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Analytical And Numerical Solution To The Partial Differential Equation Arising In Financial Modeling

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to my

MOTHER and FATHER

with love

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by

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Abstract

In this work we will present a self-contained introduction to the option pricing problem. We will introduce some basic ideas from the probability theory and stochastic differential equations. Later we will move to the partial differential equations since the option pricing problem arising in financial mathematics when asset is driven by a stochastic volatility process and assumed presence of transaction cost leads to solving non-linear partial differential equation. We will also present the complete process from deriving the desired partial differential equation to the proof of existence of a solution and also the numerical simulations. Using techniques form stochastic calculus we will derive the main equation which we are going to analyze for the rest of this work. Later we will show the existence of a solution and at last we will provide numerical results for a set of market parameters.

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Notation used

$\mathbb{R}$	set of all real numbers
$\mathbb{Q}$	set of all rational numbers
$\mathbb{Z}$	set of all integers
$\mathbb{N}$	set of all natural numbers
$\mathbb{N}_0$	set of all natural numbers including zero
$\mathbb{C}$	set of all complex numbers
$\times$	Cartesian product
$\mathbb{C}^n$	$\underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{n-times}$
$\mathbb{R}^n$	$\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n-times}$
$w.r.t.$	with respect to
$a.e.$	for almost every, expect set of measure zero.
	In probability case except events with zero probability
$Billion$	10^9
$Trillion$	10^{12}
$USD, US\$$	United States dollar
C^k	class of functions for which k^{th} derivative is continuous
PDE	Partial differential equation

Chapter 1

Introduction

Options are a vital part of a derivative market, it allows companies to hedge against future risk and change in price. In terms of money allocated, derivative market is much bigger than a stock market. The size of a derivative market was estimated at about US$ 516 trillion[7, 8] in June 2007 and US$ 684 trillion [8] in June 2008. Among derivatives, the size of option market alone is estimated to be US$ 84 trillion[8] in June 2008. The total size of the world's stock market was *'only'* US$ 54 trillion [23] in January 2007 and US$ 47 trillion [23] in January 2012. For comparison, the World GDP for year 2010 is estimated to be US$ 63 trillion [1]. From the above numbers, it is clear that option pricing plays crucial role in global economy and it is nearly the most important yet unexplored part of mathematical finance. It is not a surprise that in recent years there has been great interest in option pricing [21, 19, 18, 4, 11, 17, 20, 28, 2, 22].

As was stated before, option can serve as an insurance against future price movement and hedging. There are several types of options but in this work, we will only assume an European call option. The name *option* comes from possibility of buyer/seller to step away from contract under some conditions.

1.1 European Call Option

European call option is a two sided agreement between a buying and selling party. The buying party has an *option* (i.e. possibility) to buy asset S at a future time T for a given price K. This possibility is not for free, but buyer needs to pay seller price V (premium). The main focus of this work is to estimate the price V under certain conditions. The

following example shows a situation when European call option is used as an insurance against future price movement. For more on derivatives,options and futures see [10, 30, 29].

Example 1. *Assume that US company XYZ wants to build new subdivision in Canada. For simplicity let us assume that the construction will take 1 year and 1 million CAD will be payed by XYZ after finishing the construction. Today's exchange rate is 1 USD/CAD. Company XYZ wants to hedge against high USD/CAD exchange rate in the future, because higher exchange rate means higher cost (in USD) of building subdivision in Canada. XYZ makes agreement with a bank that XYZ will have a chance/option to buy 1 million CAD for 1 million USD one year from now. This type of agreement is called the European call option. This option has strike price K=1,000,000 and expiration date T="today's date" + 1year. Let us go over some cases of future exchange rate.*

- *Exchange rate is 1.2 USD/CAD. XYZ can execute call option, means buy 1 million CAD for 1 million USD. Regular price is 1.2 million USD for 1 million CAD. Also means that XYZ made profit equal to 0.2 million USD.*

- *Exchange rate is 1.0 USD/CAD. Executing call option or buying CAD at the foreign currency exchange market will give same result.*

- *Exchange rate is 0.8 USD/CAD. Executing call option would be bad choice since XYZ can buy 1 million CAD for just 0.8 million USD.*

This simple example illustrates that there is a certain payoff, if a strike price K (1 USD/CAD) is lower than a market price. On the other hand, if market price is lower of equal than strike price K we have no payoff.

In example 1, the cost of the call option V was neglected. It is obvious that in real world, the cost needs to be positive and depend at least on a volatility σ and a riskless rate r.

Definition 1 (Riskless rate)**.** *(also called risk-free) is an interest rate associated with risk-free investment. In other words, an investment with no possibility of financial loss.*

Treasury/government bills are considered a risk-free investment since there is very little chance of goverment defaulting.(i.e. not paying back)

The payoff function in figure 1.1 represents the amount of money that can be made by buying asset for strike price K and immediately selling for price S at maturity T (final time). The real profit is simply payoff lowered by cost of the option V (premium). The payoff P has formula

$$P = \max\{S - K, 0\}$$

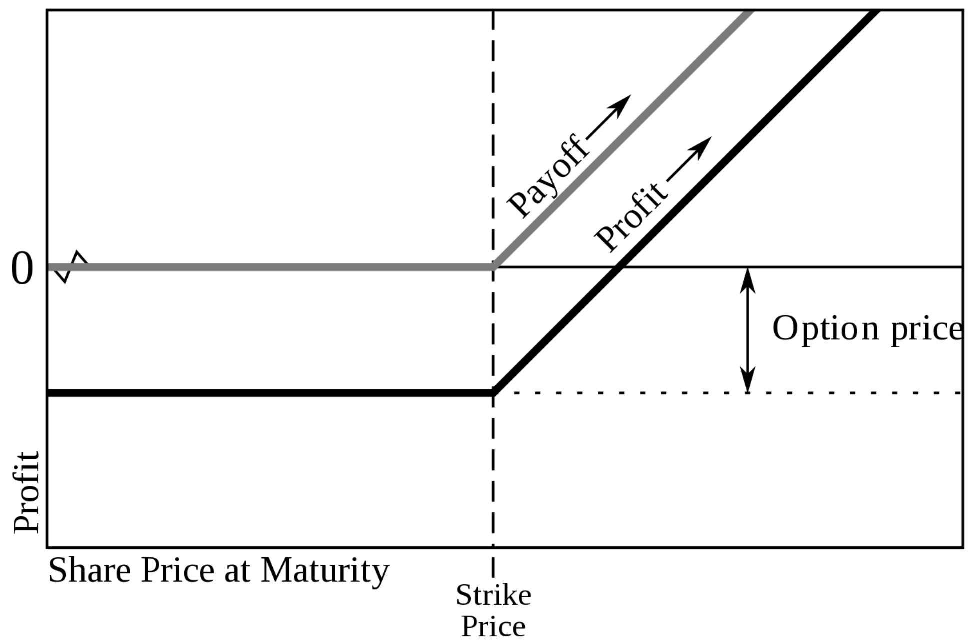


Figure 1.1: Call option payoff

Another unanswered question is what would be the option price V at time T, with maturity at time T ? In other words what would be $V(T)$. It is not hard to see that

$$V(T) = P = \max\{S - K, 0\}. \tag{1.1}$$

Consider two situations

1. $S \leq K$: $V(T)$ has value equal to zero. At expiry, nobody would possibly buy option to buy assets right now for a higher price than the market price.

2. $S > K$: $V(T) = S - K$ because an investor has an option to buy S right now for price K. Means that value of $V(T)$ has to be $S - K$.

Equation (1.1) gives us a terminal condition for backward parabolic partial differential equation derived in chapter 2. Throughout this work, we will assume that market is arbitrage free.

Definition 2 (Arbitrage free)**.** *investor can not make profit greater than risk-free without taking any risk.*

A great example of arbitrage is simultaneously buying an asset at one market and selling it at second market for a higher price. This would lead to immediate profit without taking a any risk.

Example 2. *The closing price for Apple Inc. share on March 9 was $S = 545.17$ USD. In figure 1.2 you can see the price of the call option on the y axis and price of the stock S discounted of the strike price K. As you can also see, the price of the option is converging to the line $y = \max\{S - K, 0\}$ with time getting closer to the expiry T. There are three expiry dates in figure 1.2: March 12, June 12 and October 12. The data for March 12 is really close to the payoff function. Data is publicly available at [6].*

1.2 Statement of the problem

We assume the following stochastic model for a European call option $V(S,t)$ written on an asset S in time t with an expiration date T ($T > t$) and a strike price K.

$$\begin{aligned} dS &= \mu S dt + S\sigma dX^{(a)} \\ d\sigma &= \alpha\sigma dt + \beta\sigma dX^{(b)} \\ [X^{(a)}, X^{(b)}]_t &= Cov(X_t^{(a)}, X_t^{(b)}) = \rho t\ ,\ \ |\rho| < 1 \end{aligned} \tag{1.2}$$

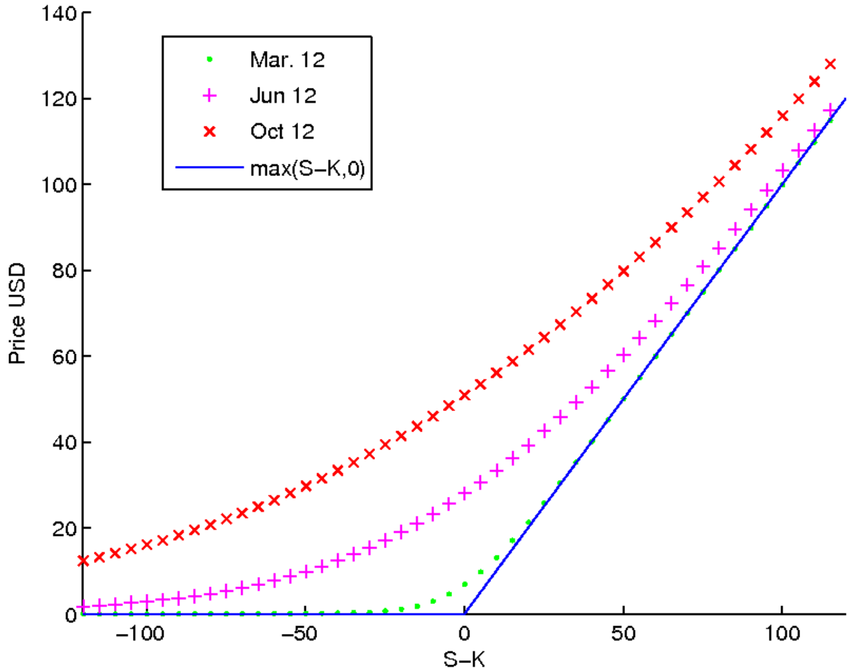


Figure 1.2: Call option value

where α, β, μ are real constants and $X^{(a)}, X^{(b)}$ are two Brownian motions. To understand what (1.2) really means we need to introduce some theorems from Stochastic processes and Stochastic differential equations in section 1.3. Section 1.3 provides only necessary explanations of the theory behind and the text should be easy to follow due to included citations with references to the page numbers.

1.3 Necessary definitions from probability theory

In this section we will provide definitions essential for understanding later chapters and covering stochastic part of this work. Following definitions are *'standard'* and can be found as similar versions in [24, 12, 27, 26, 4, 25] and many other books dealing with stochastic differential equations or stochastic processes in general.

Definition 3 (Probability space). *Probability space is a triplet* $(\Omega, \mathcal{F}, P)$ *where*

- Ω *is a sample space*
- $\mathcal{F}$ *is a* σ*-algebra on* Ω
- P *is a probability measure* $P : \mathcal{F} \to [0,1]$

Definition 4 (Random variable). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Measurable function* $X : \Omega \to \mathbb{R}^n$ *is called a random variable. Unless stated otherwise,* n *will be considered equal to 1.*

Definition 5 (Stochastic process). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. A one parametric collection* $\{X_t | t \in I\}$ *of random variables is called a stochastic process. Where* I *could be* $\mathbb{R}, [0, +\infty), \mathbb{N}$*. Unless stated otherwise, we will consider* $I = [0, +\infty)$*.*

Definition 6 (Filtration). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. We will say that a collection of* σ*-algebras* $\mathcal{F}_t$ *is a filtration provided*

- $\mathcal{F}_t$ *is increasing family of* σ*-algebras*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \quad \textit{for} \quad 0 \leq s \leq t$$

- $\forall t : \mathcal{F}_t \subset \mathcal{F}$

Definition 7 (Adapted process). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. The stochastic process* X_t *is called adapted to the filtration* $\mathcal{F}_t$ *if for every* $t \geq 0$ *is* X_t *an* $\mathcal{F}_t$*-measurable.*

Definition 8 (Generated filtration). *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* X_t *be a stochastic process. We say that* σ*-algebra* $\mathcal{F}_t^X$ *is generated by* X_t *if it is the smallest* σ*-algebra for which* X_s *is* $\mathcal{F}_t$ *measurable for every* $s \in [0, t]$*.*

Definition 9 (Conditional expectation). $\mathrm{E}[X|\mathcal{G}]$ *is a unique function from* Ω *to* $\mathbb{R}$ *for which*

- $\mathrm{E}[X|\mathcal{G}]$ *is* $\mathcal{G}$*-measurable*

- $\int_G \mathrm{E}[X|\mathcal{G}]dP = \int_G XdP$ *for all* $G \in \mathcal{G}$

Note that the definition of a conditional expectation is not constructive. It only says that conditional expectation exists and is unique but doesn't say how to get it. For more details on conditional expectation see for example [24, Appendix B].

Definition 10 (Martingale). *We call process X_t martingale w.r.t. to the filtration $\mathcal{F}_t$ if*

- $E[X_t] = \int_\Omega X_t(\omega)dP(\omega) < +\infty$
- *X_t is $\mathcal{F}_t$ adapted, in other words for every t, X_t is $\mathcal{F}_t$ measurable*
- $\mathrm{E}[X_t|\mathcal{F}_s] = X_s$, *for every* $0 \le s \le t < +\infty$

Most theorems in the stochastic calculus hold for more general class of stochastic processes than martingales. They are usually formulated for so called *semimartingales* in the most general form. For more information on semimartingales see for example [26, pg. 23][12, pg. 149][12, pg. 36] and others.

Definition 11 (Quadratic variation process). *Quadratic variation process of semimartingale X_t for which* $\mathrm{E}[X_t] < \infty$, $\forall t \ge 0$ *is a process defined the following way*

$$[X]_t = \lim_{\|\mathbb{P}\| \to 0} \sum_{\mathbb{P}} (X_{t_i} - X_{t_{i-1}})^2$$

where $\mathbb{P} = \{t_0 = 0, t_1, \ldots, t_n = t\}$ *is partition of the interval* $[0,t]$ *and* $\|\mathbb{P}\| = \sup_i |t_i - t_{i-1}|$.

Definition 12 (Quadratic covariation process). *Quadratic covariation process of semimartingales X_t, Y_t for which* $\mathrm{E}[X_t], \mathrm{E}[Y_t] < \infty$, $\forall t \ge 0$ *is process defined the following way*

$$[X,Y]_t = \lim_{\|\mathbb{P}\| \to 0} \sum_{\mathbb{P}} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$

where $\mathbb{P} = \{t_0 = 0, t_1, \ldots, t_n = t\}$ *is a partition of the interval* $[0,t]$ *and* $\|\mathbb{P}\| = \sup_i |t_i - t_{i-1}|$.

We will leave the proof of existence of above limits to [12],[26].

Definition 13 (Itō Integral)**.** *Itō integral can be defined the following way*

$$\int_0^t H_s dX_s = \lim_{\|\mathbb{P}\|\to 0} \sum_{\mathbb{P}} H_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})$$

This integral exists if H is previsible/predictable and X is semimartingale and is sometimes denoted at $H \bullet X$. For example if Y_t is semimartingale, then Y_{t-} is previsible. This is the consequence of [26, pg. 16].

Semimartingales are the most general stochastic processes for which Itō integral can be defined. It includes Brownian motion, Lévy process and also their stochastic exponential.

Example 3. *Itō integral of a constant. Let us assume semimartingale X_t and constant $c \in \mathbb{R}$ then Itō integral will be*

$$Y_t = \int_0^t c\ dX_s = c \int_0^t dX_s = c \cdot (X_t - X_0)$$

Where Y_t is also semimartingale with $Y_0 = 0$.

Definition 14 (Brownian motion)**.** *[27, pg. 1] A real valued stochastic process B_t , $t \in [0, \infty)$ is called Brownian motion if*

1. $B_0 = 0$
2. *the map $t \to B_t(\omega)$ is continuous for all $t \in [0, \infty)$ and all $\omega \in \Omega$*
3. *for any fixed $t, s \geq 0$ the random variable $B_{t+s} - B_t$ has normal distribution with mean 0 and variance s and is independent of B_l , $0 \leq l \leq t$.*

At this point, meaning of $X^{(a)}, X^{(b)}$ in (1.3) is clear. All we need to do is justify $dS, dX^{(a)}, dX^{(b)}, d\sigma$. Meaning of those will be clear in a few following pages. Let us first give two more examples on Brownian motion processes.

Example 4. *Quadratic variation of Brownian motion B at time t is equal to*

$$t = [B]_t = \lim_{\|\mathbb{P}\|\to 0} \sum \left(\underbrace{B_{t_{i+1}} - B_{t_i}}_{\Delta B_{t_i}} \right)^2$$

Proof. Previous equation holds only for a.e. $\omega \in \Omega$.

$$\begin{aligned}
\mathrm{E}\left[\left(\sum_i \Delta B_{t_i}^2 - t\right)^2\right] &= \mathrm{E}\left[\left(\sum_i \Delta B_{t_i}^2 - \Delta t_i\right)^2\right] \\
&= \sum_i \sum_j \mathrm{E}\left[(\Delta B_{t_i}^2 - \Delta t_i)(\Delta B_{t_j}^2 - \Delta t_j)\right] \\
&= \sum_i \mathrm{E}\left[\left(\Delta B_{t_i}^2 - \Delta t_i\right)^2\right] \\
&= \sum_i \mathrm{E}\left[\Delta B_{t_i}^4\right] - 2\Delta t_i \mathrm{E}[\Delta B_{t_i}^2] + \Delta t_i^2 \\
&= \sum_i 3\Delta t_i^2 - 2\Delta t_i^2 + \Delta t_i^2 \\
&= 2\sum_i \Delta t_i^2
\end{aligned}$$

As $\|\mathbb{P}\| \to 0$, the integral $\mathrm{E}\left[\left(\sum_i \Delta B_{t_i}^2 - t\right)^2\right]$ will approach zero since $\sum_i \Delta t_i^2 \to 0$. $\square$

For modified Brownian motion process with variance $\sigma^2 t$ (point 3 in definition 14), the quadratic variation will be simply $\sigma^2 t$. This can be obtained just by modifying example 4. Quadratic covariation has a similar behavior, if we have two Brownian motions B^1, B^2 and those has nonzero covariation as *'normal'* distributions $Cov(B_t^1, B_t^2) = \rho t$ then $[B^1, B^2]_t = \rho t$. This results can be also obtained by following steps of example 4. Those results will be handy in the following example.

Example 5. *Following Itō integral where B_t is the Brownian motion with variance $\sigma^2 t$*

$$Y_t = \int_0^t B_s \, dB_s$$

We need to use definition of Itō integral and properties or B_t. Let us assume equidistant partition $\mathbb{P}$ of interval $[0, t]$. Using identity

$$B_{t_j}^2 - B_{t_{j-1}}^2 = (B_{t_j} - B_{t_{j-1}})^2 + 2B_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})$$

we can rewrite

$$\sum_{\mathbb{P}} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})$$

to the form

$$\sum_{\mathbb{P}} B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) = \frac{1}{2}\underbrace{\sum_{\mathbb{P}}(B_{t_j}^2 - B_{t_{j-1}}^2)}_{A_1} - \frac{1}{2}\underbrace{\sum_{\mathbb{P}}(B_{t_j} - B_{t_{j-1}})^2}_{A_2}$$

where for $\|\mathbb{P}\| \to 0$ *we have* $A_1 = B_t^2 + \underbrace{B_0^2}_{=0} = B_t^2$ *, term* A_2 *is just the quadratic variation of* B_t *and it is equal to* $\sigma^2 t$ *for almost every* $\omega \in \Omega$*.*

$$Y_t = \int_0^t B_s \; dB_s \overset{a.e.}{=} \frac{1}{2}B_t^2 - \frac{1}{2}\sigma^2 t$$

As you can see, the stochastic integral has different behavior than classical Riemann or Lebesgue integral.

Definition 15 (Doléans exponential)**.** *[26, pg. 29] Also called stochastic exponential of a semimartingale* X *is a semimartingale* Y *(notation :* $Y = \mathcal{E}(X)$*) which is a unique solution to the following equation*

$$dY = Y_{-} \; dX \; , \; Y_0 = 1 \tag{1.3}$$

this mean nothing but

$$Y_t - 1 = \int_0^t Y_{s-} \; dX_s$$

For different initial condition $Y_0 > 0$ *we will have just :* $Y_t = Y_0 \cdot \mathcal{E}(X_t)$

Doléns exponential is just stochastic analogue of classical exponential, that is a solution to the ordinary differential equation

$$\frac{d}{dx}y = y.$$

In classical case $y = e^x$, but that is not true in the stochastic calculus. The main reason is different behavior of Itō integral. For solution of (1.3) see theorem 2.

Theorem 1 (Generalized Itō formula)**.** *[26, pg. 394] : Let f: $\mathbb{R}^n \to \mathbb{R}$, $f \in C^2$ and suppose $X = (X^1, \ldots, X^n)$ is a semimartingale in $\mathbb{R}^n$. Then*

$$df(X) = \sum_i D_i f(X_-) dX^i + \frac{1}{2} \sum_{i,j} D_{i,j} f(X_-) d[X^i, X^j]^c + \left(\Delta f(X) - \sum_i D_i f(X_-) \Delta X^i \right)$$

mean also nothing but :

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t \sum_i D_i f(X_{s-}) dX_s^i + \frac{1}{2} \int_0^t \sum_{i,j} D_{i,j} f(X_{s-}) d[X_s^i, X_s^j]^c \\ &+ \sum_{0 \le s \le t} \left(f(X_s) - f(X_{s-}) - \sum_i D_i f(X_{s-}) \Delta X_s^i \right) \end{aligned}$$

where D_i represents a partial derivative with respect to the i^{th} coordinate and 'c' in the exponent represents continuous part of a semimartingale.

Theorem 1 is one of the most important theorem in stochastic calculus. It is stochastic analogue of chain rule in ordinary calculus.

Theorem 2. *[25, pg. 84] Let X ($X_0 = 0$) be a semimartingale then Doléans exponential of X is :*

$$\begin{aligned} Y = \mathcal{E}(X) &= \exp(X_t - X_0 - \sum_{s \le t} \Delta X_s - \frac{1}{2}[X]_t^c) \prod_{s \le t} (1 + \Delta X_s) \\ &= \exp(X_t^c - X_0 - \frac{1}{2}[X]_t^c) \prod_{s \le t} (1 + \Delta X_s) \end{aligned}$$

Finally, at this point we can rewrite (1.3) to a meaningful form

$$\begin{aligned} S_t - S_0 &= \int_0^t \mu S_s ds + \int_0^t S_s \sigma_s dX_s^{(a)} \\ \sigma_t - \sigma_0 &= \int_0^t \alpha \sigma_s ds + \int_0^t \beta \sigma_s dX_s^{(b)} \end{aligned} \tag{1.4}$$

from (1.4) you can see that σ_t is Doléans exponential of continuous semimartingale $\alpha t + \beta X_t^{(b)}$. Also S_t is Doléans exponential

$$
\begin{aligned}
\sigma_t &= \mathcal{E}\left(\alpha t + \beta X_t^{(b)}\right) \\
S_t &= \mathcal{E}\left(\mu t + \int_0^t \sigma_s dX_s^{(a)}\right)
\end{aligned}
\tag{1.5}
$$

second equation might not that obvious but follows from [26, pg. 391]. In this case, the stochastic integral is well behaving and roughly speaking ' $d(\int_0^s \sigma_l dX_l^{(a)}) = \sigma_s dX_s^{(a)}$ ' in other words $S \bullet (\sigma \bullet X) = S\sigma \bullet X$ (see def. 13).

Chapter 2

Derivation

In this section we will derive the option pricing equation (2.1) following classical Black-Scholes in [2]. We will assume portfolio (2.3) and stock price movement (1.2), and we get the following PDE

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\beta^2\sigma^2\frac{\partial^2 V}{\partial \sigma^2} + \rho\beta\sigma^2 S\frac{\partial^2 V}{\partial S\partial\sigma} + rS\frac{\partial V}{\partial S} + r\sigma\frac{\partial V}{\partial \sigma} - rV \\ - \sqrt{\frac{2}{\pi\delta t}}\kappa\sigma S\sqrt{S^2(\frac{\partial^2 V}{\partial S^2})^2 + 2\rho\beta S\frac{\partial^2 V}{\partial S^2}\frac{\partial^2 V}{\partial S\partial\sigma} + \beta^2(\frac{\partial^2 V}{\partial S\partial\sigma})^2} = 0\end{aligned} \tag{2.1}$$

with price dynamics (1.2)

$$dS = \mu S dt + S\sigma dX^{(a)}$$

$$d\sigma = \alpha\sigma dt + \beta\sigma dX^{(b)}$$

$$[X^{(a)}, X^{(b)}]_t = Cov(X_t^{(a)}, X_t^{(b)}) = \rho t$$

and a terminal condition

$$V(T) = \max\{S - K, 0\}$$

If you compare (2.1) with the classical equation obtained by Black and Scholes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.2}$$

you can see that (2.2) is missing terms, associated with volatility, σ and also square root term. The square root term comes from a transaction cost which is proportional to the

number of assets an investor is buying/selling [11]. Transaction cost was neglected in original Black-Scholes [2]. Black and Scholes also assumed volatility σ to be constant during the whole time interval. This problem was originally addressed by [21] in which volatility followed stochastic processed same as in our model. To make this model even closer to the real world scenario, we would need to add jump process for S to reflect change in price. The so called *'jump'* in price S can occur, for example, when a new information is released or between closing and opening market on the next day.

2.1 PDE derivation

To derive equation (2.1) we need to compute few things. We assume a portfolio Π consisting of option V, Δ shares of S and Δ_1 shares of σ. This situation is not unreal, a good example is the S&P 500 (representing S) and the index VIX (representing σ)measuring implied volatility. In figures 2.2 and 2.1 you can see S&P 500 index and its volatility index VIX three months prior to March 9, 2012 (5 days respectively). The assumed portfolio will be

$$\Pi = V - \Delta S - \Delta_1 \sigma \tag{2.3}$$

Assuming such a portfolio is a standard procedure and allows us to get rid of the stochastic terms later on and derive the partial differential equation for V. This process is similar to the process used in [2, 11, 21]. The first step, is to apply Itō 's lemma on V, S and σ. We already have Itō 's lemma applied on S, σ and it is the equation (1.4). We can apply same Itō lemma on $V(S, \sigma, t)$

$$\begin{aligned} V_t - V_0 = &\int_0^t \frac{\partial V}{\partial S} dS_s + \int_0^t \frac{\partial V}{\partial t} ds + \int_0^t \frac{\partial V}{\partial \sigma} d\sigma_s + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial S^2} d[S,S]_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial \sigma^2} d[\sigma,\sigma]_s + \int_0^t \frac{\partial^2 V}{\partial \sigma \partial S} d[S,\sigma]_s + \int_0^t \frac{\partial^2 V}{\partial \sigma \partial S} d[t,\sigma]_s \\ &+ \int_0^t \frac{\partial^2 V}{\partial \sigma \partial S} d[t,\sigma]_s + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial t^2} d[t,t]_s \end{aligned} \tag{2.4}$$

Figure 2.1: S&P 500 and VIX three months prior to Mar. 9 [6]

As you can see in (2.4), we need to compute quadratic covariation between S, σ, t and integrate with respect to the quadratic covariation process. But for Doléans exponential, it has special form. At the end, we will end up integrating w.r.t. time. To proceed any further we need to introduce one important theorem.

Theorem 3. *[26, pg. 391] For any left continuous semimartingales M, K, H holds*

Figure 2.2: S&P 500 and VIX five days prior to March 9 [6]

1. $M \bullet (K \bullet H) = MK \bullet H$

2. $[H \bullet K, M] = \int H d[K, M] = H \bullet [K, M]$

3. $[\cdot, \cdot]_t$ *is positive semidefinite, Cauchy-Schwartz inequality holds*

Don't forget about the notation $X \bullet Y = \int X dY$.

From definition of $[\cdot,\cdot]_t$ is obvious that $[t]_t = 0$ and from Cauchy-Schwartz inequality we get $[t,S]_t = [t,\sigma]_t = 0$ since

$$[t,S]_t^2 \le [t]_t^2[S]_t^2 = 0$$
$$[t,\sigma]_t^2 \le [t]_t^2[\sigma]_t^2 = 0.$$

Knowing theorem 3 and using example 4 we can compute integrals w.r.t. $[S,S],[S,\sigma],[\sigma,\sigma]$.

$$\begin{aligned}
\underbrace{\frac{\partial^2 V}{\partial S^2}}_{H}\bullet[S,S] &= [H\bullet S,S] = [H\bullet(S\bullet(\mu t+\int\sigma dX^{(a)})),S]\\
&= [HS\bullet(\mu t+\int\sigma dX^{(a)}),S] = HS\bullet[(\mu t+\int\sigma dX^{(a)}),S]\\
&= HS\bullet[(\mu t+\int\sigma dX^{(a)}),(\mu t+\int\sigma dX^{(a)})]\\
&= HS^2\bullet[\sigma\bullet X^{(a)},\sigma\bullet X^{(a)}] = [HS^2\bullet(\sigma\bullet X^{(a)}),\sigma\bullet X^{(a)}]\\
&= [HS^2\sigma\bullet X^{(a)},\sigma\bullet X^{(a)}] = HS^2\sigma^2\bullet[X^{(a)},X^{(a)}] = HS^2\sigma^2\bullet t\\
&= \frac{\partial^2 V}{\partial S^2}S^2\sigma^2\bullet t
\end{aligned}$$

Similar steps will be done in cases of $[S,\sigma],[\sigma,\sigma]$.

$$\frac{\partial^2 V}{\partial S\partial\sigma}\bullet[S,\sigma] = \frac{\partial^2 V}{\partial S\partial\sigma}S\sigma^2\beta\bullet[X^{(a)},X^{(b)}] = \frac{\partial^2 V}{\partial S\partial\sigma}S\sigma^2\beta\rho\bullet t$$
$$\frac{\partial^2 V}{\partial\sigma^2}\bullet[\sigma,\sigma] = \frac{\partial^2 V}{\partial\sigma^2}\sigma^2\bullet[\beta X^{(b)},\beta X^{(b)}] = \frac{\partial^2 V}{\partial\sigma^2}\sigma^2\beta^2\bullet t$$

Using above results the (2.4) will look like

$$\begin{aligned}
V_t - V_0 = \int_0^t \frac{\partial V}{\partial S}\mu S ds + \int_0^t \frac{\partial V}{\partial S}\sigma S dX^{(1)} + \int_0^t \frac{\partial V}{\partial t}ds + \int_0^t \frac{\partial V}{\partial\sigma}\alpha\sigma ds + \int_0^t \frac{\partial V}{\partial\sigma}\beta\sigma dX^{(2)}\\
+ \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial S^2}S^2\sigma^2 ds + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial\sigma^2}\sigma^2\beta^2 ds + \int_0^t \frac{\partial^2 V}{\partial\sigma\partial S}S\sigma^2\beta\rho ds
\end{aligned}$$

The overall $\Pi_t - \Pi_0$ will look like

$$\begin{aligned}\Pi_t - \Pi_0 = &\int_0^t \frac{\partial V}{\partial S}\mu S ds + \int_0^t \frac{\partial V}{\partial S}\sigma S dX^{(1)} + \int_0^t \frac{\partial V}{\partial t} ds + \int_0^t \frac{\partial V}{\partial \sigma}\alpha\sigma ds + \int_0^t \frac{\partial V}{\partial \sigma}\beta\sigma dX^{(2)} \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial S^2} S^2\sigma^2 ds + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial \sigma^2}\sigma^2\beta^2 ds + \int_0^t \frac{\partial^2 V}{\partial\sigma\partial S} S\sigma^2\beta\rho ds \\ &+ \Delta\left(\int_0^t \mu S ds + \int_0^t \sigma S dX^{(1)}\right) + \Delta_1\left(\int_0^t \alpha\sigma ds + \int_0^t \beta\sigma dX^{(2)}\right)\end{aligned}$$

After rearranging coefficients we get

$$\begin{aligned}\Pi_t - \Pi_0 = &\int_0^t \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\sigma^2\beta^2 + \frac{\partial^2 V}{\partial\sigma\partial S}S\sigma^2\beta\rho\right) ds \\ &+ \int_0^t \sigma S\left(\frac{\partial V}{\partial S} + \Delta\right) dX^{(1)} + \int_0^t \beta\sigma\left(\frac{\partial V}{\partial \sigma} + \Delta_1\right) dX^{(2)} \\ &+ \int_0^t \mu S\left(\frac{\partial V}{\partial S} + \Delta\right) ds + \int_0^t \alpha\sigma\left(\frac{\partial V}{\partial \sigma} + \Delta_1\right) ds\end{aligned}$$

but the difference in portfolio prices is still driven by stochastic terms $X^{(a)}, X^{(b)}$. To make growth of portfolio deterministic and thus riskless, we need to set constants Δ, Δ_1 to

$$\begin{aligned}\Delta &:= -\frac{\partial V}{\partial S} \\ \Delta_1 &:= -\frac{\partial V}{\partial \sigma}.\end{aligned} \tag{2.5}$$

In other words, to make this portfolio deterministic, we need to have exactly $-\frac{\partial V}{\partial S}$ or asset S and $-\frac{\partial V}{\partial \sigma}$ of σ. This eliminates most of the terms in the previous equation, we are left with

$$\Pi_t - \Pi_0 = \int_0^t \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\sigma^2\beta^2 + \frac{\partial^2 V}{\partial\sigma\partial S}S\sigma^2\beta\rho\right) ds.$$

So far, we neglected the cost which is associated with continuously trading S and σ. Adding transaction cost which will be proportional to the number of stock we are

buying/selling associated with (2.5) is necessary. The first order approximation of expected change of portfolio value over the small time interval δt will be

$$\mathrm{E}[\delta\Pi] = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\sigma^2\beta^2 + \frac{\partial^2 V}{\partial\sigma\partial S}S\sigma^2\beta\rho\right)\delta t - \mathrm{E}[\kappa S|\nu|] \tag{2.6}$$

where $|\nu|$ is number or stock we are buying/selling in δt and κ is the transaction cost proportional to the stock price S. Since the portfolio is designed to be riskless, it needs to grow with riskless rate r.

$$d\Pi_t = r\Pi_t dt = r(V - \Delta S - \Delta_1\sigma)dt = r(V - S\frac{\partial V}{\partial S} - \sigma\frac{\partial V}{\partial \sigma})dt$$
$$\Pi_t - \Pi_0 = \int_0^t r\Pi_s ds$$

where r is a riskless rate. This gives us a first order approximation on expected change in portfolio Π.

$$\mathrm{E}[\delta\Pi] = r(V - \Delta S - \Delta_1\sigma)\delta t = r(V - S\frac{\partial V}{\partial S} - \sigma\frac{\partial V}{\partial \sigma})\delta t \tag{2.7}$$

Putting (2.7) and (2.6) we get almost the final version of equation (2.1)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\sigma^2\beta^2 + \frac{\partial^2 V}{\partial\sigma\partial S}S\sigma^2\beta\rho + rS\frac{\partial V}{\partial S} + r\sigma\frac{\partial V}{\partial \sigma} - rV - \kappa S|\nu| = 0$$

we need to estimate coefficient $\kappa S|\nu|$ which represents the cost associated with hedging our portfolio. Following [11] and assuming that Δ_1 is constant we can write that (for details see [11])

$$\mathrm{E}[\kappa S|\nu|] = \kappa S\sigma\sqrt{\frac{2}{\pi\delta t}}\sqrt{S^2\left(\frac{\partial^2 V}{\partial S^2}\right)^2 + 2\rho\beta S\frac{\partial^2 V}{\partial S^2}\frac{\partial^2 V}{\partial\sigma\partial S} + \beta^2\left(\frac{\partial^2 V}{\partial\sigma\partial S}\right)^2}$$

After substituting this line into the previous PDE, we finally get equation (2.1)

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\sigma^2\beta^2 + \frac{\partial^2 V}{\partial \sigma \partial S}S\sigma^2\beta\rho + rS\frac{\partial V}{\partial S} + r\sigma\frac{\partial V}{\partial \sigma} - rV \\
&\quad -\kappa S\sigma\sqrt{\frac{2}{\pi\delta t}}\sqrt{S^2\left(\frac{\partial^2 V}{\partial S^2}\right)^2 + 2\rho\beta S\frac{\partial^2 V}{\partial S^2}\frac{\partial^2 V}{\partial \sigma \partial S} + \beta^2\left(\frac{\partial^2 V}{\partial \sigma \partial S}\right)^2} = 0 \qquad (2.8)
\end{aligned}
$$

Chapter 3

Existence and uniqueness

In this chapter we are going to show an existence and uniqueness of the solution for (2.1). First of all, we are going to introduce some definitions and theorems to help us throughout this chapter. Most of definitions and theorems listed here can be found in similar or more general from in [5, 13, 15, 9, 14].

3.1 Necessary definitions and theorems from PDE theory

Definition 16. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, we define a derivative D^α of f with respect to the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ in the following way:*

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$
$$|\alpha| = \sum_i \alpha_i$$

This definition just simplifies the notation for partial derivatives.

Definition 17. *Suppose $v, u \in L^1_{loc}(U)$ (locally integrable), and α is a multiindex. We say that v is α^{th} weak partial derivative of u, written:*

$$D^\alpha u = v$$

provided

$$\int_U u D^\alpha \phi \; dx = (-1)^{|\alpha|} \int_U v\phi \; dx$$

for all $\phi \in C_c^\infty(U)$ - smooth functions with compact support.

Note that this definition follows naturally from Integration by parts theorem. Thus for function u for which α^{th} classical derivative exist we have.

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx$$

Lemma 1. *Weak α^{th} partial derivative of u, if it exists, is uniquely determined up to a set of measure zero. (For proof see [5, pg. 243])*

Definition 18. *(L^p space and norm) The normed space $L^p(\Omega)$ is defined the following way*

$$L^p(\Omega) = \{f \mid \|f\|_{L^p} < +\infty\}$$

where $\|\cdot\|_{L_p}$ for $1 \leq p < \infty$ is defined as

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$$

and for $p = +\infty$ as

$$\|f\|_{L^\infty} = \operatorname*{supess}_\Omega(|f|).$$

Throughout this text we will make no difference between set $(= \{f \mid \|f\|_{L^p} < +\infty\})$ and vector space with norm, which is set together with operations $(+, \cdot)$ and norm. The meaning should be clear from the context. In above definition we have that $L^p(\Omega) = \{f \mid \|f\|_{L^p} < +\infty\}$ which is just a set, but we will think of it as of the vector space with norm $\|\cdot\|_{L^p}$.

Definition 19. ***Sobolev space*** *$W^{k,p}(\Omega)$ on domain Ω is defined the following way:*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid (\forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k)(D^\alpha f \in L^p(\Omega))\}$$

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \|D^\alpha f\|_{L^p}^p \right)^{\frac{1}{p}} \quad , \quad 1 \le p < +\infty$$

$$\|f\|_{W^{k,p}} = \sup_{|\alpha| \le k} \|D^\alpha f\|_{L^p} \quad , \quad p = +\infty$$

where derivatives $D^\alpha f$ exists in the weak sense and belongs to $L^p(\Omega)$.

Definition 20. ***Sobolev space for parabolic problem*** $W_p^{2k,k}(\Omega)$ *on domain* $Q_T = \Omega \times I$*, where* I *is time interval, is defined the following way:*

$$W_p^{2k,k}(Q_T) = \{ f \in L^p(Q_T) \mid (\forall \alpha \in \mathbb{N}_0^n, |\alpha| \le 2k)(\forall \beta \in \mathbb{N}_0, \beta \le k)(D^\alpha \partial_t^\beta f \in L^p(Q_T)) \}$$

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \le 2k, \beta \le k} \|D^\alpha \partial_t^\beta f\|_{L^p}^p \right)^{\frac{1}{p}} \quad , \quad 1 \le p < +\infty$$

$$\|f\|_{W^{k,p}} = \sup_{|\alpha| \le 2k, \beta \le k} \|D^\alpha \partial_t^\beta f\|_{L^p} \quad , \quad p = +\infty$$

Definition 21. *Let us introduce some norms used for defining Hölder space(s) on domain* $Q_T = (0,T) \times \Omega$ *(* $\Omega \subseteq \mathbb{R}^n$ *)*

$$|u|_{0;Q_T} = [u]_{0;Q_T} = \sup_{Q_T} |u|$$

$$d(z_1, z_2) = |x_1 - x_2| + |t_1 - t_2|^{1/2} \quad , \quad z_i = (t_i, x_i)$$

$$[u]_{\delta,\delta/2;Q_T} = \sup_{\substack{z_1, z_2 \in Q_T \\ z_1 \ne z_2}} \frac{|u(t_1,x_1) - u(t_2,x_2)|}{(d(z_1,z_2))^\delta} \quad , \quad \delta \in (0,1)$$

$$|u|_{\delta,\delta/2;Q_T} = |u|_{0;Q_T} + [u]_{\delta,\delta/2;Q_T}$$

Definition 22. ***Hölder space*** $C^{\delta,\delta/2}(\overline{Q_T})$ *is set of all functions defined on* $\overline{Q_T}$ *for which* $|u|_{\delta,\delta/2;Q_T} < +\infty$.

Definition 23. ***Hölder space*** $C^{2k+\delta,k+\delta/2}(\overline{Q_T})$ *is*

$$C^{2k+\delta,k+\delta/2}(\overline{Q_T}) = \{u \mid D^\alpha \partial_t^\beta u \in C^{\delta,\delta/2}(\overline{Q_T}) \ , \ |\alpha| + 2\beta \leq 2k\}$$

This definition makes sure that for function f (and its derivatives) from Hölder space is nice close to the boundary of the domain. That is $|f(x)|$ can not go to infinity as we move x towards the boundary.

Definition 24 (C^k)**.** *We say that function u defined on U open, is of class C^k if and only if $D^\alpha u$ is continuous.*

This definition (compare with Hölder) does not make sure that function is nicely behaving close to the boundary. The following example illustrates the difference.

Example 6. *Function $f(x) = 1/x$ belongs to space $C^\infty((0,1))$ but $f \notin C^\delta([0,1])$. Definition of $C^\delta([0,1])$ is similar to the one made for parabolic domain, just without the time component.*

Proof. It is clear that

$$f^{(n)}(x) = (-1)^n x^{-n-1}$$

and this function is continuous on the interval $\Omega = (0,1)$. This also means that $f \in C^\infty(\Omega)$. On contrary

$$|f|_{0;\Omega} = \sup_\Omega |f| = +\infty$$

so $f \notin C^\delta(\overline{\Omega})$. □

Definition 25 (C^k boundary)**.** *We say that a boundary of $\Omega \subset \mathbb{R}^n$, $\partial\Omega$ is C^k if for each point $x = (x_1, \ldots, x_n) \in \partial\Omega$ there exist $r > 0$, $j \in \{1, \ldots, n\}$ and C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ s.t.*

$$\Omega \cap B(x,r) = \{x \in B(x,r) | x_j > \gamma(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\}$$

or

$$\Omega \cap B(x,r) = \{x \in B(x,r) | x_j < \gamma(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\}$$

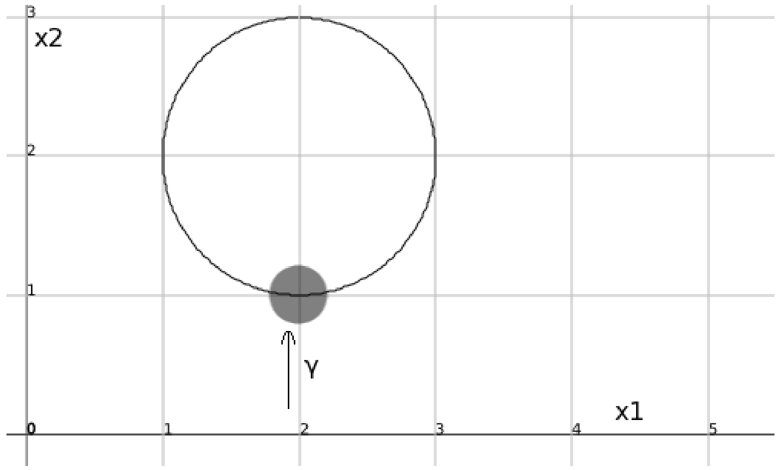


Figure 3.1: Example of $\Omega \subset \mathbb{R}^2$ with smooth boundary

Definition 26. *(Smooth domain) We say that Ω is smooth if $\partial\Omega$ is C^∞.*

Example 7. *Domain $\Omega = \{x \in \mathbb{R}^2 | (x_1 - 2)^2 + (x_2 - 2)^2 < 1\} \subset \mathbb{R}^2$ is smooth.*

Proof. In figure (3.1), you can see example of a smooth domain $\Omega = \{x \in \mathbb{R}^2 | (x_1 - 2)^2 + (x_2 - 2)^2 < 1\}$ which is inside the circle and boundary $\partial\Omega = \{x \in \mathbb{R}^2 | (x_1 - 2)^2 + (x_2 - 2)^2 = 1\}$. For point on the boundary $(2, 1)$ there is a function gamma $\gamma : \mathbb{R} \to \mathbb{R}$ having the following equation

$$\gamma(x_1) = 2 - \sqrt{1 - (x_1 - 2)^2}$$

This function is infinitely differentiable on the interval $I = (1, 2)$. This also means that for point $(2, 1)$ I can find r, say $r = 0.1$ and function γ which is infinitely differentiable and condition

$$\Omega \cap B((2, 1), 0.1) = \{x \in B((2, 1), 0.1) | x_2 > \gamma(x_1)\}$$

holds. For all points on the boundary $\partial\Omega$ I can use one of the following infinitely differen-

tiable functions

$$\gamma_1(x_1) = 2 - \sqrt{1-(x_1-2)^2}$$
$$\gamma_2(x_1) = 2 + \sqrt{1-(x_1-2)^2}$$
$$\gamma_3(x_2) = 2 - \sqrt{1-(x_2-2)^2}$$
$$\gamma_4(x_2) = 2 + \sqrt{1-(x_2-2)^2}$$

We already know that γ_1 will work for bottom half of the circle,γ_2 for top, γ_3 for the left and γ_4 for the right half. □

Definition 27 (Parabolic PDE, Uniformly parabolic PDE)**.** *The partial differential equation of second order*

$$u_t(t,x) = L(t,x)u(t,x) + f(t,x)$$
$$L(t,x) = \sum_{i,j} a^{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b^i(x,t)\frac{\partial}{\partial x_i} + c(x,t)$$

is parabolic if $L(x,t)$ is elliptic. It is uniformly parabolic if L is uniformly elliptic.

Definition 28. *The second order operator L*

$$L(t,x) = \sum_{i,j} a^{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b^i(x,t)\frac{\partial}{\partial x_i} + c(x,t)$$

is parabolic if matrix $\mathbb{A}$, $\mathbb{A}_{i,j} = a^{ij}$ is symmetric and non-negative for all x, t. In other words

$$(\forall \xi \in \mathbb{R}^n, \xi \neq 0)(\forall x,t)(\sum_{i,j} a^{ij}(x,t)\xi_i\xi_j \neq 0)$$

We say that L is uniformly elliptic if

$$(\exists \Lambda, \lambda > 0)(\forall \xi \in \mathbb{R}^n, \xi \neq 0)(\forall x,t)(\lambda\|\xi\| \leq \sum_{i,j} a^{ij}(x,t)\xi_i\xi_j \leq \Lambda\|\xi\|)$$

Definition 29 (Green's function)**.** *is a function for which the solution with zero initial and boundary conditions can be written as integration of righ-hand size and integral kernel G - the Green's function. For problem* (3.6) *it will be :*

$$v^n(x,\tau) =$$
$$\int_0^\tau \int_\Omega G(x,y,w,z,\tau,t) \left(\mathcal{G}(w,z,t, \frac{\partial^2(v^{n-1}+\varphi)}{\partial w \partial z}, \frac{\partial^2(v^{n-1}+\varphi)}{\partial w^2}, \frac{\partial(v^{n-1}+\varphi)}{\partial w}) \right) dwdzdt$$

3.2 Existence of the solution

In this section we will show the existence of the solution only for (2.1) by constructing a sequence of solutions. We consider parabolic domain $Q_T = (0,T) \times \Omega$ where Ω is a bounded smooth domain.

3.2.1 Transformation of the problem

We use the following change of variables for (2.1):

$$S = e^x \ , \ \sigma = e^y \ , \ t = T - \tau \ , \ V(S,\sigma,t) = u(x,y,\tau)$$

That gives us equation

$$\begin{aligned} -\frac{\partial u}{\partial \tau} + \frac{1}{2} e^{2y} \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right) + \frac{1}{2} \beta^2 \left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} \right) + \rho e^y \beta \frac{\partial^2 u}{\partial x \partial y} + r \frac{\partial u}{\partial x} + r \frac{\partial u}{\partial y} - ru \\ = +\kappa \sqrt{\frac{2}{\pi \delta t}} \sqrt{e^{2y} \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right)^2 + 2\rho e^y \beta \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right) \frac{\partial^2 u}{\partial x \partial y} + \beta^2 \frac{\partial^2 u}{\partial x \partial y}^2} \end{aligned} \tag{3.1}$$

This can be written in the operator form

$$\begin{aligned} -u_\tau + Lu &= \mathcal{G} && \text{in } Q_T \\ u(x,y,0) &= u_0(x,y) && \text{on } \Omega \\ u(x,y,\tau) &= g(x,y,\tau) && \text{on } \partial\Omega \times (0,T)) \end{aligned} \tag{3.2}$$

Where

$$\mathcal{G}\left(y, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial y}\right) = \kappa\sqrt{\frac{2}{\pi \delta t}}\sqrt{e^{2y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right)^2 + 2\rho e^y \beta \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x \partial y} + \beta^2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2}$$

$$L = \frac{1}{2}e^{2y}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2} + \rho e^y \beta \frac{\partial^2}{\partial x \partial y} + (r - \frac{1}{2}e^{2y})\frac{\partial}{\partial x} + (r - \frac{1}{2}\beta^2)\frac{\partial}{\partial y} - r$$

Lemma 2. *There exist constants $C, C' > 0$, independent of variables in $\mathcal{G}$ such that*

$$\left|\mathcal{G}\left(x, y, \tau, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial y}\right)\right| \leq Ce^y\left(\left|\frac{\partial^2 u}{\partial x^2}\right| + \left|\frac{\partial u}{\partial x}\right|\right) + C' \cdot \left|\frac{\partial^2 u}{\partial x \partial y}\right|$$

Proof.

$$\begin{aligned}
&\left|\mathcal{G}\left(x, y, \tau, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial y}\right)\right| \\
&= \left|\kappa\sqrt{\frac{2}{\pi \delta t}}\right|\left|\sqrt{e^{2y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right)^2 + 2\rho e^y \beta \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x \partial y} + \beta^2\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2}\right| \\
&= \left|\kappa\sqrt{\frac{2}{\pi \delta t}}\right|\left|\sqrt{e^{2y}\left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right|^2 + 2|\rho e^y \beta| \left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right|\left|\frac{\partial^2 u}{\partial x \partial y}\right| + \beta^2\left|\frac{\partial^2 u}{\partial x \partial y}\right|^2}\right|
\end{aligned}$$

Since $|\rho| < 1$ we can write

$$\begin{aligned}
&\left|\mathcal{G}\left(x, y, \tau, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial y}\right)\right| \\
&\leq \left|\kappa\sqrt{\frac{2}{\pi \delta t}}\right|\left|\sqrt{e^{2y}\left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right|^2 + 2|e^y \beta| \left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right|\left|\frac{\partial^2 u}{\partial x \partial y}\right| + \beta^2\left|\frac{\partial^2 u}{\partial x \partial y}\right|^2}\right| \\
&\leq \left|\kappa\sqrt{\frac{2}{\pi \delta t}}\right|\sqrt{\left(e^y\left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right| + |\beta|\left|\frac{\partial^2 u}{\partial x \partial y}\right|\right)^2} \\
&\leq \left|\kappa\sqrt{\frac{2}{\pi \delta t}}\right|\left(e^y\left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right| + |\beta|\left|\frac{\partial^2 u}{\partial x \partial y}\right|\right) \\
&\leq Ce^y\left(\left|\frac{\partial^2 u}{\partial x^2}\right| + \left|\frac{\partial u}{\partial x}\right|\right) + C' \cdot \left|\frac{\partial^2 u}{\partial x \partial y}\right|
\end{aligned}$$

For some $C, C' > 0$. Note that this result cannot be extended to (compare with [11])

$$\mathcal{G} \leq Ce^y\left(\left|\frac{\partial^2 u}{\partial x^2}\right| + \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial^2 u}{\partial x \partial y}\right|\right)$$

A counter-example would be $u(x,y) = xy$ and send $y \to -\infty$. $\square$

This lemma shows that right-hand side is reasonably bounded. Which is important later-on in theorem 5. We also need to know if our equation is of parabolic, elliptic or hyperbolic type in order to use theorems designed for specific type of PDE.

Lemma 3. *Suppose that* $|\rho| < 1$. *Then the equation* (3.1) *is of parabolic type.*

Proof. Note that (3.1) looks like

$$-v_\tau + \frac{1}{2}\underbrace{e^{2y}}_{\sigma^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\beta^2\frac{\partial^2 u}{\partial y^2} + \rho\underbrace{e^{y}}_{\sigma}\beta\frac{\partial^2 u}{\partial x \partial y} + \cdots = \mathcal{G}$$

Take $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$ and $\theta > 0$, we have

$$\begin{aligned}
&(\sigma^2-\theta)v_1v_1 + (\beta^2-\theta)v_2v_2 + 2\rho\sigma\beta v_1v_2 = \\
&= (\sigma^2-\theta)v_1^2 + 2\sqrt{\sigma^2-\theta}\frac{\rho\sigma\beta}{\sqrt{\sigma^2-\theta}}v_1v_2 + \frac{(\rho\sigma\beta)^2}{\sigma^2-\theta}v_2^2 + (\beta^2-\theta)v_2^2 - \frac{(\rho\sigma\beta)^2}{\sigma^2-\theta}v_2^2 \\
&= \left(\sqrt{\sigma^2-\theta}v_1 + \frac{\rho\sigma\beta}{\sqrt{\sigma^2-\theta}}v_2\right)^2 + (\beta^2-\theta)v_2^2 - \frac{(\rho\sigma\beta)^2}{\sigma^2-\theta}v_2^2 \\
&= \left[\left(\sqrt{\sigma^2-\theta}v_1 + \frac{\rho\sigma\beta}{\sqrt{\sigma^2-\theta}}v_2\right)^2 + v_2^2\left(\beta^2(1-\frac{\rho^2\sigma^2}{\sigma^2-\theta}) - \theta\right)\right]
\end{aligned}$$

and

$$\lim_{\theta\to 0}\left(\beta^2(1-\frac{\rho^2\sigma^2}{\sigma^2-\theta}) - \theta\right) = \beta^2(1-\rho^2)$$

Since $|\rho| < 1$, we have that this limit is greater than zero. Implies that $\exists \theta_1 > 0$ such that

$$\left(\beta^2(1-\frac{\rho^2\sigma^2}{\sigma^2-\theta_1}) - \theta_1\right) > 0$$

For this θ_1 and $\forall \vec{v} \in \mathbb{R}^2 \setminus \{0\}$ we have

$$(\sigma^2-\theta_1)v_1v_1 + (\beta^2-\theta_1)v_2v_2 + 2\rho\sigma\beta v_1v_2 > 0$$

Implies that

$$\sigma^2 v_1 v_1 + \beta^2 v_2 v_2 + 2\rho\sigma\beta v_1 v_2 > \theta_1(|v_1|^2 + |v_2|^2)$$

So far we proved that L is parabolic for some constant $\sigma = e^y$. But σ can take many positive values determined by 'size' of bounded domain Ω. Let us investigate condition in the above limit for $\theta \in [0, \sigma_{min})$

$$\beta^2(1 - \frac{\rho^2\sigma^2}{\sigma^2 - \theta}) - \theta = 0 \tag{3.3}$$

Take the set M of all θ satisfying (3.3) for some σ. From the form of (3.3), it is clear that $\theta = f(\sigma)$ is continuous. This set is bounded from bellow by constant $\epsilon > 0$. Assume not, there is sequence of θ in M which goes to zero. Also there is corresponding sequence of σ's. Since all σ's are in bounded set, and that can be enclosed in compact set. Then there is subsequence of σ's which converge. From continuity, we have that there exist σ (maybe on the boundary of Ω) for which θ is equal to zero ! That is not possible.

Operator L is parabolic.

$\square$

Lemma 4. *Suppose that* $|\rho| < 1$.*For any* $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$ *there exist positive constants* λ, Λ *such that for any* $(x, y) \in \Omega$ *(bounded)*

$$\lambda \cdot |\vec{v}|^2 \overset{(A)}{\leq} e^{2y} v_1 v_1 + \beta^2 v_2 v_2 + 2\rho e^y \beta v_1 v_2 \overset{(B)}{\leq} \Lambda |\vec{v}|^2$$

i.e. equation (3.1) *is* <u>*uniformly parabolic*</u>.

Proof. Part **(A)** follows from lemma 3. In part **(B)** we have β, σ constants and e^y has upper bound $\alpha = \sup_\Omega e^y$ since Ω is bounded. The inequality $|2\rho v_1 v_2| \overset{|\rho|<1}{\leq} |2v_1 v_2| \leq |v_1|^2 + |v_2|^2$ and fact that $e^y \leq \alpha$ completes the proof of **(B)**. The coefficient Λ will be chosen the following way :

$$\Lambda = 2 \cdot \max\{\alpha^2, \beta^2\}$$

$\square$

3.2.2 Existence of the classical solution

In this section we can consider a more general problem

$$
\begin{aligned}
&-u_\tau + Lu = \mathcal{G} && \text{in } Q_T = \Omega \times (0,T) \\
&u(x,y,0) = u_0(x,y) && \text{on } \Omega \\
&u(x,y,\tau) = g(x,y,\tau) && \text{on } \partial\Omega \times (0,T))
\end{aligned} \tag{3.4}
$$

$$L := \sum_{i,j} a^{i,j}(x,y,\tau)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b^i(x,y,\tau)\frac{\partial}{\partial x_i} + c(x,y,\tau)$$

Throughout this section, we will impose the following assumptions

- **A(1)** - The coefficients $a^{i,j}(x,y,\tau), b^i(x,y,\tau), c(x,y,\tau)$ belong to the Hölder space $C^{\delta,\delta/2}(\overline{Q_T})$

  (true for problem (3.2))

- **A(2)** - There exist $\lambda, \Lambda > 0$ for which

  $$\lambda|v|^2 \leq \sum_{i,j} a^{i,j}(x,y,\tau)v_i v_j \leq \Lambda|v|^2$$

  (true for problem (3.2) and the value of $|\rho| < 1$, by lemma 4)

- **A(3)** - $u_0(x,y)$ and $g(x,y,\tau)$ belong to the Hölder space $C^{2+\delta}(\mathbb{R}^2)$ and $C^{2+\delta,1+\delta}(\overline{Q_T})$ respectively.

- **A(4)** - Consistency conditions

  $$g(x,y,0) = u_0(x,y)$$

  $$g_t(x,y,0) - L(x,y,0)u_0(x) = 0$$

  are satisfied for all $(x,y) \in \partial\Omega$

- **A(5)** - Domain Ω is smooth by definition 26 and bounded.

- **A(6)** - The value of ρ less than one in absolute value.

$$|\rho| < 1$$

Lemma 5. *Suppose that **A(1) - A(5)** are satisfied. Then for any $f \in C^{\delta,\delta/2}(\overline{Q_T})$ there exists a unique solution in $C^{2+\delta,1+\delta/2}(\overline{Q_T})$ to the problem*

$$\begin{aligned} -u_\tau + Lu &= f(x,y,\tau) && \textit{in } Q_T \\ u(x,y,0) &= u_0(x,y) && \textit{on } \Omega \\ u(x,y,\tau) &= g(x,y,\tau) && \textit{on } \partial\Omega \times (0,T) \end{aligned}$$

This follows immediately from Theorem 5.2 in [15, pg. 320].

By lemma 5, there exists a unique solution to the original problem without right-hand side

$$\begin{aligned} -u_\tau + Lu &= 0 && \text{in } Q_T \\ u(x,y,0) &= u_0(x,y) && \text{on } \Omega \\ u(x,y,\tau) &= g(x,y,\tau) && \text{on } \partial\Omega \times (0,T) \end{aligned}$$

Let us denote that solution $\varphi \in C^{2+\delta,1+\delta/2}(\overline{Q_T})$ and extend it for points outside the considered domain by $\varphi(x) = 0$ for $x \notin \overline{Q_T}$. We introduce a change of variables to transform our problem (3.2) into a problem with zero boundary conditions

$$\begin{aligned} v(x,y,\tau) &= u(x,y,\tau) - \varphi(x,y,\tau) \\ v_0(x,y) &= u_0(x,y) - \varphi(x,y,0) = 0 \end{aligned}$$

then v will satisfy the initial-boundary value problem:

$$\begin{aligned} &-v_\tau + Lv = \mathcal{G}\left(y, \tfrac{\partial^2(v+\varphi)}{\partial x \partial y}, \tfrac{\partial^2(v+\varphi)}{\partial x^2}, \tfrac{\partial(v+\varphi)}{\partial x}\right) && \text{in } Q_T \\ &v(x,y,0) = 0 && \text{on } \Omega \\ &v(x,y,\tau) = 0 && \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{3.5}$$

If (3.5) has a strong solution then (3.2) has a strong solution.

Note that if $v \in C^{2+\delta,1+\delta/2}(\overline{Q_T})$ then

$$|\frac{\partial v}{\partial x}|, |\frac{\partial^2 v}{\partial x \partial y}|, |\frac{\partial^2 v}{\partial x \partial y}| \in C^{\delta,\delta/2}(\overline{Q_T})$$

and it can be showed using lemma 2 that $\mathcal{G} \in C^{\delta,\delta/2}(\overline{Q_T})$.

Now we will construct the iterative solution the following way

$$\begin{aligned}
&-v_\tau^n + Lv^n = \mathcal{G}(y, \tfrac{\partial^2(v^{n-1}+\varphi)}{\partial x \partial y}, \tfrac{\partial^2(v^{n-1}+\varphi)}{\partial x^2}, \tfrac{\partial(v^{n-1}+\varphi)}{\partial x}) && \text{in } Q_T \\
&v^n(x, y, 0) = 0 && \text{on } \Omega \\
&v^n(x, y, \tau) = 0 && \text{on } \partial\Omega \times (0, T) \qquad (3.6)
\end{aligned}$$

where v^0 is an arbitrary function from $C^{2+\delta,1+\delta/2}(\overline{Q_T})$, we can pick $v_0 = 0$. Function $\mathcal{G}$ is in $C^{\delta,\delta/2}(\overline{Q_T})$ for any index k smaller than n, therefore, by lemma 5 there exist a unique solution v^n to (3.6). To prove the existence of solution, we need to show that sequence v^n converges.

From [15] (Chapter IV. §16), there exists a Green's function $G(x, y, \tau, t)$ for (3.6) if coefficients of L belong to $C^{\delta,\delta/2}(\mathbb{R}^2 \times [0, T])$ and the derivative of coefficients with respect to the spatial variable, i.e. $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ belong also to $C^{\delta,\delta/2}(\mathbb{R}^2 \times [0, T])$. But there exists $f(y) \in C^2(\mathbb{R})$ bounded in $\mathbb{R}^2$ for which $f(y) = e^y$ on Ω. Since Ω is bounded, we can replace e^y with $f(y)$ in L.

$$\begin{aligned}
v^n(x, \tau) &= \int_0^\tau \int_\Omega G(x, y, w, z, \tau, t) \left(\mathcal{G}(w, z, t, \frac{\partial^2(v^{n-1}+\varphi)}{\partial w \partial z}, \frac{\partial^2(v^{n-1}+\varphi)}{\partial w^2}, \frac{\partial(v^{n-1}+\varphi)}{\partial w}) \right) dwdzdt \\
&+ \int_\Omega G(x, y, w, z, \tau, 0) v^n(w, z, 0) dwdz \\
&= \int_0^\tau \int_\Omega G(x, y, w, z, \tau, t) \left(\mathcal{G}(w, z, t, \frac{\partial^2(v^{n-1}+\varphi)}{\partial w \partial z}, \frac{\partial^2(v^{n-1}+\varphi)}{\partial w^2}, \frac{\partial(v^{n-1}+\varphi)}{\partial w}) \right) dwdzdt \\
&+ \int_\Omega G(x, y, w, z, \tau, 0) \cdot 0 \cdot dwdz \\
&= \int_0^\tau \int_\Omega G(x, y, w, z, \tau, t) \mathcal{H}^{n-1}(w, z, t) dwdz\ dt
\end{aligned}$$

For convenience we will denote

$$\mathcal{H}^{n-1}(w,z,t) = \underbrace{\mathcal{G}(w,z,t,\tfrac{\partial^2(v^{n-1}+\varphi)}{\partial w\partial z},\tfrac{\partial^2(v^{n-1}+\varphi)}{\partial w^2},\tfrac{\partial(v^{n-1}+\varphi)}{\partial w})}_{\mathcal{G}^{n-1}(w,z,t)}$$

The derivative with respect to x_i and x_i, x_j will be [15]

$$v^n_{x_i}(x,\tau) = \int_0^\tau \int_\Omega G_{x_i}(x,y,w,z,\tau,t)\mathcal{H}^{n-1}(w,z,t)dwdz\ dt$$
$$v^n_{x_i,x_j}(x,\tau) = \int_0^\tau \int_\Omega G_{x_i,x_j}(x,y,w,z,\tau,t)\mathcal{H}^{n-1}(w,z,t)dwdz\ dt$$

From Chapter IV in [15], we have the estimates

$$|G(x,y,w,z,\tau,t)| \le c_1(\tau-t)^{-1}\exp\left(-C_2\frac{(x-w)^2+(y-z)^2}{\tau-t}\right)$$
$$|G_{x_i}(x,y,w,z,\tau,t)| \le c_1(\tau-t)^{-3/2}\exp\left(-C_2\frac{(x-w)^2+(y-z)^2}{\tau-t}\right)$$
$$|G_{x_i,x_j}(x,y,w,z,\tau,t)| \le c_1(\tau-t)^{-2}\exp\left(-C_2\frac{(x-w)^2+(y-z)^2}{\tau-t}\right)$$

where $\tau > t$ and c_1, C_2 are constants.

If we combine everything together we get

$$\begin{aligned}
&\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} = \\
&= \|v^n(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)} + \sum_{i=1}^{2}\|v^n_{x_i}(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)} + \sum_{i=1,j=1}^{2}\|v^n_{x_i,x_j}(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)} \\
&\le \|\int_0^\tau\int_\Omega |G(x,y,w,z,\tau,t)||\mathcal{H}^{n-1}(w,z,t)|dwdz\ dt\|_{L^\infty(\Omega)} \\
&+ \sum_{i=1}^{2}\|\int_0^\tau\int_\Omega |G_{x_i}(x,y,w,z,\tau,t)||\mathcal{H}^{n-1}(w,z,t)|dwdz\ dt\|_{L^\infty(\Omega)} \\
&+ \sum_{i,j=1}^{2}\|\int_0^\tau\int_\Omega |G_{x_i,x_j}(x,y,w,z,\tau,t)||\mathcal{H}^{n-1}(w,z,t)|dwdz\ dt\|_{L^\infty(\Omega)}
\end{aligned}$$

We need to show that $\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)}$ is uniformly bounded on the interval $[0,T]$. We obtain the following estimate using lemma 2 and the form of $\mathcal{F}$.

$$\begin{aligned}\left|\mathcal{G}^{n-1}(w,z,\tau)\right| &\le Ce^y\left(\left|\frac{\partial^2(v^{n-1}-\varphi)}{\partial w^2}\right|+\left|\frac{\partial(v^{n-1}-\varphi)}{\partial w}\right|\right)+C'\cdot\left|\frac{\partial^2(v^{n-1}-\varphi)}{\partial w\partial z}\right|\\ &\le C_3\|v^{n-1}(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)}+C'_T\end{aligned}$$

by putting those two together we can write

$$\left|\mathcal{H}^{n-1}(w,z,\tau)\right| \le C_5\|v^{n-1}(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)}+C_T$$

Constant C_5 depends on spatial variables and the size of the domain and it is independent of T. But constant C_T depends on T and comes from the upper estimate of φ on $\Omega\times[0,T]$. By direct calculation we can see that [3]

$$\int_\Omega(\tau-t)^{-1}\exp\left(-C_2\frac{(x-w)^2+(y-z)^2}{\tau-t}\right)dwdz \le \left(\frac{\pi}{C_2}\right)$$

The Green's function estimate

$$\left\|\int_\Omega G_{x_i,x_j}(\cdot,\cdot,w,z,\tau,t)\right\|_{L^\infty(\Omega)} dwdz \le C_4(\tau-t)^{-\gamma} \quad , \quad \gamma\in(0,1) \tag{3.7}$$

can be found in Lemma 2.1 of [31]. The value of γ in interval $(0,1)$ is crucial in the following lines.

We also need to estimate

$$\begin{aligned}&\int_0^\tau\left\|\int_\Omega G_{x_ix_j}(x,y,z,w,\tau,\tau')\mathcal{H}^{n-1}(z,w,\tau')\,dzdw\right\|_{L^\infty(\Omega)}\\ &=\int_0^\tau\left\|\int_\Omega G_{x_ix_j}(x,y,z,w,\tau,\tau')\left(\mathcal{H}^{n-1}(z,w,\tau')-\mathcal{H}^{n-1}(x,y,\tau')+\mathcal{H}^{n-1}(x,y,\tau')\right)\,dzdw\right\|_{L^\infty(\Omega)}\\ &\le\int_0^\tau\left\|\int_\Omega G_{x_ix_j}(x,y,z,w,\tau,\tau')\left(\mathcal{H}^{n-1}(z,w,\tau')-\mathcal{H}^{n-1}(x,y,\tau')\right)\,dzdw\right\|_{L^\infty(\Omega)}+\\ &\qquad\int_0^\tau\left\|\int_\Omega G_{x_ix_j}(x,y,z,w,\tau,\tau')\left(\mathcal{H}^{n-1}(x,y,\tau')\right)\,dzdw\right\|_{L^\infty(\Omega)}.\end{aligned} \tag{3.8}$$

To estimate the second term in the right hand side of (3.8) we use (3.7). Since $\mathcal{F}^{n-1}$ is in Hölder space, for the term

$$\left(\mathcal{H}^{n-1}(z,w,\tau') - \mathcal{H}^{n-1}(x,y,\tau')\right)$$

we have inequality

$$\left|\mathcal{H}^{n-1}(z,w,\tau') - \mathcal{H}^{n-1}(x,y,\tau')\right| \leq C_5 \left(\sqrt{(z-x)^2 + (w-y)^2}\right)^{\delta} \ , \ \delta \in (0,1)$$

where C_5 is only function of T and is independent of spatial variable. Using the above result and the fact that $\mathcal{H}^{n-1}$ is in Hölder space for time variable as well, we can find some constant C_6 which is independent of both space and time variable so that

$$\left|\mathcal{H}^{n-1}(z,w,\tau') - \mathcal{H}^{n-1}(x,y,\tau')\right| \leq C_6 \left(\sqrt{(z-x)^2 + (w-y)^2}\right)^{\delta} \ , \ \delta \in (0,1). \tag{3.9}$$

We bound the first term in (3.8) using inequality (3.9) and transformation to spherical coordinates

$$\begin{aligned}
&\left|\int_\Omega G_{x_i x_j}(x,y,z,w,\tau,\tau') \left(\mathcal{H}^{n-1}(z,w,\tau') - \mathcal{H}^{n-1}(x,y,\tau')\right) \, dzdw\right| \\
&\leq \int_\Omega \left|G_{x_i x_j}(x,y,z,w,\tau,\tau')\right| \left|\left(\mathcal{H}^{n-1}(z,w,\tau') - \mathcal{H}^{n-1}(x,y,\tau')\right)\right| \, dzdw \\
&\leq \int_\Omega c_1(\tau-\tau')^{-2} \exp\left(-C_2 \frac{(x-z)^2 + (y-w)^2}{\tau-\tau'}\right) \left|\left(\mathcal{H}^{n-1}(z,w,\tau') - \mathcal{H}^{n-1}(x,y,\tau')\right)\right| \, dzdw \\
&\leq \int_0^{2\pi} \int_0^\infty c_1(\tau-\tau')^{-2} \exp\left(-C_2 \frac{r^2}{\tau-\tau'}\right) C_6 r^\delta r \, dr d\theta \\
&\leq \tilde{C} \int_0^\infty (\tau-\tau')^{-2} \exp\left(-C_2 \left(\frac{r^{2+\delta}}{(\tau-\tau')^{\frac{2+\delta}{2}}}\right)^{\frac{2}{2+\delta}}\right) r^{1+\delta} \, dr \\
&\leq \tilde{C} \int_0^\infty (\tau-\tau')^{-2} \exp\left(-C_2 s^{\frac{2}{2+\delta}}\right) (\tau-\tau')^{1+\delta/2} \, ds \\
&\leq \tilde{\tilde{C}} (\tau-\tau')^{-1+\delta/2},
\end{aligned}$$

where $\tilde{\tilde{C}}$ is a constant independent of T. Combining two previous results together, and putting them under same exponent $\gamma = \min(\beta, 1-\frac{\delta}{2})$.

Using all previous estimates we have

$$
\begin{aligned}
&\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} = \\
&= \|v^n(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)} + \sum_{i=1}^{2} \|v^n_{x_i}(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)} + \sum_{i=1,j=1}^{2} \|v^n_{x_i,x_j}(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)} \\
&\le \int_0^\tau \left((A + B(\tau-t)^{-1/2} + D(\tau-t)^{-\gamma}\right)(C_5\|v^{n-1}\|_{W^2_\infty(\Omega)} + C_T)dt \\
&= C_T\left(A\tau + 2B\tau^{1/2} + D\frac{\tau^{1-\gamma}}{1-\gamma}\right) \\
&\quad + C_5\int_0^\tau \left(A + B(\tau-t)^{-1/2} + D(\tau-t)^{-\gamma}\right)\|v^{n-1}\|_{W^2_\infty(\Omega)}dt \\
&\le C(T,\gamma) + C\int_0^\tau \left(A + B(\tau-t)^{-1/2} + D(\tau-t)^{-\gamma}\right)\|v^{n-1}\|_{W^2_\infty(\Omega)}dt
\end{aligned}
$$

Where constants A, B, C_5, D depends only on Ω, doesn't depend on T and constant $C(T,\gamma)$ depends on γ, T, Ω. Therefore we have

$$
\begin{aligned}
&\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} && (3.10)\\
&\le C(T,\gamma) + C\int_0^\tau \left(A + B(\tau-t)^{-1/2} + D(\tau-t)^{-\gamma}\right)\|v^{n-1}\|_{W^2_\infty(\Omega)}dt && (3.11)
\end{aligned}
$$

This is the point where the fact that $\gamma \in (0,1)$ is crucial. If γ would be greater or equal to 1, the integral would diverge. Observe that there exist an upper bound ϵ of the integral

$$\int_0^\tau \left(A + B(\tau-t)^{-1/2} + D(\tau-t)^{-\gamma}\right)dt$$

for $\tau \in [0,\hat{T}]$ where $\hat{T} \le T$. We need to pick $\hat{T}$ such that $|\epsilon C| < 1$. This is possible since C does not depend on T. After solving the initial problem on the interval $[0,\hat{T}]$ we can move to the next interval of length $\hat{T}$ and solve the same problem with solution $v(\hat{T})$ as an initial condition. There will be finitely many of those subintervals. From this point on we will consider $\tau \in [0,\hat{T}]$.

For v^1 we can observe from (3.10) that

$$\|v^1(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} \le C(T,\gamma)$$

similarly for v^2 we have

$$\begin{aligned}
&\|v^2(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} \\
&\le C(T,\gamma) + C\int_0^\tau \left(A + B(\tau-t)^{-1/2} + D(\tau-t)^{-\gamma}\right)\|v^1\|_{W^2_\infty(\Omega)}dt \\
&= C(T,\gamma) + C(\hat{T},\gamma)\cdot C\int_0^\tau \left(A + B(\tau-t)^{-1/2} + D(\tau-t)^{-\gamma}\right)dt \\
&= C(T,\gamma) + C(\hat{T},\gamma)\cdot C\varepsilon
\end{aligned}$$

so for v^n we have

$$\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} = C(T,\gamma)\left(1 + C\epsilon + (C\varepsilon)^2 + \cdots + (C\varepsilon)^{n-1}\right) \le \frac{C(T,\gamma)}{1-C\varepsilon}$$

From the definition of the norm $\|\cdot\|_{W^2_\infty(\Omega)}$ and the fact that $v^n \in C^{2+\delta,1+\delta/2}(\overline{Q_{\hat{T}}})$ it is clear that $|v^n(\cdot,\cdot,\tau)|, |v^n_{x_i}(\cdot,\cdot,\tau)|, |v^n_{x_i,x_j}(\cdot,\cdot,\tau)|$ are uniformly bounded on $\overline{Q_{\hat{T}}}$ for all $n \in \mathbb{N}$. From equation (3.6), for v^n_τ we have also that $|v^n_\tau|$ is also bounded on $\overline{Q_{\hat{T}}}$ for all $n \in \mathbb{N}$.

Theorem 4. *Let $\{L_m\}$ be a sequence of parabolic operators satisfying $\boldsymbol{A(2)}$ and $\|a^m_{i,j}\|_{C^{\delta,\delta/2}(\overline{Q_T})} < K_1, \|\frac{\partial}{\partial x_j}b^m_i\|_{C^{\delta,\delta/2}(\overline{Q_T})} < K_1, \|\frac{\partial^2}{\partial x_j \partial x_j}c^m\|_{C^{\delta,\delta/2}(\overline{Q_T})} < K_1$ for constant K_1 independent of m. And let $\{f_m\}$ be a sequence for of functions satisfying $\|f_m\|_{C^{\delta,\delta/2}(\overline{Q_T})} \le K_2$ where K_2 is independent of m. Suppose that $\{u_m\}$ is a sequence of functions satisfying*

$$-\frac{\partial}{\partial t}u_m + L_m u_m = f_m \quad in \quad Q_T$$

If $|u_m|_{0;Q_T} \le K_3$ where K_3 is independent of m, then for any subsequence $\{u_{m'}\}$ of $\{u_m\}$ there exist a subsequence of it, say $\{u_{m''}\}$, such that

$$u_{m''}\ ,\ D_{x_i}u_{m''}\ ,\ D_{x_i,x_j}u_{m''}\ ,\ D_t u_{m''}$$

are uniformly convergent in subdomain of D, whose closure is in Q_T, to some function u and its corresponding derivatives. Furthermore, $u \in C^{2+\delta,1+\delta/2}$.

If in particular, the coefficients of L_m converge to the corresponding coefficients of L and $\{f_m\}$ converges to f, pointwise in Q_T, then $\underline{Lu = f \text{ in } Q_T}$

Can be found in [9, pg. 80], Ch. 3, Section 6.

Theorem 5. *There exist strong solution $u \in C^{2+\delta,1+\delta/2}$ the to problem* (3.2) *on smooth bounded domain $Q_T = \Omega \times (0,T)$.*

Proof. From the previous estimates and theorem 4 there exists a solution $v \in C^{2+\delta,1+\delta/2}$ to the iterative problem (3.6) with 'zero' initial and boundary condition. Consequently there exist solution in $C^{2+\delta,1+\delta/2}$ to the original problem (3.2).

□

3.2.3 Solution in the unbounded domain $\mathbb{R}^2$

To finish the proof, we need to assume

- **A(6)** - for any bounded smooth domain Ω' and for all smooth bounded Ω, $\Omega' \subseteq \Omega$ there exist positive constant $C(\Omega')$ such that <u>for any solution u</u> to the equation (3.2) on $\Omega \times (0,T)$

$$|u| \leq C(\Omega') \text{ on } \overline{Q'_T} \tag{3.12}$$

$$|u_{x_i}| \leq C(\Omega') \text{ on } \overline{Q'_T} \tag{3.13}$$

$$\left|u_{x_i,x_j}\right| \leq C(\Omega') \text{ on } \overline{Q'_T} \tag{3.14}$$

$$|u_t| \leq C(\Omega') \text{ on } \overline{Q'_T} \tag{3.15}$$

where $Q'_T = \Omega' \times (0,T)$

Theorem 6. *Under **A(1) - A(6)**, there exist solution u to the* (3.2) *in $D_T = \mathbb{R}^2 \times (0,T)$ and $u \in C^{2,1}(D_T) \cap C^{2+\delta,1+\delta/2}_{loc}(D_T)$.*

Proof. We will prove this by using Cantor diagonal argument Consider set of solutions $\{u^M\}_{M\in\mathbb{N}}$. such that u^M solves (3.2) on $B(0,M)\times(0,T)$. By theorem 4 and assumptions **A(1) - A(6)**, there exist subsequence $\{u^{M_{k_1}}\}\subseteq\{u^M\}_{M\in\mathbb{N}}$ which converges uniformly to some function $u_1\in C^{2+\delta,1+\delta/2}(B(0,1)\times(0,T))$. In the next step we will take sequence $\{u^{M_{k_2}}\}\subseteq\{u^{M_{k_1}}\}$ which converges uniformly on $B(0,2)\times(0,T)$ to some function $u_2\in C^{2+\delta,1+\delta/2}(B(0,2)\times(0,T))$. Then step take subsequence $\{u^{M_{k_n}}\}\subseteq\{u^{M_{k_{n-1}}}\}$ s.t. we have $u_n\in C^{2+\delta,1+\delta/2}(B(0,n)\times(0,T))$ etc.

From construction of diagonal sequence it is clear that

$$u_n\Big|_{B(0,k)\times(0,T)}=u_k\ \text{ for any }\ k\le n$$

Sequence $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly on any compact subset of $\mathbb{R}^2\times(0,T)$ to the function $u\in C^{2,1}(D_T)\cap C^{2+\delta,1+\delta/2}_{loc}(D_T)$

□

Chapter 4

Numerical results

In this chapter, we are going to discretize our inverse parabolic problem (2.1) and create a linear system. We use finite differences method to get a numerical solution of our equation [16, 17].

4.1 Discretization and transformation of the problem

We rewrite (2.1) using the notation

$$V_x = \frac{\partial V}{\partial x}$$

to the following form

$$\begin{aligned} &V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\beta^2\sigma^2 V_{\sigma\sigma} + \rho\beta\sigma^2 S V_{S\sigma} + rSV_S + r\sigma V_\sigma - rV \\ &- \kappa S\sqrt{\frac{2}{\pi\delta t}}\sigma\sqrt{S^2 V_{SS}^2 + 2\rho\beta S V_{SS} V_{S\sigma} + \beta^2 V_{S\sigma}^2} = 0 \end{aligned}$$

We us the following substitution to make our problem forward parabolic,

$$V(t, S, \sigma) = W(T - t, S, \sigma)$$

this gives us

$$W_t = \frac{1}{2}\sigma^2 S^2 W_{SS} + \frac{1}{2}\beta^2\sigma^2 W_{\sigma\sigma} + \rho\beta\sigma^2 S W_{S\sigma} + rSW_S + r\sigma W_\sigma - rW \tag{4.1}$$

$$- \kappa S\sqrt{\frac{2}{\pi\delta t}}\sigma\sqrt{S^2 W_{SS}^2 + 2\rho\beta S W_{SS} W_{S\sigma} + \beta^2 W_{S\sigma}^2}. \tag{4.2}$$

In the next step we construct grid for finite spatial domain

$$\Omega = [\sigma_{min}, \sigma_{max}] \times [0, S_{max}].$$

We divide intervals $[0, S_{\max}]$ and $[\sigma_{\min}, \sigma_{\max}]$ into k+2 equidistant point, m+2 equidistant points respectively. Where $0, S_{\max}, \sigma_{\min}, \sigma_{\max}$ define boundary for our problem. Means that $S_i = ih$ where $i = 0, 1, \ldots, k+1$ such that $S_{k+1} = S_{\max}, S_0 = 0$ and $\sigma = jl + \sigma_{\min}$ where $j = 1, 2, \ldots, m+1$ such that $\sigma_0 = \sigma_{\min}, \sigma_{m+1} = \sigma_{\max}$. You can see the discretization of Ω in figure 4.1.

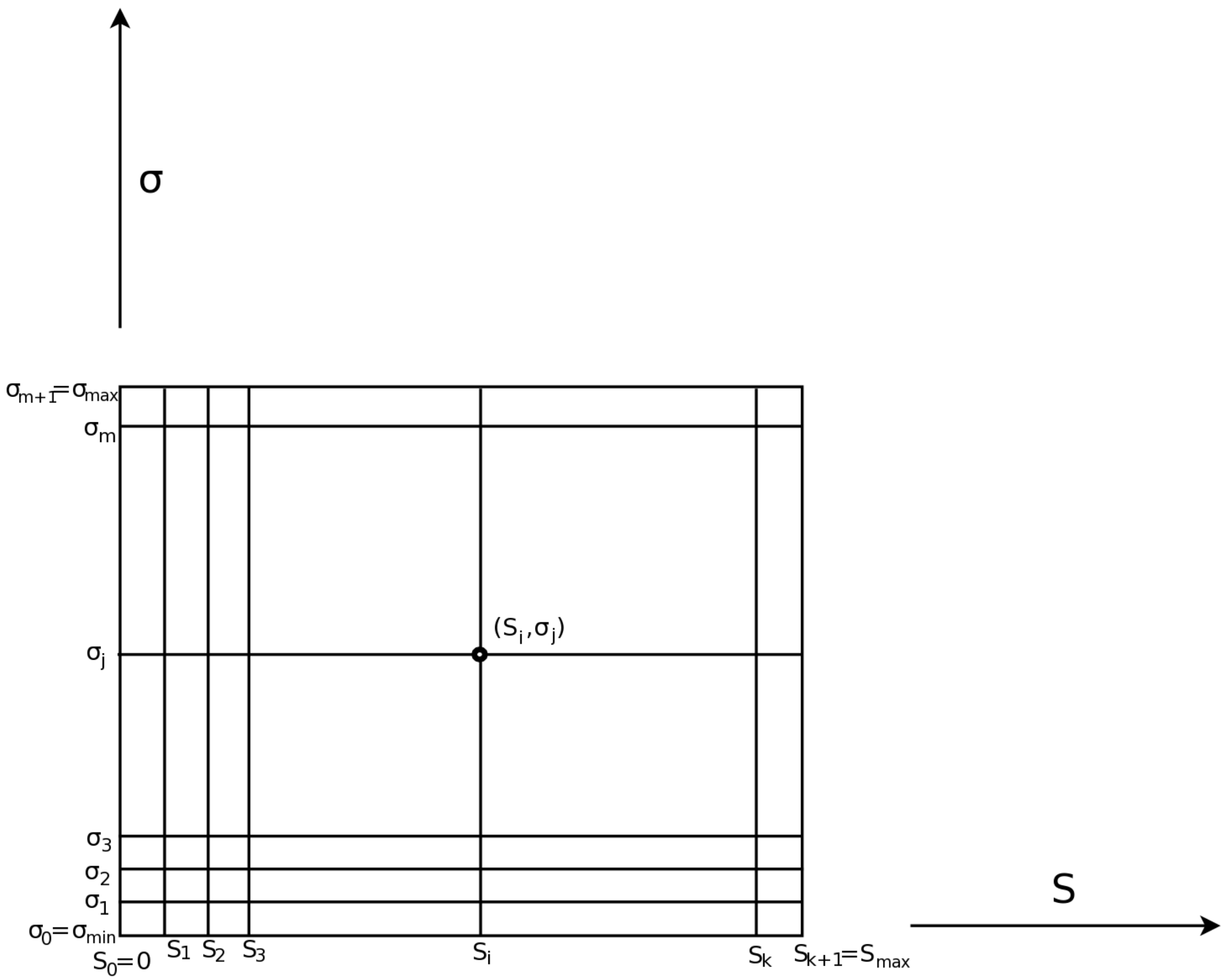


Figure 4.1: Domain discretization

Let us also define :

$$W_{i,j}^{(n)} = W(\Delta tn, S_i, \sigma_j) = W(\Delta tn, ih, jl + \sigma_{\min})$$

Where Δt is the considered time step. Approximation of derivatives at grid point (i,j) will look like

$$W_t \approx \frac{W_{i,j}^{(n+1)} - W_{i,j}^{(n)}}{\Delta t}$$

$$W_S \approx \theta \frac{W_{i+1,j}^{(n+1)} - W_{i-1,j}^{(n+1)}}{2h} + (1-\theta)\frac{W_{i+1,j}^{(n)} - W_{i-1,j}^{(n)}}{2h}$$

$$W_\sigma \approx \theta \frac{W_{i,j+1}^{(n+1)} - W_{i,j-1}^{(n+1)}}{2l} + (1-\theta)\frac{W_{i,j+1}^{(n)} - W_{i,j-1}^{(n)}}{2l}$$

$$W_{SS} \approx \theta \frac{W_{i+1,j}^{(n+1)} - 2W_{i,j}^{(n+1)} + W_{i-1,j}^{(n+1)}}{h^2} + (1-\theta)\frac{W_{i+1,j}^{(n)} - 2W_{i,j}^{(n)} + W_{i-1,j}^{(n)}}{h^2}$$

$$W_{\sigma\sigma} \approx \theta \frac{W_{i,j+1}^{(n+1)} - 2W_{i,j}^{(n+1)} + W_{i,j-1}^{(n+1)}}{l^2} + (1-\theta)\frac{W_{i,j+1}^{(n)} - 2W_{i,j}^{(n)} + W_{i,j-1}^{(n)}}{l^2}$$

$$\begin{aligned} W_{S\sigma} \approx \theta & \frac{W_{i+1,j+1}^{(n+1)} - W_{i-1,j+1}^{(n+1)} - W_{i+1,j-1}^{(n+1)} + W_{i-1,j-1}^{(n+1)}}{4lh} + \\ & (1-\theta)\frac{W_{i+1,j+1}^{(n)} - W_{i-1,j+1}^{(n)} - W_{i+1,j-1}^{(n)} + W_{i-1,j-1}^{(n)}}{4lh} \end{aligned}$$

where θ is a parameter of choice. For $\theta = 0$ we have an implicit method, explicit method for $\theta = 1$ and Crank-Nicolson method for $\theta = 0.5$. In order to construct linear system we also need to linearize term in square root containing derivatives. We consider two cases, when $\rho \approx 1$ and $\rho \approx -1$.

$$\sqrt{S^2 W_{SS}^2 + 2\rho\beta S W_{SS} W_{S\sigma} + \beta^2 W_{S\sigma}^2} \approx |SW_{SS} + \beta W_{S\sigma}| \quad \text{when} \quad \rho \approx 1$$

$$\sqrt{S^2 W_{SS}^2 + 2\rho\beta S W_{SS} W_{S\sigma} + \beta^2 W_{S\sigma}^2} \approx |SW_{SS} - \beta W_{S\sigma}| \quad \text{when} \quad \rho \approx -1$$

At the grind point *(i,j)* and time n the approximation for $\rho \approx 1$ will be :

$$\sqrt{S^2W_{SS}^2 + 2\rho\beta SW_{SS}W_{S\sigma} + \beta^2W_{S\sigma}^2} \approx |SW_{SS} + \beta W_{S\sigma}| =$$
$$\theta sgn(S_i(W_{SS})_{i,j}^{(n)} + \beta(W_{S\sigma})_{i,j}^{(n)})(S_i(W_{SS})_{i,j}^{(n+1)} + \beta(W_{S\sigma})_{i,j}^{(n+1)}) +$$
$$(1-\theta)sgn(S_i(W_{SS})_{i,j}^{(n)} + \beta(W_{S\sigma})_{i,j}^{(n)})(S_i(W_{SS})_{i,j}^{(n)} + \beta(W_{S\sigma})_{i,j}^{(n)})$$

similarly for $\rho \approx -1$:

$$\sqrt{S^2W_{SS}^2 + 2\rho\beta SW_{SS}W_{S\sigma} + \beta^2W_{S\sigma}^2} \approx |SW_{SS} - \beta W_{S\sigma}| =$$
$$\theta sgn(S_i(W_{SS})_{i,j}^{(n)} - \beta(W_{S\sigma})_{i,j}^{(n)})(S_i(W_{SS})_{i,j}^{(n+1)} - \beta(W_{S\sigma})_{i,j}^{(n+1)})$$
$$(1-\theta)sgn(S_i(W_{SS})_{i,j}^{(n)} - \beta(W_{S\sigma})_{i,j}^{(n)})(S_i(W_{SS})_{i,j}^{(n)} - \beta(W_{S\sigma})_{i,j}^{(n)})$$

Where

$$(W_{SS})_{i,j}^{(n)} = \frac{W_{i+1,j}^{(n)} - 2W_{i,j}^{(n)} + W_{i-1,j}^{(n)}}{h^2}$$
$$(W_{S\sigma})_{i,j}^{(n)} = \frac{W_{i+1,j+1}^{(n)} - W_{i-1,j+1}^{(n)} - W_{i+1,j-1}^{(n)} + W_{i-1,j-1}^{(n)}}{4lh}$$

This approximation, by almost linear term, makes sense also for ρ close to 0. Our objective is to approximate the desired square root by an almost linear expression, i.e. by absolute value of linear expression. In other words, we want approximate term of form $\sqrt{a^2 + 2\rho ab + b^2}$ by the following term $|k_a a + k_b b|$. Since quadratic root is symmetric, we would expect the same behavior from our approximation, i.e. $|k_a a + k_b b| = |k_b a + k_a b|$. This can only happen if k_a, k_b differ by factor ± 1 so we get either $k_a = k_b$ or $k_a = -k_b$. For both cases we have

$$\sqrt{a^2 + 2\rho ab + b^2} \approx k|a \pm b| \ , \ k \geq 0 \tag{4.3}$$
$$(1-k^2)a^2 + 2(\rho \mp k)ab + (1-k^2)b^2 \approx 0 \ , \ k \geq 0$$

We don't know anything about a, b but we can assume that they are of the same order. For $0 \leq \rho \leq 1$ gives us term to minimize with constrain $k \geq 0$

$$|1-k^2| + |\rho \mp k|.$$

This term is minimized for $k = 1$ and plus sign in absolute value in (4.3). By repeating previous steps, we can also get approximation for $\sqrt{a^2 + 2\rho ab + b^2}$ for $\rho \in [-1, 0)$. Conclusion is that the best approximation is

$$\begin{aligned}
\sqrt{a^2 + 2\rho ab + b^2} &\approx |a^2 + b^2| \qquad && \text{for} \qquad && 0 \leq \rho \leq 1 \\
\sqrt{a^2 + 2\rho ab + b^2} &\approx |a^2 - b^2| \qquad && \text{for} \qquad && -1 \leq \rho < 0
\end{aligned}$$

All equations above construct linear system for $k \times m$ variables. Let us show it, step by step. After using approximation above, (4.1) will be

$$\begin{aligned}
W_t = {} & \frac{1}{2}\sigma^2 S^2 W_{SS} + \frac{1}{2}\beta^2\sigma^2 W_{\sigma\sigma} + \rho\beta\sigma^2 S W_{S\sigma} + rSW_S + r\sigma W_\sigma - rW \\
& - \kappa S \sqrt{\frac{2}{\pi \delta t}}\sigma \ \cdot \ \operatorname{sgn}\left(SW_{SS} \pm \beta W_{S\sigma}\right)\left(SW_{SS} \pm \beta W_{S\sigma}\right)
\end{aligned}$$

using finite differences for point grid point (i, j) and time n we will have

$$\begin{aligned}
\frac{W_{i,j}^{(n+1)} - W_{i,j}^{(n)}}{\Delta t} = {} & \frac{1}{2} S_i^2 \sigma_j^2 \left(\theta \frac{W_{i+1,j}^{(n+1)} - 2W_{i,j}^{(n+1)} + W_{i-1,j}^{(n+1)}}{h^2} + (1-\theta)\frac{W_{i+1,j}^{(n)} - 2W_{i,j}^{(n)} + W_{i-1,j}^{(n)}}{h^2}\right) \\
& + \frac{1}{2}\beta^2\sigma_j^2 \left(\theta \frac{W_{i,j+1}^{(n+1)} - 2W_{i,j}^{(n+1)} + W_{i,j-1}^{(n+1)}}{l^2} + (1-\theta)\frac{W_{i,j+1}^{(n)} - 2W_{i,j}^{(n)} + W_{i,j-1}^{(n)}}{l^2}\right) \\
& + \rho\beta\sigma_j^2 S_i \left(\theta \frac{W_{i+1,j+1}^{(n+1)} - W_{i-1,j+1}^{(n+1)} - W_{i+1,j-1}^{(n+1)} + W_{i-1,j-1}^{(n+1)}}{4lh} + (1-\theta)\frac{W_{i+1,j+1}^{(n)} - W_{i-1,j+1}^{(n)} - W_{i+1,j-1}^{(n)} + W_{i-1,j-1}^{(n)}}{4lh}\right) \\
& + rS_i \left(\theta \frac{W_{i+1,j}^{(n+1)} - W_{i-1,j}^{(n+1)}}{2h} + (1-\theta)\frac{W_{i+1,j}^{(n)} - W_{i-1,j}^{(n)}}{2h}\right) + r\sigma_j \left(\theta \frac{W_{i,j+1}^{(n+1)} - W_{i,j-1}^{(n+1)}}{2l} + (1-\theta)\frac{W_{i,j+1}^{(n)} - W_{i,j-1}^{(n)}}{2l}\right) \\
& - r(\theta W_{i,j}^{(n+1)} + (1-\theta)W_{i,j}^{(n)})
\end{aligned}$$

$$-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\Big(\theta sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})(S_i(W_{SS})_{i,j}^{(n+1)} \pm \beta(W_{S\sigma})_{i,j}^{(n+1)})$$
$$+(1-\theta)sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})\Big)$$

after rearranging coefficients and multiplying by Δt, the equation above gives us :

$$W_{i,j}^{(n+1)}\left(1-\Delta t\theta\left(\frac{1}{2}S_i^2\frac{-2}{h^2}+\frac{1}{2}\beta^2\sigma_j^2\frac{-2}{l^2}-r-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})\frac{-2}{h^2}\right)\right)+$$
$$W_{i+1,j}^{(n+1)}\left(-\Delta t\theta\left(\frac{1}{2}S_i^2\sigma_j^2\frac{1}{h^2}+rS_i\frac{1}{2h}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})\frac{1}{h^2}\right)\right)+$$
$$W_{i-1,j}^{(n+1)}\left(-\Delta t\theta\left(\frac{1}{2}S_i^2\sigma_j^2\frac{1}{h^2}-rS_i\frac{1}{2h}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})\frac{1}{h^2}\right)\right)+$$
$$W_{i,j+1}^{(n+1)}\left(-\Delta t\theta\left(\frac{1}{2}\beta^2\sigma_j^2\frac{1}{l^2}+r\sigma_j\frac{1}{2l}\right)\right)+W_{i,j-1}^{(n+1)}\left(-\Delta t\theta\left(\frac{1}{2}\beta^2\sigma_j^2\frac{1}{l^2}-r\sigma_j\frac{1}{2l}\right)\right)+$$
$$W_{i+1,j+1}^{(n+1)}\left(-\Delta t\theta\left(\rho\beta\sigma_j^2S_i\frac{1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{1}{4lh}\right)\right)+$$
$$W_{i-1,j-1}^{(n+1)}\left(-\Delta t\theta\left(\rho\beta\sigma_j^2S_i\frac{1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{1}{4lh}\right)\right)+$$
$$W_{i+1,j-1}^{(n+1)}\left(-\Delta t\theta\left(\rho\beta\sigma_j^2S_i\frac{-1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{-1}{4lh}\right)\right)+$$
$$W_{i-1,j+1}^{(n+1)}\left(-\Delta t\theta\left(\rho\beta\sigma_j^2S_i\frac{-1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{-1}{4lh}\right)\right)$$
$$=$$
$$W_{i,j}^{(n)}\left(1+\Delta t(1-\theta)\left(\frac{1}{2}S_i^2\frac{-2}{h^2}+\frac{1}{2}\beta^2\sigma_j^2\frac{-2}{l^2}-r-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})\frac{-2}{h^2}\right)\right)+$$
$$W_{i+1,j}^{(n+1)}\left(+\Delta t(1-\theta)\left(\frac{1}{2}S_i^2\sigma_j^2\frac{1}{h^2}+rS_i\frac{1}{2h}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})\frac{1}{h^2}\right)\right)+$$
$$W_{i-1,j}^{(n+1)}\left(+\Delta t(1-\theta)\left(\frac{1}{2}S_i^2\sigma_j^2\frac{1}{h^2}-rS_i\frac{1}{2h}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)} \pm \beta(W_{S\sigma})_{i,j}^{(n)})\frac{1}{h^2}\right)\right)+$$
$$W_{i,j+1}^{(n)}\left(+\Delta t(1-\theta)\left(\frac{1}{2}\beta^2\sigma_j^2\frac{1}{l^2}+r\sigma_j\frac{1}{2l}\right)\right)+W_{i,j-1}^{(n)}\left(+\Delta t(1-\theta)\left(\frac{1}{2}\beta^2\sigma_j^2\frac{1}{l^2}-r\sigma_j\frac{1}{2l}\right)\right)+$$

$$W_{i+1,j+1}^{(n+1)}\left(+\Delta t(1-\theta)\left(\rho\beta\sigma_j^2 S_i\frac{1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)}\pm\beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{1}{4lh}\right)\right)+$$
$$W_{i-1,j-1}^{(n+1)}\left(+\Delta t(1-\theta)\left(\rho\beta\sigma_j^2 S_i\frac{1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)}\pm\beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{1}{4lh}\right)\right)+$$
$$W_{i+1,j-1}^{(n+1)}\left(+\Delta t(1-\theta)\left(\rho\beta\sigma_j^2 S_i\frac{-1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)}\pm\beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{-1}{4lh}\right)\right)$$
$$W_{i-1,j+1}^{(n+1)}\left(+\Delta t(1-\theta)\left(\rho\beta\sigma_j^2 S_i\frac{-1}{4lh}-\kappa S_i\sqrt{\frac{2}{\pi\delta t}}\sigma_j\cdot sgn(S_i(W_{SS})_{i,j}^{(n)}\pm\beta(W_{S\sigma})_{i,j}^{(n)})(\pm\beta)\frac{-1}{4lh}\right)\right) \tag{4.4}$$

which is rather a complicated expression. Using Einstein summation convention we can write that

$$L_{k,l}{}^{i,j}W_{i,j}^{(n+1)} = R_{k,l}{}^{i,j}W_{i,j}^{(n)} \tag{4.5}$$

where

$$L : \mathbb{R}^{k+2,m+2} \to \mathbb{R}^{k+2,m+2}$$
$$R : \mathbb{R}^{k+2,m+2} \to \mathbb{R}^{k+2,m+2}.$$

The equation (4.5) is not very handy for computer based calculations. Much better would be multiplication of a vector by a matrix, but there is a natural isomorphism between $\mathbb{R}^{k+2,m+2}$ and $\mathbb{R}^{(k+2)(m+2)}$. Let us call it H.

$$H : \mathbb{R}^{k+2,m+2} \to \mathbb{R}^{(k+2)(m+2)}$$
$$HW_{i,j} = W_{i+1+(k+2)\cdot j}$$

Then the system (4.5) will become

$$\underbrace{HLH^{-1}}_{\tilde{L}}\ \underbrace{HW^{(n+1)}}_{\tilde{W}^{(n+1)}} = \underbrace{HRH^{-1}}_{\tilde{R}}\ \underbrace{HW^{(n)}}_{\tilde{W}^{(n)}}$$
$$\tilde{L}\tilde{W}^{(n+1)} = \tilde{R}\tilde{W}^{(n)} \tag{4.6}$$

Equation (4.6) is suitable for computer based calculations.

4.1.1 Initial and boundary conditions

The part that need to be discussed is how to handle initial and boundary conditions. The initial condition is pretty straight forward. We just choose

$$W(0, S, \sigma) = f(S, \sigma)$$

Where f is just some, reasonably chosen function of S, σ. We already know from chapter 1 what the initial condition for W should be. The initial condition is just a payoff function for option V.

$$W(0, S, \sigma) = \max(S - K, 0)$$

In a notation of $W_{i,j}^{(n)}$

$$W_{i,j}^{(0)} = \max\{S_i - K, 0\}.$$

We can construct a linear system only for a point inside our descretized domain Ω, i.e. for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, m\}$. Points $W_{i,j}^{(n)}$ need to be replaced by boundary condition whenever $i = 0, i = k+1, j = 0, j = m+1$. Since there is no known analytical solution to this problem, we pick a boundary condition

$$g(S, \sigma, t) = max\{S - K, 0\}.$$

This also means

$$W(t, S, \sigma) = \max\{S - K, 0\}$$

and

$$W_{i,j}^{(n)} = \max\{S_i - K, 0\}\ , \quad \text{whenever } i = 0, i = k+1, j = 0, j = m+1.$$

Those conditions will replace some terms in (4.4). The boundary condition and initial condition need to be compatible, this means that the boundary function at boundary and time 0 need to match the initial condition at boundary.

$$f(S,\sigma) = g(0,S,\sigma) \quad , \quad \text{on } \partial\Omega$$

Since both f, g are the same function, the compatibility condition is satisfied.

4.2 Numerical results

For this section we developed a function in Matlab to solve the problem (4.6) given an initial, boundary condition and the set of all other parameters. In this section you will see $W(S,\sigma,t)$ computed for time $t = 5$.

We considered set of following parameters (common for all figures):

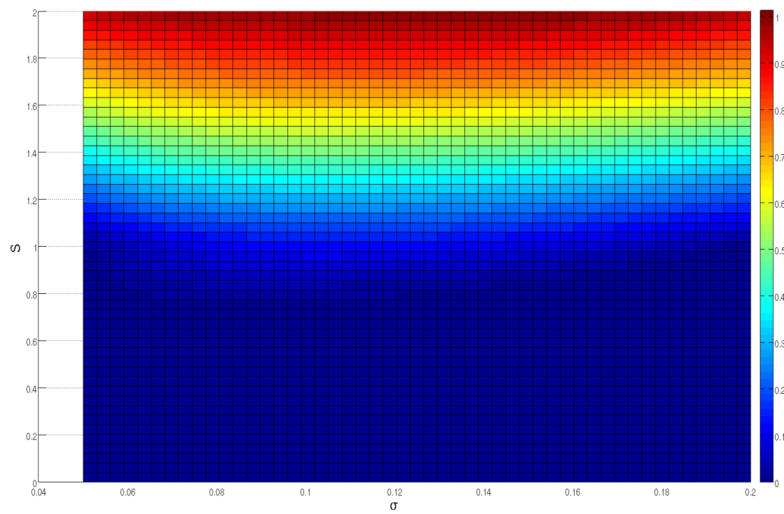
- Maximal considered underlying asset value $S_{\max} = 2$
- Minimal considered volatility $\sigma_{\min} = 0.05$
- Maximal considered volatility $\sigma_{\max} = 0.2$
- Volatility of volatility $\beta = 0.4$
- Number of inner grid points of S axis $k = 50$
- Number of inner grid points of σ axis $m = 50$
- Time increment $\Delta t = 0.5/60 = 8.33 \cdot 10^{-3}$
- Strike price $K = 1$
- Riskless rate $r = 0.05$

Rest of parameters for each one of the figures are following :

- Figure 4.2
  - Time between two re-hedging $\delta t = 1$
  - Transaction cost $\kappa = 0.01$
  - Correlation $\rho = 0.8$
  -
- Figure 4.3
  - Time between two re-hedging $\delta t = 0.1$
  - Transaction cost $\kappa = 0.01$
  - Correlation $\rho = 0.8$
  -
- Figure 4.4
  - Time between two re-hedging $\delta t = 1/52.$. Rebalancing every week.
  - Transaction cost $\kappa = 0.005.$ (0.5 %)
  - Correlation $\rho = 0.8$
  -
- Figure 4.5
  - Time between two re-hedging $\delta t = 1$
  - Transaction cost $\kappa = 0.01$
  - Correlation $\rho = -0.8$
  -
- Figure 4.6
  - Time between two re-hedging $\delta t = 0.1$

- Transaction cost $\kappa = 0.01$
- Correlation $\rho = -0.8$
-

- Figure 4.7
  - Time between two re-hedging $\delta t = 1/52$
  - Transaction cost $\kappa = 0.0005$
  - Correlation $\rho = -0.3$
  -



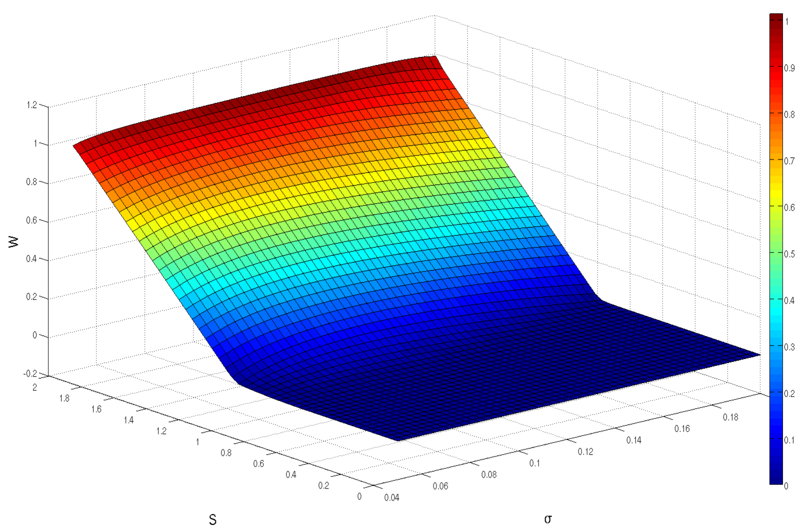


Figure 4.2: Top view and 3D view

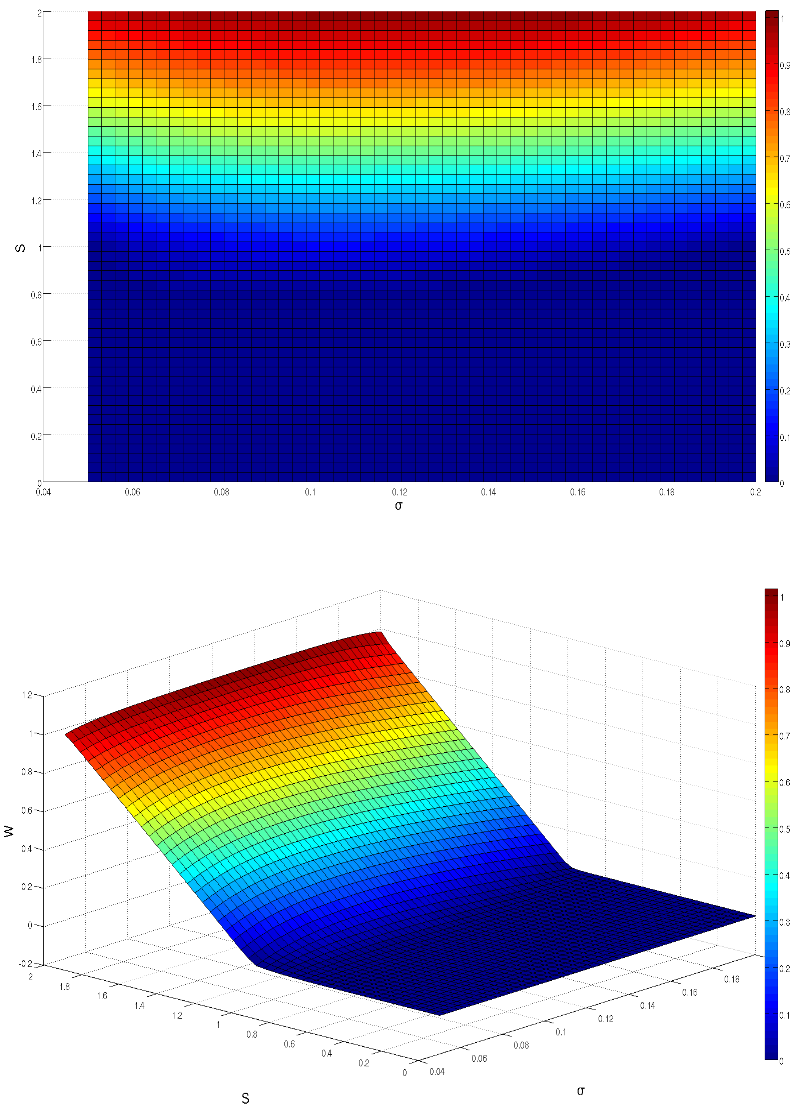


Figure 4.3: Top view and 3D view

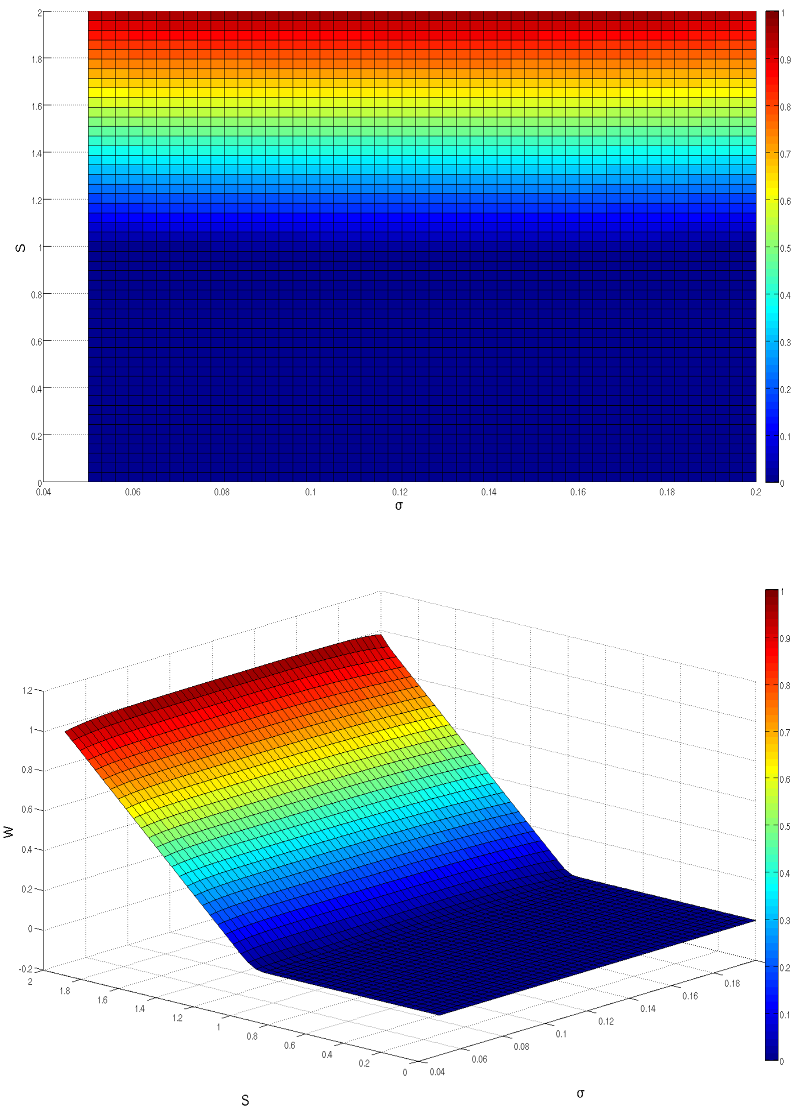


Figure 4.4: Top view and 3D view

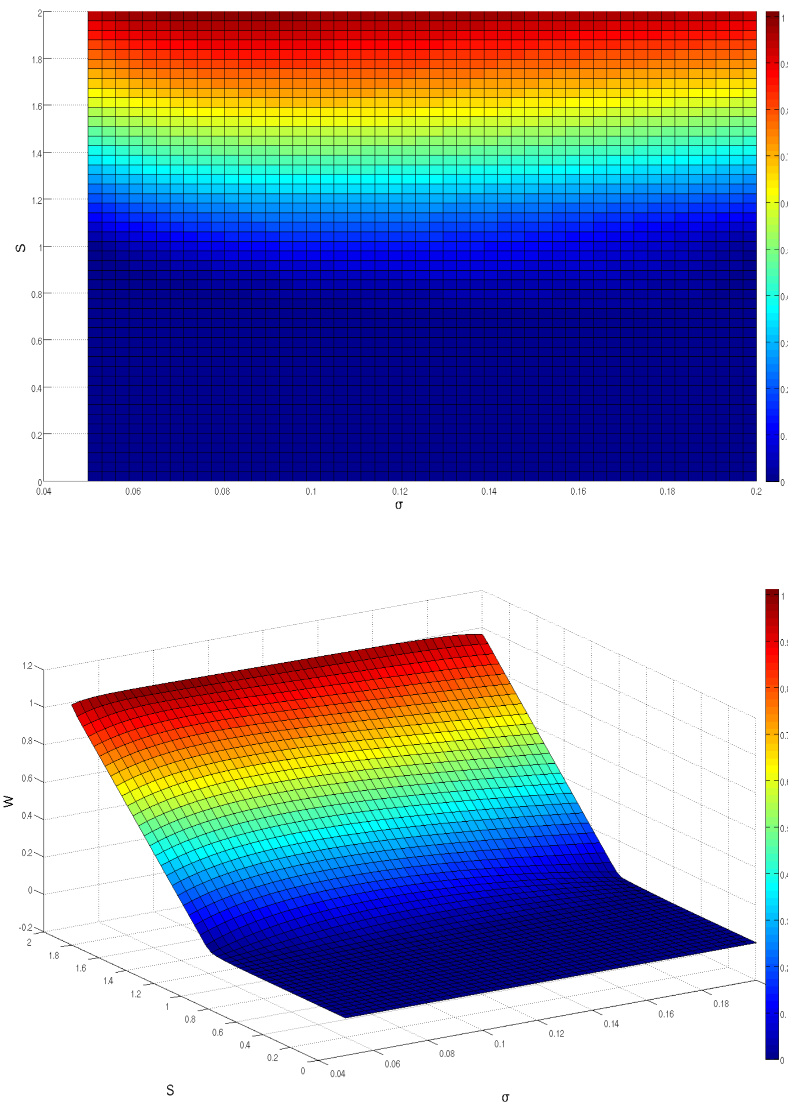
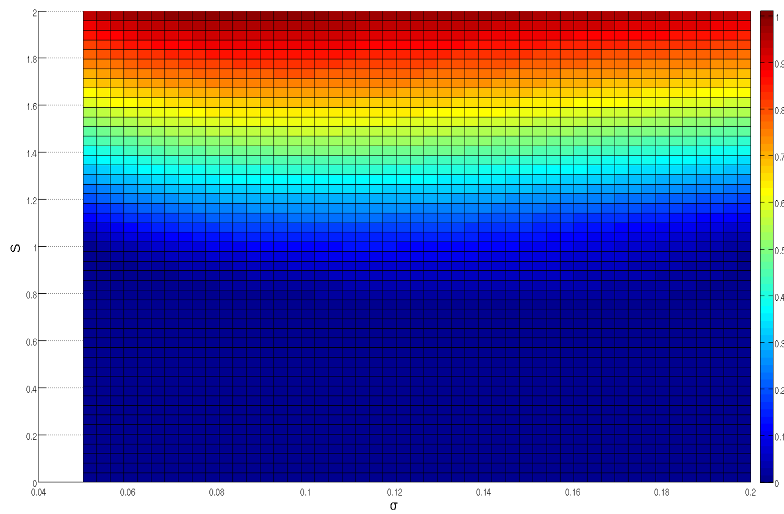


Figure 4.5: Top view and 3D view



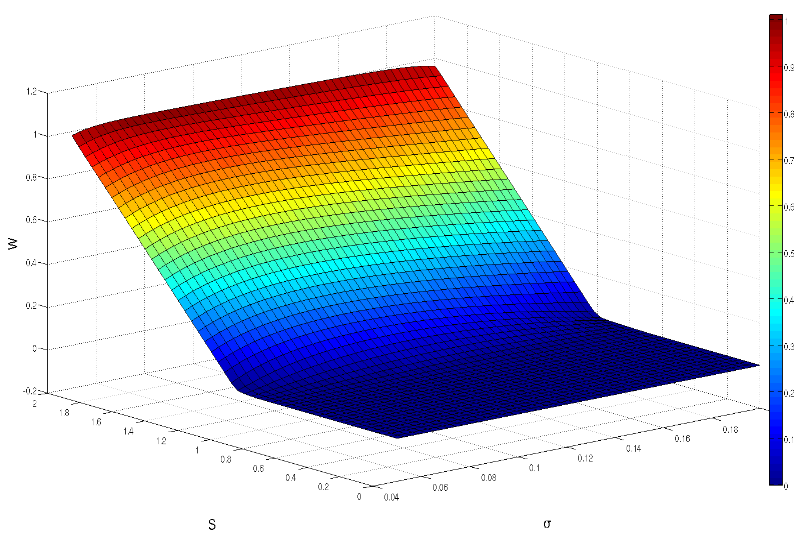
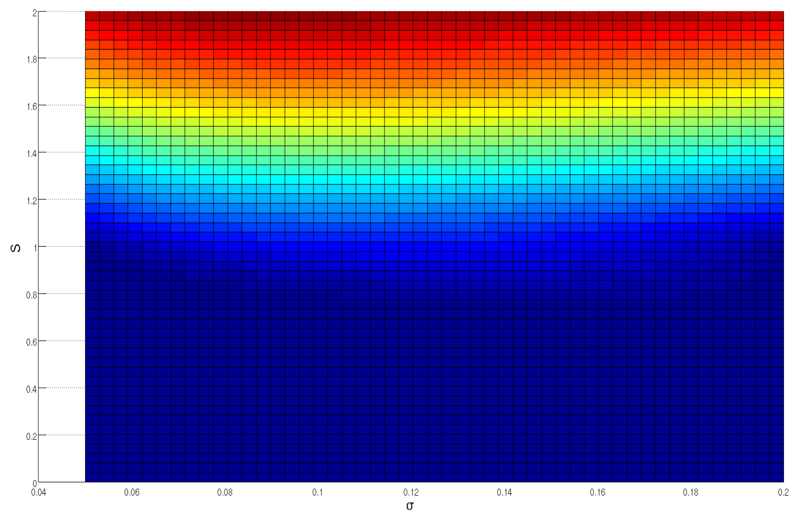


Figure 4.6: Top view and 3D view



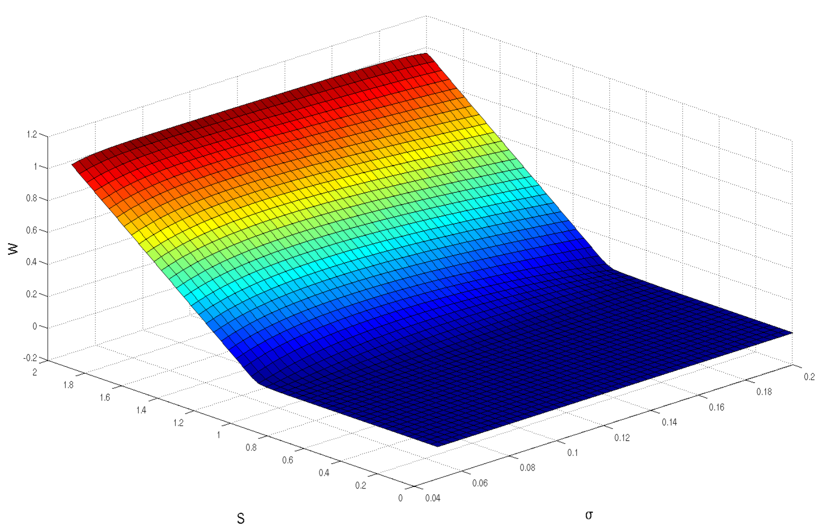


Figure 4.7: Top view and 3D view

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