12-1-2022

An Argument in Favor of Piecewise-Constant Membership Functions

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Comments:

Technical Report: UTEP-CS-22-123

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Abstract Theoretically, we can have membership functions of arbitrary shape. However, in practice, at any given moment of time, we can only represent finitely many parameters in a computer. As a result, we usually restrict ourselves to finite-parametric families of membership functions. The most widely used families are piecewise linear ones, e.g., triangular and trapezoid membership functions. The problem with these families is that if we know a nonlinear relation $y = f(x)$ between quantities, the corresponding relation between membership functions is only approximate – since for piecewise linear membership functions for x, the resulting membership function for y is not piecewise linear. In this paper, we show that the only way to preserve, in the fuzzy representation, all relations between quantities is to limit ourselves to piecewise constant membership functions, i.e., in effect, to use a finite set of certainty degrees instead of the whole interval $[0,1]$.

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1 Formulation of the Problem

Need for interpolation. In this paper, we consider fuzzy techniques; see, e.g., [2, 3, 4, 5, 6, 7]. A membership function $\mu(x)$ corresponding to a natural-language property like "small" describes, for each real number x, the expert's degree of confidence that the given value x satisfies the corresponding property – e.g., that the value x is small.

Of course, these are infinitely many real numbers x, but we cannot ask infinitely many questions to the expert about all these values x. So, in practice, we ask finitely many questions, and then apply some interpolation/extrapolation to get the values $\mu(x)$ for all other x.

Typical example: linear interpolation. The simplest interpolation is linear interpolation. So, if we know:

- that some value x_0 (e.g., $x_0 = 0$) is absolutely small – i.e., $\mu(x_0) = 1$, and
- that some values $x_- < x_0$ and $x_+ > x_0$ are definitely not small – i.e.

$$\mu(x_-) = \mu(x_+) = 0,$$

then we can use linear interpolation and get frequently used triangular membership functions. Similarly, if we know that a certain property holds for sure for two value x_0^- and x_0^+, then linear interpolation leads to trapezoid membership functions.

General case. In general, since we can only ask finitely many questions and thus, only get finitely many parameters describing a membership function, we must limit ourselves to a finite-parametric class of membership functions.

Resulting problem. This idea of a linear interpolation works well when all we have is unrelated fuzzy quantities. In this case, we can consider all quantities to be described by either triangular or trapezoid membership functions. If we have more values to interpolate from, by more general piecewise linear membership functions, i.e., functions for which there exist values $x_1 < x_2 < \ldots < x_n$ such that on each interval $(-\infty, x_1]$, $[x_1, x_2]$, $\ldots$, $[x_{n-1}, x_n]$, $[x_n, \infty)$ the function is linear; see, e.g., [1].

The situation becomes more complicated if we take into account that some quantities are related to each other by non-linear dependencies $y = f(x)$ (or $y = f(x_1, \ldots, x_n)$). Indeed, it is desirable to require that if our knowledge about a quantity x is described by a membership function from the selected family, then the resulting knowledge about the quantity $y = f(x)$ should also be described by a membership function from this family.

It is straightforward, given a membership function $\mu_X(x)$ corresponding to x, to compute a membership function $\mu_Y(y)$ corresponding to $y = f(x)$ – there is Zadeh's extension principle for this:

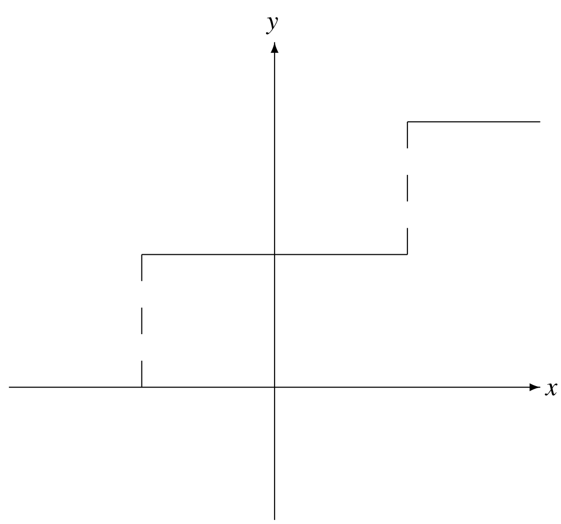
$$\mu_Y(y) = \max\{\mu_X(x) : f(x) = y\}. \quad (1)$$

The problem is that:

- when the membership function $\mu_X(x)$ is triangular, and $f(x)$ is nonlinear, then the resulting membership function for y is not piecewise linear, and,
- vice versa, if the membership for y is piecewise linear, then the corresponding membership function for $x = f^{-1}(y)$ is not piecewise linear.

So, the class of all triangular functions does not satisfy the above property. How can we find a finite-parametric class of membership functions for which for every nonlinear function $f(x)$, the result of applying this function to a membership function from this class would also lead to a membership function from this same class?

What we plan to do in this paper. In this paper, we show that the only way to reach this goal is to limit ourselves to *piecewise constant* membership functions, i.e., membership functions for which there exist values $x_1 < x_2 < \ldots < x_n$ such that on each interval $(-\infty, x_1)$, (x_1, x_2), $\ldots$, (x_{n-1}, x_n), (x_n, ∞) the function is constant:



2 Definitions

To formulate our result, let us define, in precise terms:

- what we mean by a finite-parametric family,
- what we mean by the requirement that this family should be closed under applying Zadeh's extension principle, and
- what we mean by a piecewise constant membership function.

2.1 What is a finite-parametric family

In general, a finite-parametric class of objects means that we have a continuous mapping that assigns, to each tuples of real numbers $(c_1,\ldots,c_n)$ from some open set, a corresponding object. To describe what is a continuous mapping, we need to have a metric on the set of all the objects. So, to describe what we mean by a finite-parametric family of membership functions, we need to describe a reasonable metric on the set of all membership functions, i.e., we need to describe, for each real number $\varepsilon>0$, what it means that two membership functions are ε-close.

Let us analyze what it means for two membership functions to be ε-close. Intuitively, two real numbers x_1 and x_2 are ε-close if, whenever we measure them with some accuracy ε, we may not be able to distinguish between them. Similarly, the two functions $\mu_1(x)$ and $\mu_2(x)$ are ε-close if, whenever we measure both real values x and $\mu_i(x)$ with accuracy ε, we will not be able to distinguish these functions.

In precise terms, when we only know x_1 with some accuracy ε, i.e., when we only know that $x_1\in[\widetilde{x}_1-\varepsilon,\widetilde{x}_1+\varepsilon]$ for the measurement result $\widetilde{x}_1$, then all we know is that the value $\mu_1(x_1)$ is somewhere between the minimum and the maximum of the function $\mu_1(x)$ on this interval. If we know that the function $\mu_2(x)$ is ε-close to $\mu_1(x)$, this means that the value $\mu_1(x)$ is ε-close to one of the values $\mu_2(x)$ for x from the x-interval $[\widetilde{x}_1-\varepsilon,\widetilde{x}_1+\varepsilon]$, i.e., that we have $\mu_2(x')-\varepsilon\le\mu_1(x)\le\mu_2(x'')+\varepsilon$ for some points x' and x'' from the above x-interval. Thus, we arrive at the following definition.

Definition 1. *We say that two membership functions $\mu_1(x)$ and $\mu_2(x)$ are* ε-close *if:*

- *for every real number x_1, there exist ε-close values x_2' and x_2'' for which $\mu_1(x_1)$ is ε-close to some number between $\mu_2(x_2')$ and $\mu_2(x_2'')$, and*
- *for every real number x_2 there exist ε-close values x_1' and x_1'' for which $\mu_2(x_2)$ is ε-close to some number between $\mu_1(x_1')$ and $\mu_2(x_1'')$.*

We will denote this relation by $\mu_1\approx_\varepsilon\mu_2$.

Examples. If both membership functions are continuous – e.g., if both are triangular – then we can simply take $x_2'=x_2''=x_1$.

Definition 2. *By the* distance $d(\mu_1,\mu_2)$ *between two membership functions, we mean that infimum of all the values ε for which these functions are ε-close:*

$$d(\mu_1,\mu_2)\stackrel{\text{def}}{=}\inf\{\varepsilon:\mu_1\approx_\varepsilon\mu_2\}.$$

One can prove that thus defined distance satisfies the triangle inequality and is, therefore, a metric. Now, we are ready to define what we mean by a finite-parametric family of membership functions.

Definition 3. *By a* finite-parametric family of membership functions*, we mean a continuous mapping from a open subset of $\mathbb{R}^n$ for some integer n to the class of all membership functions with the metric defined by Definition 2.*

2.2 When is a family closed under applying Zadeh's extension principle

Definition 4. *We say that a family of membership functions is* closed under transformations *if for every function $\mu_X(x)$ from this family and for every function $f(x)$, the function $\mu_X(f^{-1}(y))$ also belongs to this family.*

2.3 What is a piecewise constant function

Definition 5. *We say that a function $f(x)$ from real numbers to real numbers is* piecewise constant *if there exist values $x_1 < x_2 < \ldots < x_n$ such that on each interval $(-\infty, x_1)$, (x_1, x_2), $\ldots$, (x_{n-1}, x_n), (x_n, ∞), the function $f(x)$ is constant.*

3 Main Result

Simplifying assumption. For simplicity, let us consider non-strictly increasing membership functions. Such functions correspond to such properties as "large".

Our result can be easily extended to other membership functions that consist to two or more non-strictly increasing and non-strictly decreasing segments.

Definition 6. *We say that a membership function $\mu(x)$ is* non-strictly increasing *if $x \le x'$ implies that $\mu(x) \le \mu(x')$.*

Definition 7. *We say that a family of membership functions is* closed under increasing transformations *if for every function $\mu_X(x)$ from this family and for every increasing 1-1 function $f(x)$, the function $\mu_X(f^{-1}(y))$ also belongs to this family.*

Discussion. For a membership function $\mu_X(x)$ and for a 1-1 increasing function $f(x)$, the only value x for which $f(x) = y$ is the value $f^{-1}(y)$. So, in this case, Zadeh's extension principle means that for $y = f(x)$, the membership function $\mu_Y(y)$ has the form $\mu_Y(y) = \mu_X(f^{-1}(x))$. One can check that if the function $\mu_X(x)$ was increasing then the function $\mu_Y(y) = \mu_X(f^{-1}(x))$ will be increasing as well. Now, we are ready to formulate our main result.

Proposition. *Let F be a finite-parametric family of non-strictly increasing membership functions which is closed under increasing transformations. Then, each function from this family is piecewise constant.*

Comment. For readers' convenience, the proof is placed in a special last section.

Discussion. So, if we want to have a finite-parametric family of membership functions for which each relation $y = f(x)$ between physical quantities leads to a similar relation between the corresponding membership functions, we should use piecewise constant membership functions, i.e., functions from the real line to a finite subset of the interval $[0,1]$. This is equivalent to using this finite subset instead of the whole interval $[0,1]$, i.e., to considering a finite-valued logic instead of the usual full $[0,1]$-based fuzzy logic.

Once we limit ourself to this finite set of confidence degrees, then Zadeh's extension principle will keep the values within this set. Indeed, this principle is based on using minimum and maximum, and both operations do not introduce new values, they just select from the existing values.

4 Proof

1°. Let $\mu(x)$ be a membership function from a finite-parametric family of non-strictly increasing membership functions which is closed under increasing transformations. To prove the proposition, it is sufficient to prove that all its values form a finite set. Then, the fact that this function is piecewise constant would follow from this result and from the fact that the function $\mu(x)$ is non-strictly increasing.

We will prove this statement by contradiction. Let us assume that the set V of values of the function $\mu(x)$ is infinite.

2°. Let us pick an infinite sequence S of different values $\mu(x_1)$, $\mu(x_2)$, $\ldots$, from this set one by one. To do that, we pick one value $\mu(x_1)$ from the set V. Once we have picked the values $\mu(x_1),\ldots,\mu(x_k)$, since the set V is infinite, it has other values than these selected ones, so we pick one of them as $\mu(x_{k+1})$. This way, we get a sequence of different values $\mu(x_k)$. These values correspond to x-values $x_1,x_2,\ldots$ Since the values $\mu(x_n)$ are all different, the values x_n are also all different.

3°. It is known that if we add an infinity point to the real line and consider the usual convergence to infinity, we get a set which is topologically equivalent to a circle: after all positive numbers, we have the infinity point, and after this infinity point, we have all negative numbers. This is known as a *one-point compactification* of the real line.

A circle is a compact set. So, by the properties of a compact set, from any sequence – including the sequence $\{x_n\}_n$ – we can extract a convergent subsequence. Let us denote this subsequence by $\{c_n\}_n$, and its limit by

$$x_0 \stackrel{\text{def}}{=} \lim_{n\to\infty} c_n.$$

4°. Let us prove that this sequence $\{c_n\}_n$ either contains infinitely many elements that follow x_0 or it contains infinitely many elements which precede x_0. Here:

- for finite x_0, *follow* and *precede* simply means, correspondingly, larger and smaller;
- for $x_0=\infty$, *follow* means the values are negative and *precede* means that these values are positive.

This statement – that the sequence $\{c_n\}_n$ either contains infinitely many elements that follow x_0 or it contains infinitely many elements which precede x_0 – can be easily proven by contradiction. Indeed, if we would have only finitely elements c_n preceding x_0 and only finitely many elements c_n following x_0, then the whole sequence c_n would only have finitely many values – but we know this sequence has infinitely many different values.

5°. Let us prove that when we have infinitely many elements preceding x_0, then from the sequence $\{c_n\}_n$, we can extract a strictly increasing subsequence $\{X_n\}_n$.

Let us consider only elements of the sequence $\{c_n\}_n$ that precede x_0. Let us denote the subsequence of the sequence $\{c_n\}_n$ formed by such elements by $\{s_n\}_n$.

Then, as X_1, let us take $X_1 = s_1$. Because of the convergence $s_n \to x_0$, all the elements s_n – starting with some element – get close to x_0 and thus, become larger than X_1. Let us take the first element of the sequence s_n which is larger than X_1 as X_2. Similarly, there exists elements which are larger than X_2. Let is select the first of them by X_3, etc. Thus, we get a strictly increasing sequence $X_1 < X_2 < \ldots$ for which all the values $\mu(X_n)$ are different, i.e., for which $\mu(X_1) < \mu(X_2) < \ldots$

6°. Let us prove that when we have infinitely many elements following x_0, then from the sequence $\{c_n\}_n$, we can extract a strictly decreasing subsequence $\{X_n\}_n$.

Let us consider only elements of the sequence $\{c_n\}_n$ that follow x_0. Let us denote the subsequence formed by such elements by $\{s_n\}_n$. As X_1, let us take $X_1 = s_1$. Because of the convergence $s_n \to x_0$, all the elements s_n – starting with some element – get close to x_0 and thus, become smaller than X_1. Let us take the first element of the sequence s_n which is smaller than X_1 as X_2. Similarly, there exists elements which are smaller than X_2. Let is select the first of them by X_3, etc. Thus, we get a strictly decreasing sequence $X_1 > X_2 > \ldots$ for which all the values $\mu(X_n)$ are different, i.e., for which $\mu(X_1) > \mu(X_2) > \ldots$

7°. In both cases, we have either a strictly increasing or a strictly decreasing sequence $\{X_n\}_n$ for which the corresponding values $\mu(X_n)$ are, corresponding, either strictly increasing of strictly decreasing.

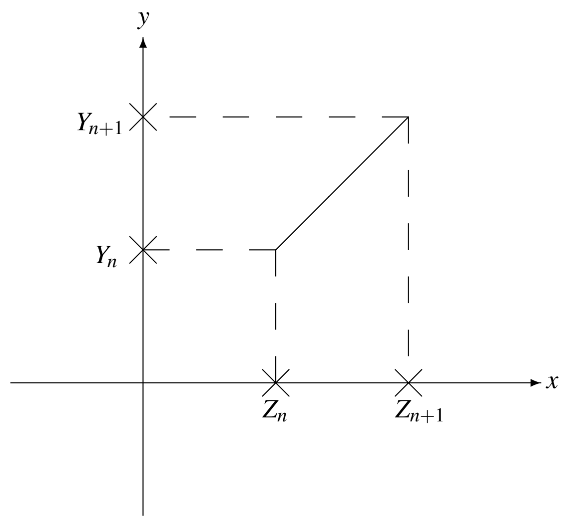
7.1°. When the sequence $\{X_n\}_n$ is strictly increasing, we can take the values

$$Y_n \stackrel{\text{def}}{=} \inf\{x : \mu(x) = \mu(X_n)\}$$

and so $Y_1 < Y_2 < \ldots$ For any other strictly increasing sequence $Z_1 < Z_2 < \ldots$ tending to the same limit, we can form a piecewise linear transformation function $f(x)$ that maps Z_n into Y_n, namely:

- for $x \le Z_1$, we take $f(x) = x + (Y_1 - Z_1)$;
- for $Z_n \le x \le Z_{n+1}$, we take

$$f(x) = Y_n + \frac{Y_{n+1} - Y_n}{Z_{n+1} - Z_n} \cdot (x - Z_n);$$



and

- if the limit x_0 is finite, then for $x \geq x_0$, we take $f(x) = x$.

For each such sequence, the transformed membership function $\mu'(x) = \mu(f^{-1}(x))$, because of closed-ness, also belongs to the given family. For this function, we have $\inf\{x : \mu'(x) = \mu(X_n)\} = Z_n$. Thus, all these functions $\mu'(x)$ are different. So, the given family contains an infinite-parametric subfamily determined by infinitely many parameters Z_n. This contradicts to our assumption that the family is finite-parametric.

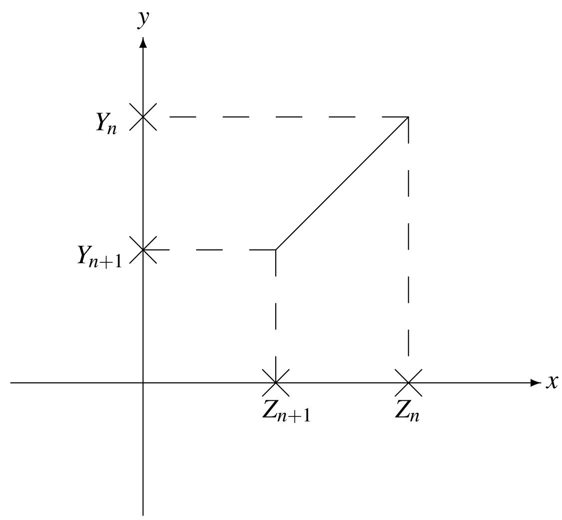
7.2°. Similarly, when the sequence $\{X_n\}_n$ is strictly decreasing, we can take the values

$$Y_n \stackrel{\text{def}}{=} \sup\{x : \mu(x) = \mu(X_n)\}$$

for which too $Y_1 > Y_2 > \ldots$ For any other decreasing sequence $Z_1 > Z_2 > \ldots$ tending to the same limit, we can form a piecewise linear transformation function $f(x)$ that maps Z_n into Y_n, namely:

- for $x \geq Z_1$, we take $f(x) = x + (Y_1 - Z_1)$;
- for $Z_{n+1} \leq x \leq Z_n$, we take

$$f(x) = Y_{n+1} + \frac{Y_n - Y_{n+1}}{Z_n - Z_{n+1}} \cdot (x - Z_{n+1});$$



and
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For each such sequence, the transformed membership function $\mu'(x) = \mu(f^{-1}(x))$, because of closed-ness, also belongs to the given family. For this function $\mu'(x)$, we have $\inf\{x : \mu(x) = \mu(X_n)\} = Z_n$. Thus, all these functions are different. So, the given family contains an infinite-parametric subfamily determined by infinitely many parameters Z_n – which contradicts to our assumption that the family is finite-parametric.

8°. In both cases, we get a contradiction. So the set of values of the membership function $\mu(x)$ cannot be infinite and must, thus, be finite. The proposition is proven.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes), and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

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