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Cognitive Constructs In Linear Algebra; Metaphors, Metonymies, Modes

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COGNITIVE CONSTRUCTS IN LINEAR ALGEBRA;
METAPHORS, METONYMIES,
MODES

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Ruben Carrizales

THESIS

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Abstract

Analysis focused on the presence of different thinking modes, metonymies, and metaphors found on the interview responses to questions related to linear independence, span, and spanning sets of three students, A12, A22 and C3, taking their first linear algebra course at the college level. Findings provide insight into how first year linear algebra students move from one thinking mode to another, and the kind of metonymies and metaphors are used to construct new knowledge. The main purpose of this research was to discover and analyze the presence of the different modes of thinking and metonymy/metaphors– in the reasoning of these three students.

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Chapter 1

Introduction

The career of students at the university level requires them to take many courses that are crucial for their field of study and for the future expansion of related topics. Many times students do not realize the difficulty level of concepts and what is required to master them. In fact matrix algebra is among those courses students tend to overlook its cognitive difficulties and its role in their subsequent studies. Linear algebra, a course that has an immense range of applications in different disciplines - is a requirement in the curricula of many subjects such as mathematics, computer science, and engineering majors.

The vast range of use of linear algebra ideas requires a close attention to the subject in the training of university level students. Students often find themselves struggling to understand, explain, and relate the theory learned while enrolled in their first linear algebra course, because of the different kinds of representations employed. It has been discovered that students do not have enough previous knowledge in mathematical structures such as algebraic skills and set theory (Dogan-Dunlap, 2006; 2010) to construct new concepts and relate them to previously learned materials (Sierpinska, 2000). This becomes a discouragement and a problem for students resulting in cutting short of the learning process or discourage from fully grasping the new material being presented.

This paper is part of an ongoing research funded by the National Science Foundation (NSF; CCLI:0737485) that focuses on the teaching methods used in first year linear algebra courses at university level, the conceptual constructs that students display while enrolled in them, and the importance that different visual representations have in the development of knowledge.

The purpose of the thesis is to reveal the different thinking modes and the examples of metonymy and metaphors exhibited by three students on their responses to a set of questions asked during one-on-one interviews conducted toward the end of their first matrix algebra course. Our goal is to further document the understanding and misconceptions that students display in light of different levels of exposure to graphical, computational, algebraic, and abstract representations of basic topics such as linear independence and dependence, span, spanning set, and vector spaces. Our goal will be achieved by addressing the following question: What are the thinking modes, metonymies and metaphors displayed by three students in their responses to interview tasks on linear independence?

1.1 LEARNING THEORIES

The study investigated the occurrence of cognitive entities— modes of thinking and metonymy/metaphors. Throughout the analysis, these cognitive entities will be recorded and materialize in the student responses.

1.1.1 Modes of Thinking

Anna Sierpinska's structure in her paper "On some aspects of students' thinking in linear algebra"(2000) will be used to document the modes of thinking displayed by the students involved in our research through the qualitative analysis of their responses in the interviews. Sierpinska documents three different thinking modes; Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural employing the origins and the characteristics of linear algebra concepts and the mathematical languages used in learner's understanding.

1.1.1.1 Synthetic-Geometric

The use of geometric representations and the lack of definitions for the concepts used in objects is a case of synthetic-geometric modes; for example, in the form of a line or a plane the properties of objects such as linear independence and span can be given by student, but they will only describe such objects, not define them (Sierpiska, 2000). In other words, using geometry in such a way in which students think visually and use it as the basis for their knowledge.

1.1.1.2 Analytic-Arithmetic

Analytic-Arithmetic mode considers an object defined which is then used to carry out computations (Sierpiska, 2000). The numerical and algebraic components of geometrical objects and the conditions they satisfy are employed. Computing the numerical and algebraic components refer to carrying out appropriate algorithms, while analytically refers to reasoning and justifying facts and algorithms confidently.

1.1.1.3 Analytic-Structural

In the analytic-structural mode objects are defined by its properties and, although algebraic structures are still considered as in the analytic-arithmetic mode, in the structural mode they are synthesized into compact structural wholes (Sierpiska, 2000). Students thinking of vectors as being part of and having characteristics of vectors spaces and proving linear independence of a set of vectors through the use of its dimension arguments are examples of the implementation of this perspective (Dogan-Dunlap, 2008; 2010; 2011 a, b, c revise the references on Dogan adding coauthors if there are coauthors on citations; I have included a few more articles on references to be cited in the body of the paper especially lit chapter ????????).

Below is a summary of the different thinking modes identified by Sierpinska, examples of their representations, and the level of competency that a student may achieve when using each one of them is presented in table 1.

Mode of Thinking	Representations/Definition	Student Competency
Synthetic-Geometric	Graphical representations Provide properties of objects readily. It describes an object but not define it.	Student is be able determine whether vectors whose graphs are provided in R^2 or R^3 are linearly independent or dependent.
Analytic-Arithmetic	Numerical Representations. Defines objects.	Student is able to construct matrix from vectors, compute its row-reduced echelon form and relate the reduced matrix to linear dependence and independence.
	Linear Combination.	Student is able to provide/refer to linear combination of vectors and determine linear independence.
Analytic-Structural	Objects are considered in a system. Defines objects.	Use of the dimension of vector spaces in determining the linear independence of vectors.

Table 1. Thinking Modes Modified from Sierpinska (obtained from Dogan-Dunlap, 2010)

It is worth mentioning that a student can and will use different modes of thinking and shift between them in order to understand and explain concepts (Sierpinska, 2000).

1.1.2. Metonymy and Metaphor

The constructs, metonymy and metaphor, are widely used primarily as examples of figurative speech or literary devices. Recently however they also been considered as cognitive entities aiding one's learning process. Presmeg (1998) is among the researchers who began document their cognitive roles in learner's cognition.

1.1.2.1 Metaphor

Defined by Webster's dictionary a metaphor is "a figure of speech in which an expression denoting one kind of object is used in place of another in order to suggest a similarity" (Webster). The use of this literary device in the learning of mathematics is employed by students when acquiring new knowledge; its application involves the comparison of two entities. There are two very important characteristics of the comparison between the two entities: the ground and the tension (Presmeg, 1998). Presmeg clarifies the grounds as the similar elements of the entities being compared, while the tension as the dissimilar elements (1998).

An example on the use of metaphors in mathematics can be seen in the statement, A is an open set; here, the definition of openness and its physical representation of being without a boundary can become a source of confusion for students since they may wrongfully interpret the "no boundary characteristic of the source concept, and come to a conclusion that the particular set is not open since it has a boundary" (Dogan-Dunlap, 2007, page 2). This example illustrates the importance of the identification of the tension and ground elements in the comparison. An additional example of the use of metaphors in the understanding of mathematical concepts is given by Presmeg through the responses of high school students who were asked to calculate the sum of the first 30 elements of the sequence $\{5, 8, 11 \dots\}$ (1998). In such study, students referred to their methods by using metaphors such as "dome" and "rainbow" to refer to the Gaussian way

of having to relate the first and last elements of the sequence and adding them, then the second and the 29th, the third and the 28th, and so on to reach their final answer.

1.1.2.2 Metonymy

Lakoff and Johnson (2000) refer to metonymy as being present when one uses "one entity to refer to another that is related to it." While Webster's dictionary definition is "a figure by which one word is put for another on account of some actual relation between the things signified". Some examples commonly used to refer to this literary device are the use of the phrases "We read "Shakespeare" when we talk about the author's work" and "Washington is talking to Moscow" when we talk about the people from these countries communicating (Webster). The principal characteristic of this figure of speech is the presence of an attribute or entity that is taken to stand for another entity (Presmeg, 1998).

1.1.2.3 The relationship between metaphor and metonymy

Both metaphor and metonymy devices are closely related to the understanding and development of new knowledge form. These two tools are widely used by students with mathematical backgrounds in which a person uses one construct to stand for another for example students using their own language or terminology to understand and relate to the new concept.

The main difference between the two devices is that through the use of metaphors, the learner would make connections based on similarity in contrast to metonymy which are made by association with no regards to similarity. This relationship is often seen when a student recently introduced to the term inverse referring to the steps of finding the inverse of a function, the word "inverse" becomes part of the concept "additive inverse" performing operations that require applying the opposite sign of a number.

1.2 RESEARCH QUESTIONS

The purpose of this thesis is to identify cognitive constructs employed in concept formation by analyzing the responses given by three students from the different modular and traditional groups to a set of questions asked during interviews. This goal will be achieved by addressing the following questions:

- What are the thinking modes displayed by three students in their responses to interview tasks on linear independence?
- What are the metonymies and metaphors displayed by the three students in such responses?

1.3 METHODOLOGY

For the purpose of this thesis, we'll analyze the responses of three undergraduate students, enrolled in their first linear algebra course, to a set of eight questions asked during their one-on-one interviews scheduled toward the end of the Spring 2009 semester. Each of these students was selected at random from a list of volunteers interviewed at the end of the semester and belonging to three different courses. Our main goal is to analyze the different aspects of learning shown by each student through the presence of the distinct modes of reasoning and the use of metaphors/metonymy as part of their responses.

1.3.1 Participants

The students that participated in this research came from three different groups taking a first year linear algebra course during the Spring 2009 semester. Due to the demographic

location of the region where this study took place, the majority of the students from the groups were Hispanic and a considerable percentage had English as their second language.

Two of the groups were enrolled in what was referred to as a modular matrix algebra course (non-traditional course), while the other was called non-modular course (traditional course). The modular matrix algebra courses enforced the use of computerized mathematical modules, accessed through the internet, that were introduced as part of the class and related homework assignments. On the other hand, the non-modular course had a traditional approach, where the professor lectured and assigned homework, but the computerized modules were not included or even mentioned.

Since these classes were taught by different professors with different teaching techniques and hence, different levels of abstraction, it's important to state that the generalization of these results to the entire population of linear algebra students had to be made carefully, as there were important differences even within the two modular courses.

1.3.2 Modular and Non-modular Section Characteristics

In the modular versions of the course, topics were presented during class to students through a formal definition, row reduced echelon operations, algebraic manipulations, and, frequently through the graphical representations. Homework from the required textbook was often assigned (but not collected) and an assignment, to be answered through the use of the computer modules, was administered and collected (students had an average of a week to work on the material). Sometimes, professors would introduce new topics by using the computer modules through an overhead projector and would explain the characteristics of the new topic and the relationship to past topics.

In the non-modular version of the course, topics were presented during the class through formal definition, row reduced echelon operations and, depending on the questions asked by students, sometimes the professor would provide a graphical representation of the topic. Homework was assigned (but not collected) and consisted on problems taken mainly from the required textbook (homework questions were given in quizzes in one of the modular groups).

The official course description for both, the modular and the non-modular sections of this class during the spring 2009 semester is as follows:

MATRIX ALGEBRA 3323: Systems of linear equations, matrices, determinants, eigenvalues and eigenvectors, diagonalization, vector spaces and linear transformations.

However, the topics chosen for this thesis is limited to:

1. The definition of linear dependence or independence in a set of vectors; identification of linear dependency in particular sets of vectors.
2. Characteristics of linearly independent/ dependent set of vectors in \mathbb{R}^2 and \mathbb{R}^3 .

1.4 ANALYSIS

A qualitative approach, namely the constant comparison method (Glaser, 1992), is used to analyze student responses on the interviews. The qualitative analysis focused on the presence and categorization of thinking modes and metaphors/metonymy in students' interview responses to questions about linear independence, span, and spanning sets.

1.4.1 Qualitative Analytical Procedures

The interviews of 3 students – one from each of the classes available during the Spring 2009 semester— were transcribed and summarized. The qualitative analyses of the transcripts were conducted by the author of this thesis, his advisor, and an additional graduate student with strong mathematics background and will consist in the identification and classification of the presence of the cognitive constructs defined previously: thinking modes and metaphors/metonymy. An inter-reliability test was conducted for each interview to rate the consensus between the raters. Discussion among the raters was done continuously to discuss the different categories identified in each interview and was completed when no additional categories emerge. Once all possible categories were listed, sample student responses and category descriptions included in each as identifiers. Afterwards, the frequency and types of thinking modes and metaphors/metonymy identified were recorded for each student in order to address our research questions.

Chapter 2

Literature Review

Research to discover the various thinking modes students display from linear algebra, has displayed to be somewhat cumbersome and limited. The teaching community believes that courses to teach linear algebra are not being presented in a format comprehensible to students. There are others who believe linear algebra will always be challenging for students no matter how it is presented (Dorier et al., 2001).

It was discovered that many students possess learning difficulties with basic linear algebra concepts (Dogan-Dunlap, 2010). Documented for several years these difficulties have help gear changes to increase the level of understanding of linear algebra students at university level. Despite these changes, radical outcomes have not been apparent because of the variety of thinking modes students display, and difficulties that still plague students today.

2.1 Epistemological Aspects of Linear Algebra

The challenge to learn mathematics for many students is by memorization. They soon discover as the level of difficulty increases this does not work. This style of learning offers a basic knowledge and may aide a student to succeed in a class, but presents future hurdles in upper level math. Memorization for many college students has been their underlying method of learning throughout their post college carrier making it challenging to create a link to their mathematical college level courses. Linear algebra is one of those courses, with a high level of abstraction; gives the presence to be particularly problematic to most students. So much so that students become discouraged not being able to comprehend the material, as a result, their familiarity structures become disjointed with absence reasoning. Dogan-Dunlap (2010) indicate

that some of the difficulties students face in linear algebra courses are the —high level of formalism” and the “axiomatic approach” for which students are not equipped to grasp.

2.1.1 The Conceptual Aspects of Linear Algebra

Dorier and Sierpienska (2001) believe there is a “necessity of cognitive flexibility” for an insightful understanding of linear algebra concepts. Many students have trouble connecting different visual representations used to represent linear algebra concepts due to their lack of logic and set theory knowledge (Dogan-Dunlap, 2006). As mentioned in Dogan-Dunlap (2010, pp. 2), Dubisky and Harrel (1997) explained that students are capable of achieving abstraction if the flexibility between the representations of the same concept is instituted. Abstraction is established if concept images and concept definitions are not contradicting each other (Dogan-Dunlap, 2010).

After 1930, a theoretical reconstruction of the methods to solve linear algebra problems initiated a new axiomatic central theory (Dorier et al, 2001). Dorier and Sierpanska (2001) state the new axiomatic central theory gave linear algebra a more universal approach and language to be used in different contexts. This new theory also involved the use of concepts and tools that were not explicitly formulated or unified, and it marked a new level in abstraction (Dorier et al, 2001). The different perspective brought a more sophisticated level in mental operations that as a result, later manifested in difficulties associated with the pre-existing related elements of knowledge from lower levels (Dorier et al, 2001).

To have a solid understanding of linear algebra concepts, students need to first ‘concretize’ these abstract objects and their representations (Dorier et al, 2001). Most linear algebra students are overwhelmed by the amount of new definitions and theorems and with the

high level of formalism students seem to have a lack of connection to what they already know (Dogan-Dunlap, 2006, 2010).

According to Hillel (2000) linear algebra can be represented with the use of three basic languages, such as geometric, algebraic, and abstract. The geometric language of two and three dimensional spaces includes line segments, points, geometric transformations, and planes. The algebraic language of the R^n space, includes n-tuples, matrices, and rank and the abstract language states to abstract theory, such as vector spaces, linear transformations of vector spaces, and the eigen value theory (Dorier et al, 2001). Hillel (2000) discovered that the way instructors used to shift from one language to the other, without any pause or attempt to alert students of the change, deprived the students of the time needed to assimilate the relationships among the concepts being learned.

2.1.2 The Cognitive Characteristics of Linear Algebra

Semiotic representations, as defined by Duval (1995) are “productions made by the use of signs belonging to a system of representation which has its own constraints of meaning and functioning”. Duval states these representations are ‘absolutely necessary’ in mathematics because some objects cannot be directly recognized and must be represented (Dorier et al, 2001). Semiotic representations play an important role in the development of mental representations, accomplishment of cognitive functions, and production of knowledge (Dorier et al, 2001). According to Duval (1995) semiosis and noesis -the highest cognitive process- are two acts that cannot be separated from each other, but they differ in that the first refers to “the comprehension or production of a representation by a sign” while the second refers to “the conceptual comprehension of an object”. Duval identified three types of cognitive activities related to

semiosis, the formation of a representation, the processing and transformation of a representation, and the conversion of a semiotic representation from one register to another (Dorier et al, 2001).

Pavlopoulou (see Dorier 2000, pp. 247-252) was able to distinguish between three registers of semiotic representation of vectors; arrows as the graphical register, columns of coordinates as the table register, and finally the axiomatic theory of vector spaces as the symbolic register. Pavlopoulou on her research also discovered confusion among the students with respect to an object and its representation and difficulty in converting from one register to another (Dorier et al, 2001). As described by Dorier and Sierpinska (2001), Alves-Dias was able to “generalize the necessity of conversions from one semiotic register to another for the understanding of linear algebra to the necessity of cognitive flexibility”. Registers of semiotic representation requires the student to be able to move from one to another (Dorier et al, 2001).

Another cognitive requirement of linear algebra students is the need for background knowledge in areas such as, set theory, logic, and proofs (Zamora, 2010). According to Dogan-Dunlap (2006), Bogomolny (2007), and Rogalski (2000) some of the problems that linear algebra students face manifest due to the lack of background knowledge in the those areas.

2.2 Principles of Teaching Linear Algebra

The United States established in 1990 a movement to develop the learning and teaching of linear algebra named the Linear Algebra Curriculum Study Group (LACSG) to report concerns of the teaching and learning of linear algebra. The LACSG, composed by sixteen mathematics educators from across the country, created a list of recommendations based on a combination of three major sources, research-based knowledge done on students' learning

processes and the optimal teaching methods of linear algebra, individual teaching experience of LACSG members, and the contribution of consultants from various disciplines who explained how linear algebra was related to their field and what kind of changes in the curriculum could benefit them (Harel, 2000).

The LACSG members made five major recommendations to improve the teaching and learning of linear algebra (Harel, 1997).

- The first course in linear algebra should not be entirely focused on proofs
- A second course of linear algebra should be part of every mathematics curriculum
- The incorporation of technology
- The introduction of linear algebra concepts in high school
- A core syllabus that included concepts such as matrix addition and multiplication, Gaussian elimination, echelon and reduced echelon form, matrix inverses, determinants, linear combinations, linear dependence and independence, subspaces of R^n , bases of R^n , matrices as linear transformations, rank, inner products, eigen vectors, eigen values, in between others (Harel, 2000).

Following these recommendations Harel (2000) developed a theoretical framework based on the three learning-teaching principles: the Concreteness Principle, the Necessity Principle, and the Generalizability Principle.

2.2.1 Concreteness Principle

After working on experiments with high school and beginning college students, Harrel (2000) found the assumption of students being able to deal with abstract structures without

extensive preparation to be unjustified. His finding led to the formulation of the Concreteness Principle (Harel, 1987) which states:

For students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student's eyes; that is to say, the student has mental procedures that can take these objects as inputs. (page 180)

This principle recommends us that students build onto their understanding of concepts more, if the context is concrete to them (Harel, 2000). Concreteness of the abstract concepts may be achieved through technology activities providing the initial mental structure needed for successful learning of topics (Dogan-Dunlap, 2010). My thesis in fact uses data from a pool of students some of whom were provided initial mental construct for basic linear algebra concepts via online web activities.

2.2.2 Necessity Principle

The key indication behind the second principle formulated by Harel (2000) is that instructors must include problem solving activities in which students can reflect abstract conceptions and apply them to solve mathematical problems that are realistic and appreciated by them, it states:

For students to learn, they must see a need for what they are intended to be taught. By „need“ it is meant an intellectual need, as opposed to a social or economic need. (page 185)

Harel (1998) states the way to transform the Necessity Principle into a more concrete teaching setting is by recognizing and identifying the intellectual need of students, allowing the

interactions of students with the problems corresponding to their intellectual needs, and by guiding students in the processes of transferring their knowledge to find a solution.

2.2.3 Generalizability Principle

The third and last principle formulated by Harel (2000) is a counterpart of the Concreteness Principle and the Necessity Principle, and it states:

When instruction is concerned with a 'concrete' model, that is a model that satisfies the Concreteness Principle, the instructional activities within this model should allow and encourage the generalizability of concepts. (page 187)

This principle proposes to assist students encapsulate concepts learned in a specific model in order to make generalizations (Harel, 2000).

2.3 The Use of Geometry in the Teaching and Learning of Linear Algebra

Controversy has plagued the use of geometry, and its use rests on the instructor's partialities. Instructors argue that geometrical representations are valuable and required to develop understanding; others claim that introducing a new concept with an over use of geometry might be damaging given geometrical representation could be perceived over metaphorically (Gueudet-Chartier, 2004). To many students linear algebra appears as an abstract subject, some of which discover it difficult to relate algebraic statements to geometric statements. Dogan-Dunlap (2010) conveyed the use of geometric representations aids students consider the different representations of a concept flexibly and allows them to move from one thinking mode to another. Reported by Pecuch-Herrero (2000), geometrical interpretations of

certain linear algebra concepts, such as the Grand-Schmidt orthogonalization process, allowed students from straying away from their calculations.

Gueudet-Chartier (2004) stated that “linear algebra cannot appear as a generalization of geometry alone; it rather must be grounded in several mathematical domains” and determined that geometry must be used prudently in linear algebra courses. Marc Rogalsky (2000) admits geometry as a significant background support for language and meaning in linear algebra and describes how it can provide images of concepts, such as subspaces, linear combinations, direct sum, solutions of systems of linear equations, etc. Geometric representations used to illustrate general situations in linear algebra are useful when used cautiously and alongside with other representations (Rogalsky, 2000) and as long as students understand how ideas can be represented symbolically, numerically, and graphically they will be able to move back and forth from one thinking mode to another (Zamora, 2010; Dogan-Dunlap, 2010), which will facilitate their understanding.

To help student’s précis concepts learned in a specific model, it was recommended by Harel to utilize geometrical representations and technology in linear algebra courses acceptable to make a broad view and obtain a higher level of understanding of abstract concepts. The aim of this thesis is mainly to document the thinking modes, metaphors and metonymies between two different sections of a matrix algebra class. Exposure to one of these sections mainly directed to concrete geometric representations providing initial mental constructs to assist effective learning. The efforts from this thesis may be taken as associations major to supplement understand the consequence of geometric representations in learning. I will now provide a brief interpretation on the frameworks used in my analysis of data.

2.4 Sierpinska's Modes of Thinking

The Sierpinska's framework is used for the purpose of this thesis. It follows the analysis on the thinking modes used by the students interviewed where three kinds of thinking modes were recognized Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural. Sierpinska (Sierpinska, 2000) states "On some aspects on students' thinking in linear algebra" She labeled these modes of reasoning in linear algebra based on their interacting language, the visual geometric, the arithmetic, and the structural language (Sierpinska, 2000). All three thinking modes interweave in linear algebra, therefore the use of one does not suggest the exclusion of another (Sierpinska, 2000). The central variance between the 'synthetic' and 'analytic' modes is that in synthetic, objects are presented unaltered for the students comprehension, and their reasoning attempts to explain them, whereas in analytic mode objects are given altered to the student, in efforts to make sense of them by the definition of properties of their elements (Sierpinska, 2000). According to Dogan-Dunlap (2010) the view of the geometric representations does not substitute one's arithmetic or algebraic modes, but promotes students to utilize multiple modes of reasoning transitionally.

2.4.1 Synthetic-Geometric

The synthetic-geometric mode employs the language of geometric figures: planes, lines, intersections, and their graphical representations. Synthetic-geometric discussions do not follow a proper linear algebra, but are experiential tools used for the conception that leads to the comprehending of an idea (Sierpinska, 2000). Geometric-synthetic modes used by students incline to describe objects without defining them (Dogan-Dunlap, 2010). Dogan-Dunlap (2009)

stated that students' geometric modes merge multiple aspects of vectors such as vector's magnitude, direction, dimension, and position within space.

2.4.2 Analytic-Arithmetic

The analytic-arithmetic mode, satisfies the written conditions sets that became from geometric figures (Sierpinska, 2000) and students relate objects with respect to their processes and procedures (Dogan-Dunlap, 2010). According to Dogan-Dunlap (2009) students' arithmetic and algebraic modes included processes such as row reduced echelon form of matrices, and the use of linear combination.

2.4.3 Analytic-Structural

In the analytic-structural thinking mode, algebraic elements of the representation are manufactured into structural wholes (Sierpinska, 2000). Students relate objects in systems and ignore processes and procedures (Dogan-Dunlap, 2010). An example could be a student referring to a theorem to suggest linear dependence. Dogan-Dunlap (2010) stated "if a student considers the characteristics of an object in the context of a system with geometric features then he/she may be applying both the structural and geometric modes".

The table below summarizes the different thinking modes recognized by Sierpinska, examples of their representations, and the levels of competency associated with each mode.

Table 2.1. Thinking Modes Modified from Sierpinska (obtained from Dogan-Dunlap, 2010)

Synthetic-Geometric	Graphical representations Provide properties of objects readily. It describes an object but not define it.	Student is be able to determine whether vectors whose graphs are provided in R^2 or R^3 are linearly independent or dependent.
Analytic-Arithmetic	Numerical Representations. Defines objects.	Student is able to construct a matrix from vectors, compute its row-reduced echelon form and relate the reduced matrix to linear dependence and independence.
	Linear Combination.	Student is able to provide/refer to linear combination of vectors and determine linear independence.
Analytic-Structural	Objects are considered in a system. Defines objects.	Use of the dimension of vector spaces in determining the linear independence of vectors.

These thinking modes are not mutually-exclusive, and they can all be used transitionally in order to maximize understanding.

To continue the understanding of the student cognition of abstract linear algebra concepts this thesis employs the framework of metonymy and metaphors. Below I provide a brief explanation of the framework.

2.5 Metonymy and Metaphor

Presmeg (2008) proposes that metaphors, metonymies, imagery, and symbolism are vital elements in the representation of mathematical concepts allowing for students to better understand the representations. “A representation does not represent by itself – it needs interpreting and, to be interpreted, and it needs an interpreter.” (Presmeg, 1998). Metonymies and metaphors are essential components that help students clear doubts symbolized in mathematical concepts (Dogan-Dunlap et al, 2010, 2011). Metonymies are formed as a result of classroom instructions and metaphors initiate from everyday experiences; both can build an understanding of mathematical concepts, such as slope (Dogan-Dunlap, 2011). Students and mathematicians alike use metonymies and metaphors, and they contribute to “an epistemology of mathematics” that enables them to form individual implications and associations (Presmeg, 1998). They can also be both influential and difficult for students in understanding complicated concepts (Zandieh, 2006).

2.5.1 Metonymy

Metonymy comes from the Greek word metonymia- denoting change of name (Presmeg, 1998). Metonymy is a figure of speech where one concept or thing is not called by its own name, and its name is replaced by a word that is associated with the concept or object. However, metonymies are not just part of language; they are extensively used in thinking (Panther et al, 2004). The words associated can represent certain attributes or characteristics of the thing or concept, and part of the concept can be used to represent the whole. Whenever we used a letter to represent any set of numbers, this is referred to as the *Fundamental Metonymy of Algebra* by Lakoff & Nunez (2000).

Panther & Thornburg (2004) defined conceptual metonymy as “a contingent relation within one conceptual domain between a source meaning and a target meaning”. In a conceptual metonymy, the source meaning provides mental access to the target meaning, which might be created on the spot, but with frequent use, it may become part of our own lexicon (Panther et al, 2004). Panther & Thornburg (2004) claimed that metonymies should not be seen as a plain substitution relation, but as a ‘reference point’ that triggers meaning, and they argued that metonymies help us determine explicit and implicit meaning.

2.5.2 Metaphor

The word metaphor comes from the Greek word *metaphora*- to transfer or to carry over (Presmeg, 1998). Presmeg (1998) states a metaphor is an implicit analogy that compares one domain or the elements belonging to it, to another domain or its elements by stating that both domains are the same, for example ‘Domain A is like domain B’. Metaphors are composed of ground and tension, ground refers to similarities and tension refers to the differences implied in the analogy (Dogan-Dunlap, 2007). Mathematics educators must be cautious with the use of metaphors because what might be tension for an instructor may become ground for students (Dogan-Dunlap, 2011). According to Dogan-Dunlap (2011) educators should make similarities and differences explicit between the examples of the source and the domain, so that students do not adopt any irrelevant aspects to form new understanding. As suggested by Presmeg (1998), metaphors should not be used to help students learn new concepts, but only to reinforce and relate ones previously learned by making connections.

2.5.3 The Relationship between Metonymy and Metaphor

Metonymies and metaphors are both literacy devices used in the process of constructing new knowledge while learning mathematical concepts (Zamora, 2010). Presmeg (1998) points out that metonymies are in contrast used to refer to an element or attribute of a class to stand for another element or the whole class, while metaphors link similarities of one domain with another domain to create a meaning of connection. The key difference is that metonymies place words by association and metaphors by similarity. Metonymy is like a horizontal chain of signifiers and the metaphor a vertical descent into meaning (Presmeg, 1998). According to Kovecses (2002), in a metaphor —the relationship is based on the similarity of the two domains, while in a metonymy —the relationship is based on the contiguity or correctness of the two entities.

Chapter 3

Methodology

The intent of this thesis is to analyze the answers given by three undergraduate students while taking their first linear algebra course, to a set of questions asked during interviews at the end of the spring 2009 semester. All of the students had a one-on-one interview. Following the same modes of reasoning presented by Sierpinska (2000), the objective was to analyze the different aspects of reasoning demonstrated by each student and their use of metonymies and metaphors as described by Presmeg (1998). Our main goal, for documentation purposes, is to determine the main characteristics of a student enrolled in their first linear algebra class that classify his cognitive use at the university level.

This study was conducted with students from a four-year southwestern university attending a junior level linear algebra class. This course, named Matrix Algebra, is the first linear algebra course offered in the Spring 2009 semester at the undergraduate level to students that satisfy the pre-requisites with a minimum grade of C in Calculus II, Calculus III, and/or Differential Equations. This course was offered in three different sections –two in the afternoon and one in the morning. At the beginning of this semester, there were a total of 95 students registered.

Two of the sections mentioned above, had a modular format, and the third had a traditional one. In the modular courses, aside of lecture and homework assignments, it included a series of constructivist assignments were completed with the aid of graphical tools available online –these tools were created by a group of instructors with extensive knowledge of the course and are referred to as modules from now on. Students in these two sections were given about a week to complete the assignments. The third and last section had a more traditional

setting where students were lectured and had homework exercises assigned from the textbook. To this particular section of students, the modules were not even mentioned nor used in the classroom.

The modular section provided assignments designed to obtain a graphical representation of the topics involved in the matrix algebra course before introducing a new concept through formal language. The module assignments are not part of the scope of this thesis.

At the end of the Spring 2009 semester students from each of the three sections volunteered to participate in an interview conducted by a professor and/or a graduate student from a southwestern university. For the purpose of this thesis, the interview responses of three students are reported and analyzed. The three interviews for the thesis were selected at random from a list of volunteers interviewed. Each interview belongs to a different section of the matrix algebra course –traditional and modular.

The results found in this study were obtained by performing a qualitative analysis to the transcripts and videos obtained from the interviews conducted with the volunteered students. The Grounded Theory, introduced by Glaser and Strauss (1967), was applied to conduct the analysis of the interviews in order to capture the thinking modes, metonymies, and metaphors revealed by students while responding to the questions asked with respect to linear algebra concepts.

A more detailed description of each class section, including the students, lecture style, assignments, and the analysis of the data is presented later in this chapter.

3.1 Participants

As explained above, the students who participated in this study belonged to three different sections of the first linear algebra course offered at a four-year southwestern university

during the Spring 2009 semester. As a result of the location of the institution where this study took place, a significant percentage of the students from each section are of Hispanic origin, and a substantial percentage has English as a second language.

The two sections in which the online modules were implemented are referred to as modular course (non-traditional course), and the third one is referred to as non-modular course (traditional course). In the modular matrix algebra courses, the use of the online modules was enforced and the integration of these was encouraged through the constructivist homework assignments. The non-modular course was more traditional. The instructor assigned homework strictly from the book, and the computerized modules were not presented to the students registered in this section.

Some key characteristics of the students of each section of the matrix algebra course in the Spring 2009 semester are briefly shown in the following tables.

Table 3.1. Demographics of Group A; modular section (obtained from Zamora, 2010)

Section A	
Question	
Gender	- Males:25 - Females:9
Ethnicity	- Hispanic/Hispanic American: 79.4% - White/Caucasian/American: 17.64% - American-Asian/Asian: 2.94%
Classification	- Freshman: 0 - Sophomore:35.29% - Junior: 41.18% - Senior: 23.53%
Major	- Mathematics: 20.59% - Computer Science: 44.12% -Electrical Engineering:23.53% - Industrial Engineering: 8.82% - Computer Engineering: 2.94%
Courses this	- Mean:4.44 - Standard Deviation:0.86 - Mode:4

semester	
Have a job?	- No: 20.59% - Yes: 79.41%
For how long?	- For Less Than a Year: 62.96% - From 1 to 3 Years: 25.93% - No Answer: 11.11%
Hours/week?	- Less Than 20: 51.85% - Exactly 20: 18.52% - More Than 20: 29.63%
English first language	- No: 44.12% Yes: 55.88%; 100% of the students that answered no to this question, reported to have Spanish as their first language (15 students).
Fluency	- 10: 33.33% - 9: 6.67% - 8.5: 6.67% - 8: 46.66% - 5: 6.67% - Standard Deviation:1.35 - Mean:8.60

At the beginning of the Spring 2009 semester there were 35 students enrolled in section A of the course, 34 of those students answered a pre-survey administered in-class. Data is taken from Zamora (2010).

Table 3.2, below, contains the demographics of section B.

Table 3.2: Demographics of Group B; modular section.

Section B	
Question	
Gender	- Males:22 - Females:6
Ethnicity	- Hispanic/Hispanic American: 75% - White/Caucasian/American: 17.86% - Mexican/Chicano:3.57% - American-Asian/Asian:3.57%
Classification	- Freshman: 0 Sophomore:7.14% Junior: 50% Senior: 42.86%
Major	- Mathematics: 3.57% - Computer Science: 35.7% - Electrical Engineering: 35.7% - Industrial Engineering: 7.14% - Mechanical Engineering: 7.14% - Philosophy: 3.57% - Physics: 3.57% - MultiDisciplinary Studies: 3.57%
Courses this semester	- Mean: 4.04 - Standard Deviation: 1.04 - Mode: 5
Have a job?	- No: 32.14% - Yes: 67.86%
For how long?	- For Less Than a Year: 26.32% - From 1 to 3 Years: 52.63% - For More than 3 Years: 21.05%

Hours/week?	- Less Than 20: 5.26% - Exactly 20: 42.11% - More Than 20: 52.63%
English first language	- No: 64.29% Yes: 35.71%; 77.78% of the students that answered no to this question, reported to have Spanish as their first language; 5.56 had Thai as their first language, and 16.66% gave no answer. .
Fluency	- 10: 16.67% - 9: 55.56% - 8: 22.22% - 7: 5.55% - Standard Deviation:1.35 - Mean:8.60

For this section, at the beginning of the semester there were 35 students enrolled, 28 answered the in-class pre-survey. Data is taken from Zamora (2010).

For the non-modular section demographics are shown below in table 3.3.

Table 3.3. Demographics of Group C; Non-modular section.

Section C	
Question	
Gender	- Males:26 - Females:9
Ethnicity	- Hispanic/Hispanic American: 80% - White/Caucasian/American: 11.43% - Mexican/Chicano: 2.86% - American-Asian/Asian: 5.71
Classification	- Freshman: 2.86% Sophomore: 17.14% Junior: 62.86% Senior: 17.14%
Major	- Mathematics: 17.14% -Computer Science: 22.86% -Electrical Engineering: 51.43% - Industrial Engineering: 5.71% - Physics: 2.86%
Courses this semester	- Mean:4.54 - Standard Deviation:1.20 - Mode:4
Have a job?	- No: 37.14% - Yes: 62.86%
For how long?	- For Less Than a Year: 31.82% - From 1 to 3 Years: 50% - No Answer: 18.18%
Hours/week?	- Less Than 20: 50% - Exactly 20: 27.27% - More Than 20: 22.73%
English first language	- No: 37.14% Yes: 62.86%; 100% of the students that answered no to this question, reported to have Spanish as their first language (13 students).
Fluency	-10: 30.77% - 9: 30.77% - 8: 23.08% - 7: 15.38% - Standard Deviation:1.09 - Mean:8.77

Finally this section with an official number of 35 enrolled at the beginning of the semester, all answered the pre-survey. Data is taken from Zamora (2010).

The students enrolled in all three Matrix algebra sections during the semester when this research was conducted, were under a number of diverse degrees plans such as engineering, mathematics, computer science, physics, philosophy, and multidisciplinary studies. There was a percentage of students from each section claiming to have had some experience with proofs in former science courses; the spread was as follows: 35.29% students in section A, 60.71% of section B, and 40% of section C (Zamora, 2010).

Sections A and C were for a period of an hour and 20 minutes twice per week, whereas section B was held for 50 minutes three times per week. The age of students was not collected, but the average age of the university population surveyed to be 26 years old (University of Texas at El Paso— UTEP, 2009).

In efforts to safeguard the identities and evade partiality in the analysis of the students enrolled in the three sections of the course surveys and interviews, students were assigned an alphanumeric-code comprised of letters and numbers depending on the section. As an example, the code name SA12 refers to the student number 12 enrolled in section A of the course. Code names ranged from SA1 through SA35 in Section A, SB1 to SB34 in section B, and from SC1 to SC36 in section C.

All students participated in the study volunteered from each section of the course. The interviews were videotaped for reference purposes, in order to be transcribed by the author of this thesis. Interviewees from each section were chosen at random for the perseverance of this thesis. No prejudices were made on the base of race, gender, age, or socioeconomic status of any

of the students while interviewing and analyzing the interview transcripts. The interview transcripts are available upon request.

3.2 Modular and Non-modular Section Characteristics

The modular versions of the course demonstrated subjects during class to students through formal definition, row reduced echelon operations, algebraic manipulations, and, regularly, static graphical representations of the topics being taught (Zamora 2010). Homework was given but not collected from the required textbook and assignments, to be completed with the aid of computer modules averaging two weeks to complete. New topics were sometimes presented by using the computer modules through an overhead projector and professors would clarify the features of the new topic and its connection with previous topics.

In the non-modular version of the course, topics were demonstrated similar to the modular version with the occasional explanation only for questions asked by students or by providing a static graphical representation of the topic. Homework, again, was assigned but not collected and consisted of problems taken mainly from the required textbook.

Quizzes consisting of assigned homework problems were administered at the beginning of some classes in both versions of the course. No major differences in the organization of the modular and non-modular sections with the exception of the modular use of the computer modules and their assignments.

The official course title for both, the modular and the non-modular sections of this class during the spring 2009 semester was:

MATRIX ALGEBRA 3323: Systems of linear equations, matrices, determinants, eigenvalues and eigenvectors, diagonalization, vector spaces and linear transformations.

However, the topics covered for this thesis:

- 1- Definition of linear dependence or independence in a set of vectors; identification of linear dependency in particular sets of vectors.
- 2- Characteristics of linearly independent/ dependent set of vectors in \mathbb{R}^2 and \mathbb{R}^3 .

The classes taught were given by different professors each with different teaching styles and is worth noting diverse levels of abstraction. Given this state, summaries of observations of each section, conducted by graduate student and research assistant Zamora and obtained from Zamora (2010), are provided. Noteworthy, Zamora (2010) conducted these observations during unannounced visits to the sections of the matrix algebra course.

3.2.1 Section A Observations.

Observations obtained from Zamora (2010). -These observations were conducted while Zamora was an active research assistant during this investigation in the Spring 2009 semester. The observations reported below were obtained from Zamora (2010) pp. 31-39.

Classroom Observation 1. 02/04/2009:

- *The presentation of the material is done in an overhead projector.*
- *Students are given a quiz on the question "Find all values of a for which the system $x_1 + ax_2 = 6$ and $ax_1 + 2ax_2 = 4$ has no solutions. Show your work." Time assigned for this activity: 20 minutes.*

- *One of the students shared his answer with the class; extra points were awarded to him for the presentation.*
- *Instructor asks questions to students looking for feedback while explaining different answers for the quiz.*
- *Instructor encourages class participation by asking students feedback about previous concepts and lectures.*
- *Some of the topics covered this day are elementary row operations, row reduced echelon form (e.g. how do you know if a matrix is in row reduced echelon form?), and Gauss-Jordan elimination process.*
- *Class participation/discussion on the following topic: Let A be an $n \times (m+1)$ matrix where A is the augmented matrix of a system. After the discussion, the instructor re-explained the definitions and properties that made the answers acceptable.*

- Discussion; is it possible to get
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \dots & 0 & a \\ 0 & 1 & 0 & \dots & 0 & b \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 & z \end{array} \right]$$
 ? Not possible to get a unique

- solution (since we don't have enough rows to get the values of the variables). Is this an inconsistent system, with infinitely many solutions (as indicated by the row of only zeroes) or no solutions (as indicated by the last row of all zeroes and a z at the end)?*
- *An additional problem is posted by the instructor: Given that A represents a consistent system and RREF of A has r non-zero rows where $r < m$, then does the system have infinitely many solutions. If $r = m$ then there is one unique solution or no solution at all?*

- Instructor uses computer module to illustrate the idea that parallel planes do not intersect and, hence, have no solution.
- What other types of answers can you have? What's the geometrical representation?
E.g. $z=f1=x+y-5$; $z=f2=2x+2y+1$. This system would have no solution. How does it look like?
- Question: Give a system of planes that has a different answer.

- Not possible for a unique solution

- Is it possible to have infinitely many solutions? Yes. Same planes— equations are multiples of one another.

Example: $z=x+y-5$; $2z=2x+2y-10$: infinitely many solutions (same plane). At the

end you'd get $\left[\begin{array}{ccc|c} 1 & \dots & 0 & \dots \\ 0 & \dots & 0 & 0 \end{array} \right]$ as the RREF.

- Infinitely many solutions: $z=x+y-5$; $z=x-2y \rightarrow$ infinitely many solutions (they

intersect at a line) $\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \dots & \dots \\ 0 & 1 & \dots & \dots \end{array} \right]$

- Instructor continues discussion on the computer module named Linear Systems developed in Geometer's Sketch pad (GSP) and explains its functionality.
- Equivalent system of linear equations:

System $A \rightarrow$ ERO (Elementary Row Operations) \rightarrow System B

System $A \neq$ System B , but both have the same set of solutions.

- *Definition: Let A, B be equivalent systems of equations. Then A and B have the same solution sets. This is explained by the instructor through the use of a module in GSP.*
- *Instructor goes over the proofs of this theorem; for simplicity purposes only 3×4 systems are considered.*

Classroom Observation 2. 02/18/2009:

- *A quiz is given to students at the beginning of class (20 minutes allowed) on the question:*

"a) Define 'consistent system'

b) Given an example of a consistent system. Explain why your example is a consistent system. "

- *Group Work. A theorem is introduced to the class; students had to work in groups to come up with a proof. Theorem given: "Let A, B, C be compatible matrices. Then $A(B+C)=AB+AC$." Distributive property of matrices. Two students provide an explanation on their reasoning to come up with the proofs.*
- *Instructor goes over the proofs provided by the students and analyzes the strengths and weaknesses of each one of them and goes over the definitions and properties previously introduced in class that make the proofs acceptable.*
- *Abstract ideas on the concepts of multiplicative inverse and identity matrix in R^n are presented by instructor through different representations of the concepts (through the use of algebraic expressions, matrices, and a numerical example with matrices).*
- *Homework form textbook is assigned to students for the next class.*

Classroom Observation 3. 04/01/2009:

- A quiz was given to students at the beginning of the class (20 minutes were assigned for this activity). The question was: "Let AB be a non-singular matrix. Use the fact that B is also a nonsingular matrix to prove that A is a nonsingular matrix."
- There is a class discussion on the quiz problem.
- Instructor uses computer module called Vector Spaces to demonstrate linear combinations of vectors graphically.
- Instructor uses overhead projector as a tool to teach the class.

Classroom Observation 4. 04/06/2009:

- A quiz was given to students at the beginning of the class (20 minutes were assigned for this activity). The question was: "Determine whether the given set of vectors is linearly independent or dependent. If the set is linearly dependent, express one vector

in the set as a linear combination of the others. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \right\}$ "

- Two students presented their answers to the class by explaining their reasoning. One answer was numerical (algebraic, by row reducing the matrix and looking at the resulting matrix and its elements) while the other skipped this part by using his calculator and analyzing the results using more abstract ideas and concepts seen in previous classes. The instructor explained why both answers were acceptable.
- Instructor goes over the concept of linear independence graphically by using a module (called Vector Spaces) in the computer.
- Students realized that all the possible linear combinations of the 4 vectors would give R^3 (through the use of the computer module).

- *Instructor uses module to show that all possible linear combinations of the first 3 vectors in the set given in the quiz would also give R^3 ; instructor uses 2 out of the first 3 vectors to show that R^3 can't be obtained by obtaining all possible linear combinations.*
- *Group discussion on: "Can you get R^3 from any set of three vectors?"*
- *Numerical examples are introduced on the use of the computer module for students to see, graphically, if the sets were dependent or independent and how they would look like.*
- *Examples of sets of vectors in R^2 are used to illustrate dependent and independent sets of vectors through the use of computer module.*
- *Instructor goes over the geometrical representation of ideas in R^2 and then moves to R^n .*

Section A Observations Summary.

Calculators were allowed only as a tool during in-class activities and instructor used a wide range of teaching techniques in order to involve students in their own learning. Techniques used: quizzes, tests, computer modules' assignments, and in-class group work, all tools for students to comprehend the class topics. Proofs introduced through lecturing allowed for group discussion among students who conjectured their own ideas. Ideas were often encouraged by instructor to present and defend by the individual or group. Answers were validated by instructor by going over the ideas covered in class (Zamora (2010) pp. 31-39).

3.2.2 Section B Observations

These observations were conducted by Zamora (2010) while attending Section B during the Spring 2009 semester in three different unannounced visits. The observations reported below were obtained from Zamora (2010) pp. 40-45.

Classroom Observation 1. 02/06/2009

- *Notes for test 1: calculators allowed; work must be included for credit.*
- *Instructor goes over the vector form of the solution. $x_1 = -4t - s - 3$, $x_2 = 2t + s + 1$, $x_3 = t$, $x_4 = 2$, $x_5 = s$, where t and s are free variables. Solution can be written as a vector.*

$$\text{Vector Solution: } \begin{bmatrix} -4t - s - 3 \\ 2t + s + 1 \\ t \\ 2 \\ s \end{bmatrix} = t \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

- *Instructor mentions that the answer is in 5 dimensional space.*

$$\text{- Vector solution: } \left\{ t \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} : t, s \in \mathfrak{R} \right\}$$

- *Instructor assigns homework from book (not to be collected).*
- *Instructor goes over the homework questions in order to guide students through the process of answering them.*
- *General questions from students are answered at the beginning of the class.*

- Instructor emphasizes the need of knowing how to solve systems of linear equations with elementary row operations since the rest of the class is based on that topic.

- Question answered in class: For what value of a is the system consistent?

$$\begin{cases} x_1 + ax_2 = 6 \\ ax_1 + 2ax_2 = 4 \end{cases}$$

- Instructor answers the question as follows: $\left[\begin{array}{cc|c} 1 & a & 6 \\ a & 2a & 4 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & a & 6 \\ 0 & a^2 - 2a & 6a - 4 \end{array} \right]$.

- Instructor asks questions to students throughout the class about what the answer would be and what is the reason behind the answer.

- Possibilities: If $a^2 - 2a \neq 0$, then the system is consistent. If $a^2 - 2a = 0 \rightarrow a(a - 2) = 0 \rightarrow a = 2$ OR $a = 0$. For $a = 0$: $6a - 4 = 6(0) - 4 = -4 \neq 0$; for $a = 2$: $6(2) - 4 = 12 - 4 = 8 \neq 0$. Therefore, for $a = 0$ and $a = 2$, $6a - 4 \neq 0 \rightarrow$ system is inconsistent if and only if $a = 0$ OR $a = 2$.

- Review for test during next course.

- Problem from book (similar to one from homework). X represents 1's digit, y represents ten's digit, and z represents 100's digit. Four equations involved: $N = z * 100 + y * 10 + x$; $N = 15(x + y + z)$; $100 * x + 10y + z = N + 396$; $x = 1 + y + z$. System with four equations and four unknowns which can be represented as:

$$\left[\begin{array}{cccc|c} 15 & 15 & 15 & -1 & 0 \\ 100 & 10 & 1 & -1 & 396 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 10 & 100 & -1 & 0 \end{array} \right] = \begin{cases} 15x + 15y + 15z - N = 0 \\ 100x + 10y + z - N = 396 \\ x - y - z = 1 \\ x + 10y + 100z - N = 0 \end{cases}$$

- Instructor states that this system can also be represented with three equations and three unknowns (by eliminating N at the beginning).

Classroom Observation 2. 02/20/2009

- Quiz is given at the beginning of the class (30 minutes of the class were used for this) on the following question: "An $n \times n$ matrix is called diagonal if for every $1 \leq i \leq n$ and $1 \leq j \leq n$ such that $i \neq j$, $a_{ij}=0$. Prove that the sum of two diagonal $n \times n$ matrices is a diagonal matrix. Proof (provided by instructor): Let $A, B \in M_{n \times n}(R)$ both be diagonal matrices. Let $C=A+B$. Let $i \neq j$, $C_{ij}=a_{ij}+b_{ij}=0+0$; since $a_{ij}=0$ and $b_{ij}=0$ then $C_{ij}=0$. Therefore, C is a diagonal matrix.

- Instructor provided an additional less abstract proof to students (with matrices instead of abstract ideas) such that they could see the different representations that

this answer could have. E.g. $A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & 0 & \dots & 0 \\ 0 & b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix}$;

$$C = A + B = \begin{bmatrix} a_{11} + b_{11} & 0 & \dots & 0 \\ 0 & a_{22} + b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} + b_{nn} \end{bmatrix} \text{ which is also a diagonal matrix.}$$

- Instructor goes over project 2 (from modules) during the class by introducing the functionality of the module and going over the questions on the project.
- The concept of non-commutativity of matrix multiplication is explained graphically with the use of a project module.
- Instructor goes over some examples that can be applied to the module and the results that can be obtained.
- Instructor goes over the concepts that are related to the ideas seen in the module.

- Students ask questions about what to do in a proof. For example: non-commutativity- show one where test fails (one that is non-commutative then you can't say all are commutative), commutativity- show condition is true for all elements.
- For homework, prove that matrix multiplication of $n \times n$ matrices is non-commutative. Hint: Look for the easier example for 2×2 matrices— using only zeroes and ones and as many ones as possible. Then generalize your idea for $n \times n$.
- Instructor shows students what they may be able to do in order to understand what is being asked and what they need in order to generalize the idea (e.g. start with a 2×2 matrix with numbers in it; see its behavior— with respect to multiplication with other matrices— then try to generalize the idea for $n \times n$ matrices).
- Instructor goes over the theorem: if $A \in M_{m \times n}(R), B \in M_{n \times r}(R)$ then $(AB)^T = B^T A^T$.
Proof: First look at the dimensions $B^T = r \times n, A^T = n \times m \rightarrow B^T A^T = (r \times n)(n \times m) = r \times m$
Now, let $A = a_{ij} \ 0 \leq i \leq m, 0 \leq j \leq n; B = b_{jk} \ 0 \leq j \leq n, 0 \leq k \leq r$. Then $(a_{ij})(b_{jk}) = (ab)_{ik}$,
 $((ab)_{ik})' = ab_{ki} \dots$

Classroom Observation 3. 04/06/2009

- Student asks questions about homework problems: $A=(1,1,-1), B=(0,1,2), C=(3,0,1)$;
 $a(x-x_0)+b(y-y_0)+c(z-z_0)=0, ax+b(y-1)+c(z-2)=0$. At point A: $a(1)+b(0)+c(-3)=0 \rightarrow a-3c=0$; at point C: $a(3)+b(-1)+c(-1)=0 \rightarrow 3a-b-c=0$.
- Instructor goes over the matrix representation of the concepts above:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 3 & -1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -8 & 0 \end{array} \right] = \begin{cases} a=3t \\ b=8t \\ c=t \end{cases} \rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} t. \text{ Now choosing } t=1,$$

would yield: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$. Hence, $3x+8(y-1)+(z-2)=0$ is the equation of the plane

containing the points A, B, and C.

- Instructor introduces the topic of determinants. He states that this is the most important numerical aspect of $n \times n$ matrices.

- Question. Given an $n \times n$ matrix A, we define its determinant $\det(A)$ (or $|A|$) as a number obtained in the following manner for $n=1,2,3$. For $n=1$: $\det[a_{11}]=a_{11}$;

$$n=2: \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$n=3: \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$

$$a_{11} * \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} * \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} * \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}. \text{ In order to calculate the}$$

determinant of 3×3 matrices, instructor mentions the use of a recursive definition in which the definition of the determinant of 2×2 matrices is used.

- Numerical example provided by instructor: $\det \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1(1) - 1(2) + 1(0) = -1$.

- Instructor mentions the fact that calculators do give you the determinant of the matrix.

- Topic introduced by instructor. Orthogonal unit vectors in a plane; instructor provides a graphical representation of the vectors represented by

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

- Definition: Given two vectors in R^3 , $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, their cross-product is defined

$$\text{as: } \vec{u} \times \vec{v} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \text{ Remember that } i, j, \text{ and } k \text{ are vectors!!}$$

$$\vec{u} \times \vec{v} = \det \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} * \vec{i} - \det \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} * \vec{j} + \det \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} * \vec{k}$$

- Instructor states that the cross product of two vectors is a linear combination of the three basic vectors i, j , and k with scalars given by the determinants.
- Instructor states that calculators are useful to obtain the cross product of the vectors u and v .
- Instructor introduces the topic of equations of a plane and its relationship with the normal vector.
- Instructor provides two numerical examples of the cross product of two vectors; using

$$\text{the vectors } i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and calculating the cross products } \vec{i} \times \vec{j} = \vec{k} \text{ and } \vec{j} \times \vec{i} = -\vec{k}.$$

One of the students mentions the fact that these cross products are not equal and hence this operation is not commutative.

- *Instructor goes over the properties of the cross product and re-assigns homework from textbook.*

Section B Observations Summary.

Calculators were allowed in class and tests and instructor introduces abstract ideas and illustrated them through the use of meaningful examples. Students took the liberty to ask questions about homework problems, in which instructor would spend a good amount of time answering those questions. Despite the fact students had all this freedom and encouragement to share their own ideas and thoughts, the level of participation was low with only a handful of students interacting during class. Geometrical interpretations used by instructor were limited to occasional drawings on the blackboard while providing more oral interpretations of them. Instructor gave clarifications of the abstract topics through the use of different ideas such as matrices, systems of linear equations, and numerical examples (Zamora 2010 pp. 40-45).

3.2.3 Section C Observations

Three unannounced visit observations were conducted by Zamora (2010) while attending Section C during the Spring 2009. The three different day observations reported below were obtained from Zamora (2010) pp. 46-51.

Classroom Observation 1. 02/05/2009

- *Instructor goes over a homework problem as requested by students (these problems were due the next class).*

$$- \begin{cases} x_2 + x_3 - x_4 = 3 \\ x_1 + 2x_2 - x_3 + x_4 = 1 \\ -x_1 + x_2 + 7x_3 - x_4 = 0 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 3 \\ 1 & 2 & -1 & 1 & 1 \\ -1 & 1 & 7 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -13 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array} \right] \quad \text{where}$$

x_4 is a free variable.

- Instructor posted the following question: is it a consistent or inconsistent system based on this method? Students expressed their opinions about this question.
- Instructor continues with the explanation of the solution of this problem: Assign a parameter to the free variable by using the information of the matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -13 \\ 0 & 1 & 0 & -2 & 17/3 \\ 0 & 0 & 1 & 1 & -8/3 \end{array} \right] = \begin{cases} x_1 + 6x_4 = -13 \\ x_2 - 2x_4 = 17/3 \\ x_3 + x_4 = -8/3 \end{cases} \text{ Now with } x_4 \text{ as the free variable,}$$

$$\text{substitute } x_4 = t \text{ in the equations: } \begin{cases} x_1 + 6x_4 = -13 \\ x_2 - 2x_4 = 17/3 \\ x_3 + x_4 = -8/3 \end{cases} = \begin{cases} x_1 + 6t = -13 \\ x_2 - 2t = 17/3 \\ x_3 + t = -8/3 \end{cases} \text{ and } x_4 = t, t \in R$$

- Instructor then talks about the geometrical representation (planes, lines, etc) but does not show it on the blackboard or computer. Instructor just mentions verbally how it would be seen in three dimensions.
- Instructor goes over the vector form of a solution to a problem. He goes over the last

$$\text{example and states the solution as being: } \begin{pmatrix} -6t - 13 \\ 2t + 17/3 \\ -t - 8/3 \\ t \end{pmatrix} = t \begin{pmatrix} -6 \\ 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -13 \\ 17/3 \\ -8/3 \\ 0 \end{pmatrix}.$$

- Instructor states verbally that a geometrical interpretation of this is not possible since the answer is in four dimensions.

- Instructor states that students are required to provide answers in vector form for quizzes and tests.
- Student asks question on homework problem. Instructor goes over it. Question: "For what value of a is the system consistent: $x_1 + ax_2 = 6$ and $ax_1 + 2ax_2 = 4$ "
- Instructor goes over problems assigned for homework and guides students on how to solve them (by providing hints).
- Example solved in class by instructor: N is a three digit number; it equals 15 times the sum of its digits. If digits are reversed, the resulting number exceeds N by 396. One's digit is one larger than the sum of the other 2. Give a linear system of 3 equations and state what N is. Four equations involved: $N = z * 100 + y * 10 + x$; $N = 15(x + y + z)$; $100x + 10y + z = N + 396$; $x = 1 + y + z$. System with four equations and four unknowns which can be represented as:

$$\begin{cases} 1x + 10y + 100z = N \\ 15(x + y + z) = N \\ 100x + 10y + 1z = 1x + 10y + 100z + 396 \\ x = y + z + 1 \end{cases} = \begin{cases} 1x + 10y + 100z = 15x + 15y + 15z \\ 100x + 10y + z = x + 10y + 100z + 396 \text{ OR} \\ x = y + z + 1 \end{cases}$$

$$\begin{cases} x + 10y + 100z = N \\ 15(x + y + z) = N \\ 100x + 10y + 1z = N + 396 \\ x = y + z + 1 \end{cases}$$

Classroom Observation 2. 02/26/2009

- Instructor lectures on the distributive law theorem for matrices. Let $A \in M_{m \times n}(R)$; $B, C \in M_{n \times r}(R)$ then $A(B + C) = AB + AC$. Instructor proves the theorem on blackboard for the group.

- Instructor goes over a homework problem. Question: Suppose $A^2=AB$, $A^2-AB=0$, $A(A-B)=0$. Then $A=0$ or $A-B=0$, but since A cannot be equal to 0, then $A=B$, which is not true, since you can actually multiply two non-zero matrices and get the zero matrix as

a result. Counter-example:
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- Instructor advises students to try to generalize the ideas of the example to get a conclusion for $n \times n$ matrices if possible.
- An example of two $n \times n$ non-zero matrices whose product is the zero matrix ($n > 1$) is given by the instructor with the help of students who provide ideas on how to come up with the proof after working on an example for 2×2 matrices. The matrices given are:

$$\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n} \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n} = 0 \text{ (the zero matrix).}$$

- Instructor multiplies matrices in reversed order to show that multiplication of matrices is non-commutative, $AB \neq BA$.
- Instructor pinpoints to students that there is an easier pair of matrices that when multiplied give the 0 matrix as a result:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n} \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n} = 0$$

- Instructor does a small review for test 2.
- Instructor states that a vector is an $n \times 1$ matrix; vectors are matrices.

- Instructor explains the definitions of vector multiplication, and norm, length, and magnitude of vectors.
- Instructor talks about three elements of vectors: orientation, direction, and length, and explains them by drawing static version of a two dimensional plane.

Classroom Observation 3. 04/09/2009:

- Instructor goes over the following problem at the beginning of the class: Given points $A_1=(1,2)$, $B_1=(0,4)$, $A_2=(0,1)$, $B_2=(-1,1)$. Find a point of intersection of lines $\overline{A_1B_1}$ and $\overline{A_2B_2}$.

- Process: $\overline{A_1B_1}$: point (0,4); vector $(1,-2)^T \rightarrow \begin{cases} x=t \\ y=-2t+4 \end{cases}$. Intersection point: (3/2,1).
 $\overline{A_2B_2}$: point (0,1); vector $(-1,0)^T \rightarrow \begin{cases} x=-s \\ y=1 \end{cases}$

- Student asks how this result was obtained; instructor goes over the procedure and formulas again: pt: (x_o, y_o) , vector: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$; $\begin{cases} x=u_1t+x_o \\ y=u_2t+y_o \end{cases}$ in 3D, add $z=u_3t+z_o$.

- Instructor goes over the general equation $a(x-x_o)+b(y-y_o)+c(z-z_o)=0$ with a point:

$$(x_o, y_o, z_o) \text{ and a normal vector } n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

- Instructor mentions how problems can be solved and what type of problems they can encounter involving these topics (e.g. intersection of a plane and a line).
- Instructor goes over an additional problem a student requested.
- Instructor encourages students to use calculators during class and tests to come up with the row reduced echelon forms of matrices.

- *Instructor illustrates statically an example of finding an angle between two planes geometrically (first, by using hands and the desk and then by drawing a plane on the blackboard) to show students how the intersection of a line and a plane would look like.*
- *Instructor goes over how to calculate a cross product and its properties.*
- *Instructor introduces the Parallelogram method to add vectors and provides a geometrical explanation.*
- *Instructor goes over a procedure for students to follow in order to check for linear independence or dependence in a set of vectors.*

1. *Recall the definition of linear independence:*

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \text{ are linearly independent vectors } \Leftrightarrow \\ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_k \vec{v}_k = \vec{0} \Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_k$$

2. *Use a matrix to check for independence by stacking the vectors as columns of a matrix. $A = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k]$*

3. *Reduce matrix A (instructor recommends, again, to use a calculator to do this).*

4. *Look for free variables (columns that are not represented by a leading 1). If there is any free variable, the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly dependent. Otherwise,*

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a linearly independent set of vectors.

- Instructor provides an example on this procedure. Take vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 5 \\ 2 \end{pmatrix} \right\}$ which

form the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In this example, there is no leading one in

the third column, and the number 2 in position (1,3) of the matrix is considered to be the free variable. Therefore, this set of vectors is considered to be linearly dependent.

- Instructor goes back to explain why this set of vectors is dependent and what it means.
- Instructor solves the same problem in a different way:

$$x \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ 1 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 5 & 0 \\ 1 & 0 & 2 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{cases} x + 2z = 0 \\ y + z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = -2z \\ y = -z \\ z = z \end{cases} \Rightarrow \text{Let } z = t : \begin{cases} x = -2t \\ y = -t \\ z = t \end{cases}. \text{ Therefore, this is a dependent set of vectors. For}$$

$$\text{example, when } t = 1 \Rightarrow \begin{cases} x = -2 \\ y = -1 \\ z = 1 \end{cases} \Rightarrow -2 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Therefore, the definition}$$

of linear independence is not met.

- Instructor assigns homework from textbook.

Section C Observations Summary

As in the other sections before the use of calculators was allowed but students were not allowed the use of the computer modules. The use of calculators during class and tests was encouraged to reduce the amount of time used for calculations. They were encouraged to share their own ideas and thoughts; the student level of participation and interaction was high in this section. Instructor took the time in answering students' questions on homework and textbook exercises. Rest of the class time was used to introduce new concepts by first providing theorems and their proofs. To provide a variety of perspectives, instructor demonstrated concepts through the use of examples and geometrical representations. The examples and visual representations were given as drawings on the board (Zamora, 2010, pp. 46-51).

3.2.4 Comparison of Sections

The use of the computer modules in Section B was the key difference between sections B and C. Section B had to complete module assignments on their own time; though, the use of the modules was limited. The greatest significance between B and C was the student level of participation and involvement during class, Section C having the highest level. These sections were examples of a traditional classroom setting where students were lectured and assigned homework. Section A on the other hand had a larger variation of activities and involvement (Zamora, 2010) and even though the use of the computer modules was also present in sections B, their use was most obvious in Section A with the instructor strongly relying on them to link ideas and to shift from geometrical to algebraic representation of abstract concepts.

3.3 Procedure

A pre-survey and consent form was administered by the instructor of each section at the beginning of the spring 2009 semester, to allow the data collected to be used for informational purposes. Copies of the pre-survey and consent form are included in the appendix.

Instructors from all sections used the months of April and May, to invite students to participate, as volunteers, in the interviews conducted. The interviews from these first linear algebra class students took part as part of the research to document their reasoning at the university level. Minuscule amount of credit was offered to all in effort to encourage participation specially those who wouldn't normally volunteer (Zamora, 2010). About a week after topics such as linear independence, span, and spanning set were covered in class, interviews began.

A schedule date and time to be interviewed was given to student by professor and a research assistant. In order to capture written responses, anxiety levels, and gestures made while orally responding to a specific set of questions, the interviews conducted were video-taped from two different angles lasting between 60 and 120 minutes. Additional questions were sometimes added during interviews in efforts to comprehend and capture the student thinking and reasoning processes.

Three randomly selected interview video tapes were chosen to transcribe by the author of this thesis. Each interview transcript was then analyzed independently by the author, mentor, and an additional graduate student. The subsequent portrayal and classification of Sierpinska (2000) on thinking modes and the portrayal of metonymies and metaphors provided by Presmeg (1998) were utilized in analysis by the three researchers mentioned above. More details on the categorization are provided in the following chapter.

The code names of the students randomly selected from the list of interviewed students used in this thesis are SA12, SA22, and SC23. These students belonged to sections A, B and C respectively. In order to determine the percentage of agreement among raters, a degree of consistency was obtained while analyzing each interview. Two raters analyzed the interviews belonging to SA12 and two raters analyzed the interview belonging to student SC22. The interview transcript belonging to student SC3 was only analyzed by the author of this thesis. Associating the information obtained from the different raters, there was a 75% accord among the opinions for student A12 and a 50% accord among the opinion for student A22 with respect to the thinking modes. This calculation was obtained by gathering the number of categories created by the two raters, their appropriate descriptions, and the number of matches among categories. The end result given the percentage of categories created by the second rater that coincide with those of the author of this thesis. The percentage of agreement of student C3 between the author of this thesis and the additional rater was 100 %.

In respect to the analysis of metaphors and metonymies, for student A12 there was a 28% accord between the raters, for student A2 there was a 32%, and finally for student C3 the accord was 52.72% of the time. These percentages are considerably low, but since the analysis of these aspects is subjective, they are significant enough for our purpose, which was to document the cognitive constructs present.

3.4 Instruments

Table 3.4 below shows the list of available assignments instructors had access for the modular and non-modular sections taken from Zamora(2010).

Table 3.4. Data Collected by section.

Section A	Section B	Section C
Pre-survey	Pre-survey	Pre-survey
Quiz 1	HW1-System Module	Test 1
Quiz 2	Test 1	Test 2
Quiz 3	HW2- Matrix Product	Test 3
HW1-System Module	HW3- Vectors	HW Linear Independence
HW2- Matrix Product	Test 2	Post-Survey
Test 1	Test 3	3 Class Observations
HW3- Vectors	HW Linear Independence	
HW4- Linear Independence	Post-Survey	
Test 2	3 Class Observations	
HW Linear Independence		
Post-Survey		
4 Class Observations		

It is worth mentioning that homework on linear independence (HW Linear Independence) was assigned in the three sections and that the use of computer modules was not required. This homework was given in efforts to tryout algebraic and numerical representations of vectors and matrices and to conjecture on the essential conditions for linear independence on variety sets of vectors in \mathbb{R}^3 . The use of the online modules was not required as part of this assignment. This thesis only used information from the pre-survey, post-survey, and the analysis of the interview

transcripts that focused on the use of metonymies and metaphors and the presence of different thinking modes given in response to the questions asked to students.

At the end of the Spring 2009 semester a total of 16 students volunteered to be interviewed. The list of students was divided according to the section of the course that each student was attending and a student was chosen at random from each section. The interviews that were analyzed were chosen at random from each section and included in this thesis correspond to student code names SA21, SA22, and SC23. Between 60 and 120 minutes took for an average of seven questions to be answered by students during interviews. The list of questions used while conducting the interview is presented below and the number of questions answered by students varied (obtained from Zamora, 2010). Changes in wording were sometimes made to the questions in efforts to collect as much as possible a student's perspective. A copy of the actual page used by the researcher during the interview is attached as an appendix.

1. Provide a definition of linear independence.
2. Provide an example of a linearly dependent set of vectors.
3. Given the set $\{u_1, u_2, u_3, u_4\}$ where vectors u_1, u_2, u_3 are on the same plane and u_4 is not, determine if the set is linearly dependent.
4. Given a linearly independent set $\{u_1, u_2, u_3, u_4\}$ in \mathbb{R}^n . Prove/Disprove that the set $\{u_1, u_2 + 5u_1, u_3, u_4\}$ is linearly independent (i.e. if it is LI, prove it is, if it is not linearly independent, explain why it is not linearly independent).
5. Given an $n \times m$ matrix A where $a_{i2} = a_{i4} + a_{i5}$, for all $0 \leq i \leq n$. Determine if the set $\{A_1, A_2, A_3, \dots, A_m\}$ (here A_j is the j th column of A) is linearly independent.
6. Given a singular 3×3 matrix A , determine if the vectors of the set $\{A_1, A_2, A_3\}$, (where A_j is the j th column of A) are all on the same plane. Explain your answer.

7. Given that the vector equation has infinitely many solutions, determine if vectors ... are on the same plane. Explain your answer.
8. Given the vector equation $a_1u_1 + a_2u_2 + a_3u_3 = 0$ with a solution $a_1=1$, $a_2=-2$ and $a_3=0$, determine linear independence of the set of u_1 , u_2 , u_3 .
9. Given that $\dim(\text{Span}(u,v,w))=1$, determine the linear independence of the set.

The actual document used in the interviews can be located as an appendix.

3.5 Online Modules Available

The modular section matrix algebra course instructors during the spring 2009 semester had available a list of online tools to be used during their lectures. Students from these modular sections were required to complete the following guided question assignments (Zamora, 2010):

1. Linear Systems Module (Equivalent Systems and Solution Sets Activity): Tool developed using Geometer's Sketch Pad (GSP) program. Provides a graphical representation of linear functions and solution sets in \mathbb{R}^2 .
2. Matrix Operations module (Matrix Product Activity): Tool developed using Geometer's Sketch Pad (GSP) program. Provides a graphical representation of matrices in \mathbb{R}^2 .
3. Vector Spaces module (Linear Combination Activity and Linear Independence Investigation): This tool was developed using Mathematica and provides a graphical representation of vectors spaces in \mathbb{R}^3 .
4. Linear Transformations Modules: The tools included in this section were developed using GSP and Mathematica programs and their main goal was to provide a graphical representation of vector spaces and linear transformations in \mathbb{R}^2 and \mathbb{R}^3 .

Each assignment was to be answered by students accessing the online modules. Copies of assignment questions used by modular section are attached as an appendix.

Chapter 4

Results

Responses to the set of questions given by the students were analyzed following a constant comparison method (Glaser, 1992), a qualitative methodology, applied. The emphasis was on the presence of the different thinking modes, metonymies, and metaphors discovered on the interview response questions given by students related to linear independence, span, and spanning sets.

4.1 Qualitative Analysis

For the purpose of this thesis three interviews were randomly chosen each enclosed the responses of a student from each section of the Matrix Algebra class during the Spring 2009 semester. All three of the interviews were conducted individually and video recorded. Each video was transcribed, and then the transcripts were independently analyzed by the author of this thesis, his advisor, and an additional graduate student. The analysis of the student responses focused on the identification and classification of cognitive constructs –thinking modes, metaphors, and metonymies.

The Grounded Theory introduced by Glaser and Strauss (1967) was referenced to produce the outcomes in this chapter. The theory's framework lead the raters to identify the thinking modes, and categorize identified key ideas present in the responses given by three students. Codes were then created in order to classify their cognitive constructs. Codes having similar constructs were merged into one category. Once created, they were clustered for further analysis (Glaser and Strauss, 1967). Categories and codes were independently created for each student.

A table of category descriptions and examples was made for all thinking modes, metaphors, and metonymies identified. The frequency of each category description was recorded separately to address the research questions.

4.2 Classification of Responses

After the individual categories were created, thinking modes were collected using quotations from the transcript as a representative of each category. Once the categories were formed by the author of this thesis, two independent researchers performed the same analysis. The discoveries of the independent analyses were then compared to the discoveries of the author of this thesis to establish credibility. The following report the percentage for reliability numbers from student A12 taken from one of the independent researchers: the Synthetic-Geometric mode had a frequency of 1 (used 5% of the time), the Analytic-Arithmetic mode had a frequency of 17 (used 85% of the time), and finally the Analytic-Structural mode had a frequency of 8 (used 40% of the time).

The following paragraphs exemplify the results found while analyzing the data obtained from the interview transcripts of the three volunteered students. The following example of a student's argument explains what aspects were taken into account and how it was categorized:

Student A12: Because in R^2 you only need two vectors to form, two linearly independent vectors to form any vector inside of R^2 so then these vectors automatically become combination of the first two, and then the same thing for R^3 but with 3 vectors and the fourth one would have to be a linear combination of the first three, if they are linearly independent.

Student A12 was focusing on the number of vectors, but instead of relying that they are scalar multiples of each other, student chose to point out the number of vectors in R^2 to determine that all vectors were linear combinations of each other. The category created for this

argument was LC –Linear Combinations Verbally - and because of the use of an arithmetic structure and the lack of definitions and/or numerical computations LC was classified into the Analytic-Arithmetic thinking mode. Another category this response belonged to is the category “number of vectors compared to dimension of R^n .”

4.2.1 Modes of Thinking

The analysis of the interview transcripts were derived from the description of thinking modes recognized by Sierpiska (2000). This category was explained in section 2.4 of chapter 2. See chapter 2 of the thesis for detailed discussion on the particular framework. The following paragraphs depict the thinking modes found in the responses of students A12, A22, and C3, and the frequency count of each category.

4.2.1.1 Student A12

Students A12 belonged to Section A of the matrix algebra course. Student was interviewed by a graduate student in the spring 2009 semester and answered a set of 9 questions. Sample responses for each category given by student A12 are reported in table 4.1 below. Table 4.1 encapsulates the thinking modes identified by the author of this thesis; each category (thinking mode) has its own code, a description, and an example taken from the transcript of Student A’s interview.

Table 4.1. Categories for responses corresponding to student A12.

Student A12			
<u>Code</u>	<u>Category</u>	<u>Description</u>	<u>Example</u>
CLC	computed linear combinations	Mention /computation of linear combinations	SA12: Because, if you multiply this by a constant you are going to have non

			zero numbers/values and then this would have to come up with the opposite non zero values and because that because is not possible they are linearly independent. The only possible solution is the zero.
①IF ②ST ③IV ④HE	①identity form ②solution type ③independent variable ④homogeneous equation	①Mention/ statement of Identity form ②Mention /statement of solution type ③Use of independent variable ④Mention and the use of equations being homogeneous	SA12: To show that they are linearly independent it shouldn't be equal to the identity matrix because if it is equal to the identity matrix then this term, first term can be rewritten in terms of the other vectors and similarly here but here we have that you can, because it is zero, zero you can have and because this is a homogeneous equation, and you can have non-trivial solutions meaning that, this variable is independent.
①IF ②IC ③LC	①identity form ②independent columns ③Verbal linear combination	①Mention/ statement of Identity form ②Use of independent variable ③Verbal statement of linear combinations, no computation included	SA12: Yes. Because when you form the identity matrix it will only take up these three slots and then this slot has to be filled with either zeros or ones or maybe not ones but numbers because those you know that,

			that values independent and umm as soon as you have a value that is independent that the other values can be written in terms of, the set becomes linearly dependent.
①LC ②NV	①linear combination ②number of vectors compared to n in R_n	①Statement of linear combinations ②Mention/statement of objects in R_n , and comparison to its dimension.	SA12: Because in R_2 you only need two vectors to form, two linearly independent vectors to form any vector inside of R_2 so then these vectors automatically become combination of the first two, and then the same thing for R_3 but with 3 vectors and the fourth one would have to be a linear combination of the first three, if they are linearly independent.
①HE ②ST	①homogeneous equation ②solution type	①Mention/ statement of homogeneous equation ②Mention /statement of solution type	SA12: Because the trivial solution is always a solution to a homogenous equation.
LC	Verbal linear combination	Verbal statement of linear combinations computed	SA12: ...if you have three vectors and they are scalar multiples of each other, then it is linearly dependent because you can re-write two of the vectors in terms of one of the others.
①IF	①identity form	①Mention/ statement of	SA12: If only one here

②ST ③RRE	②solution type ③RRE	Identity form ②Mention /statement of solution type ③Use of Gauss row reducing operations	is the trivial solution? Then if you were to put this entire thing in a umm matrix and perform Gaussian row operations then you would come up, yes you would come up with umm the identity matrix.
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The categories stated in table 4.1 are based on Sierpinska's ideas, shown in section 2.4, and can be separated into three different thinking modes recognized by Sierpinska (2000) as the ways linear algebra students think while solving problems and answering questions. The thinking modes identified by Sierpinska (2000) are Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural.

We can deduce that the categories taken from the data provided in table 2.1, overlooking exact descriptions based on the numerical and algebraic representation of objects. The categories in which objects are analyzed with the use of theorems and definitions are classified as part of the Analytic-Structural thinking modes and explanation on graphical representations can be classified into the Synthetic-Geometric thinking mode. Categories mentioned provide ideas that would allow each to be classified into more than one thinking mode. That is, categories are not mutually exclusive (Sierpinska, 2000). Students use a combination of reasoning's to provide responses.

Table 4.1 categories belonging to student A12 can be classified as follows: only one category, NV can be classified as a Synthetic-Geometric thinking mode. For this category, student A12 referred to vectors and the dimension of the space these vectors located, whose

graphs are provided to determine dependence among the vectors in the set. A total of three categories that fit into the Analytic- Structural thinking mode: IF, IV, and HE. Within these categories the use of key theorems, and definitions was present. Finally, student A12 referred to numerical representations and manipulated numerical computations to determine linear dependence or independence forming a total of five categories which are classified into the Analytic-Arithmetic thinking mode. These categories are labeled as ST, IC, LC, CLC, and RRE. The categories, IF and ST, can be classified as Analytic-Arithmetic and Analytic-Structural since student A12's reasoning included factors of both cases to determine outcomes. A total of nine different categories were created for student A12. During the interview, a total of 15 responses included in these categories were identified.

Table 4.2. Frequency of category use for student A12.

Student A12			
	<i>Category</i>	<i>Description</i>	<i>Frequency</i>
1	Identity form IF	Mention/ statement Identity form	3
2	Solution type ST	Mention /statement solution type	3
3	Independent variable IV	Mention/Use of independent variable	1
4	Homogeneous equation HE	Mention/use of equations being homogeneous	2
5	Independent columns IC	Use or mention of independent columns	1
6	Linear combinations VLC	Verbal statement of linear combinations computed	2
7	Number of vectors in R^n NV	Mention/statement of objects, and comparison to n in R^n	1
8	Row reduce equation RRE	Mention/Use of Gauss-Jordan processes	1
9	Computed linear combinations CLC	Statement of computed linear combinations	1

As table 4.2 depicts, the categories most frequently used by student A12 are IF and ST (frequency of three). Each thinking mode used by student A12 can be distributed as follows:

- Synthetic-Geometric: NV (1), with this being the only category in this graphical mode.
- Analytic-Structure: IF (3), IV (1), HE (1), and ST (3). A total of four categories with a combined frequency of eight.
- Analytic-Arithmetic: ST (3), IC (1), LC (2), RRE (1), CLC (1), and IF (3). A total of six categories with a combined frequency of 11.

Based on the data gathered the thinking mode most frequently used by student A12 was Analytic-Arithmetic, and the least used was Geometric-Synthetic.

4.2.1.2 Student A22

Student A22 belonged to Section A of the matrix algebra course during the spring 2009 semester. Student was chosen from same section as the previous student. Table 4.3 includes the thinking modes identified by the author of this thesis and an example taken from the transcript of the student interview.

Table 4.3. Categories for responses corresponding to student A22.

Student A22			
<u>Code</u>	<u>Category</u>	<u>Description</u>	<u>Example</u>
DET	Determinate	Mention of determinate	SA22: And then the determinate would be the multiplication of A times D plus B times C. So then that if that equals to zero it makes it linear independent.
ST	Solution type	Mention/ statement of solution type	I2: okay, so on this one when you have the zeros how did it imply this is linearly independent? SA22: because

			according to the definition from the book it said that X_1 , X_2 , X_3 are all equal to zero
①RRE ②IF	①RRE ②identity form	①Mention/ statement of Gauss Jordan Elimination processes. ②Use/ focus/ mention of identity form	SA22: I am going to enter the matrix and find the row reduce echelon form. SA22: for this one it gives when you do the row reduce (student writes matrix) it gives me 1 0 -1, 0 1 2, so then for me this means that it is dependent.
①RRE ②IF	①RRE ②identity form	①Mention/ state of Gauss Jordan Elimination Processes. ②Use/ focus/ mention of identity form	SA22: so I'm going to do the row reduced echelon form again. And it gives me the identity I2: okay, and it says SA22: that is independent
①PO ②Plane	①Parallel objects ②Plane	①Mention/ statement of parallel objects. ②Use of word plane/ description of plane	SA22: so like if there parallel that will never intersect SA22: I mean because of the plane or this plane over here and this plane over here contains these three maybe they can't intersect but here it is intersecting
VE	vector equation	Mention/ use of various aspects of vector equation	SA22: A hum (student is writing a set of empty brackets again and labeling them U_1 , U_2 , and U_3 on top of them,

			thinking out loud) so when you add to the matrix it looks like this and then you have $U_1 U_2 U_3$ and you multiply them and you will get this whole thing SA22: and I understand the vector equation is always equal to zero
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As before the categories stated in table 4.3 are based on Sierpinska's (2002) ideas, and are separated into three different thinking modes. The categories from table 4.3 belonging to student A22 can be classified as follows: two categories, PO and Plane can be classified as a Synthetic–Geometric thinking mode. For this category, student A22 made reference of geometrical representations. A total of two categories fit into the Analytic- Structural thinking mode: IF and VE. With these categories the use of key theorems and definitions was present. Finally, student A22 referred to numerical representations, and manipulated numerical computations to determine linear dependence or independence forming a total of three categories, which are classified into the Analytic-Arithmetic thinking mode. They are ST, DET, and RRE. The categories, IF and RRE, can be classified as Analytic-Arithmetic and Analytic-Structural since student A22 reasoning included factors of both cases to determine outcomes. A total of seven different categories were created for student A22. During the interview, a total of nine responses were identified under these categories.

Table 4.4. Frequency of category use for student A22.

Student A22			
	<i>Category</i>	<i>Description</i>	<i>Frequency</i>
1	Identity form IF	Mention/ statement of identity form	2
2	Row reduce equation RRE	Use of Gauss-Jordan Elimination Process	2
3	Solution type ST	Mention/statement of solution type	1
4	Parallel object PO	Mention/ statement of parallel objects	1
5	Plane	Use of word plane/ description of plane	1
6	Determinate DET	Mention of determinate	1
7	Vector equation VE	Mention/ use of various aspects of vector equation	1

Table 4.4 depicts the categories most frequently used by student A22 as IF and RRE (frequency of two). Each thinking mode used by student A12 can be distributed as follows:

- Synthetic-Geometric: PO (1) and Plane (1), two categories with a combined frequency of two.
- Analytic-Structure: IF (2), VE (1), and RRE (2). A total of three categories with a combined frequency of five.
- Analytic-Arithmetic: ST (1), DET (1), RRE (2), and IF (2). A total of four categories with a combined frequency of six.

Based on the data gathered the thinking mode most frequently used by student A22 was Analytic-Arithmetic, and the least used was Geometric-Synthetic. Student A22 appeared to be able to equally use more than one thinking mode in repeated occasions.

4.2.1.3 Student C3

Student C3 belonged to Section C of the matrix algebra course, the only non-modular section during the spring 2009 semester. Student C3 participated in a one to one interview, and

answered to questions 1, 2, 3, 4, 5, 6, 7, and 8 (see appendix for questions). Table 4.5 contains categories taken from the transcript to represent the thinking modes used by student C3 during the one on one interview.

Table 4.5. Categories for responses corresponding to student C3.

ST	Mention /statement of solution type	SC3: alphas are equal to zero that it's linearly independence but if they're not let's say this a one, and that is a negative one, and that it gives you non trivial answer zero so that means is linear dependent
SM	Mention /statement of singular or non-singular matrices	SC3: yes ma'am the definition had said if it's linear independent if and only if it's non-singular.
① IF ② RRE	① Mention/ statement of Identity form ② Mention/ statement of Gauss-Jordan elimination process.	SC3: linearly dependent. I guess if we would reduce to had like (student is writing a matrix) one, zero, one these would all have to be zeros (student filling matrix with one and zeros)
① ZV ② RRE	① Focus on obtaining zero vector ② Mention/ statement of Gauss-Jordan elimination process	SC3: the vector of dependence, if it was independent this would be a zero as well because it would completely row reduce row Echelon form where it's not as before it was dependent because we had X1 you would say that this would be X1, X2, X3 so X3 is equal to four I guess... (Student labels columns x1, x2 and x3 to previously drawn matrix)
① IF ② RRE ③ FV	① Mention/ statement of Identity form ② Use of Gauss-Jordan Elimination Processes ③ Mention/ statement of free variable	SC3: the vector of dependence, if it was independent this would be a zero as well because it would completely row reduce row Echelon form where it's not as before it was dependent because we had X1 you would say that this would be X1, X2, X3 so X3 is equal to four I guess. So then you would say X3 not equal to four but SC3: so as a free variable you can set X3 equals to T, so then that would make X1 equal to negative 4T, and X2 which is equal to zero I guess

① IF ② FV	① Mention/ statement of Identity form ② Mention/ statement of free variable	SC3: because this could be let's say four cause then this is a free variable, free variable and because of this it would be dependent on that, whatever that was
FV	Mention/ statement of free variable	SC3: and then it just depends on the T because it is a free variable so that is linearly dependent.
PO	Mention/ statement of parallel objects.	I3: okay so these are two vectors and you have, there is no intersections so what would that say about the vectors you had initially here SC3: that they won't ever intersect SC3: That they are parallel
① IF ② RRE	① Mention/ statement of Identity form ② Mention/ statement of Gauss-Jordan Elimination process.	SC3: okay let's just go (student is writing new matrix thinking out loud) well this can be because it can reduce completely if this goes, if this row what I understand, if this row is 0 and this goes I guess if turns out to be the answer is one zero zero (student writes new identity matrix) this last row zero zero It can be linearly independent
PO	Mention/ statement of parallel objects.	SC3: yes these vectors ... I3: okay so over here, when you said, if it is like this that means they are not parallel that is why they are linearly SC3: linearly dependent
AT	Mention/ statement of angle between vectors	SC3: then also because they are acute angles and acute angles could also possibly be linearly independent...
① ST ② LC ③ ZV	① Mention/ statement of solution type ② Verbal mention of linear combinations ③ Focus on obtaining zero vector	SC3: well in order to get the zero vector then you can, you would have the alphas would have to be something different than zero because then it will give you a nontrivial answer
FV	Mention/ statement of free variable	SC3: when I hear, right when I hear okay when the say the matrix is dependent then right away I think there's a free variables. There will have to be a free variables

<p>① IF</p> <p>② RRE</p> <p>③ SM</p> <p>④ ST</p>	<p>① Mention/ statement of Identity form</p> <p>② Use of Gauss-Jordan Elimination Process</p> <p>③ Mention/ statement of scalar multiple</p> <p>④ Mention/ statement of solution type</p>	<p>SC3: yes I believe they did because I, because the way I saw it if all these are independent then all of them can reduce so then, so then basically like any two together will reduce. These two will reduce or these two will give you independence and these two like that so then since U2 since 5U1 is just a scalar multiple of this and U2 will still be the same thing I guess if you add them it would still give you linear independent solution. So then these two are still linear independent I will say that U we will just make it up and say that is U6 I guess. So then U1 U6 U3 and U4 will then also give you a solution that is linearly independent.</p>
<p>① IF</p> <p>② RRE</p> <p>③ CV</p>	<p>① Mention/ statement of Identity form</p> <p>② Use of Gauss-Jordan Elimination Process</p> <p>③ Use of component of vectors</p>	<p>SC3: well the matrix I guess, the vectors is what they give you like anything so U1 is equal to like (student sets up U1 equal to a column xyx) $X Y Z$ it doesn't really matter and then U2 is equal to something else and U3 and then you just put into the matrix how we have been doing and you row reduce them and then its 1111 diagonal and all zeros then it's going to be independent</p>
<p>① IMS</p> <p>② EQN</p>	<p>① Mention/statement of infinitely many solutions</p> <p>② Mention/ use of equation</p>	<p>SC3: yes because basically what I get from that equation is that it's just saying that those vectors since they had infinitely many solutions that they are dependent on each other</p>
IF	Mention/ statement of Identity form	<p>SC3: give you the columns but then I guess I go back to the conclusion that since A1 A2 and A3 aren't all zero and since A1 is one and A2 is negative two then it would be dependent</p>
ZV	Focus on obtaining zero vector	<p>SC3: and then in order to get the zero vector they would have to depend on each other because but A1 makes this something so the negative two plus this will get back to zero and then zero times this will also give you zero which will give you the zero vector</p>

<p>① IF</p> <p>② CLC</p> <p>③ ZV</p>	<p>① Mention/ statement of Identity form</p> <p>② Statement of computed linear combinations.</p> <p>③ Focus on obtaining zero vector</p>	<p>SC3: where as if they were independent then this A1 A2 and A3 would always be zero cause then zero times this no matter what you get is zero and then regardless of what these two are. This is going to be this and then and let's say we combined these two and leave this out. Zero times this is still going to give you zero so the zero vector plus zero vectors going to give you the zero vector.</p>
<p>① CLC</p> <p>② ZV</p>	<p>① Statement of computed linear combinations.</p> <p>② Focus on obtaining zero vector</p>	<p>SC3: because these two added together will have to equal the zero vector so A1 U1 plus A2 U2 have together equal the zero vector because this since it already has a zero this automatically is going to go to zero but these two will have to depend because if you just have A1 U2 then this would give you this vector plus zero vector which would not give you the zero vector because this would be let's say U1 was two two two two and then U3 is zero well it does not matter what U3 is because then A3 time zero will give you a zero vector so then it would be zero zero zero and then that would give you this as your final answer which is not the zero vector so this is not equal to that</p>
<p>① CLC</p> <p>② ZV</p> <p>③ IV</p>	<p>① Statement of computed linear combinations .</p> <p>② Focus on obtaining zero vector</p> <p>③ Use of characteristics of independent vectors.</p>	<p>SC3: because it gives you values for these two already and it's already taken out the third vector out of it because it's saying its already zero so it doesn't matter what it because that's going to be zero vector so this plus this has to give you that so it's saying where zero and zero its not independent of each other so it will give you in A1 and A2 to make it where it depend on each other to reduce to a zero vector</p>
<p>① ZV</p> <p>② CLC</p>	<p>① Focus on obtaining zero vector</p> <p>② Statement of computed linear combinations.</p>	<p>SC3: okay so U1 plus U3 equals the zero vector it would be linearly independent because zero and U1 plus U3 gives you zero</p>

Once again after analyzing student C3, Sierpinski's (2000) description of thinking modes were correlated to responses. These correlations are summarized in table 4.5, and can be classified as follows: three categories, CV, PO, and AT as a Synthetic–Geometric thinking mode. These categories were identified when student C3 was making reference to geometrical representations. There were a total of six categories that fit into the Analytic-Structure thinking mode: IF, IV, ZV, FV, IMS, and EQN. With these categories student C3 made use of definitions and key theorems that led each into these categories. Lastly, the Analytic-Arithmetic thinking mode was identified when referring to numerical representations and performing numerical computations while trying to provide answers given by student C3; ST, SM, CLC, VLC, RRE, IF, ZV, and FV. Five categories, were classified as Analytic-Arithmetic and Analytic-Structural since student C3's reasoning included aspects of both modes. IF and ZV were the two categories most frequently used during this student's interview. There were a total of 15 different categories identified, and created for student C3 during the interview. Table 4.6 contains the frequency of each category used by student C3.

Table 4.6. Frequency of category use for student C3.

Student C3			
	<i>Category</i>	<i>Description</i>	<i>Frequency</i>
1	Identity form IF	Mention/ statement of Identity form	7
2	Zero Vector ZV	Focus on obtaining zero vector	7
3	Row reduce equation RRE	Use of Gauss-Jordan Elimination Process.	6
4	Computed linear combinations CLC	Statement of computed linear combinations.	4
5	Solution type ST	Mention /statement of solution type	3
6	Free variable FV	Mention/ statement of free variable	3
7	Parallel objects PO	Mention/ statement of parallel objects.	2
8	Independent vectors IV	Use of characteristics of independent vectors	1
9	Components of vectors CV	Mention/use of equations being homogeneous	1

10	Singular matrix SGM	Mention /statement of singular or non-singular matrices.	1
11	Verbal linear combination VLC	Verbal Mention of linear combinations.	1
12	Scalar multiple SM	Mention/ statement of scalar multiple.	1
13	Angle types AT	Mention/ statement of angle between vectors.	1
14	Infinitely many solutions IMS	Mention/statement of infinitely many solutions.	1
15	Equation EQN	Mention/ use of equation	1

Table 4.6 demonstrates the categories most frequently used by student C3, which are IF (frequency seven) and ZV (frequency of seven). The frequency of each thinking mode used by student C3 can be distributed as follows:

- Synthetic-Geometric: CV (1), PO (2), and AT (1). A total of 3 categories with a combined frequency of four.
- Analytic-Structural: IF (7), IV (1), ZV (7), FV (3), IMS (1), EQN (1), RRE (6), ST (3), VLC (1), CLC (4), SGM(1) and SM(1). A total of 12 categories with a combined frequency of 36.
- Analytic-Arithmetic: ST (3), CLC (4), VLC (1), RRE (6), SM (1), IF (7), ZV (7), and FV (3). A total of nine categories with a combined frequency of 32.

In contrast to the other two students, it appears that the thinking mode most frequently used by student C3 was Analytic-Structural. Student C3 appeared to be able to use more than one thinking mode in frequent instances to provide answers.

4.2.2 Metonymy and Metaphor

The analysis of the responses belonging to students A12, A22, and C3 were also conducted in order to find the metonymies and metaphors applied during their interviews. The

identification of these metonymies and metaphors are familiar to the one reported by Presmeg (1998), and are explained in section 2.5 of chapter 2.

4.2.2.1 Student A12

The author of this thesis, based on the transcript analysis, discovered a total of 22 metonymies used by student A12. Table 4.7 summarizes the metonymies used and the part of the transcript where they were found.

Table 4.7. Metonymies and Metaphors displayed by student A12 (in order of appearance).

<i>Sample Response</i>		<i>Metonymy</i>	<i>Description</i>
1	when you have linear combination vectors and the only solution is the trivial solution meaning that all coefficients are zero	trivial solution → linear independence	When stating coefficients are zero relating to trivial solution
2	SA12: but, if it is only one unique solution and you know that solution has to be the trivial solution, then the set has to be independent because you can write the vectors as a combination of the others.	① trivial solution → linear independence ② verbally stating vector combinations → dependency	① trivial solution associated to linear independence ② When stating verbally linear combinations and relating this to dependency.
3	SA12: Cause then, umm because it's a homogeneous set of equations SA12: If it's infinitely many than it has to be dependent SA12: The set of vectors	homogeneous system → infinitely many solutions → linear independence	When relating homogeneous systems to infinitely many solutions to dependency .
4	SA12: Because, if you multiply this by a constant you are going to have non zero numbers/values and then this would have to come up with the opposite non zero values and because that because is not possible they are linearly independent.	① linear combination computed → solution type ② trivial solution → linear independence	① When stating arithmetic operations (multiply, add/subtract) as linear combinations to attain solution type ② When stating or relating trivial solution to independency

5	SA12: To show that they are linearly independent it shouldn't be equal to the identity matrix because if it is equal to the identity matrix then this term, first term can be rewritten in terms of the other vectors and similarly here but here we have that you can, because it is zero, zero you can have and because this is a homogeneous equation, and you can have non-trivial solutions meaning that, this variable is independent.	① identity matrix \rightarrow linear independence ② identity matrix \rightarrow verbally stating vector combinations ③ homogeneous equation \rightarrow non trivial solution \rightarrow independent variable	① When stating or relating identity matrix to independency ② When comparing identity matrix to linear combinations ③ Comparing homogeneous to trivial solution and next to solution type to variable
6	SA12: It is just umm, basically I usually think of the first values in x and a2 as y.	a1 & a2 \rightarrow variables \rightarrow x & y	to replace every unknown with x and y
7	Because if one is umm scalar multiple of the other, then they'll be parallel and have the same slope or if they lie on the same plane then you can write one of them as a linear combination of the other two	① scalar multiple \rightarrow parallel \rightarrow having same slope ② on same plane \rightarrow linear combination of each other	① Relating scalar multiple to being parallel and having same slope ② Relating having vectors on same plane to linear combinations
8	I1: OK what if you have 4 vectors that are from R3? SA12: If I were to plot them?.. SA12: Where the first value would be the x, the second value would be the Y and third one would be the z. umm it's easier to see linear combinations. Because if one is umm scalar multiple of the other, then they'll be parallel and have the same slope or if they lie on the same plane then you can write one of them as a linear combination of the other two but umm	① scalar multiple \rightarrow parallel \rightarrow having same slope ② on same plane \rightarrow linear combination of each other ③ x \rightarrow value of first entry ④ y \rightarrow value of second entry ⑤ z \rightarrow value of third entry	① Relating scalar multiple to being parallel, and next having same slope ② Relating having vectors on same plane to linear combinations ③ ④ ⑤ to replace components of vectors with x, y and z

9	SA12: Then it depends if you u_1 and u_2 are linear combination if these two aren't scalar multiples of each other then their linearly independent because you can't write one in terms of the other and then because u_4 is not in the same plane of these two it expands the uh, it expands the scope of the linear combinations from just a plane to the entire R^3 .	① not on plane \rightarrow linear combinations in a space ② scalar multiple \rightarrow linear combinations \rightarrow linear dependence	① Relating vectors being on a plane to having linear combinations ② Relating scalar multiples to linear combinations and next to linear dependence
10	SA12: Because in R^2 you only need two vectors to form, two linearly independent vectors to form any vector inside of R^2 so then these vectors automatically become combination of the first two, and then the same thing for R^3 but with 3 vectors and the fourth one would have to be a linear combination of the first three, if they are linearly independent.	① $R^n \rightarrow$ source has n linearly independent vectors ② spanning set \rightarrow source for other vectors	① Relating dimension of vector space to the number of vectors ② Stating combinations building up to new sets.
11	SA12: Sorry, because x , y , and z have both non-trivial and trivial solutions, the vector equation is linearly, oh no I take that back the set u , v , w , is linearly dependent meaning that, and the only way if for that set of u , v , w , to be linearly dependent is if the vectors lie on the same plane.	linear dependence \rightarrow set of vectors, on same plane	Stating linear dependence because vectors are on same plane
12	SA12: ...if you have three vectors and they are scalar multiples of each other, then it is linearly dependent because you can re-write two of the vectors in terms of one of the others.	verbally stating vector combinations \rightarrow linear dependence	When stating or relating linear combination to dependency

Table 4.7 shows that the majority of metonymies used by student A12 pertained to students reasoning of linear independence and linear dependence. Student A12 had made reason

of referring to linear combination, and obtaining the trivial solution to imply linear independence. The excerpt below (obtained from the original transcript of the interview) indicates the student's reasoning based on linear combinations.

SA12: Umm, basically just when you have linear combination vectors and the only solution is the trivial solution meaning that all coefficients are zero and so that is the only way to produce the zero vector.

I1: OK, ok so when you say linear combinations of the vectors on the solution what do you refer to when you say only solution?

SA12: Oh, umm the vector equation a_1, v_1 to plus a_2, v_2 , plus etc. until a_n, v_n equals the zero vector.

I1: Oh, ok, so does it always equal to zero vector when its...

SA12: No but umm when looking at linear independence it is easier if you set it equal to the zero vector...

I1: U hum

SA12: Cause then, umm because it's a homogeneous set of equations

I1: U hum

SA12: You don't need to worry about the constants and coming up with no solutions you just need to worry about if its one unique solution or infinitely many

I1: U hum

SA12: If it's infinitely many than it has to be dependent

I1: What has to be dependent?

SA12: The set of vectors

I1: OK

SA12: because then you can write one of the vectors in that set as a combination a linear combination of the other vectors.

I1: Ok

SA12: but, if it is only one unique solution and you know that solution has to be the trivial solution, then the set has to be independent because you can write the vectors as a combination of the others.

I1: Oh I see, ok and can you elaborate on why you are saying if it is a trivial you cannot write it

SA12: because...

II: U hum, Go ahead

SA12: mmm...

II: You can give me an example if you wish to, because you said if it is the trivial solution than you cannot write any vectors as a linear combination of the others is that what you said?

SA12: Right, because we have like the vectors umm.

II: interesting you are left handed too, Andrew was left handed.. wow.

SA12: Vectors umm

(Student explains and writes $u_1 = [1\ 1]$ $u_2 = [2\ 2]$ $u_2 = 2u_1$ $a_1u_1 + a_2(u_2) = 0$)

u equal 1 and u_1 and u_2 equal 2 2 then umm you can write u_2 in terms of u_1 , where u_2 equals 2 times u_1 and so that would be linear combination and so when you write the vector equation $u_1 v_1$ plus $u_2 v_2$ equals, ok let me do that ...

II: That is ok.

SA12: equals zero, you can take this portion and you can rewrite it as plus $a_2 2u_1$ equals zero

II: so that would equal to zero, is that would it is

SA12: Right, well in the sense, and then from there you could come up with umm other possible solutions,

II: Oh I see. umm

SA12: just the fact that you can rewrite it

II: What if the set was linearly independent?

SA12: If it was linearly independent.

II: So how did you tie the trivial solution to the vector equation to the linear independence?

SA12: Umm for linear independence is the fact that you have

(student is writing $0[1;1] + 0[1;3] = [0;0]$) the only way these two can equal the zero vector is if you multiply this times zero and this times zero.

II: oh I see. Ok

SA12: Because, if you multiply this by a constant you are going to have non zero numbers/values and then this would have to come up with the opposite non zero values and because that because is not possible they are linearly independent. The only possible solution is the zero.

Student A12 in addition to his back-up reasoning of finding a linear combination, student mentions the Gauss-Jordan Elimination method in an effort to find a linear combination based on the existence of the identity matrix.

SA12: And then perform Gauss Jordan elimination and I can do it should give you ... and because that is not equivalent to the identity matrix, you can see that umm u_1 plus $2u_2$...

Another metonymy clearly identified in table 4.7 was the identity matrix, student A12 used the identity matrix and linear combination arguments interchangeably to arrive at a conclusion.

SA12: To solve each of this in terms of the other, when, wow, I completely blanked. To show that they are linearly independent it shouldn't be equal to the identity matrix because if it is equal to the identity matrix then this term, first term can be rewritten in terms of the other vectors and similarly here but here we have that you can, because it is zero, zero you can have and because this is a homogeneous equation, and you can have non-trivial solutions meaning that, this variable is independent.

Based on the statement above, it is clear that student A12 used the fact the matrix did reduce into the identity matrix to conclude that a linear combination existed and that the set was independent.

4.2.2.2 Student A22

A total of nine metonymies and one metaphor (see numbers 9, 10, 11, and 15) were derived from the transcript analysis of student A22 performed by the author of this thesis. Table 4.8 summarizes the metonymies and the metaphor used, and demonstrates the part of the transcript where they were found.

Table 4.8. Metonymies and Metaphors displayed by student A22 (in order of appearance).

<i>Sample Response</i>		<i>Metonymy</i>	<i>Description</i>
1	SA22: well when it's dependent it has and the matrix is like the identity ...	linear independence → identity form	When stating or relating linear independence to identity matrix
2	SA22: so that is how I get this. But then for when it's independent it doesn't have the identity so then the determinant is zero. So and that is how I connected them.	linear independence → identity form- >determinate zero	When stating or relating linear independence to identity matrix
3	I2: okay. Okay so when is it dependent? SA22: when it has the identity.	Identity form ->linear independence	When stating or relating identity matrix to dependency
4	SA22: and when it's the identity and then it is independent SA22: Linearly independent	identity form →linear independence	When stating or relating identity matrix to dependency
5	I2: linearly independent. What does that mean to you? When you say this thing is linearly independent. What do you mean? And you saying the matrix is linearly independent, what are you thinking when you say that word linearly independent? SA22: that each vector doesn't depend on one of these either of these two or that this one doesn't this depend on these two or this one of these two (student pointing to vectors drawn by teacher)	linear independence → lack of linear combination relation between vectors	When stating or relating linear combination to dependency

6	SA22: so I am not sure about that but I still think that it's independent I2: because SA22: because there are two different planes going on	not on same plane → linear independence	Stating or referencing on a plane and stating the formation of linear combinations of vectors on the plane
7	I2: they are parallel. So the parallel case implies what? Linearly independent or dependent SA22: independent.	parallel vectors → linear dependence	Connecting parallel to dependency
8	I2: the whole set okay, and U1, U2, U3 are SA22: dependent of each other because they are on the same plane	① on same plane → linear dependence ② dependence (Metaphor)	① Stating or referencing on a plane forming linear combinations ② Metaphor of dependence among object. "dependent of each other regardless of the criteria in which objects are defined dependent"
9	SA22: well this three are on the same plane so this will be like if I gave the example from the previous one. So these three are dependent of each other or they can be. No, they are because they are on the same plane	① on same plane → linear dependence ② dependence (Metaphor)	① Stating or referencing on a plane forming linear combinations ② Metaphor of dependence among object. "dependent of each other regardless of the criteria in which objects are defined dependent "
10	SA22: well if they are on the same plane they can be dependent of each other	① on same plane → linear dependence ② dependence (Metaphor)	① Stating or referencing on a plane forming linear combinations ② Metaphor of dependence among object. "dependent of each other regardless of the criteria in which objects are defined dependent "

11	I2: okay, and did you use the same argument here like that three you are saying are independent because are on the same plane but you are saying the third one is not that is why the set itself becomes linearly SA22: independent	① not on same plane → linear independence ② dependence (Metaphor)	① Stating or referencing on a plane forming linear combinations ② Metaphor of dependence among object. "dependent of each other regardless of the criteria in which objects are defined dependent "
12	SA22: A hum, so this one would be let's see here, (student is doing calculations in calculator) so this one turns out to be (student writes out solution) I am thinking that whenever it has zero on the bottom one it's like a sign for saying that it is independent	Identity form → linear independence	When relating identity form to dependency
13	SA22: like (student is writing on paper a matrix) I am not sure if it's the identity of when it has four vectors but then it would I think but if it saying that it's linearly independent then this set would be turned out to be the identity	linear independence → identity form	When relating linear independence to identity form
14	I2: and this says is this vector right there is this plus 3 times that, would that would that say anything about the linear independence or dependence of this set SA22: because, like this one becomes dependent of these two when because when you do these multiplications and additions it gives you this one	verbally stating linear combinations → linear dependence	When stating or relating linear combination to dependency.
15	SA22: like this one is A2 and A3 and A4 this does the same thing as these three (student continues to use and point to the empty matrices) these three but this vector and this vector isn't doing anything, then the whole set is independent like the whole is matrix independent	① dependence (Metaphor) ② set → Matrix	① Implications of Metaphor of dependence among objects regardless of the criteria in which objects are defined dependent. ② Use of Matrix in place of set

Table 4.8 can conclude that a good percentage of metonymies used by student A22 dealt with reasoning of linear independence and linear dependence. One of the metonymies frequently used by student A22 was the word identity to represent a linearly independent set of vectors. The following script (obtained from the original transcript of the interview) indicates the use of the metonymy as part of the reasoning.

SA22: well when it's dependent it has and the matrix is like the identity and everything is zero.

I2: okay.

SA22: so that is how I get this. But then for when it's independent it doesn't have the identity so then the determinant is zero. So and that is how I connected them.

I2: so let me repeat you said that if it has zero here what did you say, if this.

SA22: this with this.

I2: with this one. Yeah.

SA22: so if it doesn't have zeros here it goes with this.

I2: so if it doesn't have zeros how does it goes with this one.

SA22: well you can you can get the determinant and that's how.

I2: how did you relate to this being zero you said something about relating this being zero.

SA22: if this is zero well this its independent.

I2: what is independent?

SA22: the matrix.

I2: okay. Okay so when is it dependent?

SA22: when it has the identity.

I2: okay, so when it has the identity it's dependent. When this is zero its independent.

SA22: A hum.

Student A22 also appears to believe that vectors on a plane automatically mean dependence.

I2: the whole set okay, and $U1$, $U2$, $U3$ are

SA22: dependent of each other because they are on the same plane

I2: they are on the same, okay, so explain that to me

SA22: well this three are on the same plane so this will be like if I gave the example from the previous one. So these three are dependent of each other or they can be. No, they are because they are on the same plane

I2: okay what does that mean. How do you. What do you mean by, the reasoning they are on the same plane, yes you are saying you are sure now since they are the same plane they are dependent of each other is that what you are saying

SA22: right.

I2: so how do you see that.

SA22: well if they are on the same plane they can be dependent of each other

I2: what part of being on the same plane you think is making them dependent on each other. Are you thinking of that aspect of it

SA22: well they share the same how can, I say the, same plane they are, because of this one where like over here as well

I2: you mean the other one.

SA22: this one right here this $U1$, $U2$, $U3$ so if this one was on another plane then they couldn't be dependent the three of them

I2: okay, is this what you're describing like that (teacher is using markers as vectors to show them 3-dimensionally)

SA22: A hum.

I2: okay so that would be what

SA22: but then well for me these two are on the same plane

I2: oh okay, so what would that make

SA22: umm independent these two are independent of these two

I2: okay, how about themselves

SA22: they are dependent

I2: they are dependent. okay, How about these two?

SA22: these two?

I2: A hum.

SA22: they are still on the same plane

I2: these are on the same plane they are dependent and these two are

SA22: dependent on the same plane as well

I2: what if it's something like that

SA22: then, these two are independent but these are independent and their independent of these two as well because this once on one and this would be aligned and this one too because they are lonely vectors and this one

I2: so overall all four vectors. What is the set? Having all four, linearly independent or dependent?

Student: independent

I2: This case, because

SA22: because of those two that are in the plane

I2: even though you are saying these two are dependent

SA22: A hum

I2: okay, and did you use the same argument here like that three you are saying are independent because are on the same plane but you are saying the third one is not that is why the set itself becomes linearly

SA22: independent

In the script above, student A22 attempted to make sense of the concept of being on the same plane by relying on own metonymies of “being dependent” to arrive to a conclusion.

4.2.2.3 Student C3

A total of 25 metonymies and 5 metaphors (see numbers 7, 21, 27, 35, 38, 40) were derived from the transcript analysis of student C3 done by the author of this thesis. Table 4.9 summarizes the metonymies and the metaphor used and demonstrates the part of the transcript where they were found.

Table 4.9. Metonymies and Metaphors displayed by student C3 (in order of appearance).

<i>Sample Response</i>		<i>Metonymy</i>	<i>Description</i>
1	SC3: alphas are equal to zero that it's linearly independence but if they're not let's say this a one, and that is a negative one, and that it gives you non trivial answer zero so that means is linear dependent	trivial solution → linear independence	When stating coefficients are zero related to trivial solution
2	SC3: linearly dependent. I guess if we would reduce ... because this could be let's say four cause then this is a free variable, free variable and because of this it would be dependent on that	① Free variable → linear independence ② identity form → independence ③ column → variable	① When stating or relating free variable to dependency ② When stating or relating identity form to dependency ③ When relating a column in equation to a variable
3	SC3: the vector of dependence, if it was independent this would be a zero as well because it would completely row reduce (this referring to (student is writing a matrix) one, zero, one these would all have to be zeros (student filling matrix with one and zeros)	① independence → identity form ② dependence (Metaphor)	① When relating dependence to identity form ② Considering a general notion of dependence

4	SC3: and then it just depends on the T because it is a free variable so that is linearly dependent.	Free variable \rightarrow linear dependence	When stating or relating free variable to dependency
5	SC3: because I just see like of this is happen, like this if I have a free variable, then I automatically assume it is dependent	Free variable \rightarrow linear dependence	When stating or relating free variable to dependency
6	SC3: because if you would've completely reduced to 1, 0, 0, 0, 1, 0 (student is writing an identity matrix) all this is zeros then that would mean independence	identity form \rightarrow linear independence	When stating or relating identity form to dependency
7	SC3: just because to say how 1, 2, and 3 are on the same plane but four is not. So I would say that 1, 2, and 3 would dependent as well as four because four would depends on what those other three are	① on same plane \rightarrow linear dependence ② dependence (Metaphor)	① Stating being on same plane to linear dependency ② Considering a general notion of dependence
8	SC3: ...[student is writing new matrix] well this can be because it can reduce completely if this goes, if this row what I understand... this last row zero zero It can be linearly independent	identity form \rightarrow linear independence	When stating or relating identity matrix to independency
9	SC3: ... but if it had something else like here then that would mean let's say this was a six [student alters identity matrix] I guess then it would depend on this one, so that would be another be a free variable just mark this as the free and that would make it is dependent on something	① Identity form \rightarrow free variables \rightarrow linear dependence ② columns \rightarrow vectors/ columns \rightarrow variables ③ dependence (Metaphor)	① Comparing identity to a free variable relating to dependence ② Using columns to refer to variables or vectors ③ Considering a general notion of dependence
10	SC3: well for this, I said you can automatically just assume that since this has three vectors so there are three columns with only two rows then it can't be independent?	more columns than rows \rightarrow linear dependence	based on number of columns and rows
11	SC3: then also because they are acute angles and acute angles could also possibly be linearly independent...	acute angle \rightarrow linear independence	Determining dependency based on angles

12	SC3: no because it's saying ...three vectors are in, how we said R2 ... but since it's in R2 then it cannot be independent it would be dependent	number of vectors more than n in $R^n \rightarrow$ linear dependence	evaluating dependency based on number of vectors in R^n
13	SC3: yes, but it's on a different plane so I guess it really doesn't matter what these are because these are dependent even if u4 is independent since these are dependent so the whole thing is going to be dependent that's the way I came up with my conclusion.	① on plane \rightarrow linear dependence ② dependence (Metaphor)	① Stating being on plane to linear dependency ② Considering a general notion of dependence
14	SC3: when I hear, right when I hear okay when they say the matrix is dependent then right away I think there's a free variable. There will have to be a free variable	① dependent set \rightarrow free variable ② Matrix \rightarrow set	① When stating or relating dependency to free variable ② Using Matrix in place of set of vectors
15	SC3: yes I believe they did because I, because the way I saw it if all these are independent then all of them can reduce so then, so then basically like any two together will reduce. These two will reduce or these two will give you independence and these two like that so then since U2 since 5U1 is just a scalar multiple of this and U2 will still be the same thing I guess if you add them it would still give you linear independent solution.	① linear independence \rightarrow trivial solution ② linear combination \rightarrow solution type	① when relating linear independence to the trivial solution ② Relating linear combinations to a solution type
16	SC3: so it's still basically the same and since U2 itself will also be linearly independent then linear independence plus another linearly independent vector would add up to give you a linear independent vector itself so I guess this plus each other will give you this which is linearly independent	Independence (Metaphor)	Considering a general notion of independence

17	SC3: well the matrix I guess, the vectors is what they give you like anything so U1 is equal to like X Y Z it doesn't really matter and then U2 is equal to something else and U3 and then you just put into the matrix how we have been doing and you row reduce them and then its 1111 diagonal and all zeros then it's going to be independent	identity form \rightarrow linear independence	When relating identity form to linear dependence
18	SC3: yes because basically what I get from that equation is that it's just saying that those vectors since they had infinitely many solutions that they are dependent on each other SC3: since it's infinitely many that means that this aren't always so they dependent on each other which would make them linearly dependent so they would have to be on the same plane	① infinitely many solutions \rightarrow depend on each other \rightarrow linear dependence ② linear dependence \rightarrow on the same plane	① When relating many solutions and depending on each other to dependency ② When relating dependency to being on the same plane
19	SC3: and then in order to get the zero vector they would have to depend on each other because but A1 makes this something so the negative two plus this will get back to zero and then zero times this will also give you zero which will give you the zero vector	depend on each other \rightarrow zero vector	Performing linear combination to obtain zero and relating it to linear dependence
20	SC3: where as if they were independent then this A1 A2 and A3 would always be zero cause then zero times this no matter what you get is zero and then regardless of what these two are.	linear independence \rightarrow trivial solution	When stating or relating dependency to trivial solution
21	SC3: and then that zero vector plus zero times this vector zero times any vector will just give you another zero vector so the zero vector plus a zero vector equal the zero vector that's why these two depend on each other yeah	computed linear combination \rightarrow linear dependence	Performing linear combination to obtain zero vector and relating it to linear dependence

22	SC3: yes, yeah because this one is already a zero vector so this one is depending on these two to become a zero vector because if this zero vector adds anything else to it besides the zero vector it won't give you the zero vector so then these two have to I guess rely on the A1 and A2 which those are to give you a value to where this plus this will then give you the zero vector	①depend on each other → zero vector ②linear combinations → linear dependency	①Performing linear combination to obtain zero and relating it to linear dependence ②Stating linear combinations to linear dependency
23	SC3: that the vectors depend on each other to become a zero vector	depend on each other → zero vector	Using dependency to relate to the zero vector
24	SC3: ... but this would have to be something else those two equal the zero vector then this would have to be something else unless U2 already to begin with was zero because if it's anything else then it doesn't matter because A3 regardless of what it is A3 would make this go to zero so negative 2 times anything else besides zero vector would give you another thing it would not equal so it would depend on each other to add up to zero. So these two I would say have to be independent of each other unless the vectors are already zero vectors	①computed linear combination → linear dependence ②depend on each other → zero vector	①Performing linear combination to obtain zero and relating it to linear dependence ②Performing linear combination to obtain zero and relating it to linear dependence
25	SC3: okay so U1 plus U3 equals the zero vector it would be linearly dependent because zero and U1 plus U3 gives you zero	①linear combinations → source for new objects ②depend on each other → zero vector	①Mention combinations of linear multiples resulting in new objects ②Performing linear combination to obtain zero and relating it to linear dependence
26	SC3: ... but it would be dependent if it weren't zero because these vectors would be something like one two three plus some other	①linear combinations → source for new objects ②depend on each	①Mention combinations of linear multiples resulting in new objects. ②Performing linear

	thing so then you would have to multiply this by a scalar in order to add up to this which would make that the zero vector	other \rightarrow zero vector	combination to obtain zero and relating it to linear dependence
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Table 4.9 can conclude that the majority of metonymies used by student C3 dealt with reasoning of linear independence and linear dependence. One of the metonymies frequently used by student C3 was the word free variable to represent dependence. The following script (obtained from the original transcript of the interview) indicates the use of the metonymy as part of the reasoning.

I3: ... okay can you give me an example of a linearly dependent set

SC3: linearly dependent. I guess if we would reduce to had like (student is writing a matrix) one, zero, one these would all have to be zeros (student filling matrix with one and zeros) because this could be let's say four cause then this is a free variable, free variable and because of this it would be dependent on that, whatever that was

I3: so what would be dependent on this?

SC3: the vector of dependence, if it was independent this would be a zero as well because it would completely row reduce row Echelon form where it's not as before it was dependent because we had X_1 you would say that this would be X_1, X_2, X_3 so X_3 is equal to four I guess. So then you would say X_3 not equal to four but (Student labels columns x_1, x_2 and x_3 to previously drawn matrix) so as a free variable you can set X_3 equals to T , so then that would make X_1 equal to negative $4T$, and X_2 which is equal to zero I guess

I3: oh I see okay

SC3: and then X_3 is equal to T as well

I3: okay.

SC3: and then it just depends on the T because it is a free variable so that is linearly dependent.

I3: okay, that's meaning the matrix

SC3: yes.

I3: Okay

SC3: yes, the matrix exactly.

I3: okay, so how is $X1$, $X2$, $X3$ related to this one

SC3: $x1$, $x2$, $x3$

I3: yeah, are these related to this equation you wrote from the book

SC3: I don't believe, so well I cause I never usually do combine both of them

I3: I see.

SC3: because I just see like of this is happen, like this if I have a free variable, then I automatically assume it is dependent

We can see the repeated use of the metonymy between containing free variables and being dependent again in the following script.

I3: okay so when you said $U1$ $U3$, $U3$ are linearly dependent other than the equation and matrices when you hear linearly dependent what do you think about with respect to these vectors?

SC3: besides what?

I3: besides the equation and the matrices

SC3: if I...

I3: so yeah you label $U1$ $U3$, $U3$ vectors are linearly dependent

SC3: okay

I3: are there other things you think about or what do you think first when

SC3: when I hear, right when I hear okay when they say the matrix is dependent then right away I think there's a free variables. There will have to be a free variables

I3: so for the three there will have to be a free variable

The results reported in this chapter will be further discussed in the following chapter

Discussion and Conclusions by associating each student's thinking modes and their corresponding use of metonymies and metaphors.

Chapter 5

Discussion and Conclusion

From the data gathered analyzing the interview transcripts, there exist different thinking modes present in every student's reasoning. The benefits these thinking modes provide help the students construct their own understanding of concepts introduced in their first linear algebra course at the university level. Students were inclined to think in algebraic and arithmetic modes when answering linear independence related questions given their familiar disposition of their representations and computational methods to arrive to answers. From the analysis of the transcripts the data demonstrates obvious indications that the students were able to form their own opinions by shifting from one thinking mode to the other as done from Analytic-Arithmetic thinking mode to relate explanation on graphical representations seen in the matrix algebra course.

Examples for the students' inclinations to use a certain thinking mode, the use of metonymies and metaphors is given in this chapter. In addition, the influences taken by the students that might have led to their choice for a specific mode will be covered and their constant use of certain metonymy or metaphor. We will also discuss some of the similarities and differences among the data belonging to each student as well as the factors affecting the results, the research limitations, and the future implications.

5.1 Discussion

A comparison for students A12, A22, and C3 of the use of thinking modes by Sierpiska (2000) and the metaphors and metonymies defined by Presmeg (1998) belonging to the sections modular or non-modular of the matrix algebra course will be discussed.

5.1.1 Student A12

Referring to the data reported in chapter 4 on the classification of the categorized answers representing the thinking modes used by students during their interview, student A12 implemented nine different types of categories that can be classified into the three thinking modes presented by Sierpiska (2000). Only one category belonging to the Synthetic-Geometric mode of thinking, four categories classified into the Analytic-Structural mode, and six categories classified into the Analytic-Arithmetic mode, while two of those categories were labeled to multiple modes of thinking.

The information provided in table 4.2 demonstrates that the Synthetic-Geometric mode had a frequency of one (used 6% of the time), the Analytic-Structural mode had a frequency of eight (used 53% of the time), and finally the Analytic-Arithmetic mode had a frequency of 11 (used 73% of the time). These percentages show that student A12 appears to apply the Analytic-Arithmetic mode in his reasoning the most and not too far behind the Analytic-Structural. The following excerpt obtained from the original transcript demonstrates the use of analytic modes in order to explain his answer:

II: So what I am trying to understand is there a relation that I may not be aware of let's just use $x, y, z, 1, 2, 3$ right, you are using it that way and then we have x here plus yv plus zw equals to zero equation were vectors are u, v, w . Are you tie-in this to this part?

SA12: Yes, because this could be vector u and we could have another vector v , and another vector w , and then if you were to graph these and you come up with the only time that they form a linear combination, the only time they form this zero vector is when x, y , and z are zero then it is linearly independent.

II: Ok.

SA12: But if you find another value for x if you were to set it equal to umm zero vector if you find another value for y and z then this is linearly dependent.

I1: Ok are these the same as those x , y , z 's

SA12: Yes. It's just sometimes I confuse them and that's why I get, umm I will pause and stop to think about it.

I1: So one time you said x is for this one, y 's for this one, z 's for this one.

SA12: Yes.

I1: So is this still the case that x is for this one, and y 's is for this one, and z 's are the same for these x , y , z 's.

SA12: Yes.

I1: Yes, So if to say this is the only solution to this trivial solution here. How does that relate to this one?

SA12: If only one here is the trivial solution? Then if you were to put this entire thing in a umm matrix and perform Gaussian row operations then you would come up, yes you would come up with umm the identity matrix.

I1: Ok.

SA12: And these, each one would equal zero.

I1: So these each one meaning this x , this y , and this z .

SA12: Right.

I1: So, on each of the vector or are you just thinking one vector.

SA12: Because if you were to combine all three in to one so then it would end up with and it would just read of as x equals zero and then y equal zero and z equals zero.

Student A12 begins by stating that the set may be linearly independent because of the fact that they form linear combinations (use of Analytic-Structural mode), then confirms the answer by performing the Gauss-Jordan elimination method (use of Analytic-Arithmetic mode), one obtains the identity matrix (use of Analytic-Structural mode).

With respect to the use of metonymies and metaphors, the data obtained demonstrated a

high percentage of the metonymies used were directed to linear dependence and independence questions. From the 22 metonymies found, 10 of those were directed to linear independence. The metonymy preferred and extensively used by student A12 was –linear combination associated to linear independence (linear combination \rightarrow linear independence). Very similar to the Analytic-Arithmetic thinking mode, the metonymy among the categories classified into this mode were: ST (Solution Type), IC (Independent Columns), LC (Linear Combinations), RRE (Use of Gauss row reducing operations), and CLC (Calculated Linear Calculations). The excerpt above obtained from the original transcript also demonstrates how wide use of the metonymy (linear combination \rightarrow linear independence) was present.

5.1.2 Student A22

Seven categories used by student A22 can be classified into the three thinking modes reported by Sierpiska (2000). Two categories belonging to the Synthetic-Geometric mode of thinking, three categories classified into the Analytic-Structural mode, and finally four categories classified into the Analytic-Arithmetic mode, keeping in mind that some categories were shared.

Referring to the information provided in table 4.4, the Synthetic-Geometric mode had a frequency of two (used 22.27% of the time), the Analytic-Structural mode had a frequency of five (used 55.6% of the time), and finally the Analytic-Arithmetic mode had a frequency of six (used 66.7% of the time). The use of all thinking modes was apparent by student A22 and displays a fairly close frequency for all three thinking modes. Because the frequency of the Analytic-Arithmetic mode is slightly higher than the others, it is not the clear preferred thinking mode for student A22 based on the small outcome of categories. Student A22 clearly shifts from one thinking mode to another and in some occasions, but this is done with very few examples.

The following script obtained from the original transcript illustrates the use of the Analytic-Arithmetic mode and the Analytic-Structural mode in response to make sense of problem:

I2: yeah, you can use anything you want to, remember you're trying to show me how, how you are revealing, what kind of things you are using so calculators part of it so please

SA22: so I'm going to do the row reduced echelon form again. And it gives me the identity

I2: okay, and it says

SA22: that is independent

I2: linearly independent. What does that mean to you? When you say this thing is linearly independent. What do you mean? And you saying the matrix is linearly independent, what are you thinking when you say that word linearly independent?

SA22: that each vector doesn't depend on one of these either of these two or that this one doesn't this depend on these two or this one of these two (student pointing to vectors drawn by teacher)

I2: Oh I see.

SA22: they are not dependent on each other at all

This demonstrates how student A22 arrived to the conclusion that the set of vectors was linearly independent by performing the Gauss Jordan row elimination (use of Analytic-Arithmetic mode), it produced the identity (use of Analytic-Structural mode).

With respect to the use of metonymies and metaphors, considering the low outcome of categories we can state that the majority of the metonymies used were directly related to linear dependence and independence problems. Out of the seven metonymies found, only one was not geared to linear independence or linear dependence which is the use of matrix in place of set.

5.1.3 Student C3

Referring to the data reported in chapter 4 on the classification of the categories representing the thinking modes used by students during their interview, student C3 used 15 different types of categories that can be classified into the three thinking modes presented by Sierpinska (2000). There were three categories belonging to the Synthetic-Geometric mode of thinking, 11 categories classified into the Analytic-Structural mode, and finally there were nine categories classified into the Analytic-Arithmetic mode, with four categories sharing both Analytic-Arithmetic and Analytic-Structural thinking modes.

The information provided in table 4.6 demonstrates that the Synthetic-Geometric mode had a frequency of three (used 7.5% of the time), the Analytic-Structural mode had a frequency of 35 (used 87.5% of the time), and finally the Analytic-Arithmetic mode had a frequency of 33 (used 82.5% of the time). Based on these percentages and the amount of categories classified into both analytic modes, it can be suggested that student C3 was able to shift effortlessly from the Analytic-Arithmetic mode to the Analytic-Structural mode. There were also indications that student C3 demonstrated the ability to relate the analytic modes to the Synthetic-Geometric mode. Although section C did not have technology activities implemented, the classroom observations indicated the use of more geometric modes (by the instructor) in this section (Zamora, 2010), many of which were static geometric modes. The following script obtained from the original transcript demonstrates some of the problems in which student C3 connected all three modes to provide an answer.

I3: okay and initially with this one you said these are linearly independent (teacher points out to first vectors drawn stated to be independent) and so that makes this one linear independent. Could we use that argument for these? These are linearly independent would these?

SC3: I guess you could especially because it says it's an $R \times N$ so it doesn't say like in R^4 so even

if you do get five vectors $U1$ $U2$ $U5$ $U3$ $U4$ since it's in RN it could possibly still be linearly independent as well

I3: but you're not sure you are saying possibly

SC3: well yes I guess it could be seeing how yeah because this would just a scalar multiple of $U2$ and if its linear independent with $U1$ already then adding $U1$ two it would not affect it whatsoever

I3: oh okay but you're saying if these were a 4×4 this oh where am I, this gives you 4×4 then you're saying this will give you 4×5 is that what you're saying? The other perspective that you were looking at this one

SC3: will this would, well this could give me 5×5 but it would have to be a minimum of so because the

I3: oh I see okay so you're saying this will give you 5×5 and that would be okay

SC3: yes

I3: okay would it give you a 4×5

SC3: (student begins to draw a new matrix 4×5) no

I3: no

SC3: no because once again the M is greater than the N so it cannot

I3: it can not

SC3: yeah

I3: because of the free variable thing?

SC3: because the M 's are the columns and the rows are the N 's

With respect to the use of metonymies and metaphors, it is clear that a high percentage of the metonymies used were directly related to linear dependence and independence problems. Out of the 25 metonymies found, 14 of those represented linear independence or linear dependence of a set of vectors. The metonymies mostly used by student C3 were: Free variable \rightarrow linear dependence and identity form \rightarrow independence. The following script obtained from the interview

transcript of student C3 demonstrates how the use of metonymies helped student arrive to a conclusion based on previous knowledge.

SC3: I guess how, since what I've learned, like everything that you do is everything is always like on the same plane or on the same space. Remember how on the previous one had said like this was on the same plane but this is not we never really did anything like that so I would just automatically assume that they are on the same plane but it would be dependent

I3: they, what is they?

SC3: that the vectors are dependent on each other

I3: dependent on each other okay they are on the same plane

SC3: yeah that's what I would assume

I3: because

SC3: but the knowledge I have that we never done stuff like this. Vectors on this plane this vectors on this plane this vectors on that plane they are always on the same plane but then how we have the equations and like how I guess our problems we have to find the solutions to new problems so but never really they were on different planes

I3: they were never on different planes

SC3: no so I would assume that they are on the same plane

I3: so are you saying that it doesn't matter the information that says this equation has infinitely many solutions

SC3: yes because basically what I get from that equation is that it's just saying that those vectors since they had infinitely many solutions that they are dependent on each other

I3: okay

SC3: but it doesn't necessarily say that they are on different planes

I3: okay but it doesn't also say that they are

SC3: on the same plane

I3: it doesn't okay, if it said this equation has only the trivial solution then you would say

SC3: that they would be on the same plane because they are independent

I3: they would be

SC3: yeah

Student C3 arrived to the conclusion using his previous knowledge clearly by stating what he knew and the metonymy that being on the same plane refers to dependency. Student C3 also considers the general notion of independence as a metaphor.

5.1.4 Comparison

As reported in chapter 3 of this thesis and from the observations obtained from Zamora (2010), we can determine the level of exposure was lower in geometrical representation to the students belonging to section C than the level of exposure of students belonging to section A. Computer modules were part of the course for students belonging to sections A and B, although the use of the modules in section A were more apparent where the instructor used the computer modules in an attempt to link different representations and make sense of new concepts (Zamora, 2010).

A similarity shared by the students A12, A22, and C3 is that they all used reasoning classified into the Analytic-Arithmetic mode with a higher frequency, though student C3 had a higher frequency in the Analytic-Structural mode. They all demonstrated the ability to shift from one thinking mode to another, while student A22 displayed very few shifts in thinking.

Another similarity by all students was that most of the metonymies used dealt with linear independence and/or linear dependence of a set of vectors. The most common metonymies used were identity form \rightarrow linear independence, trivial solution \rightarrow linear independence, verbally stating vector combinations \rightarrow dependency, linear combination computed \rightarrow solution type, and linear dependence \rightarrow set of vectors, on same plane. The metonymy used by all three students

was –identity form- to stand for linearly independence, used in an Analytical-Arithmetic mode when student had to carry out computations and in an Analytical-Structural mode when no numerical computations performed just mentioned verbally.

5.2 Factors Affecting Results

Factors that may have influenced the results reported in this thesis may be the exposure given to the computer modules in sections A and B. Students under these sections had the chance to look at visual representation of some linear algebra concepts such as linear combinations, linear dependence, linear independence, span, spanning set, and vectors. The two different teaching styles may also be factors to consider (constructivist vs. traditional), in addition to the structure of the homework assignments.

5.3 Research Limitations

Limitations to this study may include the amount of time students were surveyed during this first year of linear algebra, and if given the time the results might have been different particularly if we had surveyed them in their second course of linear algebra. Objectivity of the categorization obtained from the analysis of the author of this thesis could be considered as a limitation as well. As explained in chapters 3 and 4, the interview transcripts were independently analyzed by the author of this thesis and two additional raters and the focus was on qualitative. The potential objectivity is derived from individual subjective interpretations taken from the student responses. A measure of reliability among raters was reported and explained in section 3.3 of chapter 3. A final limitation present may be the bias involving the fact that students who volunteered to participate in the interviews were offered extra credit.

5.4 Implications

This type of research at the university level gears toward future implications, students are exposed to different technological and visual aspects of learning and similar studies can be conducted with a similar process. Given the opportunity to conduct the same research again, students enrolled in the matrix algebra could benefit from tutors and/or a computer-math lab availability to enhance students understanding.

Approaches to the cognitive constructs were analyzed for the purpose of this research. While taking their first course in linear algebra at the university level, metonymies, metaphors, and thinking modes provide an insight into the students' reasoning. Worth mentioning is that some students had previous knowledge (depending on their backgrounds) of certain linear algebra concepts, such as vectors and matrices and the analysis of those students with the same backgrounds provided a better understanding of the significance of the responses in their reasoning and understanding .

5.5 Final Remarks

The sole purpose in the analysis presented in the thesis was to document the cognitive structures –metonymies, metaphors, and thinking modes- used in the three students' responses while enrolled in their first linear algebra course in the effort to make sense of the cognition of the abstract concepts presented. By no means, the number of students whose interviews were analyzed, reflect a significant sample of the students registered in the matrix algebra course during the Spring 2009. Generalizations cannot be made from this thesis sole documentation of those cognitive constructs.

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Appendix A

Pre-Survey Administered at the Beginning of the Semester

Survey

Math3323—Matrix Algebra

Spring 2009

Instructor: _____

Date: _____

Class: _____

The National Science Foundation (NSF) agency has granted funds to the Department of Mathematics of the University of Texas at El Paso (UTEP) to conduct research to identify the kind of problems and difficulties students face in learning Matrix Algebra concepts and to develop instructional tools to address these issues.

This study will help researchers to better understand the effect of technological learning devices on the learning of difficult math concepts. Our project is also interested in the effect of interventions on the learning of matrix algebra concepts among groups with various backgrounds. For this purpose, we ask your input- via this survey- to better represent the demographics of students who are taking a matrix algebra course at UTEP.

Please respond to the survey questions to the best of your knowledge.

1. Please circle your answer: Gender: Male Female

2. Please circle your answer: Ethnicity: American African-American Hispanic Asian Native-American Other: _____

3. Please circle your answer: Classification: Freshman Sophomore Junior Senior

4. Your Major: _____

5. Please provide your overall GPA at the start of the semester: _____

6. To the best of your knowledge, please list the College Mathematics courses you have taken before attending this class and the grade you earned in each course. _____

7. How many courses were you enrolled in at the start of the semester? _____

8. Did you drop any courses this semester? (Circle your answer) Yes No If yes, how many courses did you drop? _____

9. Have you had a job this semester? (Circle your answer) Yes No

10. If yes, how long have you been working/worked on the job? _____

How many hours per week are (or were) you working (on average)? Circle one. Less than 20 hrs. 20 Hours More than 20 hrs.

11. Is English your first language? (Circle your answer) Yes No If not, what is your first language? _____

12. If English is not your first language, what level of fluency in English would you say you have in a rating 1-10 (10 being the highest)? _____

13. Do you agree that language played a significant role on your learning and understanding of the topics of this course? (Circle your answer) Yes No. If yes, please explain how:

14. In a rating of 1-10 (10 being the highest level of difficulty) what level of difficulty did this class present to you? _____

15. Before this course, had you taken any classes that involved proving theorems? (Circle your answer) Yes No If yes, please provide a list of the classes you attended.

16. Assign a rating from 1-10 (10 being the highest difficulty) to each of the topics below according to the difficulty you experienced while learning, studying, and/or practicing it. Note: If a topic in the list hasn't been covered in your class yet, please indicate it by writing —NC.¶

Linear systems _____ Matrices _____ Subspaces _____ Linear Independence _____ Span & Spanning sets _____ Linear transformations _____ Eigenvalues & eigenvectors _____ Inner product spaces _____ Others: _____

17. Was there a time, while taking the matrix algebra course, you wished a topic (s) was covered differently to help you understand better? (Circle your answer) Yes No. If yes, please explain. _____

18. Do you agree that you needed some additional explanations of the topics from a different perspective while learning them –through visualization, through real life applications, etc. (Please circle your answer). Yes No If yes, please explain

19. How would you recommend the topics that were difficult for you to learn to be covered?

20. Any suggestions on how to improve the teaching and learning of matrix algebra topics?

21. Would you like to add anything else regarding the matrix algebra course? _____

Thank you for your collaboration on responding to this survey!

Appendix B

INFORMED CONSENT FORM

Interactive Online Modules and Take-Home Assignments for Inquiry-Learning to Provide First-Hand Experience in Matrix Algebra Course

You are invited to be part of research activities conducted at The University of Texas at El Paso.

The purpose of this work is to identify what role the online interactive modules and inquiry assignments play in improving student achievement. The evaluation of the impact of the activities will be done through the assessments of student performance, their responses on pre- and post-surveys as well as in clinical interviews. Furthermore, we will document student conceptualizations of basic abstract concepts through student responses on take-home assignments and class tests.

Your permission will make possible for the researcher to document the effectiveness of the proposed activities in addressing obstacles in learning basic matrix algebra concepts.

You must be 18 years of age or older to participate. Your participation is completely voluntary and you may end your participation at any time with no consequences. There are no known risks involved in your participation in this study. You are given the opportunity to ask questions concerning the procedure, and any questions will be answered to your satisfaction.

Every effort will be made to keep your data confidential. No name will be released to anyone and in any published results; to keep the identity of the participating students confidential, a random numerical/letter code will be assigned to each of the respondents. Each participant will be referred to by this numerical/letter code only in presentations and publications of qualitative or descriptive data. Neither the faculty of UTEP nor the subjects' supervisors or colleagues will be provided with the names referring to the codes.

This project, (IRB protocol number: 84840-1), has been reviewed by The University of Texas at El Paso Institutional Review Board. Any questions regarding the conduct of this research or your rights as a research participant may be directed to Lola Norton, IRB Administrator, at (915) 747-8841 or irb.orsp@utep.edu at UTEP.

If you agree to participate, you are invited to sign this consent form and receive a copy of it after thoroughly reading it and asking the researcher any questions until you understand the proposed research activities.

_____	Date _____
Student's name and signature	
_____	Date _____
Lola Norton, IRB Administrator	
_____	Date _____
Researcher's name and signature	

Appendix C

Interview Questions

INTERVIEW QUESTIONS

NSF/CCLI PROJECT

SPRING 2009

1. Define the linear independence of a set of vectors.
2. Given an example of a linearly dependent set of vectors.
3. Given the set $\{u_1, u_2, u_3, u_4\}$ where the vectors u_1, u_2, u_3 are on the same plane and u_4 is not. Determine if the set $\{u_1, u_2, u_3, u_4\}$ is linearly independent. Explain your answer.
4. Given a linearly independent set $\{u_1, u_2, u_3, u_4\}$ in \mathbb{R}^n . Determine the linear independence of the set $\{u_1, u_2 + 5u_1, u_3, u_4\}$.
5. Given an $n \times m$ matrix a where $a_{i2} = a_{i4} + 3a_{i5} \forall 1 \leq i \leq n$. Determine if the set $\{A_1, A_2, A_3, \dots, A_m\}$ (Here A_j is the j th column of A) is linearly independent. Explain your answer.
6. Given a singular 3×3 matrix a . determine if the vectors of the set $\{A_1, A_2, A_3\}$, where A_j is the j th column of A , are on the same plane. Explain your answer.
7. Given that the vector equation $xu + yv + zw = 0$ has infinitely many solutions. Determine if the vectors u, v, w are on the same plane. Explain your answer.
8. Given the vector equation $a_1u_1 + a_2u_2 + a_3u_3 = 0$ with the solution $a_1 = 1, a_2 = -2$, and $a_3 = 0$. Determine the linear independence of the set $\{u_1, u_2, u_3\}$.
9. Given that $\dim(\text{Span}\{u, v, w\}) = 1$. Determine the linear independence of the set $\{u, v, w\}$.

Vita

Ruben Carrizales was born in El Paso. He met his wife, Maria Carrizales, while attending college. He graduated with a Bachelor in Computer Science in May 2006 from the University of Texas at El Paso. As an undergraduate student, he was a work-study for the Office of Student Financial Aid. Ruben Carrizales entered the Masters of Arts in Teaching Mathematics at UTEP in the summer of 2009, and started working on his thesis in January of 2010 under the supervision of Dr. Hamide Dogan. Since August 2010, Ruben has been part of the Cross Institutional Implementation of the Supplemental Instruction UTEP-EPCC cooperative project working as a supplemental instruction leader at El Paso Community College (EPCC) Rio Grande campus. He is also a math instructor for the Upward Bound Program.

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