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Interval-Valued and Set-Valued Extensions of Discrete Fuzzy Logics, Belnap Logic, and Color Optical Computing*

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Abstract. It has been recently shown that in some applications, e.g., in ship navigation near a harbor, it is convenient to use combinations of basic colors – red, green, and blue – to represent different fuzzy degrees. In this paper, we provide a natural explanation for the efficiency of this empirical fact: namely, we show: (1) that it is reasonable to consider discrete fuzzy logics, (2) that it is reasonable to consider their interval-valued and set-valued extensions, and (3) that a set-valued extension of the 3-valued logic is naturally equivalent to the use of color combinations.

Keywords: Fuzzy logic · Set-valued extension · Interval-valued extension · Color optical computing

1 Formulation of the Problem

Color optical computing representation of fuzzy degrees. It has been recently shown that in some practical applications of fuzzy logic – e.g., in ship navigation near a harbor – it is convenient to represent different fuzzy degrees by colors, namely, by combinations of the three pure basic colors: red, green, and blue; see, e.g., [10–13]. To be more precise, these papers use $2^3 = 8$ combinations of pure colors, where each of the three basic colors is either present or not present:

– *black* corresponding to no colors at all,

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- *white* corresponding to the presence of all three basic colors,
- three pure colors corresponding to the case when only one of the three basic colors is present, and
- three combinations of two basic colors.

Question. This empirical success prompts a natural question: why is this representation efficient?

What we do in this paper. In this paper, we explain the empirical success of color optical computing representation by showing how the main ideas behind fuzzy logic naturally lead to this representation. Namely, we show:

- that it is reasonable to consider discrete fuzzy logics,
- that it is reasonable to consider interval-valued and set-valued extensions of these logics, and
- that a set-valued extension of the 3-valued logic is naturally equivalent to the use of combinations of pure colors.

We also show that the set-valued extensions of discrete fuzzy logics are related to the formalism of Belnap's logic, that allows parts of the knowledge base to be inconsistent.

2 Why Interval-Valued and Set-Valued Extensions of Discrete Fuzzy Logics

Fuzzy degrees: a brief reminder. One of the main ideas behind fuzzy logic is to assign, to each imprecise natural-language statement such as "John is tall", a degree describing to what extent this statement is true – e.g., to what extent John is tall; see, e.g., [3–5, 8, 9, 14].

Need for discrete fuzzy logic. In the original fuzzy logic, these degrees were represented by numbers from the interval $[0, 1]$. From the mathematical viewpoint, this interval contains infinitely many numbers. When the numbers are significantly different, they represent different degrees of certainty. However, when the two numbers are very close, we cannot distinguish the corresponding degrees: e.g., hardly anyone can distinguish between degrees 0.80 and 0.81.

In general, according to psychological experiments, we can meaningfully distinguish at most 7 ± 2 different degrees: some of us can only distinguish $7 - 2 = 5$ different degrees, some can distinguish $7 + 2 = 9$ different degrees; see, e.g., [6, 7]. In other words, in practice, we use, in effect, a discrete set of fuzzy degrees.

Fuzzy degrees come with uncertainty. In the ideal case, we have a single perfect expert who selects a single degree – and experts are perfect in the sense that other experts would assign the exact same degree. In practice, the situation is more complicated.

- First, an expert can be unsure what exact degree to assign. At best, the expert can provide a lower bound a and an upper bound b for this degree – just like when estimating the height of a person entering the room, the expert will not produce an exact value but rather a range of values. In this case, possible degrees form an *interval* $[a, b] \stackrel{\text{def}}{=} \{x : a \le x \le b\}$.
- Second, even if an expert produces an exact degree, other experts may produce different degrees. In this case, to describe uncertainty, it is reasonable to list all these degrees, i.e., to produce the *set* of experts' estimates. This extension of fuzzy logic is known as *hesitant* fuzzy logic.

In the following text, we will analyze such interval-valued and set-valued versions of the simplest discrete fuzzy logics, and we will show that this analysis indeed naturally leads to color optical computing.

Comment. Following this line of reasoning, it is also possible to have several experts producing intervals. This option may be worth exploring.

3 Interval-Valued and Set-Valued Extensions of 2-Valued Logic

Why 2-valued logic. In general, a discrete fuzzy logic is a finite subset of the interval $[0, 1]$ that contains both 0 ("false") and 1 ("true"). From this viewpoint, the simplest case is when this subset contains only 0 and 1, i.e., when we have a usual 2-valued logic.

Interval-values extension of 2-valued logic. In a logic consisting of two elements $0 < 1$, there are exactly three possible intervals:

- two degenerate intervals $[0, 0] = \{0\}$ and $[1, 1] = \{1\}$ consisting of a single original value, and
- a non-degenerate interval $[0, 1] = \{0, 1\}$ containing both values.

The general interpretation of interval-valued extensions – that was described in the previous section – provides the following explanation for the new truth value $[0, 1]$: this truth value corresponds to the case when we do not know whether the statement is true or false – i.e., corresponds to uncertainty. Thus, we get a usual 3-valued logic with three possible truth values: true, false, and uncertain. These values can be naturally described as 1, 0, and an intermediate value 0.5.

Set-valued extension of 2-valued logic. In a 2-valued logic with the set of truth values $\{0, 1\}$, there are four subsets:

- two 1-elements subsets $\{0\}$ and $\{1\}$;
- the original set $\{0, 1\}$, and
- the empty set $\emptyset$.

The general interpretation of set-valued extensions – that was described in the previous section – provides the following interpretation of these four subsets:

- the set $\{0\}$ means that all experts agree that the statement is false;
- the set $\{1\}$ means that all experts agree that the statement is true;
- the set $\{0, 1\}$ means that some experts believe that the statement is true, while some other experts believe that the statement is false;
- finally, the empty set means that no experts have any opinion about this statement.

Here, both the set $\{0, 1\}$ and the empty set correspond to uncertainty, but there is a difference between the two cases:

- the empty set means, in effect, that we know nothing about the statement;
- in contrast, the set $\{0, 1\}$ means, in effect, that we have some arguments in favor of the given statement, and some arguments against this statement.

How is this related to interval-valued fuzzy techniques. The need to distinguish between these two types of uncertainty is often emphasized as the need to go from the traditional fuzzy logic to its interval-valued version. Indeed, in the traditional fuzzy logic, the same value 0.5 can mean two different things:

- it can mean that we know nothing about the given statement, and
- it can also mean that we have as many arguments in favor of this statement as against it.

In the interval-valued case:

- the first situation – when we know nothing, the statement can be false or true – is naturally described by the interval $[0, 1]$ containing all possible truth values, while
- for the second situation, a value 0.5 – corresponding to the degenerate (1-point) interval $[0.5, 0.5]$ seems to be a better match.

How is this related to Belnap logic. The above four truth values have been analyzed in a non-fuzzy context, under the name of Belnap logic [1, 2]. In this context, instead of expert opinions about the truth of a statement, we consider the actual validity of this statement. In this interpretation, the set $\{0, 1\}$ corresponds to inconsistency – when our knowledge base mistakenly contains both the information that this statement is true and the information that this same statement is false.

The need to consider this logic was caused by the fact that in the usual 2-valued logic, once we have a single contradiction, we can conclude that all statements are true – and that all statements are false. So, if we use the usual logic, one wrong statement added to the database – e.g., that the train leaves at 1 pm and that this same train leaves at 1.01 pm – would make the whole knowledge base useless.

4 Interval-Valued and Set-Valued Extensions of 3-Valued Logic and Their Relation to Color Optical Computing

3-valued logic. After the simplest 2-valued logic, the next simplest is 3-valued logic, when we add, to the usual 0 (“false") and 1 (”true”), and additional intermediate degree corresponding to uncertainty. For simplicity, let us denote this degree by 0.5.

Interval-valued extension of 3-valued logic. For this logic, with 3 truth values $0 < 0.5 < 1$, there are six possible intervals:

- the degenerate interval $[0, 0] = \{0\}$ meaning that the expert believes that the given statement is false;
- the degenerate interval $[1, 1] = \{1\}$ meaning that the expert believes that the given statement is true;
- the degenerate interval $[0.5, 0.5] = \{0.5\}$ meaning that the expert is uncertain;
- the interval $[0, 0.5] = \{0, 0.5\}$ meaning the expert is uncertain but is leaning towards “false”;
- the interval $[0.5, 1] = \{0.5, 1\}$ meaning the expert is uncertain but is leaning towards “true”; and
- the interval $[0, 1] = \{0, 0.5, 1\}$ meaning that the expert is uncertain, but has some arguments in favor and against the given statement.

Comment. In the 2-valued case, the interval extension did not allow us to distinguish between two different situations:

- not having any information about a statement and
- having arguments for and argument against the statement.

To distinguish between these two cases, we had to consider set-valued extension of the 2-valued logic.

Interesting, in the 3-valued case, already the interval extension enables us to distinguish between these two situations.

Set-valued extension of 3-valued logic. In the set-valued extension of the 3-valued logic, in addition to the six sets corresponding to interval-valued extension of this logic, we have two more sets:

- the empty set $\emptyset$ corresponding to situations in which no expert has any opinion, and
- the set $\{0, 1\}$ corresponding to the polarized case when some experts strongly believe that the given statement is true while others as strongly believe that this statement is false – case typical in politics.

Set-valued extension of 3-valued logic naturally leads to color optical computing. In color optical computing, we start with three basic colors read (R), green (G), and blue (B) whose position on the spectrum is described as $R < G < B$, and we consider combinations of some of these colors, i.e., all subsets of the set $\{R, G, B\}$:

- we can have three pure colors corresponding to three 1-element sets $\{R\}$, $\{G\}$, and $\{B\}$;
- we can have white – a combination of all three basic colors – corresponding to the set $\{R, G, B\}$;
- we can have black – where there are no colors at all – corresponding to the empty set; and
- we can also have combinations of two of three colors.

These $2^3 = 8$ combinations are in natural 1-to-1 correspondence with eight subsets that form the set-valued extension of the 3-valued logic. This provides a natural explanation of the color optical interpretation of fuzzy logic.

5 Conclusions

In the classical logic, every statement is either true or false. In a computer, "true" is usually represented by 1, and "false" by 0. In many practical situations, we are unsure whether the statement is true or false. To describe different degrees of confidence in a statement, Lotfi Zadeh proposed to use real numbers between 0 and 1. From the purely mathematical viewpoint, there are infinitely many real numbers between 0 and 1. However, we humans can only meaningfully distinguish between a small number of different degrees of confidence. Thus, to make the description of degrees of confidence more adequate, it makes sense to restrict ourselves to finite (discrete) subsets of the interval $[0, 1]$.

To make this description even more adequate, it is desirable to also take into account that sometimes, experts are unsure which of the possible degrees better describe their degree of confidence. To cover such situations, we need to consider subsets of the set of possible degrees – i.e., set-valued extensions of discrete fuzzy logics. An important particular case is an interval-valued extension, when we only consider intervals – the set of all the degrees between two bounds.

It turns out that these extension ideas naturally lead to several known effective techniques – and thus, provide an explanation for their effectiveness. Namely:

- the set-theoretic extension of the 2-valued logic naturally leads to the known technique of Belnap's logic, technique that enables us to allow knowledge bases with inconsistencies, and
- the set-theoretic extension of the 3-valued discrete fuzzy logic naturally leads to color optical computing – an empirically successful way of representing and processing fuzzy degrees by different colors.

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