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An Immersed Interface Method for a 1D Poroelasticity Problem with Discontinuous Coefficients

Maranda Lee Bean

University of Texas at El Paso, mlbean@miners.utep.edu

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AN IMMERSED INTERFACE METHOD FOR A 1D POROELASTICITY
PROBLEM WITH DISCONTINUOUS COEFFICIENTS

Maranda Bean

Department of Mathematics

APPROVED:

Son-Young Yi, Chair, Ph.D.

Granville Sewell, Ph.D.

Vinod Kumar, Ph.D.

Benjamin C. Flores, Ph.D.
Dean of the Graduate School

AN IMMERSED INTERFACE METHOD FOR A 1D POROELASTICITY
PROBLEM WITH DISCONTINUOUS COEFFICIENTS

by

Maranda Bean

THESIS

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Abstract

Poroelectric models deal with the coupling of changes in stress and fluid pressure in some porous medium. For physical reasons, some of the coefficients used in the models may have discontinuities. This project focuses on applying the immersed interface method to the one dimensional Biot model [1, 3, 4, 8], in order to handle these discontinuities. This method is applied on a staggered grid. Using the immersed interface method allows us to alter only a small number of irregular grid points surrounding a discontinuity. For most of the grid points, a standard finite difference method is used. However at the irregular grid points, the method of undetermined coefficients is used to improve the standard approximation. Finally numerical experiments are carried out to confirm an expected convergence rate of $O(h + \Delta t)$.

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Chapter 1

Introduction

In simple terms, the coupling of changes in stress and changes in fluid pressure are the main focus of the subject of poroelasticity [9]. Consider some solid to be a porous framework, for example a sponge or rock. For these purposes, the solid is uniform in all orientations or isotropic. The solid is considered to contain some fluid with a specific fluid pressure. Two phenomena are responsible for poroelastic behavior [9]. First is so called “solid - to - fluid” coupling. This occurs when applied stress on the solid is changed, causing a change in the fluid pressure. The second, “fluid - to - solid” , occurs when a change in fluid pressure causes a change in the solid. These are assumed to occur instantaneously and quasistatically. In this project, a coupled system of equations will be solved so that both the fluid pressure and medium displacement are found simultaneously.

Poroelasticity has applications to a number of fields. Biot [1] developed a mathematical model to deal with soil consolidations, particularly of soils saturated with fluid. The theory has also been used in petroleum engineering and hydrology [9] to determine subsidences, and predict behavior around bore holes. It is used in geophysics to describe small deformations of fluid filled rocks [5]. Poroelasticity also has application in biomechanics [2] in dealing with applications to bones or soft tissues.

Biot’s [1] equations were developed for an incompressible fluid when the soil is not completely saturated, but according to Gaspar et al. [4] the same equations can be used for a completely saturated medium and a slightly compressible fluid. This project will focus on the following variation of Biot’s model that neglects body forces, as developed by

Gaspar et.al [4].

$$-\mu\Delta u - (\lambda + \mu) \nabla (\nabla \cdot u) + \nabla p = 0, \quad (1.1a)$$

$$\frac{\partial}{\partial t} (\gamma p + \nabla \cdot u) - \frac{\kappa}{\nu} \Delta p = g(x, t), \quad x \in \Omega, \quad 0 \leq T. \quad (1.1b)$$

In these formulations, p is the fluid pressure and u is the media displacement. λ and μ are the Lamé coefficients. The coefficient $\gamma = nB$, where n is the porosity, or a measure of the empty space in the medium, and B represents the the compressibility of the fluid. κ is the permeability of the media and ν is a measure of the viscosity of the fluid. The quantity $\nabla \cdot u$ is used to represent the change in porosity of the system, also called the dilation, and therefore the sum $\gamma p + \nabla \cdot u$ represents the variation in fluid content at a particular time. Finally, the function $g(x, t)$ is used as a source term. It represents a forced fluid extraction or injection process. To complete the problem, boundary conditions must be given. Consider the boundary, $\delta\Omega$, to be made up of disjoint subsets, Γ_1 and Γ_2 , of positive measure such that $\Gamma_1 \cup \Gamma_2 = \delta\Omega$. In the following boundary conditions, \mathbf{n} represents the unit outward normal vector.

$$u = 0, \quad \frac{\kappa}{\nu} \nabla p \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_1, \quad (1.2a)$$

$$p = 0, \quad \sigma \mathbf{n} = \mathbf{h} \quad \text{on} \quad \Gamma_2, \quad (1.2b)$$

where $\sigma = \lambda \nabla \cdot u I + 2\mu \epsilon(u)$ is the effective stress on the medium, and $\epsilon(u) = 1/2 (\nabla u + \nabla u^T)$. The physical meanings of these boundary conditions are as follows. The condition $u = 0$ on Γ_1 describes a boundary where the medium may not move. While the condition $\sigma \mathbf{n} = \mathbf{h}$ on Γ_2 describes a place where the traction force is known. On the other hand, $p = 0$ on Γ_2 gives the known pressure along a piece of the boundary, and $\frac{\kappa}{\nu} \nabla p \cdot \mathbf{n} = 0$ on Γ_1 gives a condition stating that the fluid cannot flow through this piece of the boundary. Lastly, the model will need an initial condition. The following initial condition is used to describe the case where the variation in water content is initially 0 [8].

$$(\gamma p + \nabla \cdot u) = 0 \quad \text{at} \quad t = 0. \quad (1.3a)$$

The initial state of the system, at $t = 0$, is described by

$$-\mu\Delta u - (\lambda + \mu) \nabla (\nabla \cdot u) + \nabla p = 0, \quad (1.4a)$$

$$(\gamma p + \nabla \cdot u) = 0, \quad (1.4b)$$

$$u = 0 \quad on \Gamma_1, \quad \sigma n = h \quad on \Gamma_2. \quad (1.4c)$$

In many of the examples in this project, the initial conditions are calculated from the known solutions. In Chapter 2, the one dimensional and nondimensionalized model used in the presented work will be discussed in more detail.

The Immersed Interface Method has been developed to deal with problems arising from sharp discontinuities or jumps in the domain of interest [7]. For this particular method, information about the interface and the jump conditions is assumed to be known. To describe the basics of the method, consider a simple one dimensional domain with one interface. First, the domain is divided using a grid so that the interface falls between two grid points. These two grid points are considered to be irregular. The method focuses on finding a new approximation at these irregular grid points and using a standard finite difference scheme at all the other regular grid points [7]. The details of how to apply this method will be discussed in Chapter 3. Chapter 4 will discuss the results of some numerical experiments.

Chapter 2

Problem Formulation

This project will deal with a one-dimensional version of Biot's [1] model (1.1) as follows.

$$-(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x} = 0, \quad x \in (0, 1), \quad (2.1a)$$

$$\frac{\partial}{\partial t} \left(\gamma p + \frac{\partial u}{\partial x} \right) - \frac{\kappa}{\nu} \frac{\partial^2 p}{\partial x^2} = q(x, t), \quad x \in (0, 1), 0 < t \leq T. \quad (2.1b)$$

The boundary and initial conditions are given by

$$(\lambda + 2\mu)\frac{\partial u}{\partial x} = -u_0, \quad p = 0, \quad x = 0, \quad (2.2a)$$

$$u = 0, \quad \frac{\partial p}{\partial x} = 0, \quad x = 1, \quad (2.2b)$$

$$\left(\gamma p + \frac{\partial u}{\partial x} \right)(x) = 0, \quad x \in (0, 1) \quad t = 0. \quad (2.2c)$$

2.1 Nondimensionalization

Nondimensionalization will help to simplify the process of solving these equations. We nondimensionalize following the method of Gaspar [4].

$$x \equiv \frac{x}{l}, \quad t \equiv \frac{(\lambda + 2\mu) \kappa t}{\nu l^2}, \quad p \equiv \frac{p}{u_0}, \quad u \equiv \frac{(\lambda + 2\mu) u}{u_0 l}$$

Combining these with 2.1 and 2.2 yields

$$-\frac{\partial^2 u}{\partial x^2} + \frac{\partial p}{\partial x} = 0, \quad x \in (0, 1), \quad (2.3a)$$

$$\frac{\partial}{\partial t} \left(\gamma p + \frac{\partial u}{\partial x} \right) - \frac{\partial^2 p}{\partial x^2} = f(x, t), \quad x \in (0, 1), \quad 0 < t \leq T, \quad (2.3b)$$

where $f(x, t) = \frac{\nu l^2}{\kappa u_0} q(x, t)$ is the scaled source term. The boundary and initial conditions are given by

$$\frac{\partial u}{\partial x} = -1, \quad p = 0, \quad x = 0, \quad (2.4a)$$

$$u = 0, \quad \frac{\partial p}{\partial x} = 0, \quad x = 1, \quad (2.4b)$$

$$\left(ap + \frac{\partial u}{\partial x} \right) (x) = 0, \quad x \in (0, 1), \quad t = 0. \quad (2.4c)$$

It is more convenient to work with homogeneous boundary conditions, so the transformation $u(x, t) \equiv u(x, t) + x - 1$ is used [8]. This results in homogeneous boundary conditions and $\left(ap + \frac{\partial u}{\partial x} \right) (x) = 1$ for the initial condition. Since we will be considering discontinuous coefficients, the following variation will be used [3].

$$-\frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} = 0, \quad x \in (0, 1), \quad (2.5a)$$

$$\frac{\partial}{\partial t} \left(ap + \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x} \left(k \frac{\partial p}{\partial x} \right) = f(x, t), \quad x \in (0, 1), \quad 0 < t \leq T \quad (2.5b)$$

The boundary and initial conditions are given by

$$v \frac{\partial u}{\partial x} = 0, \quad p = 0, \quad x = 0, \quad (2.6a)$$

$$u = 0, \quad k \frac{\partial p}{\partial x} = 0, \quad x = 1, \quad (2.6b)$$

$$\left(ap + \frac{\partial u}{\partial x} \right) (x) = 1, \quad x \in (0, 1), \quad t = 0 \quad (2.6c)$$

2.2 Discontinuous problem

We will allow the coefficients, $v(x)$, $a(x)$, and $k(x)$ of (2.5a) and (2.5b) to be discontinuous at just one interface point, ζ , in the domain. For simplicity, these coefficients are considered to be piecewise constant of the following form.

$$v(x) = \begin{cases} v_1 & x < \zeta \\ v_2 & x > \zeta \end{cases}, \quad a(x) = \begin{cases} a_1 & x < \zeta \\ a_2 & x > \zeta \end{cases}, \quad k(x) = \begin{cases} k_1 & x < \zeta \\ k_2 & x > \zeta \end{cases}. \quad (2.7)$$

These functions have also been nondimensionalized as follows [3];

$$v(x) = \frac{\lambda + 2\mu}{\lambda_0 + 2\mu_0}, \quad (2.8)$$

$$a(x) = \gamma (\lambda_0 + 2\mu_0), \quad (2.9)$$

$$k(x) = \frac{k(x)}{k_0}. \quad (2.10)$$

$$(2.11)$$

The notation $[g] = g^+(\zeta) - g^-(\zeta)$ will be used to denote the jump at the interface ζ . These jump conditions must be satisfied at ζ .

$$[u] = 0, \quad (2.12a)$$

$$\left[v \frac{\partial u}{\partial x} \right] = 0, \quad (2.12b)$$

$$[p] = 0, \quad (2.12c)$$

$$\left[k \frac{\partial p}{\partial x} \right] = 0. \quad (2.12d)$$

We make the additional assumption that there is at most a finite jump in the function $f(x, t)$ at ζ at all times $0 \leq t \leq T$.

Chapter 3

The Immersed Interface Method

3.1 MAC type grid

It is shown by Gaspar et al. [4] that using a MAC type or staggered grid can improve some stability and convergence limitations for this type of problem. Some standard discretization can give rise to spurious oscillations in the pressure solutions. The oscillations are attributed to the existence of sharp boundary conditions in the initial time. Although these oscillations smooth quickly as time progresses, a MAC type grid produces a smooth solution for pressure with fewer grid points than standard grids. Therefore, we use two separate grids for the spacial discretization of pressure and displacement. The interval $(0,1)$ is split into $N > 1$ equal subintervals. The size of each subinterval is $h = 2/(2N - 1)$. Define w_p to discretize the pressure:

$$\omega_p = \{y_i | y_i = ih, i = 0, 1, \dots, N-1\}.$$

To discretize the displacement, define ω_u :

$$\omega_u = \left\{ x_i | x_i = \frac{h}{2} + (i-1)h, i = 1, \dots, N \right\}.$$

On the interval $(0,1)$, the points of ω_p and ω_u alternate and are equidistant. We also require a time grid. A uniform time step, τ , is used to define

$$\omega_\tau = \{t_j | t_j = j\tau, j = 1, 2, \dots, M\}.$$

We now seek discrete solutions, $p = (p_0, p_1, \dots, p_{N-1})$ and $u = (u_1, u_2, \dots, u_N)$ in the Hilbert spaces H_{ω_p} and H_{ω_u} respectively.

$$H_{\omega_p} = \{p \mid p_i = p(y_i), y_i \in \omega_p, p_0 = 0\},$$

$$H_{\omega_u} = \{u \mid u_i = u(x_i), x_i \in \omega_u, u_N = 0\}.$$

The inner products are

$$(r, s)_{\omega_p} = h \sum_{i=0}^{N-1} r_i s_i,$$

$$(t, v)_{\omega_u} = h \sum_{i=1}^N t_i v_i,$$

and the associated norms $\|\cdot\|_{\omega_p} = \sqrt{(\cdot, \cdot)_{\omega_p}}$ and $\|\cdot\|_{\omega_u} = \sqrt{(\cdot, \cdot)_{\omega_u}}$.

We will seek to solve (2.5a) on ω_p and (2.5b) on ω_u .

3.2 Finite Difference Scheme on Standard Grid Points

The first step to applying this method is to setup the grid as done in the previous section. Now consider the case where the interface ζ falls in between the grid points x_j and x_{j+1} and between the staggered grid points y_j and y_{j+1} with $1 < j < N - 1$. According to the basic immersed interface method as described by Li and Ito [7], this makes the four grid points x_j , x_{j+1} , y_j and y_{j+1} irregular. To find the coefficients at these irregular grid points, the method of undetermined coefficients is used. This will be detailed in the next section.

A strength of immersed interface method is that it only requires changing the scheme at a few irregular grid points. That is, at all of the regular grid points a standard finite difference method, like those described by Li and Chen [6], can be used. The following standard forward, backward and central differences will be needed to make the discretization. The

standard indexless notation will be used.

$$\begin{aligned} u_{\hat{x}} &= \frac{u(x+h) - u(x)}{h}, \\ u_{\bar{x}} &= \frac{u(x) - u(x-h)}{h}, \\ u_{\dot{x}} &= \frac{u(x+h) - u(x-h)}{2h}. \end{aligned}$$

For the second derivative we use

$$vu_{\bar{x}x} = \frac{1}{h^2} \left(v \left(x + \frac{h}{2} \right) (u(x+h) - u(x)) - v \left(x + \frac{h}{2} \right) (u(x) - u(x-h)) \right).$$

These give rise to linear operators that can be written in compoenet form as

$$(Du)_i = (u_{\hat{x}})_i,$$

and

$$(Gp)_i = (p_{\bar{x}})_i.$$

We will also define operators for $-\frac{\partial^2 u}{\partial x^2}$ and $-\frac{\partial^2 p}{\partial x^2}$. These operators can be written in component form as

$$(Au)_i = -(vu_{\bar{x}x})_i,$$

and

$$(Bp)_i = -(kp_{\bar{x}x})_i.$$

Now (2.5) can be written as

$$\begin{aligned} Au(t) + Gp(t) &= 0, \\ \frac{\delta}{\delta t} (ap(t) + Du(t)) + Bp(t) &= f(t), \end{aligned} \tag{3.1}$$

for $0 < t \leq T$.

Once the problem has been discretized in space, it is possible to discretize in time using the uniform time grid ω_τ and a backward Euler method in time.

$$Au^{n+1} + Gp^{n+1} = 0, \tag{3.2a}$$

$$a \frac{p^{n+1} - p^n}{\tau} + \frac{Du^{n+1} - Du^n}{\tau} + Bp^{n+1} = f(x, t_{n+1}) \quad n = 0, 1, \dots, M-1. \tag{3.2b}$$

For a standard, internal grid point $x_i \in \omega_u$ $i \neq j, j+1$, or $y_i \in \omega_p$ $i \neq j, j+1$, we use this method [4].

3.3 Finding the Coefficients at Irregular grid points

Since three grid points are being used for every standard approximation, the following discretization are used for x_j , x_{j+1} , y_j and y_{j+1} respectively. Note that the superscript “n” is used to indicate the current time step, and the subscript of “n+1” is used to indicate the next time step.

$$\begin{aligned} &-(\gamma_{1,j}u^{n+1}(x_{j-1}) + \gamma_{2,j}u^{n+1}(x_j) + \gamma_{3,j}u^{n+1}(x_{j+1})) \\ &+(Gp^{n+1})_j = C_{1,j}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} &\frac{(ap^{n+1}(y_j) + (Du^{n+1})_j) - (ap^n(y_j) + (Du^n)_j)}{\Delta t} - \phi_{1,j}p^{n+1}(y_{j-1}) \\ &- \phi_{2,j}p^{n+1}(y_j) - \phi_{3,j}p^{n+1}(y_{j+1}) = f^{n+1}(y_j) + C_{2,j}, \end{aligned} \quad (3.4)$$

$$\begin{aligned} &-(\gamma_{1,j+1}u^{n+1}(x_j) + \gamma_{2,j+1}u^{n+1}(x_{j+1}) + \gamma_{3,j+1}u^{n+1}(x_{j+2})) \\ &+(Gp^{n+1})_{j+1} = C_{1,j+1}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} &\frac{(ap^{n+1}(y_{j+1}) + (Du^{n+1})_{j+1}) - (ap^n(y_{j+1}) + (Du^n)_{j+1})}{\Delta t} - \phi_{1,j+1}p^{n+1}(y_j) \\ &- \phi_{2,j+1}p^{n+1}(y_{j+1}) - \phi_{3,j+1}p^{n+1}(y_{j+2}) = f^{n+1}(y_{j+1}) + C_{2,j+1}. \end{aligned} \quad (3.6)$$

To determine the coefficients the local truncation error must be minimized for each of the above equations. We will begin with (3.3). The truncation error for this equation can be written as

$$\begin{aligned} T_{1,j} &= -(\gamma_{1,j}u^{n+1}(x_{j-1}) + \gamma_{2,j}u^{n+1}(x_j) + \gamma_{3,j}u^{n+1}(x_{j+1})) \\ &+(Gp^{n+1})_j - C_{1,j} \end{aligned} \quad (3.7)$$

To minimize this truncation error, we will expand each of the solutions, $u^{n+1}(x_{j-1})$, $u^{n+1}(x_j)$, and $u^{n+1}(x_{j+1})$, around the interface ζ . These Taylor expansions will then be written in terms of values from only one side of the interface. For this truncation error specifically, the left side will be used. The limiting values from the left side, $x < \zeta$, will be notated with a “-” superscript. While the limiting values from the right side, $x > \zeta$, will be notated with a “+” superscript. The Taylor expansions for $u(x_{j-1})$ and $u(x_j)$ about ζ have the form

$$u(x_l) = u^-(\zeta) + (x_l - \zeta)u_x^-(\zeta) + \frac{1}{2}(x_l - \zeta)^2u_{xx}^-(\zeta) + O(h^3), \quad (3.8)$$

where $l = j - 1, j$. However the expansion for $u^{n+1}(x_{j+1})$ about ζ takes this alternate form.

$$u(x_{j+1}) = u^+(\zeta) + (x_{j+1} - \zeta)u_x^+(\zeta) + \frac{1}{2}(x_{j+1} - \zeta)^2u_{xx}^+(\zeta) + O(h^3). \quad (3.9)$$

From now on, the notation u^\pm will be used for $u^\pm(\zeta)$. Next the jump conditions, (2.12a), (2.12b), (2.12c), and (2.12d), will be used to write this in terms of the quantities from the left side only.

$$\begin{aligned} u^+ &= u^-, \\ u_x^+ &= \frac{v^-}{v^+}u_x^-, \\ p^+ &= p^-, \\ p_x^+ &= \frac{k^-}{k^+}p_x^-. \end{aligned} \quad (3.10)$$

In addition, (2.5a) implies that

$$-v_x^+u_x^+ - v^+u_{xx}^+ + p_x^+ = -v_x^-u_x^- - v^-u_{xx}^- + p_x^-. \quad (3.11)$$

When v is a piecewise constant, $v_x^- = v_x^+ = 0$, and thus

$$\begin{aligned} -v^+u_{xx}^+ &= -v_x^-u_x^- + v_x^+u_x^+ + p_x^- - p_x^+ - v^-u_{xx}^-, \\ &= p_x^- - \frac{k^-}{k^+}p_x^- - v^-u_{xx}^-. \end{aligned}$$

Therefore we get

$$u_{xx}^+ = \frac{v^-}{v^+}u_{xx}^- - \frac{1}{v^+}\left(1 - \frac{k^-}{k^+}\right)p_x^-. \quad (3.12)$$

On the other hand (2.5a) also implies $p_x(x) = v(x)u_{xx}(x)$ and

$$\begin{aligned}
-vu_{xx} + p_x &= 0, \\
\frac{\partial}{\partial x}(-vu_x + p) &= 0, \\
\int_0^{\zeta^\pm} \frac{\partial}{\partial x}(-vu_x + p) &= 0, \\
-vu_x^\pm + p^\pm &= 0.
\end{aligned} \tag{3.13}$$

Therefore (3.12) gives us

$$u_{xx}^+ = \frac{v^-}{v^+}u_{xx}^- - \frac{v^-}{v^+}\left(1 - \frac{k^-}{k^+}\right)u_{xx}^- = \frac{v^-k^-}{v^+k^+}u_{xx}^-. \tag{3.14}$$

When all of the above equations are used with (3.9), the result is

$$u(x_{j+1}) = u^- + (x_{j+1} - \zeta)\frac{v^-}{v^+}u_x^- + \frac{1}{2}\frac{v^-}{v^+}\frac{k^-}{k^+}(x_{j+1} - \zeta)^2u_{xx}^- + O(h^3). \tag{3.15}$$

Next we will use the equations (3.8), and (3.15) to minimize the truncation error (3.7).

$$\begin{aligned}
T_{1,j}^{n+1} &= -\gamma_{1,i} \left(u^{n+1,-} + (x_{i-1} - \zeta)u_x^{n+1,-} + \frac{1}{2}(x_{i-1} - \zeta)^2u_{xx}^{n+1,-} + O(h^3) \right) \\
&\quad - \gamma_{2,i} \left(u^{n+1,-} + (x_i - \zeta)u_x^{n+1,-} + \frac{1}{2}(x_i - \zeta)^2u_{xx}^{n+1,-} + O(h^3) \right) \\
&\quad - \gamma_{3,i} \left(u^{n+1,-} + (x_{i+1} - \zeta)\frac{v^-}{v^+}u_x^{n+1,-} + \frac{1}{2}(x_{i+1} - \zeta)^2 \left(\frac{v^-}{v^+}\frac{k^-}{k^+} \right) u_{xx}^{n+1,-} + O(h^3) \right) \\
&\quad + \frac{p^{n+1}(y_j) - p^{n+1}(y_{j-1})}{h} - (-v^-u_{xx}^{n+1,-} + O(h) + p_x^{n+1}(x_j)) \\
&\quad - C_{1,j}.
\end{aligned} \tag{3.16}$$

Since the expression $\frac{p^{n+1}(y_j) - p^{n+1}(y_{j-1})}{h}$ is intended to approximate $p_x(x_j)$, We will use a Taylor series to expand $p^{n+1}(y_j)$ and $p^{n+1}(y_{j-1})$ about x_j . These expansions will take the form

$$p^{n+1}(y_l) = p^{n+1}(x_j) + (y_l - x_j)p_x^{n+1}(x_j) + O(h^2). \tag{3.17}$$

where $l = j - 1, j$.

Therefore

$$\begin{aligned}
p^{n+1}(y_j) - p^{n+1}(y_{j-1}) &= (p^{n+1}(x_j) + (y_j - x_j)p_x^{n+1}(x_j) + O(h^2)) - (p^{n+1}(x_j) + (y_{j-1} - x_j)p_x^{n+1}(x_j) + O(h^2)), \\
&= (y_j - y_{j-1})p_x^{n+1}(x_j) + O(h^2).
\end{aligned}$$

Now, using the fact that $y_j = y_{j-1} + h$,

$$\begin{aligned}
\frac{(p^{n+1}(y_j)) - (p^{n+1}(y_{j-1}))}{h} &= \frac{hp_x^{n+1}(x_j) + O(h^2)}{h}, \\
&= p_x^{n+1}(x_j) + O(h).
\end{aligned}$$

This shows that all of the pressure terms cancel out of (3.16) to leave just an $O(h)$. By collecting terms for $u^{n+1,-}$, $u_x^{n+1,-}$, and $u_{xx}^{n+1,-}$ we get the following system of equations.

$$\gamma_{1,j} + \gamma_{2,j} + \gamma_{3,j} = 0, \quad (3.18a)$$

$$\gamma_{1,j}(x_{j-1} - \zeta) + \gamma_{2,j}(x_j - \zeta) + \gamma_{3,j}(x_{j+1} - \zeta) \frac{v^-}{v^+} = 0, \quad (3.18b)$$

$$\gamma_{1,j} \frac{(x_{j-1} - \zeta)^2}{2} + \gamma_{2,j} \frac{(x_j - \zeta)^2}{2} + \gamma_{3,j} \frac{(x_{j+1} - \zeta)^2}{2} \frac{v^-}{v^+} \frac{k^-}{k^+} - v^- = 0. \quad (3.18c)$$

Now the correction term is just set to be $C_{1,j} = 0$. This system can be solved for a particular problem once the position of the interface is known. However, in the special case where $v(x)$, and $k(x)$ are continuous, then $v^+ = v^- = v$ and $k^+ = k^- = k$, and this system becomes

$$\gamma_{1,j} + \gamma_{2,j} + \gamma_{3,j} = 0, \quad (3.19a)$$

$$\gamma_{1,j}(x_{j-1} - \zeta) + \gamma_{2,j}(x_j - \zeta) + \gamma_{3,j}(x_{j+1} - \zeta) = 0, \quad (3.19b)$$

$$\gamma_{1,j} \frac{(x_{j-1} - \zeta)^2}{2} + \gamma_{2,j} \frac{(x_j - \zeta)^2}{2} + \gamma_{3,j} \frac{(x_{j+1} - \zeta)^2}{2} - v^- = 0. \quad (3.19c)$$

This system can be solved for $\gamma_{1,j}$, $\gamma_{2,j}$ and $\gamma_{3,j}$ using the fact that the grid is evenly

spaced. That is $x_j = x_{j-1} + h$. The solution gives

$$\begin{aligned}\gamma_{1,j} &= -\frac{v}{h^2}, \\ \gamma_{2,j} &= \frac{2v}{h^2}, \\ \gamma_{3,j} &= -\frac{v}{h^2}.\end{aligned}$$

This is the standard central difference scheme for a second spacial derivative.

Next we seek to reduce the truncation error in the equation (3.4). The truncation error can be written as

$$\begin{aligned}T_{2,j} &= \frac{(ap^{n+1}(y_j) + (Du^{n+1})_j) - (ap^n(y_j) + (Du^n)_j)}{\Delta t} - \phi_{1,j}p^{n+1}(y_{j-1}) \\ &\quad - \phi_{2,j}p^{n+1}(y_j) - \phi_{3,j}p^{n+1}(y_{j+1}) - (f^{n+1}(y_j)) + C_{2,j}.\end{aligned}\tag{3.20}$$

The procedure will be similar to minimizing $T_{1,j}$. This time, however, we will expand each of the solutions, $p^{n+1}(y_{j-1})$, $p^{n+1}(y_j)$, and $p^{n+1}(y_{j+1})$, around the interface ζ . Again, these will be written using expressions only from the left side of the interface. The Taylor expansions for $p^{n+1}(y_{j-1})$ and $p^{n+1}(y_j)$ about ζ have the form

$$p(y_l) = p^-(\zeta) + (y_l - \zeta)p_x^-(\zeta) + \frac{1}{2}(y_l - \zeta)^2 p_{xx}^-(\zeta) + O(h^3),\tag{3.21}$$

where $l = j-1, j$. While the expansion for $p^{n+1}(y_{j+1})$ takes the form

$$p(y_{j+1}) = p^+(\zeta) + (y_{j+1} - \zeta)p_x^+(\zeta) + \frac{1}{2}(y_{j+1} - \zeta)^2 p_{xx}^+(\zeta) + O(h^3).\tag{3.22}$$

The jump conditions are used to rewrite p^+ and p_x^+ . To find p_{xx}^+ in terms of the left side expressions, we assume that f has at most a finite jump at the interface.

$$[f] = f^+ - f^-.$$

Therefore

$$\frac{\partial}{\partial t} \left(ap^- + \frac{\partial u^-}{\partial x} \right) - k^- p_{xx}^- = \frac{\partial}{\partial t} \left(ap^+ + \frac{\partial u^+}{\partial x} \right) - k^+ p_{xx}^+ + [f].\tag{3.23}$$

Recall that from (3.13), we have that $u_x^- = p^-/v^-$ and $u_x^+ = p^+/v^+$. Using this in (3.23) yields

$$\frac{\partial}{\partial t} \left(a^- p^- + \frac{p^-}{v^-} \right) - k^- p_{xx}^- = \frac{\partial}{\partial t} \left(a^+ p^+ + \frac{p^+}{v^+} \right) - k^+ p_{xx}^+ + [f]. \quad (3.24)$$

Finally (3.10) can be used and we can solve for p_{xx}^+ as follows

$$p_{xx}^+ = \frac{1}{k^+} \frac{\partial}{\partial t} \left(\left([a] + \frac{1}{v^+} - \frac{1}{v^-} \right) p^- \right) + \frac{k^-}{k^+} p_{xx}^- + \frac{[f]}{k^+}.$$

Since neither the function a nor the function v is dependent on time, we have

$$p_{xx}^+ = \frac{1}{k^+} \left([a] + \frac{1}{v^+} - \frac{1}{v^-} \right) \frac{\partial}{\partial t} (p^-) + \frac{k^-}{k^+} p_{xx}^- + \frac{[f]}{k^+}. \quad (3.25)$$

Rewriting (3.22) using (3.10) and (3.25), we obtain

$$\begin{aligned} p(y_{j+1}) &= p^- + (y_{j+1} - \zeta) \frac{k^-}{k^+} p_x^- \\ &\quad + \frac{1}{2} (y_{j+1} - \zeta)^2 \left(\frac{1}{k^+} \left([a] + \frac{1}{v^+} - \frac{1}{v^-} \right) \frac{\partial}{\partial t} (p^-) + \frac{k^-}{k^+} p_{xx}^- + \frac{[f]}{k^+} (\zeta) \right) + O(h^3). \end{aligned} \quad (3.26)$$

We can now use the three Taylor expansions (3.21), (3.26) to write the truncation error as

$$\begin{aligned} T_{2,j} &= \frac{\left(ap^{n+1}(y_j) + \frac{u^{n+1}(x_{j+1}) - u^{n+1}(x_j)}{h} \right) - \left(ap^n(y_j) + \frac{u^n(x_{j+1}) - u^n(x_j)}{h} \right)}{\Delta t} \\ &\quad - \phi_{1,j} \left(p^{n+1,-} + (y_{j-1} - \zeta) p_x^{n+1,-} + \frac{1}{2} (y_{j-1} - \zeta)^2 p_{xx}^{n+1,-} + O(h^3) \right) \\ &\quad - \phi_{2,j} \left(p^{n+1,-} + (y_j - \zeta) p_x^{n+1,-} + \frac{1}{2} (y_j - \zeta)^2 p_{xx}^{n+1,-} + O(h^3) \right) \\ &\quad - \phi_{3,j} \left(p^{n+1,-} + (y_{j+1} - \zeta) \frac{k^-}{k^+} p_x^{n+1,-} \right) \\ &\quad - \phi_{3,j} \left(\frac{1}{2} (y_{j+1} - \zeta)^2 \left(\frac{1}{k^+} \left([a] + \frac{1}{v^+} - \frac{1}{v^-} \right) \frac{\partial}{\partial t} (p^{n+1,-}) + \frac{k^-}{k^+} (p_{xx}^{n+1,-}) + \frac{[f]}{k^+} \right) + O(h^3) \right) \\ &\quad - \left(\frac{\partial}{\partial t} ap^{n+1}(y_j) + \frac{\partial}{\partial t} u_x^{n+1}(y_j) - k p_{xx}^{n+1,-} + O(h) \right) \\ &\quad - C_{2,j}. \end{aligned} \quad (3.27)$$

Consider the terms $ap^{n+1}(y_j) - ap^n(y_j)$. We will use a Taylor series to expand $p^n(y_j)$ about t^{n+1} as follows.

$$p^n(y_j) = p^{n+1}(y_j) + (t^n - t^{n+1}) \frac{\partial}{\partial t} p^{n+1}(y_j) + O(\Delta t^2). \quad (3.28)$$

Note that $t^n - t^{n+1} = -\Delta t$. Therefore

$$\begin{aligned} a \frac{p^{n+1}(y_j) - p^n(y_j)}{\Delta t} &= \frac{a \Delta t \frac{\partial}{\partial t} p^{n+1}(y_j) + O(\Delta t^2)}{\Delta t}, \\ &= a \frac{\partial}{\partial t} p^{n+1}(y_j) + O(\Delta t). \end{aligned}$$

This cancels with the time derivative of p coming from the function f to leave just an $O(\Delta t)$ term. Next, consider the expression $\frac{u(x_{j+1}) - u(x_j)}{h}$, just as before. This is intended to be an approximation to the first derivative of u at the point y_j . So, we expand $u(x_{j+1})$ and $u(x_j)$ about y_j with Taylor series of the form

$$u(x_l) = u(y_j) + (x_l - y_j)u_x(y_j) + \frac{1}{2}(x_l - y_j)^2 u_{xx}(y_j) + \frac{1}{6}(x_l - y_j)^3 u_{xxx}(y_j) + O(h^4), \quad (3.29)$$

where $l = j, j+1$. Plugging in this expansion yields

$$\begin{aligned} \frac{u(x_{j+1}) - u(x_j)}{h} &= \frac{1}{h}(x_{j+1} - x_j)u_x(y_j) + \frac{1}{2h}((x_{j+1} - y_j)^2 - (x_j - y_j)^2)u_{xx}(y_j) \\ &\quad + \frac{1}{6h}((x_{j+1} - y_j)^3 - (x_j - y_j)^3)u_{xxx}(y_j) + \frac{O(h^4)}{h}. \end{aligned} \quad (3.30)$$

Note that $x_{j+1} - y_j = \frac{h}{2}$ and $x_j - y_j = -\frac{h}{2}$. Therefore (3.53) gives us

$$\frac{u(x_{j+1}) - u(x_j)}{h} = u_x(y_j) + \frac{h^2}{24}u_{xxx}(y_j) + O(h^3). \quad (3.31)$$

We now use (3.53) in both the “n+1” and the “n” time steps.

$$\frac{\left(\frac{u^{n+1}(x_{j+1}) - u^{n+1}(x_j)}{h}\right) - \left(\frac{u^n(x_{j+1}) - u^n(x_j)}{h}\right)}{\Delta t} = \frac{u_x^{n+1}(y_j) - u_x^n(y_j) + \frac{h^2}{24}(u_{xxx}^{n+1}(y_j) - u_{xxx}^n(y_j)) + O(h^3)}{\Delta t}. \quad (3.32)$$

Next, we will expand the term $u_x^n(y_j)$ in the time grid about t^{n+1} to get

$$u_x^n(y_j) = u_x^{n+1}(y_j) + (t^n - t^{n+1}) \frac{\partial}{\partial t} u_x^{n+1}(y_j) + O(\Delta t^2). \quad (3.33)$$

Similarly, we expand the term $u_{xxx}^n(y_{j+1})$ in the time grid about t^{n+1} to get

$$u_{xxx}^n(y_j) = u_{xxx}^{n+1}(y_j) + (t^n - t^{n+1}) \frac{\partial}{\partial t} u_{xxx}^{n+1}(y_j) + O(\Delta t^2). \quad (3.34)$$

Now (3.32) becomes

$$\begin{aligned} & \frac{-(t^n - t^{n+1}) \frac{\partial}{\partial t} u_x^{n+1}(y_j) - (t^n - t^{n+1}) \frac{h^2}{24} \frac{\partial}{\partial t} u_{xxx}^{n+1}(y_j) + O(\Delta t^2) + O(h^3)}{\Delta t} \\ &= \frac{\partial}{\partial t} u_x^{n+1}(y_j) + O(h^2) + O(\Delta t) + O\left(\frac{h^3}{\Delta t}\right). \end{aligned} \quad (3.35)$$

This cancels with the time derivative of u_x coming from the function f to leave just the $O(\Delta t) + O(h^2) + O(\frac{h^3}{\Delta t})$ terms. Note that when we consider the largest of the error terms in (3.27) we expect a rate of convergence of $O(h + \Delta t)$.

By collecting terms for $p^{n+1,-}$, $p_x^{n+1,-}$, and $p_{xx}^{n+1,-}$, we get the following system of equations.

$$\phi_{1,j} + \phi_{2,j} + \phi_{3,j} = 0, \quad (3.36a)$$

$$\phi_{1,j}(y_{j-1} - \zeta) + \phi_{2,j}(y_j - \zeta) + \phi_{3,j}(y_{j+1} - \zeta) \frac{k^-}{k^+} = 0, \quad (3.36b)$$

$$\phi_{1,j} \frac{(y_{j-1} - \zeta)^2}{2} + \phi_{2,j} \frac{(y_j - \zeta)^2}{2} + \phi_{3,j} \frac{(y_{j+1} - \zeta)^2}{2} \frac{k^-}{k^+} - k^- = 0. \quad (3.36c)$$

Now we can set the correction term to

$$C_{2,j} = -\phi_{3,j} \left(\frac{1}{2} (y_{j+1} - \zeta)^2 \left(\frac{1}{k^+} \left([a] + \frac{1}{v^+} - \frac{1}{v^-} \right) \frac{\delta}{\delta t} (p^{n+1,-}) + \frac{[f]^{n+1}}{k^+} \right) \right).$$

We will approximate the time derivative of $p^{n+1,-}$ with the standard backward Euler scheme, so that the correction term is approximated as

$$C_{2,j} = -\phi_{3,j} \left(\frac{1}{2} (y_{j+1} - \zeta)^2 \left(\frac{1}{k^+} \left([a] - \frac{[v]}{v^+ v^-} \right) \left(\frac{(p^{n+1,-}) - (p^{n,-})}{\Delta t} + O(\Delta t) \right) + \frac{[f]}{k^+} \right) \right).$$

The system of equations, (3.36), can again be solved once the position of an interface is known. As above, this system reduces to the standard central difference scheme for second derivatives in the special case where $v(x)$ and $k(x)$ are continuous.

The same procedure is applied to (3.5) and (3.6), except that this time we will need to write every term using only quantities from the right or “+” side. To achieve this, we will need to obtain expressions for terms on the “-” side in term of the “+” side. When solved for the “-” side, the jump conditions, (3.10), become

$$\begin{aligned} u^- &= u^+, \\ u_x^- &= \frac{v^+}{v^-} u_x^+, \\ p^- &= p^+, \\ p_x^- &= \frac{k^+}{k^-} p_x^+. \end{aligned} \tag{3.37}$$

Next, we consider the equation (3.11). This time, however, we solve for u_{xx}^- to get

$$u_{xx}^- = \frac{v^+}{v^-} u_{xx}^+ + \frac{1}{v^-} \left(1 - \frac{k^-}{k^+}\right) p_x^+ = \frac{v^+}{v^-} \frac{k^+}{k^-} u_{xx}^+. \tag{3.38}$$

Recall the Taylor expansions for $u(x_i)$ is shown in (3.8). We can now rewrite it in terms of the “+” side using the jump conditions, (3.37), and (3.38)

$$u(x_j) = u^+ + (x_j - \zeta) \frac{v^+}{v^-} u_x^+ + \frac{1}{2} (x_j - \zeta)^2 \frac{v^+}{v^-} \frac{k^+}{k^-} u_{xx}^+ + O(h^3). \tag{3.39}$$

The Taylor expansions of the other terms, $u(x_{j+1})$ and $u(x_{j+2})$ take the form

$$u(x_l) = u^+(\zeta) + (x_l - \zeta) u_x^+(\zeta) + \frac{1}{2} (x_l - \zeta)^2 u_{xx}^+(\zeta) + O(h^3). \tag{3.40}$$

where $l = j + 1, j + 2$. As above, we consider truncation error for (3.5) to be

$$\begin{aligned} T_{1,j+1} &= -(\gamma_{1,j+1} u^{n+1}(x_j) + \gamma_{2,j+1} u^{n+1}(x_{j+1}) + \gamma_{3,j+1} u^{n+1}(x_{j+2})) \\ &\quad + (Gp^{n+1})_{j+1} - C_{1,j+1}. \end{aligned} \tag{3.41}$$

Plugging in the appropriate Taylor expansions, (3.39) and (3.40), yields

$$\begin{aligned}
T_{1,j+1} = & -\gamma_{1,j+1} \left(u^{n+1,+} + (x_j - \zeta) \frac{v^+}{v^-} u_x^{n+1,+} + \frac{1}{2} (x_j - \zeta)^2 \frac{v^+}{v^-} \frac{k^+}{k^-} u_{xx}^{n+1,+} + O(h^3) \right) \\
& - \gamma_{2,j+1} \left(u^{n+1,+} + (x_{j+1} - \zeta) u_x^{n+1,+} + \frac{1}{2} (x_{j+1} - \zeta)^2 u_{xx}^{n+1,+} + O(h^3) \right) \\
& - \gamma_{3,j+1} \left(u^{n+1,+} + (x_{j+2} - \zeta) u_x^{n+1,+} + \frac{1}{2} (x_{j+2} - \zeta)^2 u_{xx}^{n+1,+} + O(h^3) \right) \\
& + \frac{p^{n+1}(y_{j+1}) - p^{n+1}(y_j)}{h} - (-v^+(u_{xx})^{n+1,+} + O(h) + p_x^{n+1}(x_{j+1})) \\
& - C_{1,j+1}.
\end{aligned} \tag{3.42}$$

Since the expression $\frac{p^{n+1}(y_{j+1}) - p^{n+1}(y_j)}{h}$ is intended to approximate $p_x(x_j)$, We will use a Taylor series to expand $p^{n+1}(y_{j+1})$ and $p^{n+1}(y_j)$ about x_{j+1} , similar to the procedure used when dealing with (3.16). These expansions will take the form

$$p^{n+1}(y_l) = p^{n+1}(x_{j+1}) + (y_l - x_{j+1}) p_x^{n+1}(x_{j+1}) + O(h^2), \tag{3.43}$$

where $l = j, j+1$.

Therefore

$$\begin{aligned}
& p^{n+1}(y_j) - p^{n+1}(y_{j-1}) \\
& = (p^{n+1}(x_{j+1}) + (y_{j+1} - x_{j+1}) p_x^{n+1}(x_{j+1}) + O(h^2)) - (p^{n+1}(x_{j+1}) + (y_j - x_{j+1}) p_x^{n+1}(x_{j+1}) + O(h^2)), \\
& = (y_{j+1} - y_j) p_x^{n+1}(x_{j+1}) + O(h^2).
\end{aligned}$$

Now, using the fact that $y_{j+1} = y_j + h$,

$$\begin{aligned}
\frac{p^{n+1}(y_j) - p^{n+1}(y_{j-1})}{h} &= \frac{h p_x^{n+1}(x_{j+1}) + O(h^2)}{h}, \\
&= p_x^{n+1}(x_{j+1}) + O(h).
\end{aligned}$$

This shows that all of the pressure terms cancel out of (3.42) to leave just an $O(h)$.

Combining like terms yields the following system for the coefficients $\gamma_{1,j+1}$, $\gamma_{2,j+1}$ and $\gamma_{3,j+1}$.

$$\gamma_{1,j+1} + \gamma_{2,j+1} + \gamma_{3,j+1} = 0, \quad (3.44a)$$

$$\gamma_{1,j+1}(x_j - \zeta) \frac{v^+}{v^-} + \gamma_{2,j+1}(x_{j+1} - \zeta) + \gamma_{3,j+1}(x_{j+2} - \zeta) = 0, \quad (3.44b)$$

$$\gamma_{1,j+1} \frac{(x_j - \zeta)^2}{2} \frac{v^+}{v^-} \frac{k^+}{k^-} + \gamma_{2,j+1} \frac{(x_{j+1} - \zeta)^2}{2} + \gamma_{3,j+1} \frac{(x_{j+2} - \zeta)^2}{2} - v^+ = 0. \quad (3.44c)$$

We can let $C_{1,j+1} = 0$. This reduces to the standard case when the coefficients are continuous. Finally we apply these ideas to the last equation (3.6). Its truncation error becomes

$$\begin{aligned} T_{2,j+1} &= \frac{(ap^{n+1}(y_{j+1}) + (Du^{n+1})_{j+1}) - (ap^n(y_{j+1}) + (Du^n)_{j+1})}{\Delta t} - \phi_{1,j+1}p^{n+1}(y_j) \\ &\quad - \phi_{2,j+1}p^{n+1}(y_{j+1}) - \phi_{3,j+1}p^{n+1}(y_{j+2}) - f^{n+1}(y_{j+1}) - C_{2,j+1}. \end{aligned} \quad (3.45)$$

Recall the form of the Taylor expansion of $p^{n+1}(y_j)$ about ζ from (3.21). To write this in in terms of the “+” side, recall (3.24). This can be solved for p_{xx}^- using (3.37) as follows.

$$p_{xx}^- = \frac{1}{k^-} \left(-[a] + \frac{1}{v^-} - \frac{1}{v^+} \right) \frac{\partial}{\partial t} (p^+) + \frac{k^+}{k^-} p_{xx}^+ - \frac{[f]}{k^-}. \quad (3.46)$$

Using (3.37) and (3.46) in (3.21) yields

$$\begin{aligned} p(y_j) &= p^+ + (y_j - \zeta) \frac{k^+}{k^-} p_x^+ \\ &\quad + \frac{1}{2} (y_j - \zeta)^2 \left(\frac{1}{k^-} \left(-[a] + \frac{1}{v^-} - \frac{1}{v^+} \right) \frac{\partial}{\partial t} (p^+) + \frac{k^+}{k^-} p_{xx}^+ - \frac{[f]}{k^-} \right) + O(h^3). \end{aligned} \quad (3.47)$$

The Taylor expansions of the remaining terms, $p(y_{j+1})$ and $p(y_{j+2})$, about ζ take the form

$$p(y_l) = p^+(\zeta) + (y_l - \zeta) p_x^+(\zeta) + \frac{1}{2} (y_l - \zeta)^2 p_{xx}^+(\zeta) + O(h^3). \quad (3.48)$$

where $l = j + 1, j + 2$. Using the Taylor expansions, (3.48) and (3.47), in (3.45) Gives

$$\begin{aligned}
T_{2,j+1} = & \frac{\left(ap^{n+1}(y_{j+1}) + \frac{u^{n+1}(x_{j+2}) - u^{n+1}(x_{j+1})}{h} \right) - \left(ap^n(y_{j+1}) + \frac{u^n(x_{j+2}) - u^n(x_{j+1})}{h} \right)}{\Delta t} \\
& - \phi_{1,j+1} \left(p^+ + (y_j - \zeta) \frac{k^+}{k^-} p_x^+ \right) \\
& - \phi_{1,j+1} \left(\frac{1}{2} (y_j - \zeta)^2 \left(\frac{1}{k^-} \left(-[a] + \frac{1}{v^-} - \frac{1}{v^+} \right) \frac{\partial}{\partial t} (p^+) + \frac{k^+}{k^-} p_{xx}^+ - \frac{[f]}{k^-} \right) + O(h^3) \right) \\
& - \phi_{2,j+1} \left(p^+ + (y_{j+1} - \zeta) p_x^+ + \frac{1}{2} (y_{j+1} - \zeta)^2 p_{xx}^+ + O(h^3) \right) \\
& - \phi_{3,j+1} \left(p^+ + (y_{j+2} - \zeta) p_x^+ + \frac{1}{2} (y_{j+2} - \zeta)^2 p_{xx}^+ + O(h^3) \right) \\
& - \left(\frac{\partial}{\partial t} ap^{n+1}(y_{j+1}) + \frac{\partial}{\partial t} u_x^{n+1}(y_{j+1}) - kp_{xx}^{n+1,-} + O(h) \right) \\
& - C_{2,j+1}.
\end{aligned} \tag{3.49}$$

Again, we use a Taylor series to expand $p^n(y_{j+1})$ about t^{n+1} as follows.

$$p^n(y_{j+1}) = p^{n+1}(y_{j+1}) + (t^n - t^{n+1}) \frac{\partial}{\partial t} p^{n+1}(y_{j+1}) + O(\Delta t^2). \tag{3.50}$$

Recall that $t^n - t^{n+1} = -\Delta t$. Therefore

$$\begin{aligned}
a \frac{p^{n+1}(y_{j+1}) - p^n(y_{j+1})}{\Delta t} &= \frac{a \Delta t \frac{\partial}{\partial t} p^{n+1}(y_{j+1}) + O(\Delta t^2)}{\Delta t}, \\
&= a \frac{\partial}{\partial t} p^{n+1}(y_{j+1}) + O(\Delta t).
\end{aligned}$$

This cancels with the time derivative of p coming from the function f to leave just an $O(\Delta t)$ term. Next, consider the expression $\frac{u(x_{j+2}) - u(x_{j+1})}{h}$. This is intended to be an approximation to the first derivative of u at the point y_{j+1} . So, we expand $u(x_{j+2})$ and $u(x_{j+1})$ about y_{j+1} with Taylor series of the form

$$\begin{aligned}
u(x_l) &= u(y_{j+1}) + (x_l - y_{j+1}) u_x(y_{j+1}) + \frac{1}{2} (x_l - y_{j+1})^2 u_{xx}(y_{j+1}) \\
&+ \frac{1}{6} (x_l - y_{j+1})^3 u_{xxx}(y_{j+1}) + O(h^4),
\end{aligned} \tag{3.51}$$

where $l = j + 1, j + 2$. Note that $x_{j+2} - y_{j+1} = \frac{h}{2}$, $x_{j+1} - y_{j+1} = -\frac{h}{2}$ and $x_{j+2} - x_{j+1} = h$.

$$\begin{aligned} u(x_{j+2}) - u(x_{j+1}) &= (x_{j+2} - x_{j+1})u_x(y_{j+1}) + \frac{1}{2}((x_{j+2} - y_{j+1}))^2 - (x_{j+1} - y_{j+1})^2 u_{xx}(y_{j+1}) \\ &\quad + \frac{1}{6}((x_{j+2} - y_{j+1}))^3 - (x_{j+1} - y_{j+1})^3 u_{xxx}(y_{j+1}) + O(h^4), \\ &= hu_x(y_{j+1}) + \frac{h^3}{24}u_{xxx}(y_{j+1}) + O(h^4). \end{aligned} \quad (3.52)$$

As above in the discussion about (3.27), this yields

$$\begin{aligned} \frac{u(x_{j+2}) - u(x_{j+1})}{h} &= \frac{hu_x(y_{j+1}) + \frac{h^3}{24}u_{xxx}(y_{j+1}) + O(h^4)}{h}, \\ &= u_x(y_{j+1}) + \frac{h^2}{24}u_{xxx}(y_{j+1}) + O(h^3). \end{aligned} \quad (3.53)$$

Next, we will expand the terms $u_x^n(y_{j+1})$ and $u_{xxx}^n(y_{j+1})$ in the time grid about t^{n+1} , as above, to get

$$u_x^n(y_{j+1}) = u_x^{n+1}(y_{j+1}) + (t^n - t^{n+1}) \frac{\partial}{\partial t} u_x^{n+1}(y_{j+1}) + O(\Delta t^2) \quad (3.54)$$

and

$$u_{xxx}^n(y_{j+1}) = u_{xxx}^{n+1}(y_{j+1}) + (t^n - t^{n+1}) \frac{\partial}{\partial t} u_{xxx}^{n+1}(y_{j+1}) + O(\Delta t^2). \quad (3.55)$$

So

$$\begin{aligned} &\frac{\left(\frac{u^{n+1}(x_{j+2}) - u^{n+1}(x_{j+1})}{h} \right) - \left(\frac{u^n(x_{j+2}) - u^n(x_{j+1})}{h} \right)}{\Delta t} \\ &= \frac{u_x^{n+1}(y_{j+1}) - u_x^n(y_{j+1}) + \frac{h^2}{24}(u_{xxx}^{n+1}(y_{j+1}) - u_{xxx}^n(y_{j+1})) + O(h^3)}{\Delta t}, \\ &= \frac{\Delta t \frac{\partial}{\partial t} u_x^{n+1}(y_{j+1}) + \frac{h^2}{24} \Delta t \frac{\partial}{\partial t} u_{xxx}^{n+1}(y_{j+1}) + O(\Delta t^2) + O(h^3)}{\Delta t}, \\ &= \frac{\partial}{\partial t} u_x^{n+1}(y_{j+1}) + O(h^2) + O(\Delta t) + O\left(\frac{h^3}{\Delta t}\right). \end{aligned} \quad (3.56)$$

This cancels with the time derivative of u_x coming from the function f to leave just the $O(h^2) + O(\Delta t) + O(\frac{h^3}{\Delta t})$ terms. Again we expect to see a convergence rate of $O(h + \Delta t)$.

By collecting terms for $p^{n+1,+}$, $p_x^{n+1,+}$, and $p_{xx}^{n+1,+}$ we get the following system of equations.

$$\phi_{1,j+1} + \phi_{2,j+1} + \phi_{3,j+1} = 0, \quad (3.57a)$$

$$\phi_{1,j+1}(y_j - \zeta) \frac{k^+}{k^-} + \phi_{2,j+1}(y_{j+1} - \zeta) + \phi_{3,j+1}(y_{j+2} - \zeta) = 0, \quad (3.57b)$$

$$\phi_{1,j+1} \frac{(y_j - \zeta)^2 k^+}{2 k^-} + \phi_{2,j+1} \frac{(y_{j+1} - \zeta)^2}{2} + \phi_{3,j+1} \frac{(y_{j+2} - \zeta)^2}{2} - k^+ = 0. \quad (3.57c)$$

Now we can set the correction term to

$$C_{2,j+1} = -\phi_{1,j+1} \left(\frac{1}{2}(y_j - \zeta)^2 \left(\frac{1}{k^-} \left(-[a] + \frac{[v]}{v^+ v^-} \right) \left(\frac{p^{n+1,+} - p^{n,+}}{\Delta t} + O(\Delta t) \right) - \frac{[f]}{k^-} \right) \right).$$

This reduces to the standard case when the coefficients are continuous.

3.4 Applying the method

Recall the problem in the form of (3.2). In this equation, Au is an approximation to $\frac{\delta^2}{\delta x^2}(u)$ on ω_u and is using a standard central difference formula at regular grid points. At the end point x_1 , the central difference takes the form

$$v_1 \frac{u(x_1 + h) - 2u(x_1) + u(x_1 - h))}{h^2},$$

since this point is to the left of the interface, ζ , and $v(x) = v_1$ for $x < \zeta$. Since the point $x_1 - h$ does not fall on the staggered grid, we use the the boundary condition, $\frac{\partial u}{\partial x} = 0$ at $x = 0$, to make the approximation

$$0 = \frac{u(x_1) - u(x_1 - h))}{h},$$

at y_0 . Thus $u(x_1) = u(x_1 - h)$, and the approximation at x_1 becomes

$$v_1 \frac{u(x_1 + h) - u(x_1)}{h^2} = \frac{-v_1}{h} (u_{\hat{x}})_1.$$

So Au takes the form

$$(Au)_l = \begin{cases} \frac{-v_1}{h}(u_{\hat{x}})_l & l = 1, \\ -(vu_{\bar{x}x})_l & l = 2, \dots, j-1, \\ -(\gamma_{1,j}u(x_{j-1}) + \gamma_{2,j}u(x_j) + \gamma_{3,j}u(x_{j+1})) & l = j, \\ -(\gamma_{1,j+1}u(x_j) + \gamma_{2,j+1}u(x_{j+1}) + \gamma_{3,j+1}u(x_{j+2})) & l = j+1, \\ -(vu_{\bar{x}x})_l & l = j+2, \dots, N-1, \\ 0 & l = N. \end{cases} \quad (3.58)$$

Gp is an approximation to $\frac{\partial p}{\partial x}$ on ω_u and takes the form

$$(Gp)_l = \begin{cases} (p_{\bar{x}})_l & l = 1, 2, \dots, N-1, \\ 0 & l = N. \end{cases} \quad (3.59)$$

Since every point in the grid ω_u , except for x_N , is surrounded by points from the ω_p grid, we use the values of p at the two surrounding grid points in ω_p to approximate the first derivative at a point in ω_u . For the second equation, we approximate on ω_p . In this equation Du is an approximation to $\frac{\partial u}{\partial x}$ on ω_p and takes the form

$$(Du)_l = \begin{cases} 0 & l = 0, \\ (u_{\hat{x}})_l & i = 1, 2, \dots, N-1. \end{cases} \quad (3.60)$$

As with Gp , Du uses the surrounding points of ω_u to make an approximation in ω_p . Finally, Bp is an approximation to $\frac{\partial^2}{\partial x^2} p$ on ω_p and takes the form

$$(Bp)_l = \begin{cases} 0 & l = 1, \\ -(kp_{\bar{x}x})_l & l = 2, \dots, j-1, \\ -(\phi_{1,j}p(x_{j-1}) + \phi_{2,j}p(x_j) + \phi_{3,j}p(x_{j+1})) & l = j, \\ -(\phi_{1,j+1}p(x_j) + \phi_{2,j+1}p(x_{j+1}) + \phi_{3,j+1}p(x_{j+2})) & l = j+1, \\ -(kp_{\bar{x}x})_l & l = i+2, \dots, N-1, \\ \frac{k_2}{h}(p_{\bar{x}})_l & i = N. \end{cases} \quad (3.61)$$

The last line of this matrix, $(Bp)_N$ reflects the boundary condition $\frac{\partial p}{\partial x} = 0$ at $x = 1$. At the end point y_N . Here the central difference formula for second derivatives becomes

$$k_2 \frac{p(y_N + h) - 2p(y_N) + p(y_N - h)}{h^2},$$

since this point is to the right of the interface, ζ , and $k(x) = k_2$ for $x > \zeta$. The point $x_N + h$ does not fall on the staggered grid. Therefore we use the the boundary condition

$$0 = \frac{p(y_N + h) - p(y_N)}{h}.$$

Therefore $p(y_N + h) = p(y_N)$, and the approximation at y_N becomes

$$k_2 \frac{-p(y_N) + p(y_N - h)}{h^2} = \frac{k_2}{h} p_{\bar{x}}.$$

On the right hand side, $f(t)$ can be found at each point in ω_p by just plugging in each grid point, except at the points y_j and y_{j+1} . At these two points the correction terms $C_{2,j}$ and $C_{2,j+1}$ must be added, respectively. Call the corrected formed $f_c(t)$.

The problem now becomes that of solving the linear system

$$\begin{bmatrix} A & G \\ \frac{D}{\tau} & \frac{a}{\tau} + B \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ f_c^{n+1} + \frac{ap^n}{\tau} + \frac{Dp^n}{\tau} \end{bmatrix}. \quad (3.62)$$

for $n = 0, 1, \dots, M - 1$.

Finally we must account for the remaining boundary conditions. Place a “1” in the N^{th} row and N^{th} column of left side matrix and a “0” in the N^{th} row of the right side vector to account for the boundary condition $u = 0$ if $x = 1$. Similarly, we place a “1” in the $(N + 1)^{\text{th}}$ row and $(N + 1)^{\text{th}}$ column of left side matrix and a “0” in the $(N + 1)^{\text{th}}$ row of the right side vector to account for the boundary condition $p = 0$ if $x = 0$.

Chapter 4

Numerical Experiments and Future Work

Although analytical solutions to (2.5) are not common, two test problems with analytic solutions were found [3, 4]. For these tests the relative error in the L_2 - norm is calculated.

$$\|\epsilon_w\|_{L_2} = \frac{\sum_{x_i \in \omega_w} h |\omega^{ex}(x_i, t_j) - \omega_i^{app}|}{max_{\omega_w} |\omega^{ex}(x_i, t_j)|},$$

where ω^{ex} stands for the exact solution, ω^{app} stands for the approximate solution, and $\omega = \{u, p\}$.

4.1 Test Problem 1

Numerical experiments were first conducted on test problem with continuous coefficients. For the purposes of this experiment an artificial interface was placed at $x = \frac{1}{6}$. In this test problem $v(x) = 1$, $k(x) = 1$ and $a(x) = 0$. The function $f(x, t)$ is chosen so that the exact solution will be

$$u(x, t) = - \sum_{\infty}^{n=0} \frac{2}{M^2} \cos(Mx) e^{(a-M^2)t}, \quad (4.1a)$$

$$p(x, t) = \sum_{\infty}^{n=0} \frac{2}{M} \sin(Mx) e^{(a-M^2)t}, \quad (4.1b)$$

Where

$$M = \frac{\pi(2n+1)}{2}.$$

Table 4.1: Example 1: Convergence in L_2 -norm at the final time $T=1$

h	Δt	$\ \epsilon_u\ $	Ratio	$\ \epsilon_p\ $	Ratio
5.128E-02	2.500E-02	4.898E-02	—	4.879E-02	—
2.532E-02	6.250E-03	1.227E-02	3.992	1.223E-02	3.991
1.258E-02	1.563E-03	3.068E-03	3.998	3.058E-03	3.998
6.269E-03	3.906E-04	7.670E-04	4.0	7.644E-04	4.0
3.130E-02	9.765E-05	1.916E-04	4.0	1.910E-04	4.0
Rate	—	—	2.0	—	2.0

Since this problem has continuous coefficients, this method reduces to a standard finite difference scheme. Using a Crank-Nicholson scheme for the time discretization, Gaspar et al. [4], were able to prove that this method will have a convergence rate of $O(h^2 + \Delta t^2)$. This project has used a backward Euler method instead. So to see the $O(h^2)$ part of the convergence rate in Table 4.1, h is being reduced by a factor of 2 in each trial, while Δt is being reduced by a factor of 4 in each trial.

The rate in the final line of Table 4.1 is calculated by

$$rate = \frac{\ln \frac{\|\epsilon_1\|}{\|\epsilon_2\|}}{\ln \frac{h_1}{h_2}} \quad (4.2)$$

where ϵ_1 , and ϵ_2 are errors calculated at grids with step sizes h_1 and h_2 respectively.

4.2 Test problem 2

In this test problem, Ewing et al. [3] set the interface at $\zeta = 1/6$, and use the following

$$v(x) = \begin{cases} v_1 = 1 & \text{if } x < \zeta \\ v_2 = \frac{\tan \frac{1}{12} \tan \frac{10\pi}{3}}{(8\pi)} & \text{if } x > \zeta \end{cases},$$

Table 4.2: Example 2: Convergence in L_2 -norm at the final time $T=1$

h	Δt	$\ \epsilon_u\ $	Ratio	$\ \epsilon_p\ $	Ratio
2.531E-02	2.50E-02	6.827E-03	—	1.327E-01	—
6.269E-03	6.250E-03	2.250E-03	3.0	3.777E-02	3.57
1.563E-03	1.563E-03	5.911E-04	3.8	9.771E-03	3.87
3.907E-04	3.906E-04	1.495E-04	4.0	2.464E-03	4.0
Rate	—	—	1.0	—	1.0

$$k(x) = \begin{cases} k_1 = 1 & \text{if } x < \zeta \\ k_2 = \frac{1}{(8\pi \tan \frac{1}{12} \tan \frac{10\pi}{3})} & \text{if } x > \zeta \end{cases},$$

$$a(x) = \begin{cases} a_1 = 0 & \text{if } x < \zeta \\ a_2 = 0 & \text{if } x > \zeta \end{cases},$$

and

$$f(x, t) = 0.$$

Using these functions yields an exact solution to (2.5), but with different initial conditions. That is, the boundary conditions (2.6a), and (2.6b), remain as stated, but the initial condition (2.6c) is no longer valid. The new initial conditions are found by calculating the exact solutions at $t = 0$. These exact solutions are

$$p(x, t) = \begin{cases} \cos\left(\frac{10\pi}{3}\right) \sin(.5x)e^{-.25t} & \text{if } x \leq \zeta \\ \sin\left(\frac{1}{12}\right) \cos(4\pi(1-x))e^{.25t} & \text{if } x > \zeta \end{cases},$$

$$u(x, t) = \begin{cases} -2 \cos\left(\frac{10\pi}{3}\right) \cos(.5x)e^{-.25t} & \text{if } x \leq \zeta \\ -\frac{2 \cos\left(\frac{1}{12}\right)}{\tan\left(\frac{10\pi}{3}\right)} \sin(4\pi(1-x))e^{.25t} & \text{if } x > \zeta \end{cases}.$$

Table 4.2 indicates a convergence rate of $O(h + \Delta t)$ as expected.

4.3 Future Work

Better approximations to the order of convergence may be obtained by including higher order terms in the analysis. For example, including higher order terms in the expansions about t^{n+1} discussed in Chapter 3 may improve the estimates. Furthermore, the rate of convergence may have some dependence on the placement of the interface. In some of the numerical tests the error was not reducing at a regular rate when h was reduced by different factors. For example, if h is reduced by a factor of 2 in each trial, the convergence rate appears to fluctuate. Further research is required to discover the cause of this.

Futhtermore, in this project only the second derivative terms were being corrected. Correcting the first derivative terms should improve the method. Another possible way to improve the results is to try using more points to approximate each derivative. However, as the system of equations would be quite large, this would likely encounter additional complications.

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Curriculum Vitae

Maranda Bean

7117 Orizaba Ave.

El Paso, Texas 79912

Phone: (915) 539-9567

E-mail: mlbean@utep.edu

Education

B.S. in Physics minor in Mathematics, New Mexico Tech May 2007

Employment

Graduate Teaching Assistant, University of Texas at El Paso Jan. 2011- Dec. 2012
New Mexico Tech Advancement Office Jan. 2004 - May 2007

Academic Honors

New Mexico Tech Silver Scholarship Fall 2003 - May 2007
New Mexico Tech Scholar Spring 2005- May 2007