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Why Empirical Membership Functions Are Well-Approximated by Piecewise Quadratic Functions: Theoretical Explanation for Empirical Formulas of Novak's Fuzzy Natural Logic

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

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Why Empirical Membership Functions Are Well-Approximated by Piecewise Quadratic Functions: Theoretical Explanation for Empirical Formulas of Novak's Fuzzy Natural Logic

Olga Kosheleva and Vladik Kreinovich

Abstract Empirical analysis shows that membership functions describing expert opinions have a shape that is well described by a smooth combination of two quadratic segments. In this paper, we provide a theoretical explanation for this empirical phenomenon.

1 Formulation of the problem

What we membership functions: a brief reminder. A significant part of human knowledge is formulate not in precise terms, but by using imprecise ("fuzzy") natural-language words. To describe such knowledge in precise terms, Lotfi Zadeh proposed a technique that he called *fuzzy logic*; see, e.g., [1, 4, 5, 7, 9, 13].

In this technique, to describe an imprecise property like "small", we assign:

- to each possible value x of the corresponding quantity,
- a degree $m(x)$ – from the interval $[0,1]$ – to which, in the expert's opinion, this value satisfies the given property (e.g., is small).

Such a function $m(x)$ is called a *membership function*.

What are the shapes of membership functions: a brief history. During the first few decades of fuzzy research, many researchers tried to empirically capture the shape of the membership function. However, it turned out that in most practical applications, the results of using fuzzy techniques do not depend much on this shape.

Olga Kosheleva
Department of Teacher Education, University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA, e-mail: olgak@utep.edu

Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA, e-mail: vladik@utep.edu

As a result, most applications of fuzzy techniques use the simplest shapes: triangular and trapezoid.

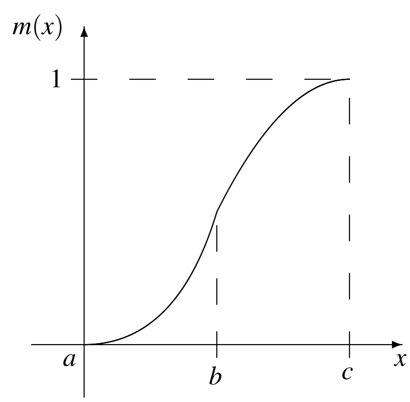
However, in some applications, capturing the exact shape is important [8]. Also, while it may not be very useful in other applications, finding the right shape is interesting from the viewpoint of understanding human reasoning – an area that has led to many breakthroughs in computing.

So what are these shapes? So what are the shapes of membership functions? A large amount of empirical data about shapes of membership function has been summarized in Chapter 5 of [8].

Usually, a membership function consists of segments in which it is either constant 0, or constant 1, or increases from 0 to 1, or decreases from 1 to 0. As we have mentioned, in most current applications, the behavior of $m(x)$ on each non-constant segment is described by a linear function. However, empirical shapes are different. It turns out that empirical shapes are well described by smooth (differentiable) piecewise quadratic functions.

In this approximation, on a segment $[a,c]$ on which the function increases from 0 to 1:

- we start with a quadratic function $m_1(x)$ for which $m_1(a) = m_1'(a) = 0$, where $m_1'(x)$, as usual, denotes the derivatives;
- we continue, at some intermediate point b, with a quadratic function $m_2(x)$ for which $m_2(c) = 1$ and $m_2'(c) = 0$; and
- the transition from $m_1(x)$ to $m_2(x)$ is smooth: $m_1(b) = m_2(b)$ and $m_1'(b) = m_2'(b)$.



Why these shapes? A natural question is: why these shapes? In this paper, we provide a possible explanation for these emprical shapes.

2 Our explanation

Preliminary comment. The numerical values of the quantity x depend on the choice of the measuring unit and on the choice of the starting point. If we change the measuring unit to a new one which is λ times smaller and then the starting point to the one that is v value smaller, we get a new numerical value $\lambda \cdot x + v$. A classical example of such a transformation is the transformation of temperature from the metric Celsius scale (C) to the US-used Fahrenheit (F) scale: $t_F = 1.8 \cdot t_C + 32$.

By using such a linear transformation, we can always transform a given interval $[a,c]$ into $[0,1]$: this can be done by the transformation with

$$\lambda = \frac{1}{c-a} \text{ and } v = -\frac{a}{c-a}.$$

Thus, without losing generality, we can consider only increasing segments $m(x)$ located in the interval $[0,1]$ for which the membership function increases from $m(0) = 0$ to $m(1) = 1$.

Similarly, decreasing segments can be transformed into increasing ones if we change the sign of the quantity x. This transformation also makes sense in many practical situations: e.g., we can consider gain instead of loss and vice versa. Because of this possibility, in the following text, we will only consider increasing segments for which $m(0) = 0$ and $m(1) = 1$.

From the decision making viewpoint, what is the meaning of $m(1) = 1$ and $m(0) = 0$? The ultimate objective of our knowledge is to make appropriate decisions. From this viewpoint, the fact that for $x = 1$, we have $m(x) = 1$, means, in effect, that for $x = 1$, we have a decision that perfectly fits this input.

Similarly, the fact that for $x = 0$, we have $m(x) = 0$ means that the perfect-for-($x = 1$) decision is absolutely inappropriate for $x = 0$. This, in turn, usually means that we have $n(x) = 1$ for some other membership function, i.e., that there is some other decision that works perfectly well when x is exactly equal to 0.

From this viewpoint, what is the meaning of $m(x)$ for x between 0 and 1? In these terms, for each value x from the interval $[0,1]$, the degree $m(x)$ can be viewed as a degree to which:

- the 1-decision – i.e., the decision that is perfect when the value of the quantity is 1 – is appropriate for this value x,
- as opposed to the alternative 0-decision, the decision that is perfect when the value of the quantity is 0.

The value $m(x)$ must be determined by the corresponding losses. When the value x is different from 0 and 1, neither the 1-decision nor the 0-decision are perfect. Whichever of these two decisions we pick, there will be losses – in comparison to the ideal situation when the available decision perfectly fits the value of the corresponding quantity. So, the degree $m(x)$ should be determined by these two losses:

- the loss $L_0(x)$ corresponding to using the 0-decision, and

- the loss $L_1(x)$ corresponding to using the 1-decision.

In other words, $m(x) = F(L_0(x), L_1(x))$ for some function $F(y,z)$ of two variables.

The degree $m(x)$ should not depend on the measuring unit for losses. Like many other quantities, we can describe losses in different units. If we are talking about monetary losses, we can use US dollars, Euros, etc. If we replace the original monetary unit with a new unit which is λ times smaller, then the numerical values of all the losses will multiply by λ: we will get $\lambda \cdot L_0(x)$ and $\lambda \cdot L_1(x)$ instead of the original values $L_0(x)$ and $L_1(x)$.

The choice of a monetary unit is a question of convenience. Our decisions – and related values $m(x)$ – should not depend on the choice of a monetary unit: the value $F(y,z)$ corresponding to the original monetary unit should be the same as the decision $F(\lambda \cdot y, \lambda \cdot z)$ corresponding to the new units. In other words, for all y, z, and λ, we shall have

$$F(y,z) = F(\lambda \cdot y, \lambda \cdot z). \tag{1}$$

So, the value $m(x)$ must depend only on the ratio of the two losses. Let us show that the equality (1) implies that the value $F(y,z)$ is uniquely determined by the ratio y/z, i.e., that the value $m(x)$ depends only on the ratio of the two losses.

Indeed, for any y and z, we can take $\lambda = 1/z$. For this λ, the formula (1) takes the form $F(y,z) = F(y/z, 1)$, i.e., the form $F(y,z) = G(y/z)$, where we denoted $G(t) \stackrel{\text{def}}{=} F(t,1)$. For $y = L_0(x)$ and $z = L_1(x)$, this means that

$$m(x) = G(t), \text{ where } t \stackrel{\text{def}}{=} \frac{L_0(x)}{L_1(x)}. \tag{2}$$

Let us use the fact that we have two meaningful 1-D scales to describe the decision situation. The fact that the value $m(x)$ is uniquely determined by the ratio of two losses means that we have two 1-D scales to describe the decision situation:

- the scale in which the situation is described by the degree $m(x)$ – that can take any value from the interval $[0,1]$, and
- the scale in which the situation is described the ratio of two losses – the ratio that can take value from 0 to infinity.

Both are meaningful scales, with a meaningful transformation between them – meaningful as opposed to a generic mathematical transformation that may not any meaningful interpretation with respect to our decision making situation. We may have other meaningful scales and meaningful transformations. For example, we can apply a linear transformation and us degrees not from 0 to 1, but from -1 to 1 or from 0 to 5. Let us consider the class of all possible meaningful transformations.

If we have a meaningful transformation from scale A to scale B, then an inverse transformation – from scale B to scale A – should also be meaningful. Similarly, if we have a meaningful transformation from scale A to scale B and a meaningful transformation from scale B to scale C, then the composition of these two transformations is a meaningful transformation from scale A to scale C. Thus, the class of all possible meaningful transformations should be closed under taking the inverse and under taking the composition. In mathematics, classes with this property are known as *transformation groups*.

Another reasonable requirement on this class is that, since the main objective of fuzzy techniques in general is to use computers, we want all meaningful transformations to be implementable on a computer. In a single computer, we can only store finitely many numbers. Thus, the class of all transformations that can be implemented on a computer must depend on finitely many parameters. In mathematical terms, this means that this class (i.e., this transformation group) must be *finite-dimensional*. So, we are looking for finite-dimensional transformation groups that contain all linear transformations.

Interestingly, there is a full classification of all such transformation groups – including not only the 1-D case when we are talking about functions from real numbers to real numbers, but also multi-D case when we are talking about transformations of a multi-D space. This classification result was conjectured by Norbert Wiener, the father of cybernetics (see [12]). Wiener's conjecture was proven in [3, 10]. In particular, for 1-D transformations, the classification result says that all meaningful transformations must be fractional-linear, i.e., they must have the form

$$m(x) = G(t) = \frac{c_0 + c_1 \cdot t}{c_2 + c_3 \cdot t}, \tag{3}$$

for some coefficients c_i.

Comment. The proof of the general multi-D classification is lengthy and complicated, but sincd, for our purposes, we are interested only in the 1-D case, it should be mentioned that for this case, a much shorter and simpler proof is possible; see, e.g., [6].

So what is the relation between $m(x)$ and the ratio of losses? For $x = 0$, there is no loss in using 0-decision, so $L_0(x) = 0$ and the ratio t is equal to 0. In this case, there is no sense in using 1-decision, so we should have $m(0) = 0$. So, for $t = 0$, the formula (3) should lead to $G(0) = 0$. Substituting the expression (3) into this equality, we conclude that $c_0 = 0$, i.e., that

$$m(x) = G(t) = \frac{c_1 \cdot t}{c_2 + c_3 \cdot t}. \tag{4}$$

For $x = 1$, there is no loss in using 1-decision, so $L_1(x) = 0$ and thus, the ratio t is equal to infinity. For $x = 1$, there is no sense in using 0-decision, so we should have $m(1) = 1$. So, for $t = \infty$, the formula (3) should lead to $G(\infty) = 1$.

Due to $G(\infty) = 1 \neq 0$, we cannot have $c_1 = 0$, since if $c_1 = 0$, we would have $G(t) = 0$ for all t. Since $c_1 \neq 0$, we can divide both the numerator and the denominator of the expression (4) by c_1 and get a simplified formula

$$m(x) = G(t) = \frac{t}{c_2' + c_3' \cdot t}, \tag{5}$$

where we denoted $c_i' \stackrel{\text{def}}{=} c_i/c_1$. For $t \to \infty$, the expression (5) tends to $1/c_3'$, so $G(\infty) = 1$ implies that $1/c_3' = 1$, i.e., $c_3' = 1$. Hence, the formula (5) takes the form

$$m(x) = G(t) = \frac{t}{t + c_2'}. \tag{6}$$

Substituting $t = L_0(x)/L_1(x)$ into this formula and multiplying both the numerator and the denominator by $L_1(x)$, we get the following expression:

$$m(x) = \frac{L_0(x)}{L_0(x) + c_2' \cdot L_1(x)}. \tag{7}$$

How can we estimate the losses? Due to the formula (7), in order to find the exact expression for the membership function, we need to find out how the losses $L_i(x)$ depend on the value x. A natural idea – which is actively and successfully used in physics (see, e.g., [2, 11]) is to expand the unknown dependence in Taylor series and, as a good first approximation, to use the first non-trivial term in this expansion.

Let us first look for the expression for the loss function $L_0(x)$ that describes losses caused by using 0-decision. First we expand this function in Taylor series:

$$L_0(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots \tag{8}$$

By the definition of 0-decision, when x is equal to 0, this decision is perfect, there is no loss. So, we must have $L_0(0) = 0$. For the formula (8), this condition implies that $a_0 = 0$.

For $x = 0$, the loss function $L_0(x)$ attains its minimum value 0. Thus, for $x = 0$, the derivative of this function is equal to 0: $L_0'(0) = 0$. For the formula (8), we have $L_0'(x) = a_1 + 2a_2 \cdot x + \ldots$, so the fact that $L_0'(0) = 0$ implies that $a_1 = 0$. Thus, the first non-zero term in the expansion (8) is quadratic. So, in the above-mentioned first approximation, we have

$$L_0(x) = a_2 \cdot x^2, \tag{9}$$

for some value a_2.

Similarly, we can expend the loss function $L_1(x)$ (corresponding to using 1-decision) in Taylor series around the point $x = 1$:

$$L_1(x) = b_0 + b_1 \cdot (x - 1) + b_2 \cdot (x - 1)^2 + \ldots \tag{10}$$

For $x = 1$, there is no loss, so $L_1(1) = 0$ and thus, $b_0 = 0$. For $x = 1$, the loss function $L_1(x)$ attains its minimum value 0. Thus, for $x = 1$, the derivative of this function is equal to 0: $L_1'(1) = 0$. Thus, we get $b_1 = 0$. Thus, the first non-zero term in the expansion (10) is quadratic. So, in the first approximation, we have

$$L_1(x) = b_2 \cdot (1-x)^2, \tag{11}$$

for some value b_2.

Towards the final expression for the membership function. Substituting the expressions (9) and (11) into the formula (7), we get

$$m(x) = \frac{a_2 \cdot x^2}{a_2 \cdot x^2 + a_3' \cdot b_2 \cdot (1-x)^2}. \tag{12}$$

We can simplify this expression if we divide both the numerator and the denominator of this expression by a_2. Then we get

$$m(x) = \frac{x^2}{x^2 + \alpha \cdot (1-x)^2}, \tag{13}$$

where we denoted $c \stackrel{\text{def}}{=} a_3' \cdot b_2 / a_2$.

This expression has the right behavior for $x \approx 0$ and $x \approx 1$. Let us show that this expression satisfies the empirical conditions described in [8]. Indeed, this expression is smooth for all $x \in (0,1)$, and satisfies the conditions $m(0) = 0$ and $m(1) = 1$.

For small x, this expression has the form

$$m(x) = \frac{x^2}{\alpha - 2\alpha \cdot x + 2 \cdot x^2} = \alpha^{-1} \cdot x^2 + o(x^2), \tag{14}$$

so we have $m'(0) = 0$.

To describe the asymptotic behavior of the expression (13) for $x \approx 1$, we can take into account that

$$1 - m(x) = 1 - \frac{x^2}{x^2 + \alpha \cdot (1-x)^2} = \frac{\alpha \cdot (1-x)^2}{x^2 + \alpha \cdot (1-x)^2}. \tag{15}$$

Thus, we have

$$m(x) = 1 - (1 - m(x)) = 1 - \frac{\alpha \cdot (1-x)^2}{x^2 + \alpha \cdot (1-x)^2}. \tag{16}$$

For x close to 1, we can reformulate this expression in terms of the difference $d \stackrel{\text{def}}{=} 1 - x$, for which $x = 1 - d$. Substituting $x = 1 - d$ into the formula (16), we conclude that

$$m(x) = 1 - \frac{\alpha \cdot d^2}{(1-d)^2 + \alpha \cdot d^2} = 1 - \frac{\alpha \cdot d^2}{1 - 2 \cdot d + (1+\alpha) \cdot d^2} =$$
$$1 - \alpha \cdot d^2 + o(d^2). \tag{17}$$

So, we have $m'(1) = 0$ as well.

What about the general case? The formula (13) corresponds to the interval $[0,1]$ on which the membership function is increasing. By using the above-mentioned linear transformation relating the interval $[0,1]$ with a generic interval $[a,c]$, we can conclude that on an interval $[a,c]$, the increasing membership function takes the form

$$m(x) = \frac{(x-a)^2}{(x-a)^2 + \beta \cdot (c-x)^2} \tag{18}$$

for some value $\beta > 0$. Similarly, we can conclude that an increasing membership function takes the form

$$m(x) = \frac{(c-x)^2}{(c-x)^2 + \beta \cdot (x-a)^2}. \tag{19}$$

Conclusion. So, our conclusion is that natural requirements indeed lead to the expressions (18) and (19) for the membership function, expression that fits very well with the empirical data.

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