval of interest.

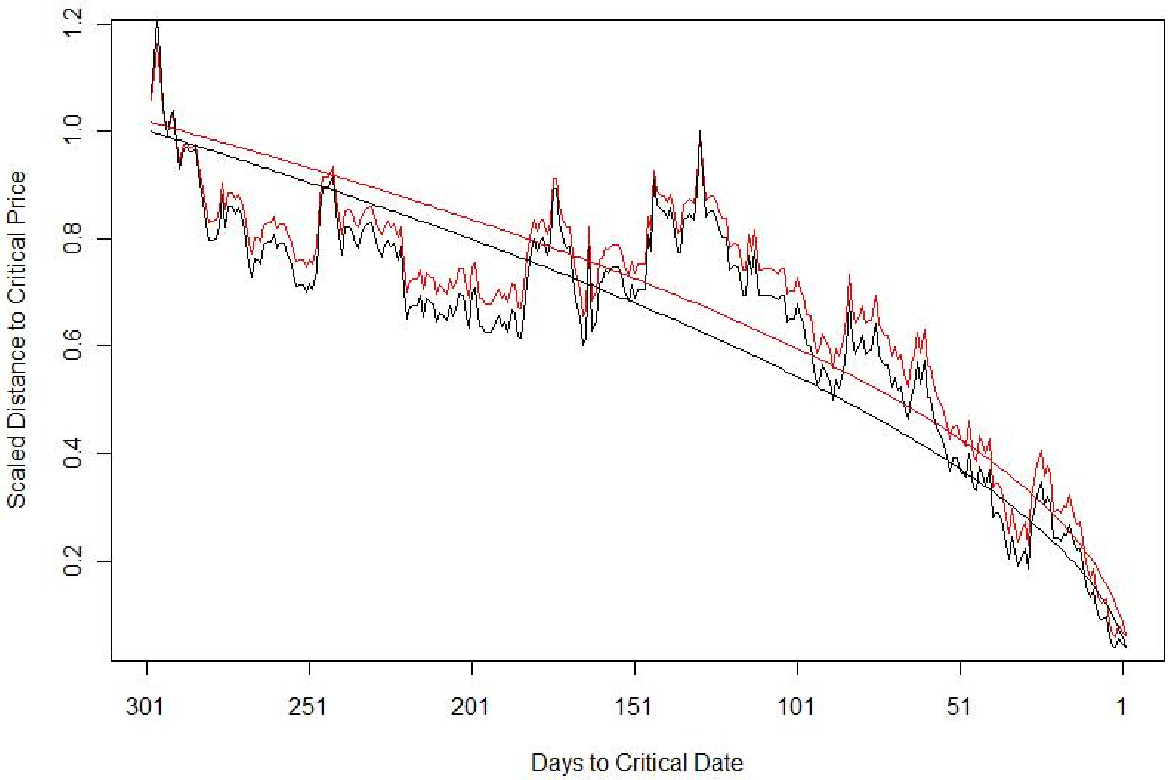

Figure A.13: A plot of the distance to critical price for 301 days of the sets of exchange rates ISK/CHF (red) and CHF/ISK (black) scaled to lie on the interval (0,1], along with each set's governing power law determined by least squares fitting. The sum of squared errors for each set is 3.98 for ISK/CHF and 4.30 for CHF/ISK

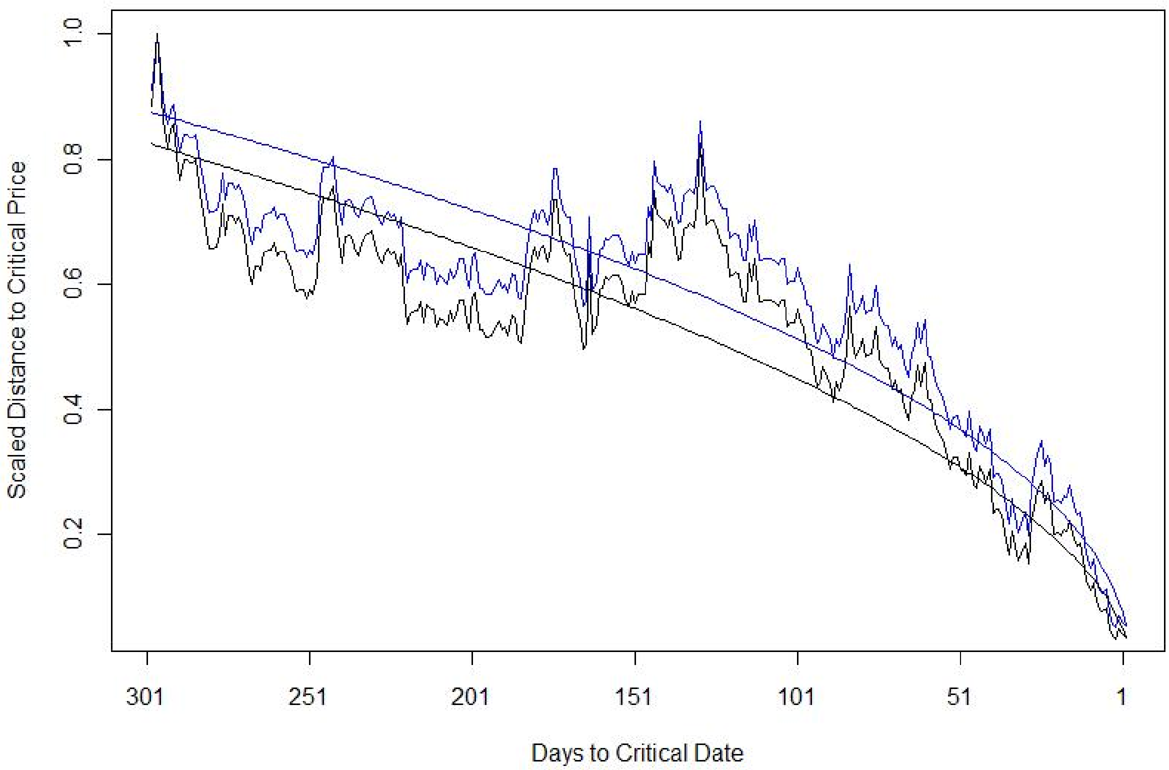

Figure A.14: A plot of the distance to critical price for 301 days of the sets of exchange rates ISK/USD (blue) and USD/ISK (black) scaled to lie on the interval (0,1], along with each set's governing power law determined by least squares fitting. The sum of squared errors for each set is 2.56 for ISK/USD and 2.75 for USD/ISK

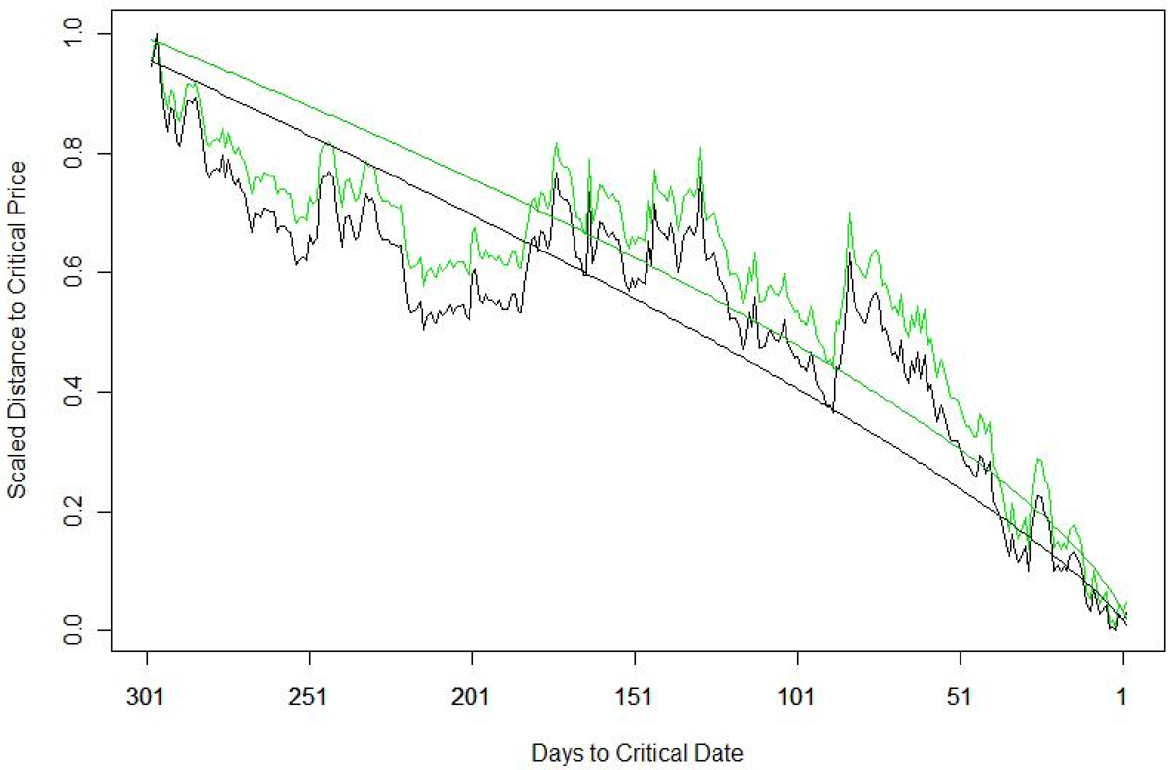

Figure A.15: A plot of the distance to critical price for 301 days of the sets of exchange rates ISK/JPY (green) and JPY/ISK (black) scaled to lie on the interval (0,1], along with each set's governing power law determined by least squares fitting. The sum of squared errors for each set is 3.90 for ISK/JPY and 4.22 for JPY/ISK

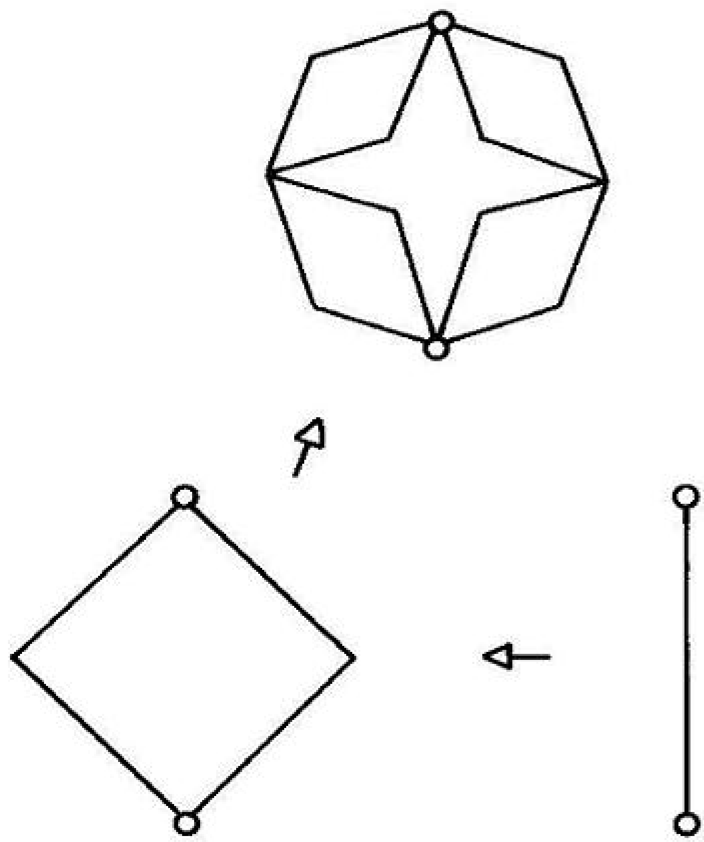

Figure A.16: A simple demonstration of the concept behind the hierarchal Diamond Lattice in the Ising model, taken from [19].

Appendix B

Contour Plots of the Analysis

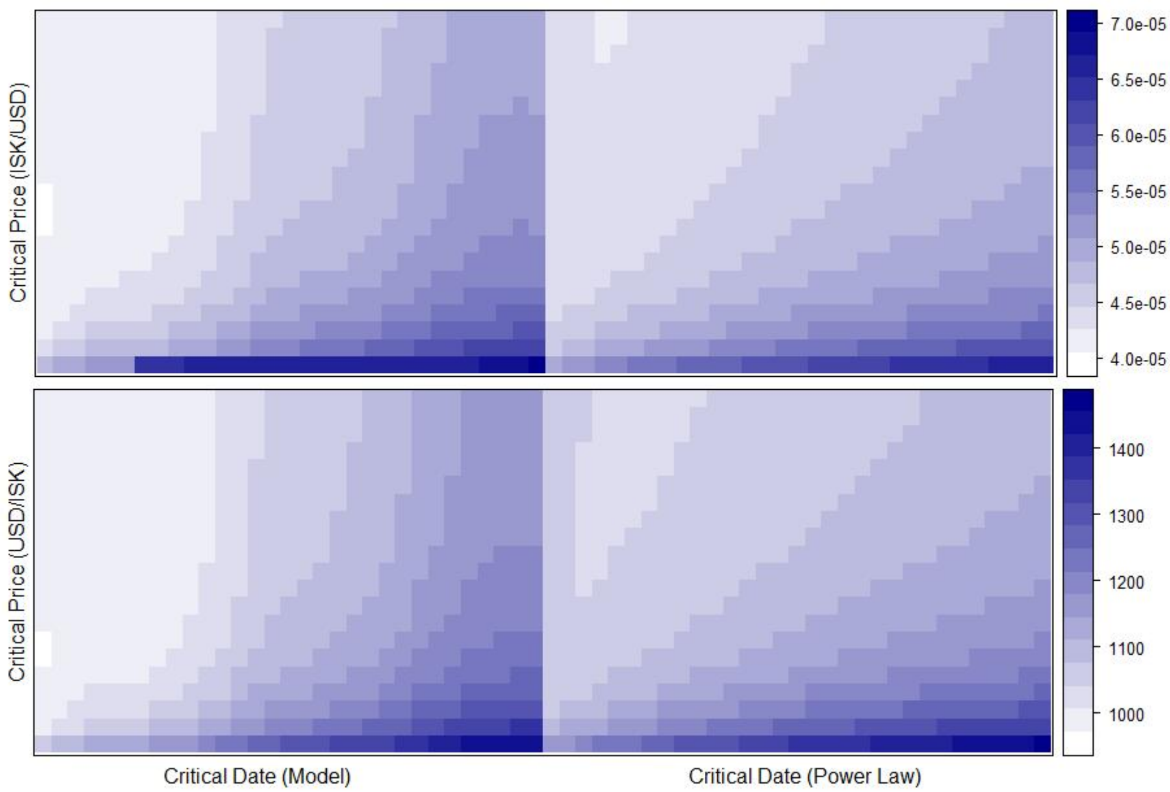

Figure B.1: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the U.S. dollar with 365 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

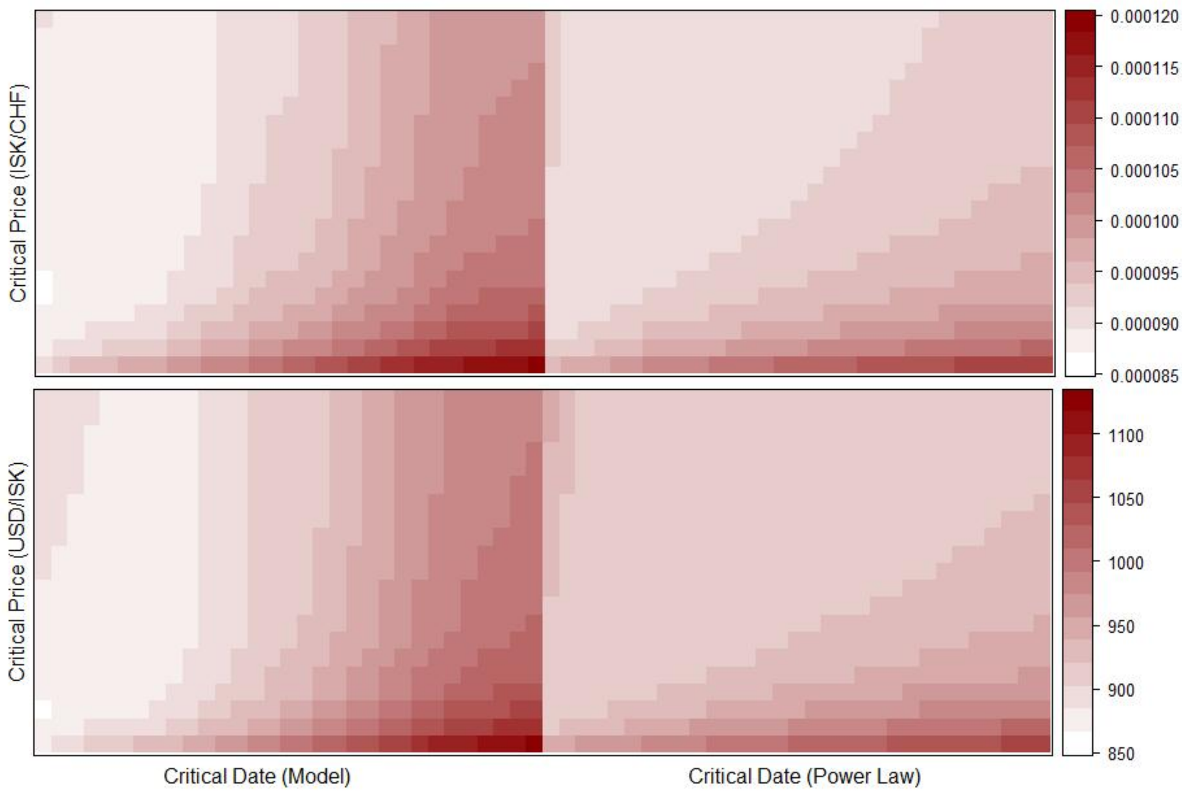

Figure B.2: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Swiss franc with 365 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

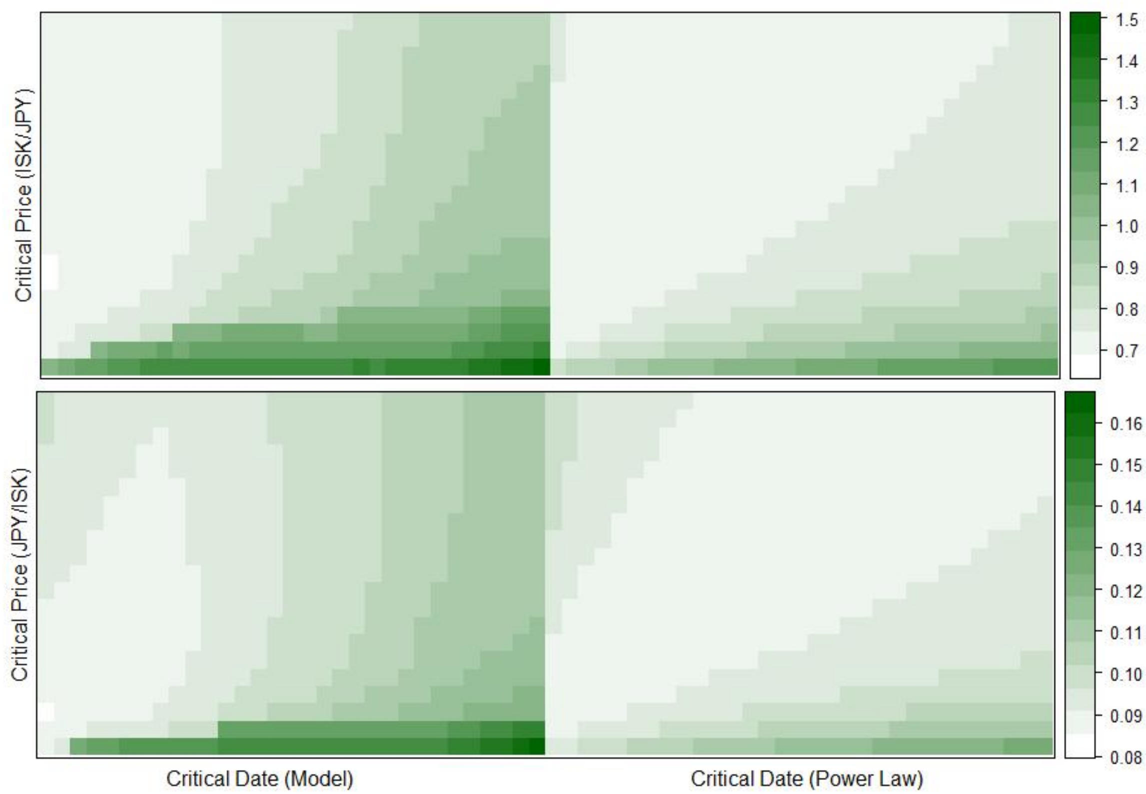

Figure B.3: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Japanese yen with 365 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

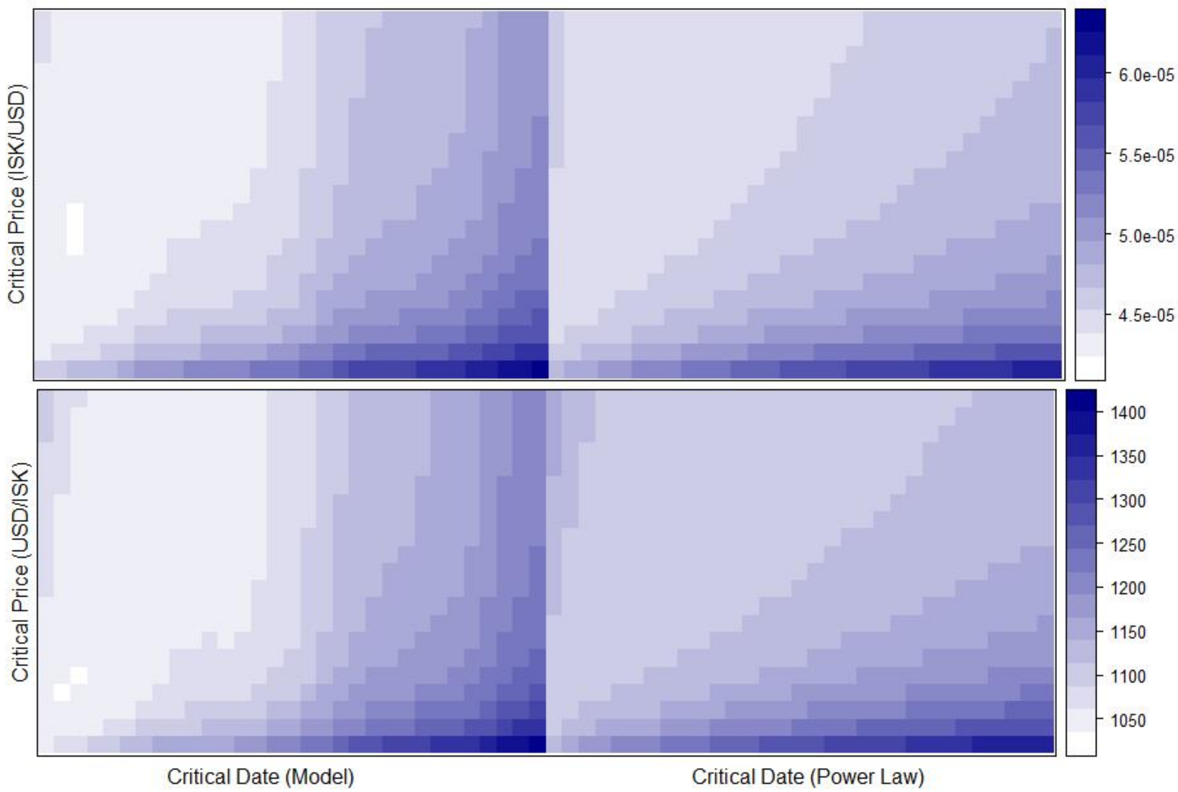

Figure B.4: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the U.S. dollar with 365 observations alongside a fitting to a pure power-law. $t_c = 07/13/2007$

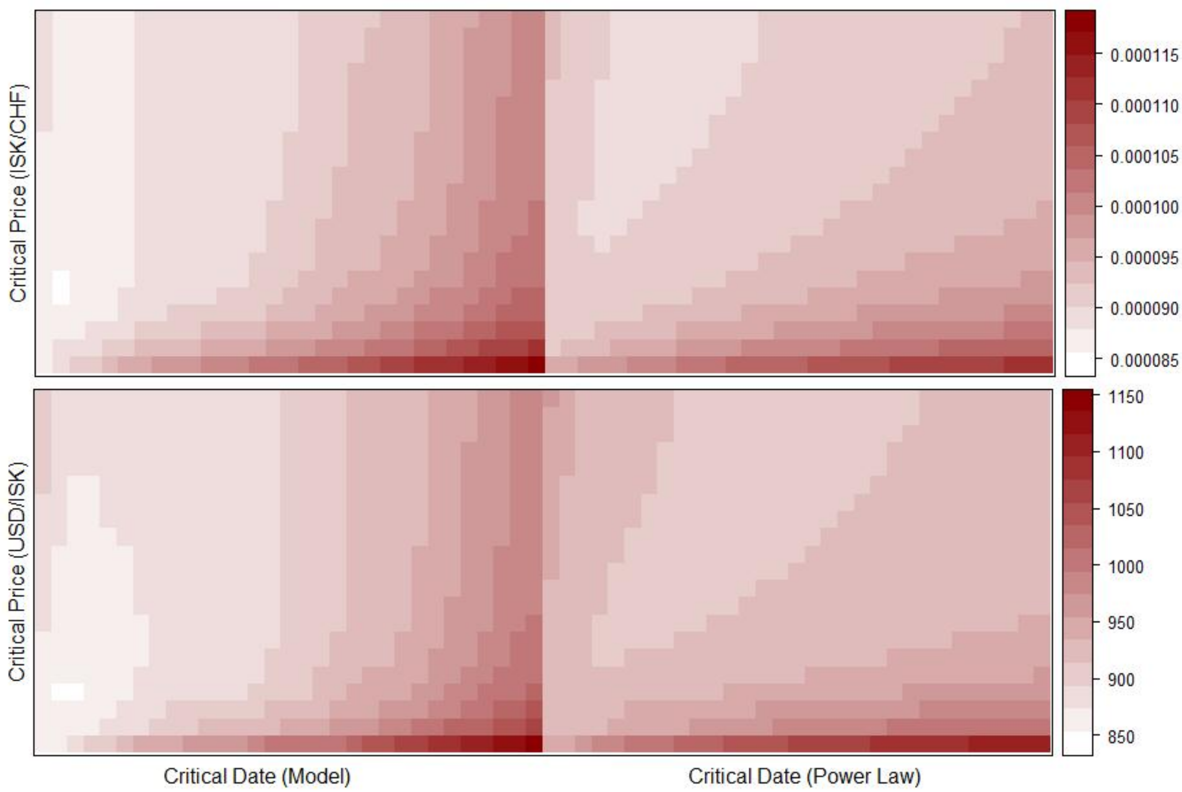

Figure B.5: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Swiss franc with 365 observations alongside a fitting to a pure power-law. $t_c = 07/13/2007$

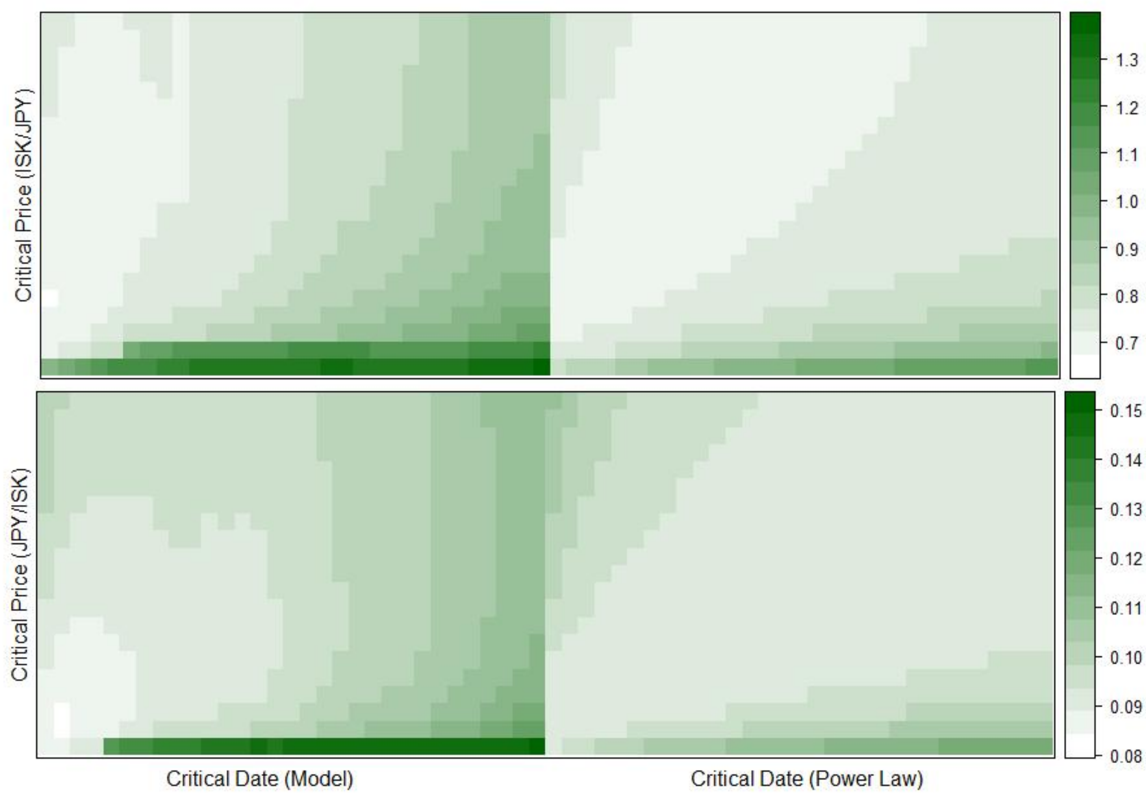

Figure B.6: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Japanese yen with 365 observations alongside a fitting to a pure power-law. $t_c = 07/13/2007$

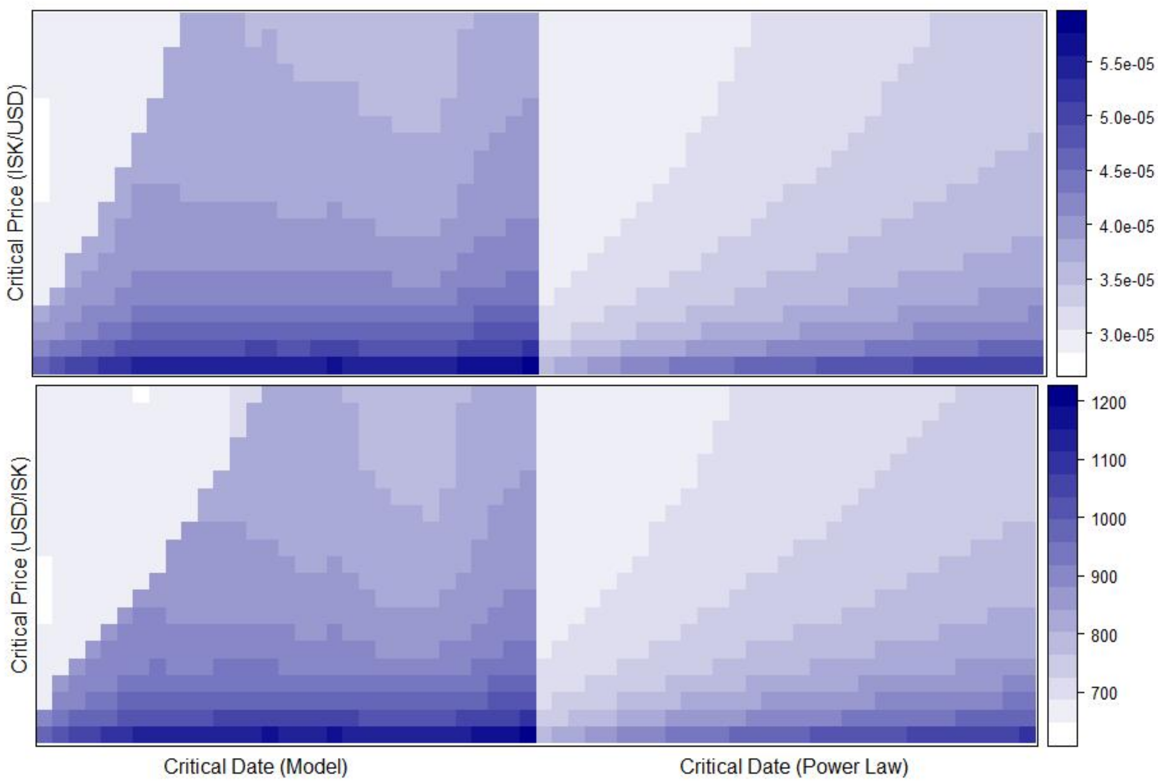

Figure B.7: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the U.S. dollar with 305 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

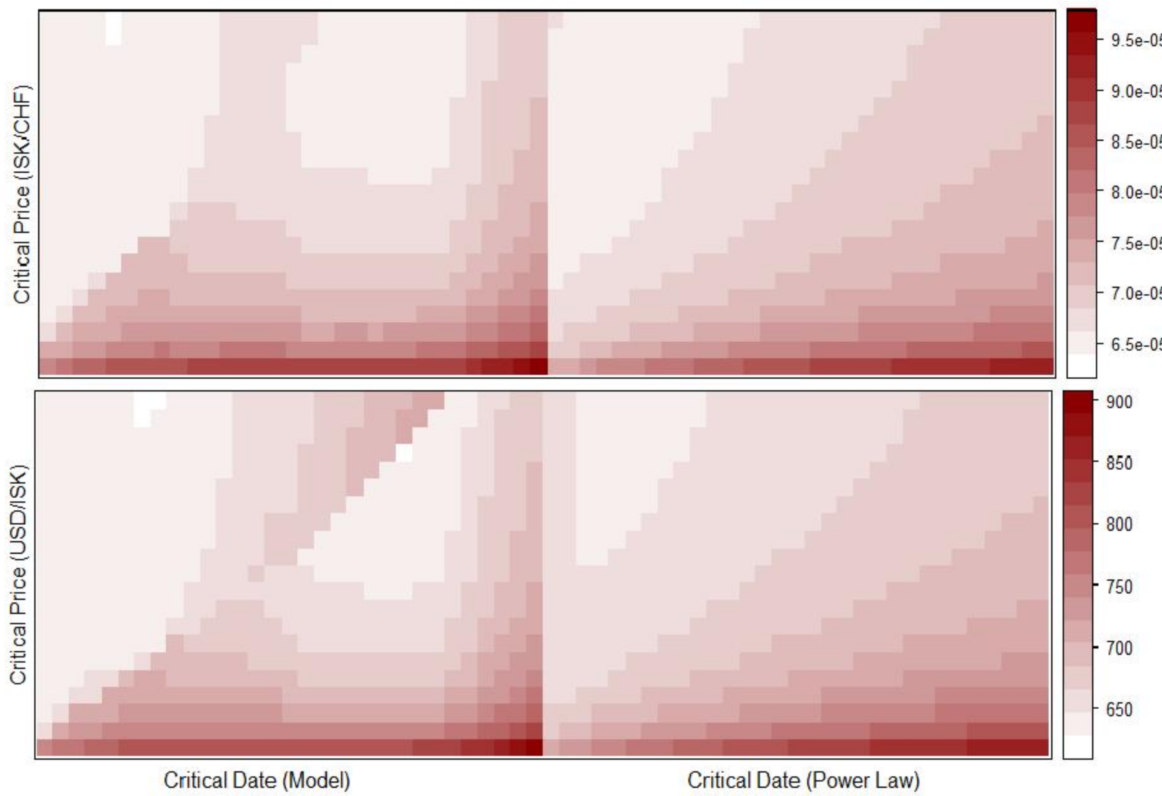

Figure B.8: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Swiss franc with 305 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

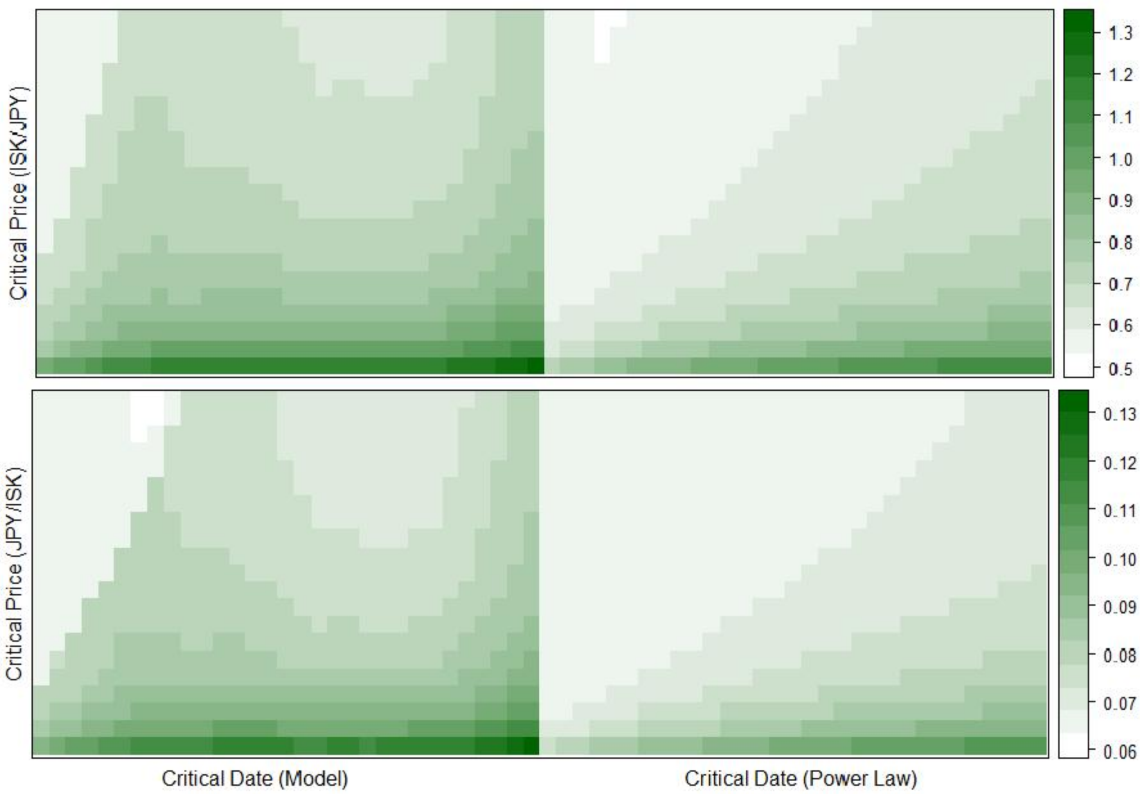

Figure B.9: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Japanese yen with 305 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

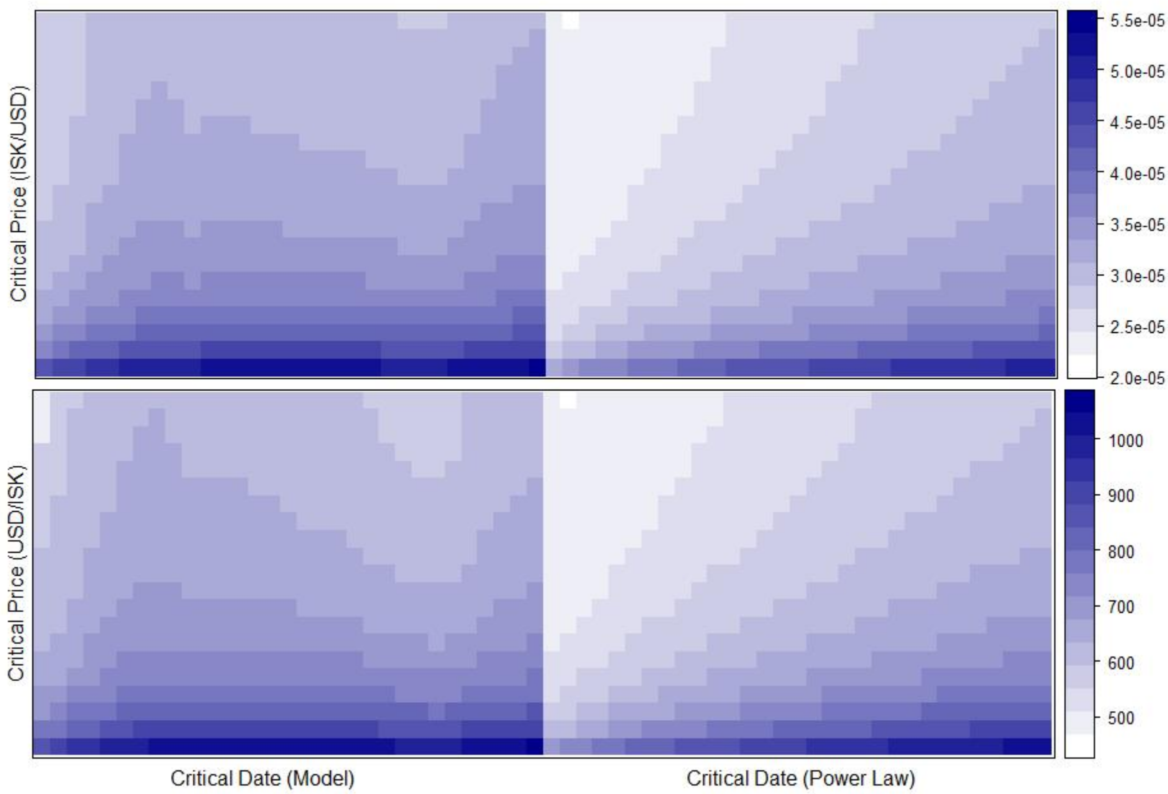

Figure B.10: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the U.S. dollar with 275 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

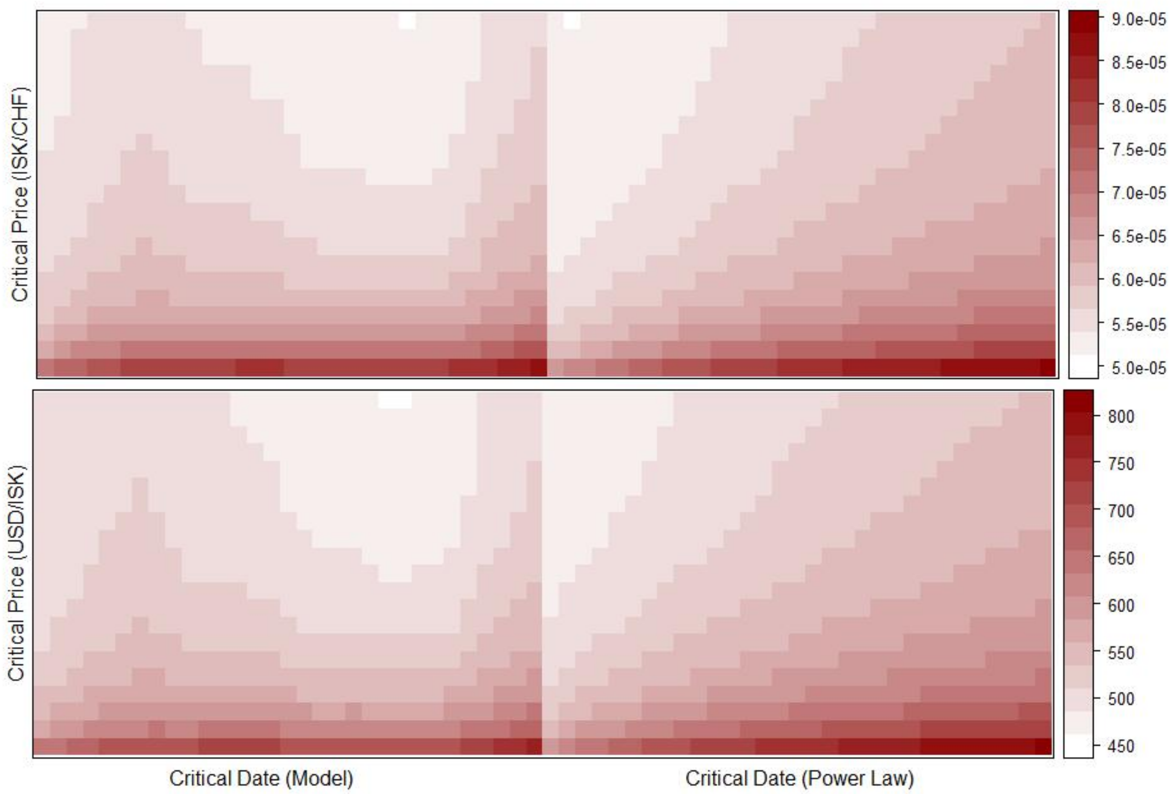

Figure B.11: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Swiss franc with 275 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

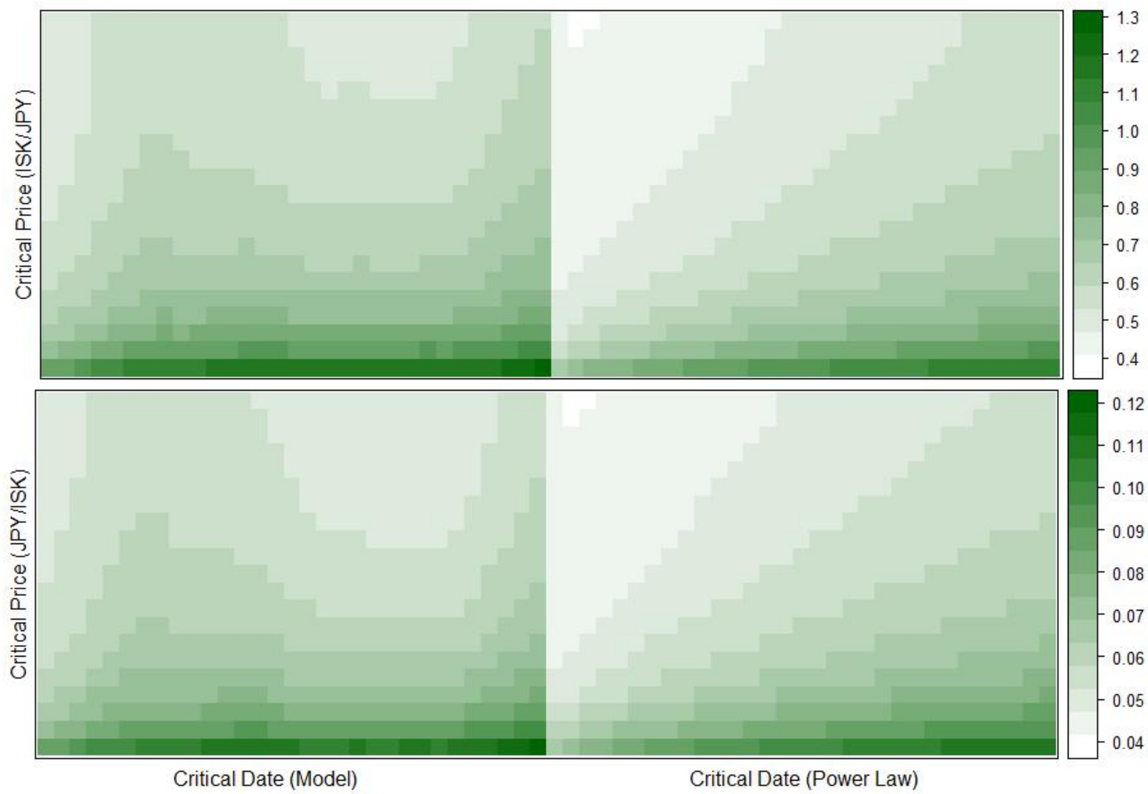

Figure B.12: A contour plot of the sum of squared errors of the fitted equation on exchange rates based on the Japanese yen with 275 observations alongside a fitting to a pure power-law. $t_c = 07/20/2007$

Curriculum Vitae

Graduate student preparing to complete Master's level degree in Mathematics, with an undergraduate degree in Economics. I am well spoken and curious and would like to continue my studies.

Rice University | August 2005-May 2009

Bachelor of Arts, Economics

University of Texas El Paso | Fall 2009-Present

Master of Science in Mathematics