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# Fuzzy Mathematics under Non-Minimal "And"-Operations (t-Norms): Equivalence leads to Metric, Order Leads to Kinematic Metric, Topology Leads to Area or Volume

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## Abstract

Most formulas analyzed in fuzzy mathematics assume – explicitly or implicitly – that the corresponding "and"-operation (t-norm is the simplest minimum operation. In this paper, we analyze what happens if instead, we use other "and"-operations. It turns out that for such operations, a fuzzification of a mathematical theory naturally leads to a more complex mathematical setting: fuzzification of equivalence relation leads to metric, fuzzification of order leads to kinematic metric, and fuzzification of topology leads to area or volume.

**Keywords:** fuzzy mathematics, equivalence relation, metric, order, kinematic metric, topology, area, volume.

## 1 Formulation of the Problem

**Need for fuzzy techniques: a brief reminder.** In our everyday life, we often make decisions based on imprecise ("fuzzy") natural-language statements: whether rain is *light* or *heavy*, whether the car in front us is *close*, etc. To help us make good decisions, it is therefore desirable to take this imprecise knowledge into account. To be able to do this, we need to translate this knowledge into computer-understandable – i.e., numerical – form. Techniques for such translation are known as *fuzzy* techniques; see, e.g., [1, 5, 6, 8, 9, 12].

In these techniques, pioneered by Lotfi Zadeh, one of the main ideas is to use numbers to describe the correspondingly uncertainty. For example:

- a car 4 meters in front is definitely close,
- a car 1 km away is definitely not close,
- but for many intermediate values of the distance, we may be not sure whether it is close or not: we can say that it is somewhat close, reasonably close, etc.

In the computer:

- "true" is usually represented as 1, and
- "false" is usually represented as 0.

It is therefore reasonable to represent intermediate degrees of confidence by numbers intermediate between 0 and 1.

To get such a degree, we can ask the expert to mark his/her degree of confidence on a scale from 0 to 1. In the current culture, this is a common procedure:

- this is how we reply to what extent we are satisfied with a service,
- this is how students reply to what extent they are satisfied with our teaching skills, etc.

**Need for "and"-operations (t-norms): a brief reminder.** In decision making, we often use composite statements. For example, if we need to go out into the rain, how we dress and whether we take an umbrella depends not only on whether the rain is light or heavy but also whether the wind is strong: if the rain is heavy and the wind is strong, a usual umbrella will not work, a raincoat will work better.

We have already mentioned that to find the degree to which rain of different strength is heavy, we can ask an expert. Similarly, to find the degree to which wind of different speed is strong, we can ask experts. All this is feasible. But, in view of the use of composite statements, we also need to find, for each possible combination of rain intensity and wind speed, to what extend the phrase "the rain is heavy and the wind is strong" holds true. Even if we could potentially ask the expert about all possible combinations of these two quantities, what if we have 3? 4? 5? 7 such quantities – and in serious decisions, we do take into account values of many quantities.

In such situations, since we cannot elicit the corresponding degrees directly from the expert, we need to estimate these degrees based on the available information that we have already elicited. In other words, we need to be able:

- given the expert's degree of confidence $a$ and $b$ in statements $A$ and $B$,
- to provide an estimate for the expert's degree of certainty in the composite statement $A\,\&\,B$.

Functions $f_\&(a,b)$ providing such an estimate are known as *"and"-operations*, or, for historical reasons, *t-norms*.

The simplest t-norm – introduced in the first Zadeh's paper on fuzzy logic – is the *min t-norm* $f_\&(a,b) = \min(a,b)$ [12]. However, there are many other "and"-operations, starting with the product operation $f_\&(a,b) = a \cdot b$ introduced by Zadeh in that same paper.

*Comment.* Similarly, there exists fuzzy "or"-operations $f_\vee(a,b)$ (also known as t-conorms) that transform the expert's degees of certainty $a$ and $b$ in statements $A$ and $B$ into an estimate for the expert's degree of certainty in a composite statement $A \vee B$.

**Fuzzy mathematics: a brief reminder.** Usually, fuzzy techniques are used to describe our uncertainty about the numerical values. However, people also use fuzzy words to describe more complex mathematical objects.

For example, an engineer can say that the function $y = f(x)$ describing the dependence between the two quantities is "rather smooth" or "very smooth", or that the solution to the corresponding system of equation is "usually unique", etc.

To formalize such use of "fuzzy" natural-language words, researchers have extended fuzzy techniques to more general *fuzzy mathematics*, where fuzzy degrees can be applied to general mathematical objects.

**Fuzzy mathematics mostly uses the min t-norm.** Mathematics is not easy, and extending mathematics to fuzzy objects does not make it easier. Because of this complexity, most results of fuzzy mathematics limit themselves to the simplest t-norm, i.e., the min t-norm.

**A natural question.** A natural question is: what will happen if we use more general "and"-operations?

**What we do in this paper.** In this paper, we show, on several examples, that the use of non-minimal "and"-operations naturally leads to new mathematical concepts:

- equivalence leads to metric,
- order leads to kinematic metric, and
- topology leads to volume.

## 2 First Example: Equivalence Leads to Metric

**What is equivalence: reminder.** In mathematics, an relation $\sim$ is called an *equivalent relation* if it satisfies the following three conditions:

- it is *reflexive*, i.e., $x \sim x$ for all $x$;
- it is *symmetric*, i.e., $x \sim y$ is equivalent to $y \sim x$ for all $x$ and $y$, and
- it is *transitive*, i.e., if $x \sim y$ and $y \sim z$, then $x \sim z$.

**Towards a fuzzy analogue of the equivalence relation.** In 2-valued logic, for every two objects $x$ and $y$, either $x$ is equivalent to $y$ or not. A natural fuzzy analogue is when for every two objects $x$ and $y$, we have a *degree* $d(x,y) \in [0,1]$ to which these two objects are equivalent. Let us reformulate the above three properties in these terms.

- Reflexivity means that each object is absolutely equivalent to itself, i.e., that $d(x,x) = 1$ for all $x$.
- Symmetry means that the degree to which $x$ is equivalent to $y$ should be the same as the degree to which $y$ is equivalent to $x$, i.e., that $d(x,y) = d(y,x)$ for all $x$ and $y$.
- Finally, transitivity means that our degree of belief that $x$ is equivalent to $z$ should be at least a large as our belief in the statement

  "$x$ is equivalent to $y$ and $y$ is equivalent to $z$",

  i.e., that we should have

  $$d(x,z) \geq f_\&(d(x,y), d(y,z)) \qquad (1)$$

  for all $x$, $y$, and $z$.

**Fuzzy version naturally leads to metric.** It is known (see, e.g., [7]) that for every $\varepsilon > 0$, every "and"-operation can be approximated, with accuracy $\varepsilon$, by a *strictly Archimedean "and"-operation*, i.e., by an "and"-operation of the type

$$f_{\&}(a,b) = \psi^{-1}(\psi(a)\cdot\psi(b)) \quad (2)$$

for some strictly increasing 1-1 function $\psi(a)$ from the interval $[0,1]$ to itself.

In practice, we always get the degrees with some accuracy. Thus, from the practical viewpoint, it is safe to assume that the actual "and"-operation is strictly Archimedean, i.e., has the form (2). For such an "and"-operation, transitivity (1) means that

$$d(x,z) \ge \psi^{-1}(\psi(d(x,y))\cdot\psi(d(y,z))).$$

Applying the increasing 1-1 function $\psi(a)$ to both sides of this inequality, we will get an equivalent inequality

$$\psi(d(x,z)) \ge \psi(d(x,y))\cdot\psi(d(y,z)). \quad (3)$$

Similarly, by applying the function $\psi(a)$ to equalities describing reflexivity and symmetry, we conclude that:

- $\psi(d(x,x)) = 1$ for all $x$ and
- $\psi(d(x,y)) = \psi(d(x,y))$ for all $x$ and $y$.

Let us see how we can simplify the above properties. The inequality (3) involves multiplication. From the computational viewpoint, multiplication is more complex than addition, but if can be reduced to addition if we take logarithms, since $\ln(a\cdot b) = \ln(a)+\ln(b)$: this is exactly what logarithms were invented for, to simplify computing the products – and this is what the slide rule, the main engineering computational device for many centuries, was based on.

It is therefore reasonable to apply logarithms to both sides of the inequality (3). This way, we get an equivalent inequality

$$\ln(\psi(d(x,z))) \ge \ln(\psi(d(x,y)))+\ln(\psi(d(y,z))). \quad (4)$$

This is better, but not perfect, since when we apply logarithm to the degree $\psi(d(x,y))$ from the interval $[0,1]$, we get negative numbers (or 0). However, it is easier for us to deal with positive numbers, since we are more accustomed to them. So, to make the resulting expression more convenient, let us change the signs of both sides, and consider the following non-negative expression:

$$\rho(x,y) \stackrel{\text{def}}{=} -\ln(\psi(d(x,y))). \quad (5)$$

Since we changed the signs of both sides,we need to replace $\ge$ with $\le$. Thus, for the expression (5), the inequality (4) takes the form

$$\rho(x,z) \le \rho(x,y)+\rho(y,z).$$

This is a well-known *triangle inequality* – one of the properties describing a general metric. Similarly, by applying $-\ln(a)$ to both sides of equalities describing reflexivity and symmetry, we get $\rho(x,x) = 0$ and $\rho(x,y) = \rho(y,x)$ for all $x$ and $y$ – the two remaining properties of the metric.

So, fuzzification of equivalence relation indeed leads to a metric.

*Comment.* Usually, the metric also needs to satisfy an additional property: that if $\rho(x,y) = 0$ then $x = y$. This property can also be derived if we additionally require that if $d(x,y) = 1$ then $x$ must be equal to $y$.

## 3 Second Example: Order Leads to Kinematic Metric

**Strict order relation: a brief reminder.** In this example, we will consider a *strict order* relation, i.e., a relation $<$ which is:

- *anti-reflexive*: $x \not< x$ for all $x$;
- *anti-symmetric*: if $x < y$, then $y \not< x$; and
- *transitive*: if $x < y$ and $y < z$, then $x < z$.

An example of such a relation is "$x$ is smaller than $y$".

*Comment.* Strictly speaking, we do not need to postulate the second property, since it follows from the other two. Indeed, if we had $x < y$ and $y < x$, then by transitivity, we would have $x < x$, which contradicts anti-reflexivity.

However, we explicitly added this property to keep this section as similar to the previous one as possible.

**Where fuzziness enters.** For some objects, $x$ is definitely not smaller than $y$. However when $x$ *is* smaller than $y$, then we can distinguish between different degrees of this smallness:

- we can have cases when $x$ slightly smaller than $y$,
- we can have cases when $x$ much smaller than $y$, etc.

The relation "much smaller" has the following natural property:

if $x < y < z$, then whenever $x$ is much smaller than $y$ or $y$ is much smaller than $z$, then $x$ is much smaller than $z$.

If we use the usual notation $x \ll y$ for "$x$ is much smaller than $y$", then this property takes the following form:

if $x < y < z$ and either $x \ll y$ or $y \ll z$, then $x \ll z$.

To describe the relation $\ll$ in precise terms, to each pair of objects $x$ and $y$ for which $x < y$, we assign a degree $d(x,y)$ to which $x$ is much smaller than $y$. When $x$ is not smaller than $y$, then, of course, $d(x,y) = 0$. In terms of this degree, the above "or"-property, in fuzzy terms, takes the following form:

$$\text{if } x < y < z \text{ then}$$

$$d(x,z) \geq f_\vee(d(x,y), d(y,z)). \quad (6)$$

**This can be transformed into a kinematic metric.** Let us simplify the above inequality. For this purpose, we need to use the known fact that there is a 1-1 correspondence between "and"- and "or"-operations:

- if $f_\&(a,b)$ is an "and"-operation, then
$$1 - f_\&(1-a, 1-b)$$
is an "or"-operation, and
- vice versa, if $f_\vee(a,b)$ is an "and"-operation, then
$$1 - f_\vee(1-a, 1-b)$$
is an "and"-operation.

Because of this correspondence, each "or"-operation has the form

$$f_\vee(a,b) = 1 - f_\&(1-a, 1-b) \quad (7)$$

for some "and"-operation $f_\&(a,b)$. Substituting this expression into the inequality (6), we conclude that

$$d(x,z) \geq 1 - f_\&(1 - d(x,y), 1 - d(y,z)). \quad (8)$$

Subtracting both sides of this inequality from 1, we conclude that

$$1 - d(x,z) \leq f_\&(1 - d(x,y), 1 - d(y,z)). \quad (9)$$

We have already mentioned that, from the practical viewpoint, we can safely assume that the "and"-operation has the form (2), Substituting the expression (2) for the "and"-operation into the inequality (9), we conclude that

$$1 - d(x,z) \leq$$

$$\psi^{-1}(\psi(1 - d(x,y)) \cdot \psi(1 - d(y,z))). \quad (10)$$

If we apply the function $\psi(a)$ to both sides of this inequality, we conclude that

$$\psi(1 - d(x,z)) \leq \psi(1 - d(x,y)) \cdot \psi(1 - d(y,z)). \quad (11)$$

Similarly to the previous section, we can simplify this inequality if we apply $-\ln(a)$ to both sides. Then, for the values

$$\tau(x,y) \stackrel{\text{def}}{=} -\ln(\psi(1 - d(x,y)),$$

we get

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z). \quad (12)$$

Here, if $x \not< y$, then $d(x,y) = 0$ and thus,

$$\tau(x,y) = -\ln(\psi(1 - d(x,y)) = 0.$$

The inequality (12) – which differs from the triangle inequality by its sign – is known as the *anti-triangle inequality*. It is true in physics, where $\tau(x,y)$ indicates the longest time needed to go from space-time event $x$ to space-time event $y$; see, e.g., [3, 11], This longest time corresponds to rest or inertial motion. From this viewpoint, the inequality (12) described the so-called *twin paradox* of relativity theory: that a twin that stays on Earth – and thus, stays on the rest trajectory from $x$ to $z$ – grows older than the twin that first traveled to a faraway star ($y$) and then came back.

In general, functions $\tau(x,y)$ that satisfy this inequality as known as *kinematic metrics*; they form the basis of the study of space-time; see, e.g., [2, 4, 10]. So, fuzzification of order indeed leads to a kinematic metric.

## 4 Third Example: Topology Leads to Area or Volume

**How topology is usually defined.** In mathematics, topology is usually defined as a class of open sets, i.e., sets that contain, with each point, some neighborhood of this point. One the most frequent ways to define topology is to describe its *basis*, i.e., describe a class of open sets – e.g., on a plane, small circles. Then, we say that a set is open if it is a union of some sets from the basis - or of their finite intersections.

**A simplified description.** To analyze the possible use of fuzzy, let us consider the simplified case when we have a grid formed by points with integer coordinates $c = (c_1, \ldots, c_n)$, and the basis consists of semi-open boxes attached to each such point $c$. For each grid point $c$, the corresponding box $B(c)$ consists of all the points $x = (x_1, \ldots, x_n)$ for which $c_i \leq x_i < c_i + 1$ for all $i$.

In this scheme, the only open sets are finite unions of such boxes.

**What fuzzy adds to this description.** One of the main ideas behind fuzzy techniques is that everything is a matter of degree. In particular, this means that in the general fuzzy case, the boxes are not *absolutely* open, but rather open *with some degree of confidence d*. In this case, the union $S = B_1 \cup \ldots \cup B_k$ of $k$ boxes $B_i$ is open if and only if each of these $k$ boxes is open, i.e., if the box $B_1$ is open, *and* the box $B_2$ is open, etc. For each of these $k$ statements, the degree of confidence is $d$. Thus, our degree of confidence $d(S)$ that all $k$ boxes are open – and thus, that their union $S$ is open – can be obtained by applying the "and'-operation to these $k$ values:

$$d(S) = f_\&(d, \ldots, d) \ (k \text{ times}).$$

**This leads to area or volume.** If we now substitute the expression (2) for the generic "and"-operation, we thus conclude that

$$d(S) = \psi^{-1}(\psi(d) \cdot \ldots \cdot \psi(d)) \ (k \text{ times}),$$

i.e.,

$$d(S) = \psi^{-1}((\psi(d))^k).$$

If we apply the function $\psi$ to both sides of this equality, we get

$$\psi(d(S)) = (\psi(d))^k.$$

If we now apply the function $-\ln(a)$ to both sides, then for the resulting expression $V(S) \stackrel{\text{def}}{=} -\ln(\psi(d(S)))$, we get

$$V(S) = k \cdot V_0,$$

where we denoted $V_0 \stackrel{\text{def}}{=} -\ln(\psi(d))$.

So, by fuzzifying topology, we indeed get a value which is proportional to the number of boxes forming the set $S$ – i.e., depending on the dimension, to the area or to the volume of this set.

*Comments.*

- If we decrease the size of the boxes, then, in the limit when this size tends to 0, we get the actual area or volume.
- In our description, we simplified this example so that computations will be clear. In doing this, we uses boxes which are *not* actually open in the usual Euclidean topology. However, a similar result holds if we use intersecting open small boxes instead.

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