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Paradox of Causality and Paradoxes of Set Theory

Alondra Baquier, Bradley Beltran, Gabriel Miki-Silva, Olga Kosheleva, and Vladik Kreinovich

Abstract Logical paradoxes show that human reasoning is not always fully captured by the traditional 2-valued logic, that this logic’s extensions – such as multi-valued logics – are needed. Because of this, the study of paradoxes is important for research on multi-valued logics. In this paper, we focus on paradoxes of set theory. Specifically, we show their analogy with the known paradox of causality, and we use this analogy to come up with similar set-theoretic paradoxes.

1 Introduction

Paradoxes: a brief reminder. Informal intuitive reasoning often leads us to paradoxes. For example, there is a famous heap paradox. It is based on the following simple reasoning:

• one grain of sand does not form a heap, and
• if we have several grains of sand that do not form a heap, and we add one more grain, we will also not get a heap.

By mathematical induction, we can easily conclude that no matter how many grains of sand we combine, we will never get a heap – but in reality, it is easy to form a heap.

Paradoxes and multi-valued logics. Logical paradoxes show that human reasoning is not always fully captured by the traditional 2-valued logic, that this logic’s extensions – such as multi-valued logics (an example of which is fuzzy logic; see, e.g., [1, 2, 3, 4, 5, 6]) – are needed.

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For example, in the heap case, a usual solution to this paradox is that “being a heap” is not a crisp 2-valued notion. Yes, one grain of sand is definitely not a heap, and big heaps are definitely heaps, but on the intermediate stage, we have a configuration about which we are not certain. By adding a single grain of sand, we never jump from “definitely not a heap” to “definitely a heap”. Instead, we slightly increase our degree of confidence that the resulting collection of grains of sand is a heap.

**It is therefore important to study paradoxes.** Because of this relation between paradoxes and multi-valued logics, the study of paradoxes is important for research on multi-valued logics.

**What we do in this paper.** In this paper, we focus on paradoxes of set theory. Specifically:
- we show their analogy with the known paradox of causality, and
- we use this analogy to come up with modified paradoxes of set theory.

**The structure of this paper.** In Section 2, we recall the paradox of causality. In Section 3, we recall the most well-known paradox of set theory – Russell’s paradox. In Section 4, we describe the relation between these paradoxes. Finally, in Section 5, we use this relation to come up with modified versions of Russell’s paradox.

### 2 Paradox of causality: a brief reminder

**Paradox of causality: in brief.** Anyone who watched a movie about time travel or read a book about it knows that time travel leads to paradoxes. By using a time machine, a person can go into the past and kill his/her grandfather before the time traveler’s father was conceived. Then:
- the traveler’s father could not have been born and thus, the traveler him/herself could not have been born,
- however, the time traveler was actually born, he/she is here.

This is clearly a paradox.

**Let us describe this paradox in more precise terms.** This paradox describes the situation when:
- the event $e_1$ – the time traveler making a decision to use the time machine – affects the event $e_2$ – life of his/her grandfather, and
- at the same time the event $e_2$ clearly affects the existence of the time traveler and thus, affects the event $e_1$.

Let us denote the relation “event $e_1$ affects the event $e_2$” by $e_1 < e_2$. In this notation, we get $e_1 < e_2$ and $e_2 < e_1$. Here we have a loop $e_1 < e_2 < e_1$ of size 2:
- the event $e_1$ affects the event $e_2$, and
the event $e_2$ affects the event $e_1$.

**We can consider longer loops.** A similar paradox appears also when we have a longer loop:

- the event $e_1$ affects the event $e_2$,
- the event $e_2$ affects some other event $e_3$ – for example, the birth of the time traveler’s father, and
- the event $e_3$ affects the event $e_1$.

So, here, $e_1 < e_2 < e_3 < e_1$.

We can have even longer loops.

### 3 Paradox of intuitive (“naive”) set theory

**What this paradox is about.** In set theory, there is a similar paradox – discovered by the famous philosopher Bertrand Russell – that is related to the loop of size 1, namely, to the possibility that a set $x$ may be an element of itself: $x \in x$. Examples of such sets are easy to find: for example:

- the set $U$ of all sets is a set and
- thus, this set is its own element: $U \in U$.

**What exactly is the paradox.** The paradox appears if we consider the set

$$S = \{ x : x \not\in x \}$$

of all the sets that do not belong to itself.

Specifically, the paradox appears when we check whether this set $S$ belongs to itself.

**Why is this a paradox.** Indeed, we have only two possible options:

- either the set $S$ belongs to itself: $S \in S$,
- or the set $S$ does not belong to itself $S \not\in S$.

Let us show that in both cases, we get a contradiction.

**What is $S \in S$?** Let us assume that the set $S$ belongs to itself. By definition, the set $S$ includes all the sets $x$ that do not belong to themselves.

- Since the set $S$ itself is an element of $S$, it means that the set $S$ *does not* belong to itself.
- However, we assumed that it *does* belong to itself.
So, in this case, we have a contradiction.

**What is** $S \not\in S$? Let us now assume that the set $S$ does not belong to itself. This means that the set $S$ does not have the property that describes all elements $x$ of the set $S$ – that each of these elements $x$ does not belong to itself. Since the set $S$ *does not* have this property, this means that the set $S$ *does* belong to itself. However, we assume that it *does not* belong to itself.

So, in this case, we also have a contradiction.

**So, we have a paradox.** In both possible cases – when $S \in DS$ and when $S \not\in S$ – we get a contradiction – which means that none of these two cases is possible. So, we indeed have a paradox.

### 4 How are these paradoxes related?

**How are these paradoxes similar.** In both cases – for time travel and for sets – we have a paradox. Both paradoxes relate to closed loops.

**How are these paradoxes different.** The main difference between these two paradoxes is as follows:

- in the case of time travel, the paradox appears for loops of any size, while
- for sets, the paradox is only known for a loop consisting of a single element:

$$x \in x.$$

**A natural question.** So, a natural question is: what if in the set case, we have a longer loop, will we still get a paradox?

**Our answer.** Our answer to this question is: yes, it is possible to have versions of Russell’s paradox that include loops of any size. We will show the details in the next section.

### 5 Resulting modifications of Russell’s paradox

**What is the desired paradox.** In this section, we show that yes, in set theory we can get a similar paradox if we consider loops of any length.

In other words, we get a paradox if we consider the possibility that:

- a set $x$ is an element of a set $x_1$,
- the set $x_1$ is an element of set $x_2$, and so on,
- and at the end, the set $x_n$ is an element of the set $x$ with which we started.

To show that there is a paradox let us consider a new set $S_n$: the set of all elements $x$ for which such a loop does not exist:
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\[ S_n = \{ x : \neg \exists x_1, \ldots, x_n (x \in x_1 \in x_2 \in \ldots \in x_n \in x) \} . \]

Just like before, let us check whether this set \( S_n \) belongs to itself.

**Why is this a paradox.** Of course, the set \( S_n \) either belongs to itself or it does not. Let us prove that in both cases, we get a contradiction.

**What if \( S_n \in S_n \)?** Let us first assume that the set \( S_n \) belongs to itself: \( S_n \in S_n \). By definition of the set \( S_n \), this means that we cannot have a loop in which \( S_n \) belongs to some set \( x_1 \), \( x_1 \) belongs to some set \( x_2 \), and so on, until we get a set \( x_n \) that belongs to \( S_n \):

\[ \neg \exists x_1, \ldots, x_n \left( S_n \in x_1 \in x_2 \in \ldots \in x_n \in S_n \right) . \]

But we do have such a loop – we can take \( x_1, x_2, \) and so on all equal to \( S_n \), then we have

\[ S_n \in x_1 = S_n \in x_2 = S_n \in \ldots \in x_n = S_n \in S_n . \]

So, in this case, we get a contradiction.

**What if \( S_n \notin S_n \)?** On the other hand, let us consider the case when \( S_n \) is not an element of itself, i.e., when \( S_n \notin S_n \). This means that the set \( S_n \) does not have the property that defines this set – that no loop exists containing this set. The fact that this property is not satisfied means that for \( S_n \) such a loop does exist. In other words, there exist sets \( x_1, \ldots, x_n \) such that:

- the set \( S_n \) belongs to \( x_1 \),
- the set \( x_1 \) belongs to \( x_2 \), and so on, and
- the set \( x_n \) belongs to \( S_n \):

\[ S_n \in x_1 \in x_2 \in \ldots \in x_n \in S_n . \]

Here:

- Since the set \( x_n \) belongs to \( S_n \), this means that for this \( x_n \), the property describing the set \( S_n \) is true, i.e., for the set \( x_n \), such a loop is not possible.
- However, we can build such a loop: indeed, \( x_n \) belongs to \( S_n \), \( S_n \) belongs to \( x_1 \), and so on, and at the end, \( x_{n-1} \) belongs to \( x_n \):

\[ x_n \in S_n \in x_1 \in x_2 \in \ldots \in x_{n-1} \in x_n . \]

So, in this case, we also get a contradiction.

**So, we have a paradox.** In both possible cases – when \( S_n \in S_n \) and when \( S_n \notin S_n \) – we get a contradiction. This means that none of these two cases is possible – so, we indeed have a paradox.

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