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Stochastic Dominance: Cases of Interval and P-Box Uncertainty

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Abstract Traditional decision theory recommendation about making a decision assume that we know both the probabilities of different outcomes of each possible decision, and we know the utility function – that describes the decision maker's preferences. Sometimes, we can make a recommendation even when we only have partial information about utility. Such cases are known as cases of stochastic dominance. In other cases, in addition to not knowing the utility function, we also only have partial information about the probabilities of different outcomes. For example, we may only known bounds on the outcomes (case of interval uncertainty) or bounds on the values of the cumulative distribution function (case of p-box uncertainty). In this paper, we extend known stochastic dominance results to these two cases.

1 Formulation of the problem

Decision making according to decision theory: a brief reminder. Decision theory (see, e.g., [2, 3, 6, 9, 12, 13, 14]) describes decisions of a *rational* decision maker, i.e., of a decision maker whose decisions satisfy commonsense conditions of rationality. For example, if a rational decision maker prefers A to B and prefers B to C, then this decision maker should prefer A to C. It is known that preferences of a rational decision maker can be described by a function $u(a)$ called *utility* such that:

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- the decision maker prefers a to b if and only if $u(a) > u(b)$, and
- the utility of a situation in which we can get outcome a_1 with probability p_1, outcome a_2 with probability $p_2, \ldots,$ and outcome a_n with probability p_n is equal to $p_1 \cdot u(a_1) + \ldots + p_n \cdot u(a_n)$.

Stochastic dominance theory (see, e.g., [8, 15]) deals with the case when each outcome a_i is characterized by a numerical value – monetary gain. In this case, each alternative is described by a probabilistic uncertainty, which means that we have a probability distribution on the set of all real numbers. For example, in the above case, we have a probability distribution that is located on the set $\{a_1, \ldots, a_n\}$ and for which the probability of each value a_i is equal to p_i.

In general, we can describe each probability distribution by a cumulative distribution function $F(x) = \text{Prob}(a \le x)$. In this case, the utility of this alternative is equal to the expected value $\int u(x)\,dF(x)$ of the utility function with respect to this probability distribution.

Stochastic dominance: a brief reminder. In some practical situations, we do not know the decision maker's utility function, we only have some information about the utility function. In some such cases, we can sometimes conclude that one alternative is better than the other one. Such cases are known as cases of *stochastic dominance.*

In some cases, all we know about the utility function is that this function is (non-strictly) increasing: if $a \le b$ then $u(a) \le u(b)$. In this case, the following known result holds:

Proposition 1. *For every two probability distributions $F(x)$ and $G(x)$, the following two conditions are equivalent:*

- *for every non-strictly increasing function $u(x)$, the utility corresponding to $F(x)$ is larger than or equal to the utility corresponding to $G(x)$;*
- *for all x, we have $G(x) \le F(x)$.*

Sometimes, we also know that the decision maker is risk-averse, i.e., that for each lottery in which the person gets amounts x_i with probabilities p_i, the decision would prefer to receive the expected value $p_1 \cdot x_1 + \ldots + p_n \cdot x_n$ than to participate in this lottery. In view of the above-described relation between expected utility and decisions, this means that for all such cases, we have

$$u(p_1 \cdot x_1 + \ldots + p_n \cdot x_n) \ge p_1 \cdot u(x_1) + \ldots + p_n \cdot u(x_n),$$

i.e., that the utility function $u(x)$ is *concave.* In this case, the following known result holds:

Proposition 2. *For every two probability distributions $F(x)$ and $G(x)$, the following two conditions are equivalent:*

- *for every non-strictly increasing concave function $u(x)$, the utility corresponding to $F(x)$ is larger than or equal to the utility corresponding to $G(x)$;*
- *for all x, we have*
$$\int_{-\infty}^{x} G(t)\,dt \le \int_{-\infty}^{x} F(t)\,dt.$$

Need to consider interval and p-box uncertainty. The above results assume that for each possible action, we know the probabilities of different outcomes. In practice, we often also have only partial information about these probabilities.

Sometimes, we have no information at all about the probabilities, we only know the range $[\underline{x},\overline{x}]$ of possible values. This case is known as the case of *interval uncertainty*; see, e.g., [5, 7, 10, 11].

Sometimes, we have partial information about the cumulative distribution function $F(x)$. Uncertainty usually means that for each x, instead of the exact value $F(x)$, we only know the range $\left[\underline{F}(x),\overline{F}(x)\right]$ of possible values. In this case, for each tuple of values $(x_1,\ldots,x_n)$ the set of possible values of the corresponding tuple $(F(x_1),\ldots,F(x_n))$ is a box

$$\left[\underline{F}(x_1),\overline{F}(x_1)\right]\times\ldots\times\left[\underline{F}(x_n),\overline{F}(x_n)\right].$$

This box is called *probability box*, or *p-box*, for short. Because of this, this case is known as the case of *p-box uncertainty*; see, e.g., [1].

How to make decisions under interval uncertainty. According to decision theory [4, 6, 9], to make decisions under interval uncertainty, a decision maker has to select his/her degree α_H of optimism-pessimism – a number from the interval $[0,1]$. Then, the utility of an interval $[\underline{x},\overline{x}]$ is determined as

$$\alpha_H\cdot u(\overline{x})+(1-\alpha_H)\cdot u(\underline{x}).$$

The name of this degree comes from the fact that when $\alpha_H=1$, the utility is equal to $u(\overline{x})$. In this case, we only take into account the best-case scenario, and we ignore the possibility that the outcome can be worse. This is clearly the case of pure optimism. On the other hand, when $\alpha_H=0$, the utility is equal to $u(\underline{x})$. In this case, we only take into account the worst-case scenario, and we ignore the possibility that the outcome can be better. This is clearly the case of pure pessimism. Of course, these two are extreme cases. In practice, decision makers take both the best-case and the worst-case scenarios into account, i.e., they select values $\alpha_H\in(0,1)$.

How to make decisions under p-box uncertainty. As we have mentioned, for a given probability distribution $F(x)$, the utility is equal to $\int u(x)\,dF(x)$. By using integration by part, we can reduce this integral to the form

$$-\int u'(x)\cdot F(x)\,dx.$$

Since the function $u(x)$ is non-strictly increasing, we have $u'(x)\ge 0$. Thus, if $F(x)\le G(x)$, then we have

$$\int u(x)\,dF(x)=-\int u'(x)\cdot F(x)\,dx\ge-\int u'(x)\cdot G(x)\,dx=\int u(x)\,dG(x).$$

As a result, when $\underline{F}(x)\le F(x)\le\overline{F}(x)$, we have

$$\int u(x)\,d\underline{F}(x) \geq \int u(x)\,dF(x) \geq \int u(x)\,d\overline{F}(x).$$

Thus, in the p-box case, possible values of utility form an interval

$$\left[\int u(x)\,d\overline{F}(x), \int u(x)\,d\underline{F}(x)\right].$$

So, according to decision theory, we need to select an alternative for which the value

$$\alpha_H \cdot \int u(x)\,d\underline{F}(x) + (1-\alpha_H) \cdot \int u(x)\,d\underline{F}(x)$$

is the largest.

What we do in this paper. In this paper, we extend the known stochastic dominance results to the cases of interval and p-box uncertainty: namely, we describe all the case with interval and p-box uncertainty when we can make a decision without knowing the exact shape of the utility function.

2 Case of interval uncertainty

Proposition 3. *Let $\alpha_H \in (0,1)$ be given. Then, for every two intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = \left[\underline{y}, \overline{y}\right]$, the following two conditions are equivalent:*

- *for every non-strictly increasing function $u(x)$, the utility corresponding to $\mathbf{x}$ is larger than or equal to the utility corresponding to $\mathbf{y}$;*
- *we have $\underline{x} \geq \underline{y}$ and $\overline{x} \geq \overline{y}$.*

Proposition 4. *Let $\alpha_H \in (0,1)$ be given. Then, for every two intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = \left[\underline{y}, \overline{y}\right]$, the following two conditions are equivalent:*

- *for every non-strictly increasing concave function $u(x)$, the utility corresponding to $\mathbf{x}$ is larger than or equal to the utility corresponding to $\mathbf{y}$;*
- *we have $\underline{x} \geq \underline{y}$ and $\alpha_H \cdot \overline{x} + (1-\alpha_H) \cdot \underline{x} \geq \alpha_H \cdot \overline{y} + (1-\alpha_H) \cdot \underline{y}$.*

Proof of Proposition 3.

1°. To prove the desired equivalence, we will prove the following two statements:

- that if the second condition is satisfied, then the first condition is also satisfied, and
- that if the second condition is not satisfied, then the first condition is also nor satisfied.

Let us prove them one by one.

2°. Let us first prove that if the second condition is satisfied, then the first condition is also satisfied. Indeed, from $\underline{x} \geq \underline{y}$, we conclude that

$$(1-\alpha_H)\cdot \underline{x} \geq (1-\alpha_H)\cdot \underline{y}.$$

Similarly, from $\overline{x} \geq \overline{y}$, we conclude that

$$\alpha_H \cdot \overline{x} \geq \alpha_H \cdot \overline{y}.$$

By adding the two centered inequalities, we conclude that

$$u(\mathbf{x}) = \alpha_H \cdot \overline{x} + (1-\alpha_H)\cdot \underline{x} \geq \alpha_H \cdot \overline{y} + (1-\alpha_H)\cdot \underline{y} = u(\mathbf{y}).$$

3°. Let us now prove that if the second condition is not satisfied, then the first condition is also not satisfied. The second condition consists of two inequalities; thus, the fact that it is not satisfied means that one of these inequalities is false, i.e., either $\underline{x} < \underline{y}$ or $\overline{x} < \overline{y}$. We will consider these two cases one by one.

3.1°. Let us first consider the case when $\underline{x} < \underline{y}$. In this case, we can consider the following non-strictly increasing utility function:

- for $x < \underline{y}$, we have $u(x) = 0$, and
- for $x \geq \underline{y}$, we take $u(x) = 1$.

In this case, since $\overline{y} \geq \underline{y}$, we have $u(\overline{y}) = u\left(\underline{y}\right) = 1$ and thus,

$$u(\mathbf{y}) = \alpha_H \cdot u(\overline{y}) + (1-\alpha_H)\cdot u\left(\underline{y}\right) = \alpha_H \cdot 1 + (1-\alpha_H)\cdot 1 = 1.$$

On the other hand, since $\underline{x} < \underline{y}$, we have $u(\underline{x}) = 0$, thus

$$(1-\alpha_H)\cdot u(\underline{x}) = 0.$$

For our utility function, we have $u(x) \leq 1$ for all x, in particular, we have $u(\overline{x}) \leq 1$, thus

$$\alpha_H \cdot u(\overline{x}) \leq \alpha_H < 1.$$

By adding the two centered inequalities, we conclude that

$$u(\mathbf{x}) = \alpha_H \cdot u(\overline{x}) + (1-\alpha_H)\cdot u(\underline{x}) \leq \alpha_H < 1 = u(\mathbf{y}).$$

So, in this case, the first condition is also not satisfied.

3.2°. Let us now consider the case when $\overline{x} < \overline{y}$. In this case, we can consider the following non-strictly increasing utility function:

- for $x < \overline{y}$, we have $u(x) = 0$, and
- for $x \geq \overline{y}$, we take $u(x) = 1$.

In this case, we have $u(\overline{y}) = 1$ and thus,

$$\alpha_H \cdot u(\overline{y}) = \alpha_H > 0.$$

For this utility function, $u(x) \geq 0$ for all x, in particular, $u\left(\underline{y}\right) \geq 0$, thus

$$(1-\alpha_H)\cdot u\left(\underline{y}\right)\ge 0.$$

By adding the two last centered inequalities, we conclude that

$$u(\mathbf{y})=\alpha_H\cdot u(\overline{y})+(1-\alpha_H)\cdot u\left(\underline{y}\right)\ge \alpha_H>0.$$

On the other hand, since $\underline{x}\le\overline{x}<\overline{y}$, we have $u(\underline{x})=u(\overline{x})=0$, thus

$$u(\mathbf{x})=\alpha_H\cdot u(\overline{x})+(1-\alpha_H)\cdot u(\underline{x})=0<\alpha_H\le u(\mathbf{y}).$$

So, in this case, the first condition is also not satisfied.

The proposition is proven.

Proof of Proposition 4.

1°. To prove the desired equivalence, we will prove the following two statements:

- that if the second condition is satisfied, then the first condition is also satisfied, and
- that if the second condition is not satisfied, then the first condition is also nor satisfied.

Let us prove them one by one.

2°. Let us first prove that if the second condition is satisfied, then the first condition is also satisfied. For this purpose, we will consider two possible cases: when $\overline{x}\ge\overline{y}$ and when $\overline{x}<\overline{y}$. We will consider these two cases one by one.

2.1°. If $\overline{x}\ge\overline{y}$, then, taking into account that $\underline{x}\ge\underline{y}$ and that the utility function $u(x)$ is non-strictly increasing, we conclude that $u(\overline{x})\ge u(\overline{y})$ and $u(\underline{x})\ge u\left(\underline{y}\right)$. Multiplying the first inequality by $\alpha_H>0$ and the second one by $1-\alpha_H>0$ and adding the resulting inequalities, we conclude that

$$\alpha_H\cdot\overline{x}+(1-\alpha_H)\cdot\underline{x}\ge\alpha_H\cdot\overline{y}+(1-\alpha_H)\cdot\underline{y},$$

i.e., indeed, that the utility of the interval $\mathbf{x}$ is larger than or equal to the utility of the interval $\mathbf{y}$.

2.2°. Let us now consider the case when $\overline{x}<\overline{y}$, i.e., when $\underline{y}\le\underline{x}\le\overline{x}<\overline{y}$ and when

$$\alpha_H\cdot\overline{x}+(1-\alpha_H)\cdot\underline{x}\ge\alpha_H\cdot\overline{y}+(1-\alpha_H)\cdot\underline{y}.$$

Here,

$$\underline{x}=\underline{y}+\frac{\underline{x}-\underline{y}}{\overline{y}-\underline{y}}\cdot(\overline{y}-\underline{y})=\frac{\underline{x}-\underline{y}}{\overline{y}-\underline{y}}\cdot\overline{y}+\frac{\overline{y}-\underline{x}}{\overline{y}-\underline{y}}\cdot\underline{y}.$$

Since the utility function is concave, we conclude that

$$u(\underline{x})\ge\frac{\underline{x}-\underline{y}}{\overline{y}-\underline{y}}\cdot u(\overline{y})+\frac{\overline{y}-\underline{x}}{\overline{y}-\underline{y}}\cdot u\left(\underline{y}\right),$$

i.e., that

$$u(\underline{x}) \geq u\left(\underline{y}\right) + \frac{\underline{x} - \underline{y}}{\overline{y} - \underline{y}} \cdot \left(u(\overline{y}) - u\left(\underline{y}\right)\right). \quad (1)$$

Similarly,

$$\overline{x} = \underline{y} + \frac{\overline{x} - \underline{y}}{\overline{y} - \underline{y}} \cdot (\overline{y} - \underline{y}) = \frac{\overline{x} - \underline{y}}{\overline{y} - \underline{y}} \cdot \overline{y} + \frac{\overline{y} - \overline{x}}{\overline{y} - \underline{y}} \cdot \underline{y}.$$

Since the utility function is concave, we conclude that

$$u(\overline{x}) \geq \frac{\overline{x} - \underline{y}}{\overline{y} - \underline{y}} \cdot u(\overline{y}) + \frac{\overline{y} - \overline{x}}{\overline{y} - \underline{y}} \cdot u\left(\underline{y}\right),$$

i.e., that

$$u(\overline{x}) \geq u\left(\underline{y}\right) + \frac{\overline{x} - \underline{y}}{\overline{y} - \underline{y}} \cdot \left(u(\overline{y}) - u\left(\underline{y}\right)\right). \quad (2)$$

Multiplying the inequality (2) by α_H and the inequality (1) by $1 - \alpha_H$ and adding the resulting inequalities, we conclude that

$$u(\mathbf{x}) \geq u\left(\underline{y}\right) + \frac{x - \underline{y}}{\overline{y} - \underline{y}} \cdot \left(u(\overline{y}) - u\left(\underline{y}\right)\right), \quad (3)$$

where we denoted $x \stackrel{\text{def}}{=} \alpha_H \cdot \overline{x} + (1 - \alpha_H) \cdot \underline{x}$. We have assumed that

$$x \geq y \stackrel{\text{def}}{=} \alpha_H \cdot \overline{y} + (1 - \alpha_H) \cdot \underline{y}.$$

Thus, from (3), we can conclude that

$$u(\mathbf{x}) \geq u\left(\underline{y}\right) + \frac{y - \underline{y}}{\overline{y} - \underline{y}} \cdot \left(u(\overline{y}) - u\left(\underline{y}\right)\right). \quad (4)$$

Here, by definition of y, we have

$$\frac{y - \underline{y}}{\overline{y} - \underline{y}} = \alpha_H.$$

Thus, the inequality (4) takes the form

$$u(\mathbf{x}) \geq u\left(\underline{y}\right) + \alpha_H \cdot (u(\overline{y}) - u\left(\underline{y}\right)) = \alpha_H \cdot u(\overline{y}) + (1 - \alpha_H) \cdot u\left(\underline{y}\right),$$

i.e., that indeed $u(\mathbf{x}) \geq u(\mathbf{y})$.

3°. Let us now prove that if the second condition is not satisfied, then the first condition is also not satisfied. The second condition consists of two inequalities; thus, the fact that it is not satisfied means that one of these inequalities is false, i.e., either $\underline{x} < \underline{y}$ or

$$\alpha_H \cdot \overline{x} + (1 - \alpha_H) \cdot \underline{x} < \alpha_H \cdot \overline{y} + (1 - \alpha_H) \cdot \underline{y}.$$

We will consider these two cases one by one.

3.1°. Let us first consider the case when $\underline{x} < \underline{y}$. In this case, we can consider the following non-strictly increasing concave utility function:

- for $x \le \underline{y}$, we have $u(x) = x$, and
- for $x \ge \underline{y}$, we take $u(x) = \underline{y}$.

In this case, since $\underline{y} \ge \overline{y}$, we have $u\left(\underline{y}\right) = u(\overline{y}) = \underline{y}$ and thus,

$$u(\mathbf{y}) = \alpha_H \cdot u(\overline{y}) + (1-\alpha_H) \cdot u\left(\underline{y}\right) = \alpha_H \cdot \underline{y} + (1-\alpha_H) \cdot \underline{y} = \underline{y}.$$

On the other hand, by definition of our utility function, since $\underline{x} < \underline{y}$, we have $u(\underline{x}) = \underline{x} < \underline{y}$. Since $\alpha_H \in (0,1)$, we have $1 - \alpha_H > 0$ and thus,

$$(1-\alpha_H) \cdot u(\underline{x}) < (1-\alpha_H) \cdot \underline{y}.$$

For our utility function, we have $u(x) \le \underline{y}$ for all x, in particular, $u(\overline{x}) \le \underline{y}$, thus

$$\alpha_H \cdot u(\overline{x}) \le \alpha_H \cdot \underline{y}.$$

By adding two centered inequalities, we conclude that

$$u(\mathbf{x}) = \alpha_H \cdot u(\overline{x}) + (1-\alpha_H) \cdot u(\underline{x}) < \alpha_H \cdot \underline{y} + (1-\alpha_H) \cdot \underline{y} = \underline{y} = u(\mathbf{y}),$$

i.e., indeed, the first condition is not satisfied.

3.2°. Let us now consider the case when

$$\alpha_H \cdot \overline{x} + (1-\alpha_H) \cdot \underline{x} < \alpha_H \cdot \overline{y} + (1-\alpha_H) \cdot \underline{y}.$$

In this case, we can consider the utility function $u(x) = x$. This function is increasing and concave, but for this function, the above inequality means that $u(\mathbf{x}) < u(\mathbf{y})$. Thus, in this case, the first condition is also not satisfied.

The proposition is thus proven.

3 Case of p-box uncertainty

Proposition 5. *Let $\alpha_H \in (0,1)$ be given. Then, for every two p-boxes $\mathbf{F}(x) = \left[\underline{F}(x), \overline{F}(x)\right]$ and $\mathbf{G}(x) = \left[\underline{G}(x), \overline{G}(x)\right]$, the following two conditions are equivalent:*

- *for every non-strictly increasing function $u(x)$, the utility corresponding to $\mathbf{F}(x)$ is larger than or equal to the utility corresponding to $\mathbf{G}(x)$;*
- *for all x, we have $G(x) \le F(x)$, where*

$$F(x) \stackrel{\text{def}}{=} \alpha_H \cdot \underline{F}(x) + (1-\alpha_H) \cdot \overline{F}(x)$$

and

$$G(x) \stackrel{\text{def}}{=} \alpha_H \cdot \underline{G}(x) + (1-\alpha_H) \cdot \overline{G}(x).$$

Proposition 6. *Let $\alpha_H \in (0,1)S$ be given. Then, for every two p-boxes $\mathbf{F}(x) = \left[\underline{F}(x), \overline{F}(x)\right]$ and $\mathbf{G}(x) = \left[\underline{G}(x), \overline{G}(x)\right]$, the following two conditions are equivalent:*

- *for every non-strictly increasing concave function $u(x)$, the utility corresponding to $\mathbf{F}(x)$ is larger than or equal to the utility corresponding to $\mathbf{G}(x)$;*
- *for all x, we have*

$$\int_{-\infty}^{x} G(t)\,dt \le \int_{-\infty}^{x} F(t)\,dt,$$

  where

$$F(x) \stackrel{\text{def}}{=} \alpha_H \cdot \underline{F}(x) + (1-\alpha_H) \cdot \overline{F}(x)$$

  and

$$G(x) \stackrel{\text{def}}{=} \alpha_H \cdot \underline{G}(x) + (1-\alpha_H) \cdot \overline{G}(x).$$

Proof of Propositions 5 and 6. According to decision theory, the utility of a p-box $\left[\underline{F}(x), \overline{F}(x)\right]$ is equal to

$$\alpha_H \cdot \int u(x)\,d\underline{F}(x) + (1-\alpha_H) \cdot \int u(x)\,d\overline{F}(x).$$

As we have mentioned earlier, by applying integration by parts, we can transform each of these integrals into an equivalent form

$$-\int u(x) \cdot \underline{F}(x)\,dx \text{ and } -\int u(x) \cdot \overline{F}(x)\,dx.$$

Thus, the utility of a p-box takes the form

$$\alpha_H \cdot \left(-\int u(x) \cdot \underline{F}(x)\,dx\right) + (1-\alpha_H) \cdot \left(-\int u(x) \cdot \overline{F}(x)\,dx\right).$$

Using the fact that the integral of the sum is equal to the sum of the corresponding integrals, we can conclude that this expression is equal to

$$-\int u(x) \cdot (\alpha_H(x) \cdot \underline{F}(x) + (1-\alpha_H) \cdot \overline{F}(x))\,dx,$$

i.e., by definition of the combination $F(x)$, the form

$$-\int u(x) \cdot F(x)\,dx = \int u(x)\,dF(x).$$

Thus, the utility of a p-box $\left[\underline{F}(x), \overline{F}(x)\right]$ is equal to the utility corresponding to the probability distribution

$$F(x) = \alpha_H \cdot \underline{F}(x) + (1-\alpha_H) \cdot \overline{F}(x).$$

Because of this, Propositions 5 and 6 dealing with p-boxes follow from Propositions 1 and 2 that deal with probability distributions.

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References

1. S. Ferson, V. Kreinovich, L. Ginzburg, D. S. Myers, and K. Sentz, *Constructing Probability Boxes and Dempster-Shafer Structures*, Sandia National Laboratories, Report SAND2002-4015, January 2003.
2. P. C. Fishburn, *Utility Theory for Decision Making*, John Wiley & Sons Inc., New York, 1969.
3. P. C. Fishburn, *Nonlinear Preference and Utility Theory*, The John Hopkins Press, Baltimore, Maryland, 1988.
4. L. Hurwicz, *Optimality Criteria for Decision Making Under Ignorance*, Cowles Commission Discussion Paper, Statistics, No. 370, 1951.
5. L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics*, Springer, London, 2012.
6. V. Kreinovich, "Decision making under interval uncertainty (and beyond)", In: P. Guo and W. Pedrycz (eds.), *Human-Centric Decision-Making Models for Social Sciences*, Springer Verlag, 2014, pp. 163–193.
7. B. J. Kubica, *Interval Methods for Solving Nonlinear Constraint Satisfaction, Optimization, and Similar Problems: from Inequalities Systems to Game Solutions*, Springer, Cham, Switzerland, 2019.
8. H. Levy, *Stochastic Dominance: Investment Decision Making under Uncertainty*, Springer Verlag, New York, 2006.
9. R. D. Luce and R. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.
10. G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.
11. R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.
12. H. T. Nguyen, O. Kosheleva, and V. Kreinovich, "Decision making beyond Arrow's 'impossibility theorem', with the analysis of effects of collusion and mutual attraction", *International Journal of Intelligent Systems*, 2009, Vol. 24, No. 1, pp. 27–47.
13. H. T. Nguyen, V. Kreinovich, B. Wu, and G. Xiang, *Computing Statistics under Interval and Fuzzy Uncertainty*, Springer Verlag, Berlin, Heidelberg, 2012.
14. H. Raiffa, *Decision Analysis*, McGraw-Hill, Columbus, Ohio, 1997.
15. S. Sriboonchitta, W.-K. Wong, S. Dhompongsa, and H. T. Nguyen, *Stochastic Dominance and Applications to Finance, Risk and Economics Hardcover*, CRC Press, Boca Raton, Florida, 2009.