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Recommended Citation
Escamilla, Jeffrey and Kreinovich, Vladik, "How to Make a Decision under Interval Uncertainty If We Do Not Know the Utility Function" (2024). Departmental Technical Reports (CS). 1871.
https://scholarworks.utep.edu/cs_techrep/1871

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How to Make a Decision under Interval Uncertainty If We Do Not Know the Utility Function

Jeffrey Escamilla and Vladik Kreinovich

Abstract Decision theory describes how to make decisions, in particular, how to make decisions under interval uncertainty. However, this theory’s recommendations assume that we know the utility function – that describes the decision maker’s preferences. Sometimes, we can make a recommendation even when we do not know the utility function. In this paper, we provide a complete description of all such cases.

1 Formulation of the problem

How a rational person should make a decision: a brief reminder. According to decision theory (see, e.g., [1, 2, 4, 5, 6, 7, 8]), decisions of a rational person – i.e., e.g., a person who when preferring $A$ to $B$ and $B$ to $C$ always prefers $A$ to $C$ – are described by a function $u(a)$ called utility.

In this description, an alternative in which we gain amount $a$ is better than the alternative in which we gain amount $b$ if and only if $u(a) > u(b)$.

In business situations, when different outcomes can be described by monetary gains, the utility function is (non-strictly) increasing of the corresponding gain:

- if $a \leq b$
- then $u(a) \leq u(b)$.

Need to take uncertainty into account. In practice, we only know the consequence of each action with uncertainty.

Case of interval uncertainty: what is it and what should we do. In many cases, all we know is the bounds on possible gain, i.e., the interval $[a,\overline{a}]$ of possible values of the gain.

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In this case, according to decision theory (see, e.g., [3, 4, 5]), the decision maker should:

• select some value \( \alpha_H \in [0, 1] \) describing the decision maker’s degree of optimism-pessimism, and then
• select the alternative for which the value \( \alpha_H \cdot u(\pi) + (1 - \alpha_H) \cdot u(\varphi) \) is the largest.

Remaining problem: what if we do not know the utility function? Recommendations of decision theory are based on the assumption that we know the utility function of a decision maker. However, sometimes, we do not know the utility function.

Can we sometimes recommend a decision even if we do not know the utility function? And if yes, when? When can we still conclude that \([a, \pi]\) is better than \([b, \varphi]\)?

Sometimes, the solution is easy. The answer is easy:

• when \( \alpha_H = 1 – \) then we select the alternative with the larger \( \pi \), and
• when \( \alpha_H = 0 – \) then we select the alternative with the larger \( \varphi \).

But in general, the problem remains open. But what can we suggest in realistic situations, when \( 0 < \alpha_H < 1 \)?

What we do in this paper. In this paper, we provide a complete answer to this challenge.

Structure of this paper. In Section 2, we formulate and prove our main result. Two auxiliary results form Section 3.

2 Main result

It turns out that the only case when we can make a recommendation without knowing the utility function is when both bounds of the \( a \)-interval are larger than the corresponding bounds of the \( b \)-interval. Let us describe this result in precise terms.

Main result. Let \( \alpha_H \in (0, 1) \) be a real number. Then, for every two intervals \([a, \pi]\) and \([b, \varphi]\), the following two conditions are equivalent:

1. for all non-strictly increasing functions \( u(a) \), we have
   \[ \alpha_H \cdot u(\pi) + (1 - \alpha_H) \cdot u(\varphi) \geq \alpha_H \cdot u(\varphi) + (1 - \alpha_H) \cdot u(b); \]
2. \( a \geq b \) and \( \pi \geq \varphi \).

Proof. To prove the desired equivalence, it is sufficient to prove that:

• if the Condition 2. is satisfied, then the Condition 1. is also satisfied, and
• if the Condition 2. is not satisfied, then the Condition 1. is also not satisfied.
1°. Let us first prove that if the Condition 2. is satisfied, then the Condition 1. is also satisfied.

Indeed, let us assume that the Condition 2. is satisfied. Then, due to monotonicity of the utility function:

- since \( a \geq b \), we get \( u(a) \geq u(b) \); and
- since \( \bar{a} \geq \bar{b} \), we get \( u(\bar{a}) \geq u(\bar{b}) \).

If we multiply both sides of the first inequality by \( \alpha_H > 0 \) and both sides of the second inequality by \( 1 - \alpha_H > 0 \), then we conclude that

\[
\alpha_H \cdot u(a) \geq \alpha_H \cdot u(b) \quad \text{and} \quad (1 - \alpha_H) \cdot \bar{a} \geq (1 - \alpha_H) \cdot \bar{b}.
\]

If we add these two inequalities, we get the desired Condition 1.

2°. Let us prove that if the Condition 2. is not satisfied, i.e., if \( a < b \) or \( a < \bar{b} \), then the Condition 1. is violated for some increasing function \( u(a) \).

We will consider the two possibilities \( a < b \) or \( a < \bar{b} \) separately.

2.1°. Let us first consider the case when \( a < b \).

In this case, let us consider the following utility function \( u(a) \):

- we take \( u(a) = 0 \) for \( a \leq a \), and
- we take \( u(a) = 1 \) for all \( a > a \).

Let us compute the right- and left-hand sides of Condition 1. for this utility function.

- Since \( \bar{b} \geq b \) and \( b > a \), we have \( \bar{b} > a \). Thus, \( u(\bar{b}) = u(b) = 1 \), and the right-hand side of the Inequality 1. is equal to \( \alpha_H + (1 - \alpha_H) = 1 \).
- On the other hand, since \( a \leq a \), we have \( u(a) = 0 \), so the left-hand side is equal to \( \alpha_H \cdot u(\bar{a}) \). Here, \( u(\bar{a}) \leq 1 \), so \( \alpha_H \cdot u(\bar{a}) \leq \alpha_H < 1 \).

Thus, the Inequality 1. is not satisfied.

2.2°. Let us now consider the case when \( a < \bar{b} \).

In this case, let us consider the following utility function \( u(a) \):

- we take \( u(a) = 0 \) for \( a < \bar{b} \), and
- we take \( u(a) = 1 \) for \( a \geq \bar{b} \).

Let us compute the right- and left-hand sides of Condition 1. for this utility function.

- Here, \( \bar{b} \geq \bar{b} \), so \( u(\bar{b}) = 1 \). Thus, the right-hand side is greater than or equal to \( \alpha_H \cdot u(\bar{b}) = \alpha_H > 0 \).
- In this case, we have \( a < \bar{b} \), so \( u(a) = 0 \). Since \( a \leq a \) and \( a < \bar{b} \), we have \( a < \bar{b} \) and thus, \( u(a) = 0 \). Thus, the left-hand side of the Inequality 1. is equal to 0.

Thus, the Inequality 1. is not satisfied.

The proposition is proven.
3 Auxiliary results

Discussion. In the previous section, we considered the case when we know $\alpha_H$ and but we do not know $u(a)$. It is reasonable to consider two other cases:

- when we know $u(a)$ but we do not know $\alpha_H$; and
- when we do not know neither $\alpha_H$ nor $u(a)$.

In this case, we can prove two similar results.

Comment. In the first case, we have to additionally assume that the utility function $u(a)$ is strictly increasing, i.e., that $a < b$ implies that $u(a) < u(b)$. For such functions, not only $a \geq b$ implies that $u(a) \geq u(b)$, but also, vice versa, $u(a) \geq u(b)$ implies that $a \geq -$ because otherwise, if we had $a < b$, then we would have $u(a) < u(b)$.

First auxiliary result. Let $u(a)$ be a strictly increasing function. Then, for every two intervals $[a, \bar{a}]$ and $[b, \bar{b}]$, the following two conditions are equivalent:

1. for all $\alpha_H \in [0, 1]$, we have $$\alpha_H \cdot u(a) + (1 - \alpha_H) \cdot u(\bar{a}) \geq \alpha_H \cdot u(b) + (1 - \alpha_H) \cdot u(\bar{b});$$

2. $a \geq b$ and $\bar{a} \geq \bar{b}$.

Proof.

1°. Similarly to the proof of the main result, we can prove that the Condition 2. implies the Condition 1.

2°. Let us now assume that the Condition 1. is satisfied for all $\alpha_H \in [0, 1]$. In particular, this means that this condition is satisfied for $\alpha_H = 0$ and for $\alpha_H = 1$.

- For $\alpha_H = 0$, the Condition 1. means that $u(a) \geq u(b)$. Since the function $u(a)$ is strictly increasing, this implies that $a \geq b$.
- For $\alpha_H = 1$, the Condition 1. means that $u(\bar{a}) \geq u(\bar{b})$. Since the function $u(a)$ is strictly increasing, this implies that $\bar{a} \geq \bar{b}$.

Thus, the Condition 2. is indeed satisfied.

The proposition is proven.

Second auxiliary result. For every two intervals $[a, \bar{a}]$ and $[b, \bar{b}]$, the following two conditions are equivalent:

1. for all $\alpha_H \in (0, 1)$ and for all non-strictly increasing functions $u(a)$, we have $$\alpha_H \cdot u(\bar{a}) + (1 - \alpha_H) \cdot u(\bar{a}) \geq \alpha_H \cdot u(b) + (1 - \alpha_H) \cdot u(\bar{b});$$

2. $a \geq b$ and $\bar{a} \geq \bar{b}$. 
Proof.

1°. Similarly to the proof of the main result, we can prove that the Condition 2. implies the Condition 1.

2°. Let us now assume that the Condition 1. is satisfied for all $\alpha_H \in (0, 1)$. In particular, this means that this condition is satisfied for $\alpha_H = 0.5$. Then, by our Main result, we can conclude that the Condition 2. is satisfied.

The proposition is proven.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395 (Center for Collective Impact in Earthquake Science C-CIES), and by the AT&T Fellowship in Information Technology.

It was also supported by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

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