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Which Fuzzy Implications Operations Are Polynomial? A Theorem Proves That This Can Be Determined by a Finite Set of Inequalities

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Abstract

To adequately represent human reasoning in a computer-based systems, it is desirable to select fuzzy operations that are as close to human reasoning as possible. In general, every real-valued function can be approximated, with any desired accuracy, by polynomials; it is therefore reasonable to use polynomial fuzzy operations as the appropriate approximations. We thus need to select, among all polynomial operations that satisfy corresponding properties – like associativity – the ones that best fit the empirical data. The challenge here is that properties like associativity mean satisfying infinitely many constraints (corresponding to infinitely many possible triples of values), while most effective optimization techniques assume that the number of equality or inequality constraints is finite. Thus, it is desirable to find, for each corresponding family of infinitely many constraints, an equivalent finite set of constraints. Such sets have been found for many fuzzy operations – e.g., for implication operations represented by polynomials of degree 4. In this paper, we show that such equivalent finite sets always exist, and we describe an algorithm for generating these sets.

Keywords: fuzzy logic, fuzzy operations, polynomial fuzzy operations, Tarski-Seidenberg theorem

1 Formulation of the Problem

Need for fuzzy techniques: a brief reminder. In the early 1960s, Lotfi Zadeh, then one of the world's leading specialists in automatic control, noticed that in many situations, expert human controllers achieved better results that the supposedly optimal automatic controllers. The general reason for this situation was clear:

- automatic controllers were based on the available models of the controlled plants,
- so the fact that the resulting control did not work perfectly meant that these models were not fully adequate, that expert human controllers had some additional knowledge about the situation, knowledge that was not incorporated into the available models.

The problem was *not* that the experts were hiding some information: many experts were willing to share this additional knowledge and to explain what could be improved in the control generated by the automatic controllers. The difficulty was that their knowledge was described:

- not in precise mathematical easy-to-automate terms,
- but rather by using imprecise ("fuzzy") words from natural language like "small".

To incorporate this knowledge, Zadeh started designing techniques for translating this imprecise natural-language knowledge into computer-understandable numerical form. He named techniques providing such fuzzy-to-precise transformation *fuzzy techniques*; see, e.g., [3, 8, 10, 11, 12, 17].

One of his main ideas was that:

- in contrast to precise properties like "smaller than 0.1", properties that are always either true or false,
- fuzzy properties like "small" are usually only true to some degree.

So, Zadeh suggested to describe each such property by assigning, to each possible value x of the corresponding quantity, the degree – from the interval $[0,1]$ – to which this value x satisfies the given property. Many experts were able to provide such degrees by marking them on the scale – just like we can gauge the quality of a customer experience on a 0 to 10 scale and how students gauge the quality of our teaching. The resulting function $\mu(x)$ mapping real values x into the degree $\mu(x)$ was called a *membership function* or, alternatively, a *fuzzy set*.

Since this idea involves, in effect, expanding the set of "truth values" – one of the basic concepts of logic – fuzzy techniques are also called *fuzzy logic techniques*.

Need for fuzzy operations. One of the main objectives of the original fuzzy techniques was to transforms natural-language expert rules like the one below into a precise control strategy:

if temperature t is high and humidity h is low, then set the strength s of the air conditioner on medium.

To translate such statements, first we need to ask the experts to provide membership functions corresponding to temperature, to humidity, and to strength. However, this is not enough: to come up with an appropriate control, we need to come up the degrees of confidence in complex statements – like the statement above.

In the ideal world, we should determined the degree of certainty of each such statement for all possible combinations of inputs. However, this is not realistic: even if we have 10 different values of each of the three inputs, we would need to ask the expert 10^3 questions about each rule. This may be feasible, but what if we have 5 inputs? 7 inputs? This is typical for complex control systems, and this would require asking 10^7 = 10 million questions to the expert – this is clearly not feasible.

Since we cannot get the estimates of complex statements directly from the experts, we need to generate these estimates based on whatever information we have – i.e., based on the expert's degrees of confidence that t is high, that h is now, and that s is medium.

In other words, for logical connectives $\odot$ such as "and", "or", "implies", etc., we need to be able, given our degree of confidence x and y in the component statements X and Y, to provide an estimate for our degree of confidence in the complex statement $X \odot Y$. The function $f_\odot(x,y)$ that provides such estimates is called a *fuzzy operation*; for example:

- functions $f_\&(x,y)$ corresponding to "and" are known as *"and"-operations* (or, for historical reasons, *t-norms*);
- functions $f_\vee(x,y)$ corresponding to "or" are known as *"or"-operations* (or, for historical reasons, *t-conorms*);
- functions $f_\rightarrow(x,y)$ corresponding to implication are known as *implication operations*, etc.

Such operations need to satisfy some reasonable properties. For example:

- since $X \,\&\, Y$ means the same as $Y \,\&\, X$, these two expressions should lead to the same estimate, i.e., we should have $f_\&(x,y) = f_\&(y,x)$; in mathematical terms, this means that the "and"-operation should be commutative;
- since $X \,\&\, (Y \,\&\, Z)$ means the same as $(X \,\&\, Y) \,\&\, Z$, these two expressions should also lead to the same estimate, i.e., we should have $f_\&(x, f_\&(y,z)) = f_\&(f_\&(x,y), z)$; in mathematical terms, this means that the "and"-operation should be associative, etc.

Need to approximate fuzzy operations. Fuzzy operations should reflect the expert reasoning. The more accurately these operation reflect expert reasoning, the better we capture the expert knowledge. We should therefore be prepared to capture the actual human reasoning as accurately as possible – whatever the reasoning will be.

Need for polynomial approximations. Techniques for capturing all kinds of empirical dependence is well known in science. For example, in physics (see, e.g., [7, 16]) the usual way to capture such a dependence is based on the fact that many dependencies are analytical, so they can be expanded into an infinite Taylor series. For example:

- for functions of one variable, we have
$$f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots;$$
- for functions of two variables, we have
$$f(x,y) = a_{00} + a_{10} \cdot x +$$
$$a_{01} \cdot y + a_{20} \cdot x^2 + a_{11} \cdot x \cdot y + a_{02} \cdot y^2 + \ldots$$

To get a good approximation, we keep only the few first terms in this expansion:

- in the first approximation, we only keep linear terms, i.e., approximate the desired function by a linear polynomial
$$f_1(x,y) = a_{00} + a_{01} \cdot x + a_{10} \cdot y;$$

- to get a more accurate approximation, we also keep quadratic terms, i.e., approximate the desired function by a quadratic polynomial

$$f_2(x,y) = a_{00} + a_{10} \cdot x + a_{01} \cdot y +$$
$$a_{20} \cdot x^2 + a_{11} \cdot x \cdot y + a_{02} \cdot y^2;$$

- to get an even more accurate approximation, we can add cubic terms, i.e., approximate the desired function by a quadratic polynomial, etc.

Which of these polynomial functions are fuzzy operations: a challenge. We want an operation that satisfies the corresponding properties – e.g., we want an "and"-operation to be commutative and associative. So, when we try to match the empirical data about these operations – coming from experts:

- we should not consider all possible polynomial functions,
- we should only consider functions that satisfy the corresponding properties.

So, among all the tuples coefficients a_i of the corresponding polynomials for which the resulting polynomial is a fuzzy operation, we need to select the tuple that best fits the available data. The problem is that the condition "is a fuzzy operation" is very complicated, it means that some equality should be satisfied for all possible values x and y (or even x, y, and z). Since there are infinitely many posisble values x and y, we thus have infinitely many constraints that need to be satisfied.

While there exist numerical (and even analytical) methods for optimizing a function under a finite number of constraints, there are, in general, no available methods for optimizing a function under infinitely many constraints. An ideal solution to this problem would be to translate the complex constraint – describing that a polynomial function is the corresponding fuzzy operation – into a finite sequence of inequalities.

What is known: a brief (incomplete) overview. Such a translation into a finite set of inequalities is known for several operations and for several degrees of the corresponding polynomial. For example, for polynomial degrees up to order 4, such a translation is known for describing whether a polynomial function is a fuzzy implication, and whether it is an implication that satisfies certain general properties; see, e.g., [1, 9].

In [1, 9], implication operation is defined as a function $f(x,y)$ for which $f(0,0) = f(1,1) = 1$, $f(1,0) = 0$, and for which the following properties are satisfied:

$$\forall x \forall y \forall z\, (x \le y \Rightarrow f(x,z) \ge f(y,z)) \text{ and}$$

$$\forall x \forall y \forall z\, (y \le z \Rightarrow f(x,y) \ge f(x,z)).$$

Finite sets of inequalities are also described for each of the following additional properties:

$$\forall y\, (f(1,y) = y);$$
$$\forall x \forall y \forall z\, (f(x, f(y,z)) = f(y, f(x,z));$$
$$\forall x \forall y\, (x \le y \Leftrightarrow f(x,y) = 1);$$
$$\forall x \forall y\, (f(x, f(x,y)) = f(x,y)); \text{ and}$$
$$\forall x\, (f(x,x) = 1).$$

These papers also consider properties relating implication operation with other polynomial fuzzy operations – e.g., with the "and"-operation $t(x,y)$, with negation operations, etc., such as

$$\forall x \forall y \forall z\, (f(t(x,y),z) = f(x, f(y,z)).$$

Remaining challenge. While there has been a lot of progress in this translation into a finite set of inequalities, it has not yet even been clear which properties of fuzzy operations can be translated into an equivalent finite set of inequalities.

What we do in this paper. In this paper:

- we prove that a translation into an equivalent finite set of inequalities is always possible, and
- we provide an algorithm for this translation.

Word of caution. While the algorithm is available, it is not always practically useful: in some cases, it requires double exponential time.

2 Main Result

Definition 1. *Let:*

- $c_1, \ldots, c_m$ *be symbols for real-valued constants,*
- $n_1, \ldots, n_k$ *be positive integers,*
- $f_1, \ldots, f_k$ *be symbols of functions, so that each* f_i *is a function of* n_i *real variables, and*
- $x, x_1, x_2, \ldots, y, \ldots, z, \ldots$ *be real-valued variables.*

A term *is defined as follows:*

- *every symbol* c_i *and evert variable* x_i *is a term;*
- *if* $t_1, \ldots, t_{n_i}$ *are terms, then* $f_i(t_1, \ldots, t_{n_i})$ *is a term.*

A formula *is defined as follows:*

- *if t and t' are terms, then $t = t'$, $t < t'$, $t > t'$, $t \le t'$, $t \ge t'$, and $t \ne t'$ are formulas;*
- *if F and G are formulas, then $F \,\&\, G$, $F \vee G$, $F \Rightarrow G$, $F \Leftrightarrow G$, and $\neg F$ are formulas;*
- *if F is a formula as x is a variable, then $\forall x F$ and $\exists x F$ is a formula.*

A formula is closed *is every variable is covered by some quantifier.*

Comment. In view of this definition, expressions like $f(x, f(y,z))$ and $f(t(x,y),z)$ are terms, and all the above-descrbed properties from [1, 9] are closed formulas.

Definition 2.

- *By a* polynomial equality*, we mean an expression of the type $P = Q$, where P and Q are polynomials.*
- *By a* polynomial inequality*, we mean an expression of one of the type $P < Q$, $P > Q$, $P \le Q$, $P \ge Q$, or $P \ne Q$, where P and Q are polynomials.*
- *By a* system of polynomial equalities and inequalities*, we mean a finite set of polynomial equalities and inequalities. We say that a tuple* satisfies *this system if it satisfies all equalities and inequalities from this system.*
- *By a* set of systems of polynomial equalities and inequalities*, we mean a finite set of systems of polynomial equalities and inequalities. We say that a tuple* satisfies *this set if it satisfies one of the systems from this set.*

Example. One of the conditions from [1] has the form

$$\text{If } \alpha < 0 \text{ and } -2\alpha \le \beta - \delta \le -4\alpha, \text{ then} \\ (\beta - \delta)^2 + 2\alpha \cdot (\delta + \varepsilon + 1) \le 0.$$

In general, implication "if A then B" means that either B is true or A is false. So:

- either we have $(\beta - \delta)^2 + 2\alpha \cdot (\delta + \varepsilon + 1) \le 0$
- or the condition

  "$\alpha < 0$ and $-2\alpha \le \beta - \delta$ and $\beta - \delta \le -4\alpha$"

  is false.

In general, the fact that the conjunction "A and B and C" is false means that either A is false, or B is false, or C is false, i.e., in our case, that either $\alpha \ge 0$ or $-2\alpha > \beta - \delta$ or $\beta - \delta > -4\alpha$. Thus, the above condition means that the tuple $(\alpha, \beta, \ldots)$ satisfies the following set of four systems, each of which consists of a single inequality:

- $(\beta - \delta)^2 + 2\alpha \cdot (\delta + \varepsilon + 1) \le 0$;
- $\alpha \ge 0$;
- $-2\alpha > \beta - \delta$;
- $\beta - \delta > -4\alpha$.

Similarly, all other conditions from [1, 9] can be described in these terms.

Definition 3. *Let $x_1, x_2, \ldots$ be variables.*

- *A* monomial *is an expression of the type*

  $$x_{i_1}^{v_1} \cdot \ldots \cdot x_{i_m}^{v_m},$$

  where $i_1 < \ldots < i_m$ and v_i are positive integers. The sum $v_1 + \ldots + v_m$ is called the degree *of the monomial.*
- *By a* polynomial of degree *d, we means an expression of the type $a_1 \cdot M_1 + \ldots + a_N \cdot M_N$, where a_i are real numbers called* coefficients *and $M_1, \ldots, M_n$ are all possible monomials of degree d or smaller.*

Examples. For two variables:

- a polynomial of degree 1 is an expression

  $$a_1 + a_2 \cdot x_1 + a_3 \cdot x_2;$$
- a polynomial of degree 2 is an expression

  $$a_1 + a_2 \cdot x_1 + a_3 \cdot x_2 + a_4 \cdot x_1^2 + a_5 \cdot x_1 \cdot x_2 + a_6 \cdot x_2^2.$$

Proposition. *Let $d_1, \ldots, d_k$ be positive integers. Then:*

- *for each closed formula F, this formula – when limited to the case when each function f_i is a polynomial of degree d_i – is equivalent to a set of systems of polynomial equalities and inequalities in terms of constants c_i and coefficients of the polynomials f_i;*
- *there exists an algorithm that, given a closed formula F – limited to the case when each function f_i is a polynomial of degree d_i – returns the equivalent set of systems of polynomial equalities and inequalities.*

Comment. As one see from the proof below, a similar result holds if instead of polynomials of a given degree d_i, we consider:

- rational functions of this degree, i.e., ratios of polynomials of degree d_i,

- or, more generally, algebraic functions of degree d_i, i.e., functions $f_i(x_1,\dots,x_{n_i})$ whose values are determined by a polynomial equation $P(x_1,\dots,x_{n_i},f_i(x_1,\dots,x_{n_i}))=0$ of degree d_i.

Algebraic functions include expressions with square roots, cubic roots, etc.

Proof. Our proof is based on the Tarski-Seidenberg theorem (see, e.g., [2, 4, 5, 6, 13, 14, 15]) about the so-called first order theory of real numbers; we will denote this theory by T, after Tarski. In this theory T, we start with real-valued constants x_i and real-valued variables x_j. Then:

- T-terms are polynomials in terms of c_i and x_j;
- elementary T-formulas are equalities and inequalities between terms; and
- a general T-formula is a formula obtained from elementary T-formulas by using logical connectives &, $\vee$, etc., and quantifiers $\forall x$ and $\exists x$ over real numbers.

The Tarski-Seidenberg theorem states that:

- each closed T-formula F is equivalent to a set of systems of polynomial equalities and inequalities in terms of constants c_i; and
- there exists an algorithm that, given a closed T-formula F, returns the equivalent set of systems of polynomial equalities and inequalities.

To apply this theorem to our case, we need to represent our formulas in terms of the theory T. By comparing our definition of a formula with the definition of the T-formula, one can see that the only difference is between terms as defined above and T-terms. So, the only thing we need for such a translation is to transform each term into the T-term form, i.e., into a polynomial. This can be done by induction over the inductive definition of a term. Indeed:

- constants and variables are already polynomials;
- if the terms $t_1,\dots,t_{n_i}$ are polynomials, and the function f_i is a polynomial, then the expression $f_i(t_1,\dots,t_{n_i})$ is also a polynomial.

For example, if $f(x,y)=a_0+a_1\cdot x+a_1\cdot y$, then

$$f(x,f(x,y))=a_0+a_1\cdot x+a_2\cdot(a_0+a_1\cdot x+a_2\cdot y))=$$

$$a_0+a_1\cdot x+a_2\cdot x+a_2\cdot a_0+a_2\cdot a_1\cdot x+a_2\cdot a_2\cdot y.$$

The proposition is thus proven.

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