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Which Interval-Valued Alternatives Are Possibly Optimal If We Use Hurwicz Criterion

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Abstract In many practical situations, for each alternative i, we do not know the corresponding gain x_i, we only know the interval $[\underline{x}_i, \overline{x}_i]$ of possible gains. In such situations, a reasonable way to select an alternative is to choose some value α from the interval $[0,1]$ and select the alternative i for which the Hurwicz combination $\alpha \cdot \overline{x}_i + (1-\alpha) \cdot \underline{x}_i$ is the largest possible. In situations when we do not know the user's α, a reasonable idea is to select all alternatives that are optimal for some α. In this paper, we describe a feasible algorithm for such a selection.

1 Formulation of the Problem

Need to make decisions under interval uncertainty. In the ideal case, when we know the exact expected gain of different investments, a natural idea is to select the investment for which the expected gain is the largest. In practice, however, we rarely know the exact values of the gains. At best, for each alternative investment i, we know the interval $[\underline{x}_i, \overline{x}_i]$ of possible values of the gain. We therefore need to make a decision based on this incomplete information.

Hurwicz approach. To make a decision, we need to announce the price that we are willing to pay for each interval-valued alternative. For this, we need to select a function $v([\underline{x}, \overline{x}])$ that assigns a numerical value to each interval.

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This function must satisfy two natural condition. The first is that this price must be somewhere between the lower bound $\underline{x}_i$ and the upper bound $\overline{x}_i$: it makes no sense to pay more than we are expecting to gain.

The second condition is related to the fact that some investments consist of two independent parts. We can view these two parts, with interval gains $[\underline{x}_1, \overline{x}_1]$ and $[\underline{x}_2, \overline{x}_2]$, separately. This way we pay the price $v([\underline{x}_1, \overline{x}_1])$ for the first part, and we pay the price $v([\underline{x}_2, \overline{x}_2])$ for the second part, to the total of $v([\underline{x}_1, \overline{x}_1]) + v([\underline{x}_2, \overline{x}_2])$.

For example, if in the first part of the investment, we gain between 1 and 2 dollars, and in the second part of the investment, we also gain between 1 and 2 dollars, then overall, in both parts, our gain is between 2 and 4 dollars.

Alternative, we can view the two parts as a single investment, with the interval of possible gains

$$\{x_1 + x_2 : x_1 \in [\underline{x}_1, \overline{x}_1] \text{ and } x_2 \in [\underline{x}_2, \overline{x}_2]\} = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2].$$

In this case, the price will be equal to $v([\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2])$.

These are the two ways to describe the same overall investment. So it makes sense to require that the resulting overall price of this investment should not depend on how we describe it – as a single investment or as an investment consisting of two parts. So, we should have

$$v([\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2]) = v([\underline{x}_1, \overline{x}_1]) + v([\underline{x}_2, \overline{x}_2]).$$

It turns that every function $v([\underline{x}, \overline{x}])$ that satisfies these two conditions has the form

$$v([\underline{x}, \overline{x}]) = \alpha \cdot \overline{x} + (1 - \alpha) \cdot \underline{x}, \tag{1}$$

for some value $\alpha \in [0, 1]$; see, e.g., [1, 2, 3].

So, to make a decision under interval uncertainty, we should select some $\alpha \in [0, 1]$ and then select an alternative for which the corresponding α-combination is the largest. This recommendation was first proposed by an economist Leo Hurwicz – who later received a Nobel Prize for it – and it is thus called *Hurwicz criterion.*

Remaining problem. If we know the customer's α, we can easily find the optimal alternative. However, in some cases, we need to make a recommendation without knowing α. In this case, the only thing we can do is come up with a list of *possibly optimal* alternatives, i.e., alternatives which are optimal for some $\alpha \in [0, 1]$.

In this paper, we describe an algorithm for generating such a list. To describe this algorithm, we first, in Section 2, explain the current idea of such a selection, and provide an example explaining that this idea is not sufficient – sometimes it returns alternatives which are not possibly optimal. Then, in Section 3, we describe an algorithm that returns all possibly optimal alternatives – and only them.

2 Current Idea and Its Limitations

Idea. If for two alternatives $[\underline{x}_i,\overline{x}_i]$ and $[\underline{x}_j,\overline{x}_j]$, we have $\underline{x}_i < \underline{x}_j$ and $\overline{x}_i < \overline{x}_j$, then, clearly, for every $\alpha \in [0,1]$, we will have

$$\alpha \cdot \overline{x}_i + (1-\alpha)\cdot \underline{x}_i < \alpha \cdot \overline{x}_j + (1-\alpha)\cdot \underline{x}_j.$$

Thus, the alternative i will never be optimal. So, a reasonable idea is to dismiss all alternatives i which are *dominated* by some other alternative j, i.e., for which $\underline{x}_i < \underline{x}_j$ and $\overline{x}_i < \overline{x}_j$.

After this dismissal, the remaining list will contain all possibly optimal alternatives. But are all alternatives in the remaining list possibly optimal? Sometimes they are, but, as we will show, sometimes they are not. Let us describe an example when one of the remaining alternatives is not possibly optimal.

An example of a remaining alternative which is not possibly optimal. Let us assume that we have three alternatives, with interval gains $[\underline{x}_1,\overline{x}_1] = [0,10]$, $[\underline{x}_2,\overline{x}_2] = [1,8]$, and $[\underline{x}_3,\overline{x}_3] = [2,7]$. Let us prove, by contradiction, that the alternative $[\underline{x}_2,\overline{x}_2] = [1,8]$ cannot be optimal.

Indeed, suppose that this alternative is optimal for some α. This means, in particular, that for this α, this alternative is better than (or is of the same quality as) the alternative $[0,10]$. This means that

$$\alpha \cdot 8 + (1-\alpha)\cdot 1 \ge \alpha \cdot 10 + (1-\alpha)\cdot 0,$$

i.e.,

$$8\alpha + 1 - \alpha \ge 10\alpha,$$

so $3\alpha \le 1$ and $\alpha \le 1/3$.

Similarly, the fact that this alternative is better than $[2,7]$ means that

$$\alpha \cdot 8 + (1-\alpha)\cdot 1 \ge \alpha \cdot 7 + (1-\alpha)\cdot 2,$$

i.e.,

$$8\alpha + 1 - \alpha \ge 7\alpha + 2 - 2\alpha,$$

so $2\alpha \ge 1$ and $\alpha \ge 1/2$. A number cannot be at the same time larger than or equal to 1/2 and smaller then or equal to 1/3, so our assumption that the alternative $[\underline{x}_2,\overline{x}_2] = [1,8]$ can be optimal leads to a contradiction. Thus, this alternative cannot be optimal.

3 Algorithm for Selecting Possibly Optimal Alternatives

General analysis of the problem. Let us first weed out all dominated alternatives. We need to decide, for each of the remaining alternatives i, whether this alternative is possibly optimal, i.e., whether it is optimal for some $\alpha \in [0,1]$.

This optimality happens if for all $j \neq i$, we have

$$\alpha \cdot \overline{x}_i + (1-\alpha) \cdot \underline{x}_i \geq \alpha \cdot \overline{x}_j + (1-\alpha) \cdot \underline{x}_j,$$

i.e., equivalently, that

$$\underline{x}_i + \alpha \cdot w_i \geq \underline{x}_j + \alpha \cdot w_j,$$

where, for each interval $[\underline{x}_k, \overline{x}_k]$, by w_k, we denote its width $w_k \stackrel{\text{def}}{=} \overline{x}_k - \underline{x}_k$. This inequality can be equivalently reformulated as

$$\underline{x}_i - \underline{x}_j \geq \alpha \cdot (w_j - w_i). \tag{2}$$

Here, we have three options: $w_j - w_i = 0$, $w_j - w_i > 0$, and $w_j - w_i < 0$. Let us consider them one by one.

First case. Let us first consider the case when $w_j - w_i = 0$, i.e., when $w_i = w_j$. In this case, the right-hand side of the inequality (2) is equal to 0, so this inequality takes the form

$$\underline{x}_i - \underline{x}_j \geq 0. \tag{2a}$$

Let us prove, by contradiction, that when $w_i = w_j$, the inequality (2a) is always satisfied after we filter out dominated alternatives.

Indeed, suppose that the inequality (2a) is not satisfied, i.e., that $\underline{x}_i - \underline{x}_j < 0$. In this case, we would have $\underline{x}_i < \underline{x}_j$ and thus,

$$\overline{x}_i = \underline{x}_i + w_i < \underline{x}_j + w_i = \underline{x}_j + w_j = \overline{x}_j.$$

So, the i-th alternative is dominated by the j-th alternative – but we assumed that we have already dismissed all dominated alternatives. So, the assumption that $\underline{x}_i - \underline{x}_j < 0$ leads to a contradiction. This contradiction shows that when $w_i = w_j$. the inequality (2a) is always satisfies for the after-filtering-out intervals. In other words, in situations when $w_i = w_j$, the inequality (2) is satisfied for all α.

Second case. Let us now consider the case when $w_j - w_i > 0$, i.e., $w_j > w_i$. In this case, dividing both sides of the inequality (2) by the positive number $w_j - w_i$, we get an equivalent inequality

$$\alpha \leq \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}.$$

This inequality must be satisfied for all j for which $w_j > w_i$, so we must have

$$\alpha \leq \min_{j:\, w_j > w_i} \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}.$$

We must also have $\alpha \le 1$, so we must have

$$\alpha \le \min\left(1, \min_{j:\ w_j > w_i} \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}\right). \tag{3}$$

Third case. When $w_j - w_i < 0$, i.e., when $w_j < w_i$, then, dividing both sides of the inequality (2) by the negative number $w_j - w_i$, we get the opposite inequality

$$\alpha \ge \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}.$$

This inequality must be satisfied for all j for which $w_j < w_i$, so we must have

$$\alpha \ge \max_{j:\ w_j < w_i} \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}.$$

We must also have $\alpha \ge 0$, so we must have

$$\alpha \ge \max\left(0, \max_{j:\ w_j < w_i} \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}\right). \tag{4}$$

The only possibility to have the value α that satisfies both inequalities (3) and (4) is when the lower bound for α is smaller than or equal to the upper bound. Thus, we arrive at the following algorithm.

Algorithm. First, we dismiss all alternatives $[\underline{x}_i, \overline{x}_i]$ which are dominated, i.e., for which, for some j, we have $\underline{x}_i < \underline{x}_j$ and $\overline{x}_i < \overline{x}_j$.

From the remaining list, we select as possibly optimal all alternatives i for which

$$\max\left(0, \max_{j:\ w_j < w_i} \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}\right) \le \min\left(1, \min_{j:\ w_j > w_i} \frac{\underline{x}_i - \underline{x}_j}{w_j - w_i}\right). \tag{5}$$

Example. In the above 3-interval example, we have $w_1 = 10 - 0 = 10$, $w_2 = 8 - 1 = 7$, and $w_3 = 7 - 2 = 5$. Thus, for the second interval $i = 2$, the left-hand of the inequality (5) takes the form

$$\max\left(0, \frac{\underline{x}_2 - \underline{x}_3}{w_3 - w_2}\right) = \max\left(0, \frac{1-2}{5-7}\right) = \max\left(0, \frac{1}{2}\right) = \frac{1}{2},$$

while the right-hand side of this inequality has the form

$$\min\left(1, \frac{\underline{x}_2 - \underline{x}_1}{w_1 - w_2}\right) = \min\left(1, \frac{1-0}{10-7}\right) = \min\left(1, \frac{1}{3}\right) = \frac{1}{3}.$$

Here, the lower bound is larger than the upper bound, so the inequality (5) is not satisfied and thus, the alternative 2 is not possibly optimal.

For alternatives 1 and 3, similar arguments show that they are possibly optimal.

What is the computational complexity of this algorithm. For each alternative i, we need to consider all other alternatives, so we need $O(n)$ steps for check whether the alternative i is possibly optimal. We need to repeat this check for all n alternatives, so the overall time is $n \cdot O(n) = O(n^2)$.

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