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## Graph Approach to Uncertainty Quantification

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# Graph Approach to Uncertainty Quantification

Hector A. Reyes, Cliff Joslyn, and Vladik Kreinovich

**Abstract** Traditional analysis of uncertainty of the result of data processing assumes that all measurement errors are independent. In reality, there may be common factor affecting these errors, so these errors may be dependent. In such cases, the independence assumption may lead to underestimation of uncertainty. In such cases, a guaranteed way to be on the safe side is to make no assumption about independence at all. In practice, however, we may have information that a few pairs of measurement errors are indeed independent – while we still have no information about all other pairs. Alternatively, we may suspect that for a few pairs of measurement errors, there may be correlation – but for all other pairs, measurement errors are independent. In both cases, unusual pairs can be naturally represented as edges of a graph. In this paper, we show how to estimate the uncertainty of the result of data processing when this graph is small.

## 1 Introduction

**What is the problem and what we do about it: a brief description.** Estimating uncertainty of the result of data processing is important in many practical applications. Corresponding formulas are well known for two extreme cases:

- when all measurement errors are independent, and
- when we have no information about the dependence.

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These cases are indeed ubiquitous, but often, the actual cases are somewhat different; e.g.:

- most pairs of inputs are known to be independent, but
- there are a few pairs for which we are not sure.

Alternatively, for almost all pairs, we may have no information about the dependence, but for a few pairs of inputs, we know that the corresponding measurement errors are independent. Such unusual pairs can be naturally represented as edges of a graph. It is desirable to analyze how the presence of this graph changes the corresponding estimates.

In this paper, we start answering this question for all graphs of sizes 2, 3, and 4. We hope that our results will be extended to larger-size graphs.

**Structure of the paper.** In Section 2, we provide a detailed description of the general problem, and describe how uncertainty is estimated in the above-described two extreme cases. In the following sections, we present our results about situations in which the deviation from one of these extreme cases is described by a small-size graph.

## 2 Detailed Formulation of the Problem

**Need for data processing.** One of the main objectives of science is to describe the current state of the world and to predict future events based on what we know about the current and past states. In general, the state of a system is characterized by the values of corresponding quantities.

Some quantities we can measure directly – e.g., we can directly measure the temperature in the room or the distance between two campus buildings. However, some quantities cannot (yet) be measured directly: we cannot directly measure the temperature inside a star or a distance to this star. Since we cannot measure such quantities directly, the only way we can estimate the values of these quantities is by measuring them indirectly: i.e., by measuring related quantities  $x_1, \dots, x_n$  that are related by  $y$  by a known dependence  $y = f(x_1, \dots, x_n)$ . Once we know such related quantities, we can measure their values, and use the measurement results  $\tilde{x}_1, \dots, \tilde{x}_n$  to compute the estimate  $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$  for  $y$ . Computing this estimate is an important case of *data processing*.

Data processing is also needed for *predictions*. For example, we may want to predict the future location of a near-Earth asteroid or tomorrow’s weather. The future state can be described if we describe the future values of all the quantities characterizing this state. For example, tomorrow’s weather can be characterized by temperature, wind speed, etc. To be able to make this prediction, for each of the quantities describing the future state, we need to know the relation  $y = f(x_1, \dots, x_n)$  between the future value  $y$  of this quantity and the current and past values  $x_i$  of related quantities. Once we know this relation, then we can use it to transform the

measured values  $\tilde{x}_i$  of the quantities  $x_i$  into the estimate  $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$  for the desired quantity  $y$ . Computing  $\tilde{y}$  based on the measured values  $\tilde{x}_i$  is another important case of data processing.

**Need for uncertainty quantification.** Measurement results  $\tilde{x}$  are, in general, somewhat different from the actual (unknown) value  $x$  of the corresponding quantity; see, e.g., [5]. In other words, the difference  $\Delta x \stackrel{\text{def}}{=} \tilde{x} - x$  is usually non-zero. This difference is known as the *measurement error*.

Since the inputs  $\tilde{x}_i$  to the algorithm  $f$  are, in general, different from the actual values  $x_i$ , the resulting estimate  $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$  is, in general, different from the actual value  $y = f(x_1, \dots, x_n)$  that we would have gotten if we knew the exact values  $x_i$ . In practical situations, it is important to know how big this difference can be. For example, suppose we predict that the asteroid will pass at a distance of 150,000 km from the Earth; then:

- if the accuracy of this estimate is  $\pm 200,000$  km, then this asteroid may hit the Earth, while
- if the the accuracy is  $\pm 20,000$  km, this particular asteroid is harmless.

Estimating the accuracy of our estimates is an important case of *uncertainty quantification*.

**What we know about measurement errors.** In similar situations, with the exact same value of the measured quantity, the same measuring instrument can produce different results. This is well known to anyone who has ever repeatedly measured the same quantity: the results are always somewhat different, whether it is a current or body temperature or blood pressure. In this sense, measurement errors are *random*. Each random variable has an average (mean) value, and its actual values deviate from this mean.

Measuring instruments are usually calibrated: the measurement results provided by this instrument are compared with measurement results provided by a much more accurate (“standard”) measuring instrument. If the mean difference is non-zero – i.e., in statistical terms, if the measuring instrument has a *bias* – then we can simply subtract this bias from all the measurement results and thus, make the mean error equal to 0. For example, if a person knows that his/her watch is 5 minutes ahead, this person can always subtract 5 minutes from the watch’s reading and get the correct time. So, we can safely assume that the mean value  $E[\Delta x]$  of each measurement error  $\Delta x$  is 0:  $E[\Delta x] = 0$ .

The deviations from the mean value are usually described by the mean squared deviation – which is known as the *standard deviation*  $\sigma$ . Instead of the standard deviation  $\sigma$ , it is sometimes convenient to use its square  $V \stackrel{\text{def}}{=} \sigma^2$  which is called the *variance*. In precise terms, the variance is the mean value of the square of the difference between the random variable and its mean value:  $V[X] = E[(X - E[X])^2]$ . For measurement error, the mean is  $E[\Delta x] = 0$ , so we get a simplified formula  $V[\Delta x] = E[(\Delta x)^2]$ .

For each measuring instrument, the standard deviation is also determined during the calibration. So, we can assume that for each measuring instrument:

- we know that the mean value of its measurement error is 0, and
- we know the standard deviation of the measurement error.

**In many cases, distributions are normal.** In most practical cases, there are many factors that contribute to the measurement error. For example, if we measure voltage, the measuring instrument is affected not only by the current that we measure but also by the currents of multiple devices present in the room, including the computer used to process the data, the lamps in the ceiling, etc. Each of these factors may be relatively small, but there are many of them, and thus, the resulting measurement error is much larger than each of them.

It is known – see, e.g., [6] – that the probability distribution of the joint effect of a large number of small random factors is close to Gaussian (normal). Thus, in such cases, we can safely assume that the measurement errors are normally distributed.

**Possibility of linearization.** In general, the estimation error is equal to  $\Delta y \stackrel{\text{def}}{=} \tilde{y} - y$ . Here,  $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$  and  $y = f(x_1, \dots, x_n)$ , so

$$\Delta y = f(\tilde{x}_1, \dots, \tilde{x}_n) - f(x_1, \dots, x_n),$$

By definition of the measurement error  $\Delta x_i$  as the difference  $\Delta x_i = \tilde{x}_i - x_i$ , we have  $x_i = \tilde{x}_i - \Delta x_i$ . Thus, the above expression for the approximation error takes the form

$$\Delta y = f(\tilde{x}_1, \dots, \tilde{x}_n) - f(\tilde{x}_1 - \Delta x_1, \dots, \tilde{x}_n - \Delta x_n). \quad (1)$$

Measurement errors are usually relatively small, so that terms quadratic in these errors can be safely ignored. For example, if the measurement error is 10%, its square is 1%, which is much smaller. Thus, we can expand the right-hand side of the equality (1) in Taylor series and keep only linear terms in this expansion. As a result, we conclude that

$$\Delta y = \sum_{i=1}^n c_i \cdot \Delta x_i, \quad (2)$$

where we denoted

$$c_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i} \Big|_{x_1=\tilde{x}_1, \dots, x_n=\tilde{x}_n}.$$

In other words, the desired estimation error  $\Delta y$  is a linear combination of measurement errors  $\Delta x_i$ .

**Case when all measurement errors are independent.** It is known that the variance of the sum of the several random variables is equal to the sum of their variances. It is also known that if we multiply a random variable by a constant, then its standard deviation is multiplied by the absolute value of this constant. So, if we denote the standard deviation of the  $i$ -th measuring instrument by  $\sigma_i$ , then the standard deviation of the product  $c_i \cdot \Delta x_i$  is equal to  $|c_i| \cdot \sigma_i$  and thus, its variance is equal to  $(|c_i| \cdot \sigma_i)^2 = c_i^2 \cdot \sigma_i^2$ . Thus, the variance of the sum  $\Delta y$  is equal to the sum of these variances:

$$\sigma^2 = \sum_{i=1}^n c_i^2 \cdot \sigma_i^2, \quad (3)$$

and thus, the standard deviation is equal to

$$\sigma = \sqrt{\sum_{i=1}^n c_i^2 \cdot \sigma_i^2}. \quad (4)$$

**Towards the general case: a known geometric interpretation of random variables.** We have  $n$  random variables  $v_i \stackrel{\text{def}}{=} c_i \cdot \Delta x_i$ . For each variable, we know its standard deviation  $|c_i| \cdot \sigma_i$ , and we are interested in estimating the standard deviation of the sum  $\Delta y = v_1 + \dots + v_n$  of these variables.

It is known (see, e.g., [6]) that we can interpret each variable – and, correspondingly, each linear combination of the variables – as vectors  $\mathbf{a}$ ,  $\mathbf{b}$  in an  $n$ -dimensional space, so that the length  $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$  of each vector (where  $\mathbf{a} \cdot \mathbf{b}$  means dot (scalar) product) is equal to the standard deviation of the corresponding variable. In these terms, independence means that the two vectors are orthogonal. Indeed:

- In statistical terms, independence implies that the variance of the sum is equal to the sum of the variances.
- For the sum  $\mathbf{a} + \mathbf{b}$  of two vectors, the square of the length has the form

$$\|\mathbf{a} + \mathbf{b}\|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{b}.$$

Here,  $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ ,  $\mathbf{b} \cdot \mathbf{b} = \|\mathbf{b}\|^2$ , and  $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos(\theta)$ , where  $\theta$  is the angle between the two vectors, so the above expression takes the form

$$\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\|\mathbf{a}\| \cdot \|\mathbf{b}\| \cdot \cos(\theta).$$

So, the variance  $\|\mathbf{a} + \mathbf{b}\|^2$  of the sum is equal to the sum  $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$  of the variances if and only if  $\cos(\theta) = 0$ , i.e., if only if the angle is  $90^\circ$ , and the vectors are orthogonal.

In the independent case,  $n$  vectors  $\mathbf{v}_i$  corresponding to individual measurements are orthogonal to each other, so, similarly to the above argument, one can show that the length of their sum is equal to the square root of the sum of the squares of their lengths:

$$\|\mathbf{v}_1 + \dots + \mathbf{v}_n\|^2 = \|\mathbf{v}_1\|^2 + \dots + \|\mathbf{v}_n\|^2.$$

Let us use this geometric interpretation to estimate the uncertainty in situations when we know nothing about correlation between different measurement errors.

**What if we have no information about correlations: analysis of the problem and the resulting formula.** In this case, we still have  $n$  vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of given lengths  $\|\mathbf{v}_i\| = |c_i| \cdot \sigma_i$ . The main difference from the independent case is that these vectors are not necessarily orthogonal, we can have different angles between them.

In this case, in contrast to the independent case, the length of the sum is not uniquely determined. For example:

- if two vectors of equal length are parallel, the length of their sum is double the length of each vector, but
- if they are anti-parallel  $\mathbf{b} = -\mathbf{a}$ , then their sum has length 0.

In such cases, it is reasonable to find the worst possible standard deviation, i.e., the largest possible length.

One can easily check that the sum of several vectors of given length is the largest when all these vectors are parallel and oriented in the same direction. In this case, the length of the sum is simply equal to the sum of the lengths, so we get

$$\sigma = \sum_{i=1}^n |c_i| \cdot \sigma_i, \quad (5)$$

and

$$\sigma^2 = \left( \sum_{i=1}^n |c_i| \cdot \sigma_i \right)^2. \quad (6)$$

**Precise mathematical formulation of this result.** The above result can be presented in the following precise form.

**Definition 1.**

- Let  $s = (\sigma_1, \dots, \sigma_n)$  be a tuple of non-negative real numbers.
- Let  $D$  denote the class of all possible multi- $D$  distributions  $(\Delta x_1, \dots, \Delta x_n)$  for which, for each  $i$ , we have  $E[\Delta x_i] = 0$  and  $\sigma[\Delta x_i] = \sigma_i$ .
- Let  $\mathcal{S}$  be a subset of the set  $D$ ; we will denote it, as usual, by  $\mathcal{S} \subseteq D$ .
- Let  $c = (c_1, \dots, c_n)$  be a tuple of real numbers.
- For each distribution from  $D$ , let  $\Delta y$  denote  $\Delta y \stackrel{\text{def}}{=} c_1 \cdot \Delta x_1 + \dots + c_n \cdot \Delta x_n$ .

Then, by  $\sigma(c, s, \mathcal{S})$  we denote the largest possible value of the standard deviation  $\sigma_\rho[\Delta y]$  over all distributions from the set  $\mathcal{S}$ :

$$\sigma(c, s, \mathcal{S}) \stackrel{\text{def}}{=} \max_{\rho \in \mathcal{S}} \sigma_\rho[\Delta y].$$

In these terms, the above result takes the following form:

**Proposition 1.** For the set  $\mathcal{S} = D$  of all possible distributions, we have

$$\sigma(c, D, c) = \sum_{i=1}^n |c_i| \cdot \sigma_i.$$

Similarly, the previous result – about independent case – takes the following form.

**Definition 2.** By  $I \subset D$ , we denote the class of all distributions for which, for all  $i$  and  $j$ , the variables  $\Delta x_i$  and  $\Delta x_j$  are independent. We will call  $I$  independent set.

**Proposition 2.** For the independent set  $I$ , we have

$$\sigma(s, I, c) = \sqrt{\sum_{i=1}^n c_i^2 \cdot \sigma_i^2}.$$

*Comment.* Interestingly, the formula (5) is similar to what we get in the linearized version of the interval case (see, e.g., [1, 2, 3, 4, 5]), i.e., the case when we only know the upper bound  $\Delta x_i$  on the absolute value of the measurement error. In other words, this means that:

- the measurement error is located somewhere in the interval  $[-\Delta x_i, \Delta x_i]$ , and
- we have no information about the probability of different values from this interval.

In this case, the largest possible value of the estimation error

$$\Delta y = c_1 \cdot \Delta x_1 + \dots + c_n \cdot \Delta x_n$$

is equal to  $|c_1| \cdot \Delta_1 + \dots + |c_n| \cdot \Delta_n$ . This is indeed the same expression as our formula (5).

### 3 What If a Few Pairs of Measurement Errors Are Not Necessarily Independent

#### 3.1 Description of the situation

**Descriptions of the situation.** In the previous text, we considered two extreme cases:

- when we know that all measurement errors are independent, and
- when we have no information about their dependence.

Such cases are indeed frequent, but sometimes, situations are similar but not exactly the same. For example, we can have the case of “almost independence”, when for most pairs, we know that they are independent, but for a few pairs, we do not have this information. This is the situation that we describe in this section.

*Comment.* The opposite situations, when we only have independence information about a few pairs, is described in the next section.

**Graph representation of such situations.** To describe such situations, we need to know for which pairs of measurement errors, we do not have information about



independence. A natural way to represent such information is by an undirected graph in which:

- measurement errors are vertices and
- an edge connects pairs for which we do not have information about independence.

We only need to know which vertices are connected. So, it makes sense to include, in the description of the graph, only vertices that are connected by some edge, i.e., only measurement errors that may not be independent with respect to others. In this case, we arrive at the following definition.

**Definition 3.**

- Let  $G = (V, E)$  be an undirected graph with the set of vertices  $V \subseteq \{1, \dots, n\}$  for which every vertex is connected to some other vertex. Here,  $E$  is a subset  $E \subseteq V \times V$  for which:
  - for each  $a \in V$ , we have  $(a, a) \notin E$ ,
  - for each  $a$  and  $b$ ,  $(a, b) \in E$  if and only if  $(b, a) \in E$ , and
  - for each  $a \in V$ , we have  $(a, b) \in E$  for some  $b \in V$ .
- By  $I_G \subseteq D$ , we mean the class of all distributions for which for all pairs  $(i, j) \notin E$  the variables  $\Delta x_i$  and  $\Delta x_j$  are independent.

**Discussion.** Our objective is to find the value  $\sigma(s, I_G, c)$  for different graphs  $G$ . In this paper, we only consider the simplest graphs: all graphs with 2, 3, or 4 vertices. We hope that this work will be extended to larger-size graphs.

### 3.2 General results

Let us first present some general results. For this purpose, let us introduce the following notations. For any set  $S \subseteq \{1, \dots, n\}$ , by a restriction  $c_S$  we mean sub-tuples consisting only of elements  $c_i$  for which  $i \in S$ . For example, for  $c = (c_1, c_2, c_3, c_4)$  and  $S = \{1, 3\}$ , we have  $c_S = (c_1, c_3)$ . Similarly, we can define the restriction  $s_S$ . It is then relatively easy to show that the following result holds:

**Proposition 3.** For each graph  $G = (V, E)$ , we have:

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + (\sigma(s_V, I_G, c_V))^2.$$

*Comments.*

- In other words, it is sufficient to only consider measurement errors from the exception set  $V$  – which are not necessarily independent, then all other measurement errors can be treated the same way as in the case when all measurement errors are independent.

- For reader's conveniences, all the proofs are placed in a special Proofs section.

Another easy-to-analyze important case is when the graph  $G$  is disconnected, i.e., consists of several connected components.

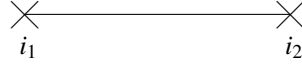
**Proposition 4.** *When the graph  $G = (V, E)$  consists of several connected components  $G_1 = (V_1, E_1), \dots, G_k = (V_k, E_k)$  with for which  $V = V_1 \cup \dots \cup V_k$ , then*

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + \sum_{j=1}^k (\sigma(s_{V_j}, \mathcal{S}_{G_j}, c_{V_j}))^2.$$

*Comment.* In view of this result, it is sufficient to estimate the value  $\sigma(s, I_G, c)$  for connected graphs. We consider connected graphs with 2, 3, or 4 vertices.

### 3.3 Connected graph of size 2

There is only one connected graph of size 2: two vertices  $i_1$  and  $i_2$  connected by an edge, so that  $V = \{i_1, i_2\}$  and  $E = \{(i_1, i_2), (i_2, i_1)\}$ .



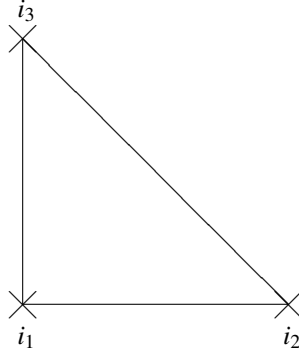
**Proposition 5.** *When the graph  $G = (V, E)$  consists of two vertices  $i_1$  and  $i_2$  connected by an edge, then*

$$\sigma^2(s, I_G, c) = \sum_{i \neq i_1, i_2} c_i^2 \cdot \sigma_i^2 + (|c_{i_1}| \cdot \sigma_{i_1} + |c_{i_2}| \cdot \sigma_{i_2})^2.$$

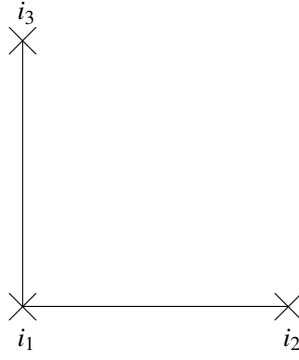
### 3.4 Connected graphs of size 3

In a connected graph of size 3, two vertices are connected, and the third vertex is:

- either connected to both of them – in this case we have a connected 3-element graph,



- or to only one of them.



So, modulo isomorphism, there are two different connected graphs of size 3. For these graphs, we get the following results:

**Proposition 6.** *When  $G = (V, E)$  is a complete 3-element graph with  $V = \{i_1, i_2, i_3\}$ , then*

$$\sigma^2(s, I_G, c) = \sum_{i \in V} c_i^2 \cdot \sigma_i^2 + (|c_{i_1}| \cdot \sigma_{i_1} + |c_{i_2}| \cdot \sigma_{i_2} + |c_{i_3}| \cdot \sigma_{i_3})^2.$$

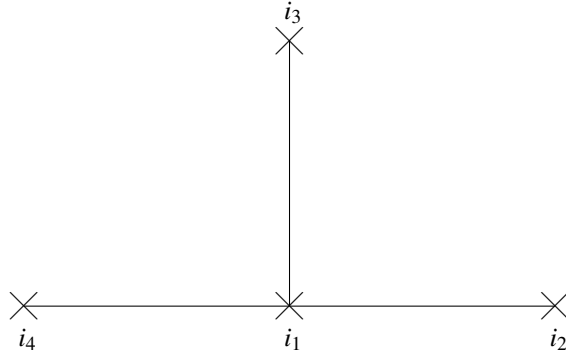
**Proposition 7.** *For a 3-element graph with  $V = \{i_1, i_2, i_3\}$  in which  $i_1$  is connected to  $i_2$  and  $i_3$  but  $i_2$  and  $i_3$  are not connected, we have:*

$$\sigma^2(s, I_G, c) = \sum_{i \in V} c_i^2 \cdot \sigma_i^2 + \left( |c_{i_1}| \cdot \sigma_{i_1} + \sqrt{c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2} \right)^2.$$

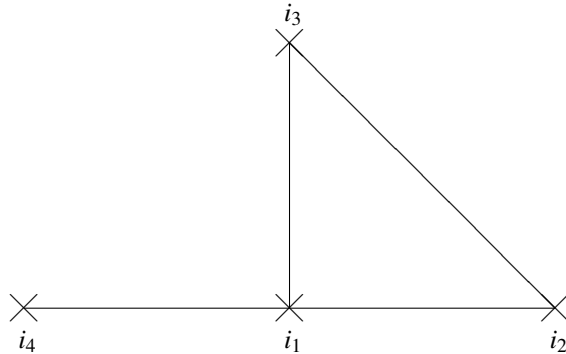
### 3.5 Connected graphs of size 4

Let us first consider graphs of size 4 for which there is a vertex (we will denote it  $i_1$ ) connected with all three other vertices. In this case, there are four possible options:

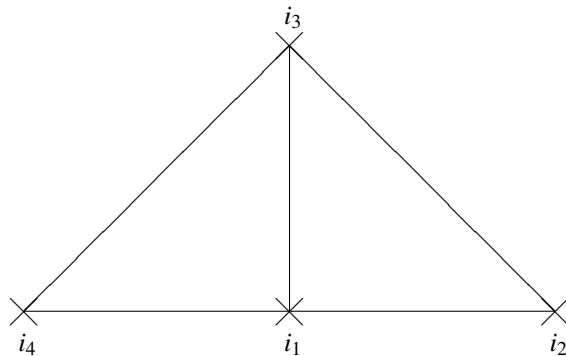
- when connections between  $i_1$  and all three other vertices are the only connections:



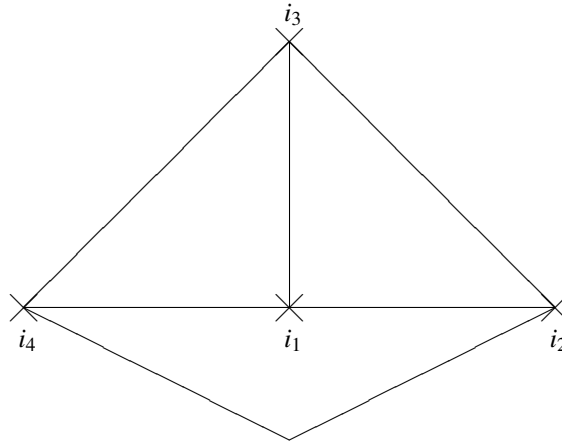
- when, in addition to edges connecting  $i_1$  to three other vertices, there is also one connection between two of these other vertices:



- when, in addition to edges connecting  $i_1$  to three other vertices, there are two connections between these other vertices:



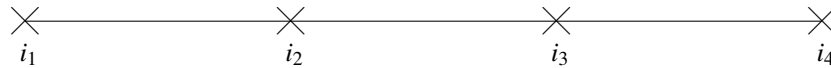
- and when we have a complete 4-element graph:



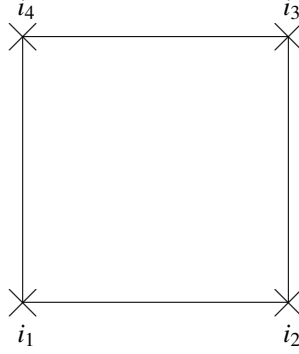
Finally, let us consider graphs in which each vertex is connected to no more than two others. If each vertex is connected to only one vertex, then a vertex  $i_1$  is connected to some vertex  $i_2$ , and there is no space for each of them to have any other connection – so the 4-element graph containing vertices  $i_1$  and  $i_2$  cannot be connected. Thus, there should be at least one vertex connected to two others.

Let us denote one such vertex by  $i_2$ , and the two vertices to which  $i_2$  is connected by  $i_1$  and  $i_3$ . Since the graph is connected, the fourth vertex  $i_4$  must be connected to one of the previous three vertices. The vertex  $i_4$  cannot be connected to  $i_2$  – because then  $i_2$  should be connected to three other vertices, and we consider the case when each vertex is connected to no more than two others. Thus,  $i_4$  is connected to  $i_1$  and/or  $i_3$ . If it is connected to  $i_1$ , then we can swap the names of vertices  $i_1$  and  $i_3$ , and get the same configuration as when  $i_4$  is connected to  $i_3$ . If  $i_4$  is connected to both  $i_1$  and  $i_3$ , then the resulting graph is uniquely determined. Thus, under the assumption that each vertex is connected to no more than two others, we have two possible graphs:

- a “linear” graph:



- and a “square graph”:



For all these graphs, we have the following results:

**Proposition 8.** For a 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , in which  $i_1$  is connected to  $i_2$ ,  $i_3$ , and  $i_4$ , but  $i_2$ ,  $i_3$ , and  $i_4$  are not connected to each other, we have:

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + \left( |c_{i_1}| \cdot \sigma_{i_1} + \sqrt{c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2} \right)^2.$$

**Proposition 9.** For a 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , in which  $i_1$ ,  $i_2$ , and  $i_3$  form a complete graph, and  $i_4$  is connected only to  $i_1$ , we have:

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + \left( |c_{i_1}| \cdot \sigma_{i_1} + \sqrt{(|c_{i_2}| \cdot \sigma_{i_2} + |c_{i_3}| \cdot \sigma_{i_3})^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2} \right)^2.$$

**Proposition 10.** For a 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , in which  $i_1$ ,  $i_2$ , and  $i_3$  form a complete graph, and  $i_4$  is connected to  $i_1$  and  $i_3$ , we have:

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + \left( |c_{i_1}| \cdot \sigma_{i_1} + |c_{i_3}| \cdot \sigma_{i_3} + \sqrt{c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2} \right)^2.$$

**Proposition 11.** For a complete 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , we have:

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + (|c_{i_1}| \cdot \sigma_{i_1} + |c_{i_2}| \cdot \sigma_{i_2} + |c_{i_3}| \cdot \sigma_{i_3} + |c_{i_4}| \cdot \sigma_{i_4})^2.$$

**Proposition 12.** For a linear 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , the value  $\sigma(s, I_G, c)$  has the following form:

- if  $|c_{i_2}| \cdot \sigma_{i_2} \cdot |c_{i_3}| \cdot \sigma_{i_3} > |c_{i_1}| \cdot \sigma_{i_1} \cdot |c_{i_4}| \cdot \sigma_{i_4}$ , then

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + \left( \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_2}^2 \cdot \sigma_{i_2}^2} + \sqrt{c_{i_3}^2 \cdot \sigma_{i_3}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2} \right)^2;$$

- otherwise, we get

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + \max(V_2, V_3, V_0), \text{ where}$$

$$V_2 = c_{i_3}^2 \cdot \sigma_{i_3}^2 + \left( \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_2}^2 \cdot \sigma_{i_2}^2} + |c_{i_4}| \cdot \sigma_{i_4} \right)^2,$$

$$V_3 = c_{i_2}^2 \cdot \sigma_{i_2}^2 + \left( |c_{i_1}| \cdot \sigma_{i_1} + \sqrt{c_{i_3}^2 \cdot \sigma_{i_3}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2} \right)^2, \text{ and}$$

$$V_0 = (|c_{i_1}| \cdot \sigma_{i_1} + |c_{i_3}| \cdot \sigma_{i_3})^2 + (|c_{i_2}| \cdot \sigma_{i_2} + |c_{i_4}| \cdot \sigma_{i_4})^2.$$

**Proposition 13.** For a square 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , we have:

$$\sigma^2(s, I_G, c) = \sum_{i \notin V} c_i^2 \cdot \sigma_i^2 + \left( \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2} + \sqrt{c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2} \right)^2.$$

## 4 What If Only a Few Pairs of Measurement Errors Are Known to Be Independent

### 4.1 Description of the situation

**Graph representation of such situations.** To describe such situations, we need to know for which pairs of measurement errors, we have information about independence. A natural way to represent such information is by an undirected graph in which:

- measurement errors are vertices and
- an edge connects pairs for which we have information about independence.

For simplicity, we can only consider vertices that are connected by some edge, i.e., only measurement errors that are known to be independent with respect to others. In this case, we arrive at the following definition.

**Definition 4.**

- Let  $G = (V, E)$  be an undirected graph with the set of vertices  $V \subseteq \{1, \dots, n\}$ . Here,  $E$  is a subset  $E \subseteq V \times V$  for which:
  - for each  $a \in V$ , we have  $(a, a) \notin E$ , and
  - for each  $a$  and  $b$ ,  $(a, b) \in E$  if and only if  $(b, a) \in E$ .

- By  $D_G \subseteq D$ , we mean the class of all distributions for which for all pairs  $(i, j) \in E$  the variables  $\Delta x_i$  and  $\Delta x_j$  are independent.

**Discussion.** Our objective is to find the value  $\sigma(s, D_G, c)$  for different graphs  $G$ . In this paper, we only consider the simplest graphs: all graphs with 2, 3, or 4 vertices. We hope that this work will be extended to larger-size graphs.

## 4.2 General results

Let us first present some general results. It is then relatively easy to show that the following result holds:

**Proposition 14.** For each graph  $G = (V, E)$ , we have:

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sigma(s_V, D_G, c_V).$$

*Comment.* In other words, it is sufficient to only consider measurement errors from the exception set  $V$  – which are not necessarily independent; then all other measurement errors can be treated the same way as in the case when we have no information about dependence.

Another easy-to-analyze important case is when the graph  $G$  is disconnected, consisting of several connected components.

**Proposition 15.** When the graph  $G = (V, E)$  consists several connected components  $G_1 = (V_1, E_1), \dots, G_k = (V_k, E_k)$  with for which  $V = V_1 \cup \dots \cup V_k$ , then

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sum_{j=1}^k \sigma(s_{V_j}, \mathcal{L}_{G_j}, c_{V_j}).$$

*Comment.* In view of this result, it is sufficient to estimate the value  $\sigma(s, D_G, c)$  for connected graphs. In this paper, we consider all connected graphs with 2, 3, or 4 vertices.

## 4.3 Connected graph of size 2

**Proposition 16.** When the graph  $G = (V, E)$  consists of two vertices  $i_1$  and  $i_2$  connected by an edge, then

$$\sigma^2(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_2}^2 \cdot \sigma_{i_2}^2}.$$



#### 4.4 Connected graphs of size 3

**Proposition 17.** *When  $G = (V, E)$  is a complete 3-element graph with  $V = \{i_1, i_2, i_3\}$ , then*

$$\sigma^2(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2}.$$

**Proposition 18.** *For a 3-element graph with  $V = \{i_1, i_2, i_3\}$ , in which  $i_1$  is connected to  $i_2$  and  $i_3$  but  $i_2$  and  $i_3$  are not connected, we have:*

$$\sigma^2(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + (|c_{i_2}| \cdot \sigma_{i_2} + |c_{i_3}| \cdot \sigma_{i_3})^2}.$$

#### 4.5 Connected graphs of size 4

**Proposition 19.** *For a 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , in which  $i_1$  is connected to  $i_2, i_3$ , and  $i_4$ , but  $i_2, i_3$ , and  $i_4$  are not connected to each other, we have:*

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + (|c_{i_2}| \cdot \sigma_{i_2} + |c_{i_3}| \cdot \sigma_{i_3} + |c_{i_4}| \cdot \sigma_{i_4})^2}.$$

**Proposition 20.** *For a 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , in which  $i_1, i_2$ , and  $i_3$  form a complete graph, and  $i_4$  is connected only to  $i_1$ , we have:*

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + \left( \sqrt{c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2} + |c_{i_4}| \cdot \sigma_{i_4} \right)^2}.$$

**Proposition 21.** *For a 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , in which  $i_1, i_2$ , and  $i_3$  form a complete graph, and  $i_4$  is connected to  $i_1$  and  $i_3$ , we have:*

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2 + (|c_{i_2}| \cdot \sigma_{i_2} + |c_{i_4}| \cdot \sigma_{i_4})^2}.$$

**Proposition 22.** *For a complete 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , we have:*

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2}.$$

**Proposition 23.** *For a linear 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ :*

- if  $|c_{i_1}| \cdot \sigma_{i_1} \cdot |c_{i_4}| \cdot \sigma_{i_4} > |c_{i_2}| \cdot \sigma_{i_2} \cdot |c_{i_3}| \cdot \sigma_{i_3}$ , then

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_2}^2 \cdot \sigma_{i_2}^2} + \sqrt{c_{i_3}^2 \cdot \sigma_{i_3}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2}.$$

- otherwise, we get

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{\max(\sigma_2^2, \sigma_3^2, \sigma_0^2)}, \text{ where}$$

$$\sigma_2^2 = v_{i_3}^2 + \left( \sqrt{v_{i_1}^2 + v_{i_2}^2} + v_{i_4} \right)^2,$$

$$\sigma_3^2 = v_{i_2}^2 + \left( v_{i_1} + \sqrt{v_{i_3}^2 + v_{i_4}^2} \right)^2, \text{ and}$$

$$\sigma_0^2 = (v_{i_1} + v_{i_3})^2 + (v_{i_2} + v_{i_4})^2.$$

**Proposition 24.** For a square 4-element graph with  $V = \{i_1, i_2, i_3, i_4\}$ , we have:

$$\sigma(s, D_G, c) = \sum_{i \notin V} |c_i| \cdot \sigma_i + \sqrt{(|c_{i_1}| \cdot \sigma_{i_1} + |c_{i_3}| \cdot \sigma_{i_3})^2 + (|c_{i_2}| \cdot \sigma_{i_2} + |c_{i_4}| \cdot \sigma_{i_4})^2}.$$

**Hypothesis.** On all these cases, the desired bounds can be obtained if we combine the values  $|c_i| \cdot \sigma_i$  by using either formulas corresponding to independence, or formulas corresponding to the possibility of dependence. For example, in the case covered by Proposition 24:

- we first combine  $|c_{i_1}| \cdot \sigma_{i_1}$  and  $|c_{i_3}| \cdot \sigma_{i_3}$  by using the formula corresponding to the possibility of dependence,
- then, we combine  $|c_{i_1}| \cdot \sigma_{i_1}$  and  $|c_{i_3}| \cdot \sigma_{i_3}$  by using the formula corresponding to the possibility of dependence,
- and finally, we combine the two previous results by using the formula corresponding to independence.

In some cases – e.g., in the case covered by Proposition 12 – we have subcases in which different formulas of this type should be used. In such situations, the selection of these subcases can be also described by inequalities relating such formulas. Indeed, in the case of Proposition 12, the inequality

$$|c_{i_2}| \cdot \sigma_{i_2} \cdot |c_{i_3}| \cdot \sigma_{i_3} > |c_{i_1}| \cdot \sigma_{i_1} \cdot |c_{i_4}| \cdot \sigma_{i_4}$$

describing which subcase to use is equivalent to

$$(|c_{i_2}| \cdot \sigma_{i_2} + |c_{i_3}| \cdot \sigma_{i_3})^2 + c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_4}^2 \cdot \sigma_{i_4}^2 >$$

$$c_{i_2}^2 \cdot \sigma_{i_2}^2 + c_{i_3}^2 \cdot \sigma_{i_3}^2 + (|c_{i_1}^2| \cdot \sigma_{i_1} + |c_{i_4}| \cdot \sigma_{i_4})^2.$$

It is natural to conjecture that formulas corresponding to general graphs will have similar structure.

## 5 Proofs

**Proof of Proposition 3.** The proof of this proposition naturally follows from the geometric interpretation, in which we associate a vector to each random variable, and we are looking for a configuration in which the sum of these vectors has the largest length. Here, the sum  $\mathbf{v}$  of all the corresponding vectors  $\mathbf{v} = \mathbf{v}_1 + \dots + \mathbf{v}_n$  can be represented as

$$\sum_{i \notin V} \mathbf{v}_i + \mathbf{a},$$

where

$$\mathbf{a} \stackrel{\text{def}}{=} \sum_{j \in V} \mathbf{v}_j.$$

Vectors  $\mathbf{v}_i$  corresponding to “normal” errors ( $i \notin V$ ) are orthogonal (since the corresponding measurement errors are independent), and since each of them is orthogonal to each of the “abnormal” vectors  $\mathbf{v}_j$ , it is also orthogonal to the sum  $\mathbf{a}$  of these abnormal vectors. Thus, the square of the length of the sum  $\mathbf{v}$  is equal to the sum of the squares of the lengths of the “normal” vectors  $\mathbf{v}_i$  and of the vector  $\mathbf{a}$ :

$$\|\mathbf{v}\|^2 = \sum_{i \notin V} \|\mathbf{v}_i\|^2 + \|\mathbf{a}\|^2.$$

The values  $\|\mathbf{v}_i\|^2$  are given: they are equal to  $c_i^2 \cdot \sigma_i^2$ . Thus, the largest possible value of  $\|\mathbf{v}\|^2$  is attained when the length  $\|\mathbf{a}\|^2$  is the largest. This largest length is what in Definitions 1 and 3 we denoted by  $\sigma(s_V, I_G, c_V)$ . Thus, we get the desired formula.

The proposition is proven.

**Proof of Proposition 5.** For this graph, the value  $\sigma(s, I_G, c)$  follows from Proposition 1 – it is equal to  $|c_{i_1}| \cdot \sigma_{i_1} + |c_{i_2}| \cdot \sigma_{i_2}$ . Thus, by Proposition 3, we get the desired result.

**Proof of Propositions 6** is similar to the proof of Proposition 5.

**Proof of Proposition 7.** Since the vertices  $i_2$  and  $i_3$  are not connected, this means that the measurement errors corresponding to these vertices are independent, so the length of  $\mathbf{v}_{i_2} + \mathbf{v}_{i_3}$  is equal to  $\sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_3}\|^2}$ .

The vertex  $i_1$  is connected to both  $i_2$  and  $i_3$ , which means that we know nothing about the dependence between the corresponding measurement errors. Thus, as we have described earlier, the largest possible length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} = \mathbf{v}_{i_1} + (\mathbf{v}_{i_2} + \mathbf{v}_{i_3})$$

can be obtained by adding the lengths of  $\mathbf{v}_{i_1}$  and of  $\mathbf{v}_{i_2} + \mathbf{v}_{i_3}$ :

$$\|\mathbf{v}_{i_1}\| + \sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_3}\|^2}.$$

The desired result now follows from Proposition 3.

**Proof of Proposition 8** is similar to the proof of Proposition 7.

**Proof of Proposition 9.** We have no restriction on vectors  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_3}$ , so the largest possible length of their sum  $\mathbf{v}_{i_2} + \mathbf{v}_{i_3}$  is the sum of their lengths:  $\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_3}\|$ . There is no edge between  $i_4$  and the group of vertices  $(i_2, i_3)$ , this means that the measurement errors corresponding to  $i_4$  are independent from the measurement errors  $\Delta x_{i_2}$  and  $\Delta x_{i_3}$ . Thus, the vector  $\mathbf{v}_{i_4}$  is orthogonal to vectors  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_3}$  and is, thus, orthogonal to their sum  $\mathbf{v}_{i_2} + \mathbf{v}_{i_3}$ . So, the maximum length of the sum

$$\mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = (\mathbf{v}_{i_2} + \mathbf{v}_{i_3}) + \mathbf{v}_{i_4}$$

is equal to the square root of the sums of their lengths:

$$\sqrt{(\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_3}\|)^2 + \|\mathbf{v}_{i_4}\|^2}.$$

Since  $i_1$  is connected to all the three other vertices, this means that there is no restriction on the relation between the vector  $i_1$  and three other vectors. So, the maximum length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = \mathbf{v}_{i_1} + (\mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4}).$$

Thus, the maximum length of this sum is equal to the sum of the lengths:

$$\|\mathbf{v}_{i_1}\| + \sqrt{(\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_3}\|)^2 + \|\mathbf{v}_{i_4}\|^2}.$$

The use of Proposition 3 completes the proof.

**Proof of Proposition 10.** Vectors  $i_2$  and  $i_4$  are independent, so the length of the sum  $\mathbf{v}_{i_2} + \mathbf{v}_{i_4}$  is equal to  $\sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_4}\|^2}$ . Now, there are no restrictions on the relation between  $\mathbf{v}_{i_1}$ ,  $\mathbf{v}_{i_3}$ , and  $\mathbf{v}_{i_2} + \mathbf{v}_{i_4}$ . Thus, the maximum length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = \mathbf{v}_{i_1} + \mathbf{v}_{i_3} + (\mathbf{v}_{i_2} + \mathbf{v}_{i_4})$$

is equal to the sum of the lengths:

$$\|\mathbf{v}_{i_1}\| + \|\mathbf{v}_{i_3}\| + \sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_4}\|^2}.$$

The use of Proposition 3 completes the proof.

**Proof of Proposition 11** is similar to the proofs of Propositions 5 and 6.

**Proof of Proposition 12.** The given graph means that between the four vertices, the only independent pairs are those which are not connected by an edge, i.e., pairs

$(i_3, i_1)$ ,  $(i_1, i_4)$ , and  $(i_4, i_2)$ . One can easily see that these vertices also form a linear graph. For this case, the largest value of the sum of the four vectors is computed in the proof of Proposition 23. The use of Proposition 3 completes the proof.

**Proof of Proposition 13.** Since the vertices  $i_1$  and  $i_3$  are not connected, this means that the measurement errors corresponding to these vertices are independent, so the length of  $\mathbf{v}_{i_1} + \mathbf{v}_{i_3}$  is equal to  $\sqrt{\|\mathbf{v}_{i_1}\|^2 + \|\mathbf{v}_{i_3}\|^2}$ . Similarly, since the vertices  $i_2$  and  $i_4$  are not connected, this means that the measurement errors corresponding to these vertices are independent, so the length of  $\mathbf{v}_{i_2} + \mathbf{v}_{i_4}$  is equal to  $\sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_4}\|^2}$ .

Each of the vertices  $i_1$  and  $i_3$  is connected to both  $i_2$  and  $i_4$ , which means that we know nothing about the dependence between the corresponding measurement errors. Thus, as we have described earlier, the largest possible length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = (\mathbf{v}_{i_1} + \mathbf{v}_{i_3}) + (\mathbf{v}_{i_2} + \mathbf{v}_{i_4})$$

can be obtained by adding the lengths of  $\mathbf{v}_{i_1} + \mathbf{v}_{i_3}$  and of  $\mathbf{v}_{i_2} + \mathbf{v}_{i_4}$ :

$$\sqrt{\|\mathbf{v}_{i_1}\|^2 + \|\mathbf{v}_{i_3}\|^2} + \sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_4}\|^2}.$$

The desired result now follows from Proposition 3.

**Proof of Proposition 14.** The proof of this proposition naturally follows from the geometric interpretation, in which we associate a vector to each random variable, and we are looking for a configuration in which the sum of these vectors has the largest length. Here, the sum  $\mathbf{v}$  of all the corresponding vectors  $\mathbf{v} = \mathbf{v}_1 + \dots + \mathbf{v}_n$  can be represented as

$$\sum_{i \notin V} \mathbf{v}_i + \mathbf{a},$$

where

$$\mathbf{a} \stackrel{\text{def}}{=} \sum_{j \in V} \mathbf{v}_j.$$

We do not have any restrictions on the relative orientation of the vectors  $\mathbf{v}_i$  corresponding to “normal” errors ( $i \notin V$ ) and of the vector  $\mathbf{a}$ . Thus, the largest possible value of the length of the sum  $\mathbf{v}$  is equal to the sum of the lengths of the “normal” vectors  $\mathbf{v}_i$  and of the vector  $\mathbf{a}$ :

$$\max \|\mathbf{v}\| = \sum_{i \notin V} \|\mathbf{v}_i\| + \|\mathbf{a}\|.$$

The values  $\|\mathbf{v}_i\|$  are given: they are equal to  $|c_i| \cdot \sigma_i$ . Thus, the largest possible value of  $\|\mathbf{v}\|$  is attained when the length  $\|\mathbf{a}\|$  is the largest. This largest length is what in Definitions 1 and 4 we denoted by  $\sigma(s_V, D_G, c_V)$ . Thus, we get the desired formula.

The proposition is proven.

**Proof of Proposition 16.** For this graph, the value  $\sigma(s, D_G, c)$  follows from Proposition 2 – it is equal to  $\sqrt{c_{i_1}^2 \cdot \sigma_{i_1}^2 + c_{i_2}^2 \cdot \sigma_{i_2}^2}$ . Thus, by Proposition 12, we get the desired result.

**Proof of Proposition 17** is similar to the proof of Proposition 16.

**Proof of Proposition 18.** Since the vertices  $i_2$  and  $i_3$  are not connected, this means that we do not have any restrictions on the relative location of the vectors  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_3}$ , so the largest possible value of the length of the sum  $\mathbf{v}_{i_2} + \mathbf{v}_{i_3}$  is equal to the sum of the lengths  $\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_3}\|$ .

The vertex  $i_1$  is connected to both  $i_2$  and  $i_3$ , which means that the measurement error corresponding to  $i_1$  is independent of the errors corresponding to  $i_2$  and  $i_3$ . Thus, as we have described earlier, the largest possible length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} = \mathbf{v}_{i_1} + (\mathbf{v}_{i_2} + \mathbf{v}_{i_3})$$

is equal to

$$\sqrt{\|\mathbf{v}_{i_1}\|^2 + (\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_3}\|)^2}.$$

The desired result now follows from Proposition 14.

**Proof of Proposition 19** is similar to the proof of Proposition 18.

**Proof of Proposition 20.** The vectors  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_3}$  are independent, so the length of their sum  $\mathbf{v}_{i_2} + \mathbf{v}_{i_3}$  is equal to  $\sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_3}\|^2}$ . There is no edge between  $i_4$  and the group of vertices  $(i_2, i_3)$ , this means there is no restriction on the relation between  $\mathbf{v}_{i_4}$  and  $\mathbf{v}_{i_2} + \mathbf{v}_{i_3}$ . Thus, the largest possible length of the sum

$$\mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = (\mathbf{v}_{i_2} + \mathbf{v}_{i_3}) + \mathbf{v}_{i_4}$$

is equal to the sum of their lengths:

$$\sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_3}\|^2} + \|\mathbf{v}_{i_4}\|.$$

Since  $i_1$  is connected to all the three other vertices, this means that this vector is orthogonal to three other vectors – and thus, to their sum. So, the maximum length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = \mathbf{v}_{i_1} + (\mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4}).$$

is equal to

$$\sqrt{\|\mathbf{v}_{i_1}\|^2 + \left(\sqrt{\|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_3}\|^2} + \|\mathbf{v}_{i_4}\|\right)^2}.$$

The use of Proposition 11 completes the proof.

**Proof of Proposition 21.** There is no constraint on the vectors  $i_2$  and  $i_4$ , so the maximum length of the sum  $\mathbf{v}_{i_2} + \mathbf{v}_{i_4}$  is equal to the sum of their length:  $\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_4}\|$ . Now, the three vectors  $\mathbf{v}_{i_1}$ ,  $\mathbf{v}_{i_3}$ , and  $\mathbf{v}_{i_2} + \mathbf{v}_{i_4}$  are independent. Thus, the maximum length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = \mathbf{v}_{i_1} + \mathbf{v}_{i_3} + (\mathbf{v}_{i_2} + \mathbf{v}_{i_4})$$

is equal to:

$$\sqrt{\|\mathbf{v}_{i_1}\|^2 + \|\mathbf{v}_{i_3}\|^2 + (\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_4}\|)^2}.$$

The use of Proposition 11 completes the proof.

**Proof of Proposition 22** is similar to the proof of Propositions 16 and 17.

**Proof of Proposition 23.** In accordance with Proposition 14, we need to compute  $\sigma \stackrel{\text{def}}{=} \sigma(s_V, D_G, c_V)$ , the largest possible length of the vector  $\mathbf{v} = \mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4}$ . In general, the square  $\|\mathbf{v}\|^2$  of the length  $\|\mathbf{v}\|$  of the sum  $\mathbf{v} = \mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4}$  of the four vectors is equal to

$$\begin{aligned} \|\mathbf{v}\|^2 &= \|\mathbf{v}_{i_1}\|^2 + \|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_3}\|^2 + \|\mathbf{v}_{i_4}\|^2 + \\ &2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_2} + 2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_3} + 2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_4} + 2\mathbf{v}_{i_2} \cdot \mathbf{v}_{i_3} + 2\mathbf{v}_{i_2} \cdot \mathbf{v}_{i_4} + 2\mathbf{v}_{i_3} \cdot \mathbf{v}_{i_4}. \end{aligned}$$

For the situation described by the given graph, vector  $\mathbf{v}_{i_1}$  is orthogonal to  $\mathbf{v}_{i_2}$ , the vector  $\mathbf{v}_{i_2}$  is orthogonal to  $\mathbf{v}_{i_3}$ , and the vector  $\mathbf{v}_{i_3}$  is orthogonal to  $\mathbf{v}_{i_4}$ . Thus, we have

$$\|\mathbf{v}\|^2 = \|\mathbf{v}_{i_1}\|^2 + \|\mathbf{v}_{i_2}\|^2 + \|\mathbf{v}_{i_3}\|^2 + \|\mathbf{v}_{i_4}\|^2 + 2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_3} + 2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_4} + 2\mathbf{v}_{i_2} \cdot \mathbf{v}_{i_4}.$$

The length of each vector  $\|\mathbf{v}_{i_j}\|$  is fixed  $\|\mathbf{v}_{i_j}\|^2 = v_{i_j}^2$ , so to maximize the length of the sum, we need to maximize the sum of the remaining terms:

$$2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_3} + 2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_4} + 2\mathbf{v}_{i_2} \cdot \mathbf{v}_{i_4}.$$

Let us denote the half of this sum by  $J$ , then the sum itself becomes equal to  $2J$ .

We need to maximize the sum  $2J$  under the constraints

$$\|\mathbf{v}_{i_j}\|^2 = v_{i_j}^2 \text{ for all } j, \mathbf{v}_{i_1} \cdot \mathbf{v}_{i_2} = 0, \mathbf{v}_{i_2} \cdot \mathbf{v}_{i_3} = 0, \text{ and } \mathbf{v}_{i_3} \cdot \mathbf{v}_{i_4} = 0.$$

By using the Lagrange multiplier method, we can reduce the above-described conditional optimization problem to the following unconstrained optimization problem:

$$2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_3} + 2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_4} + 2\mathbf{v}_{i_2} \cdot \mathbf{v}_{i_4} + \sum_{j=1}^4 \lambda_j \cdot \|\mathbf{v}_{i_j}\|^2 + \sum_{j=1}^3 \mu_j \cdot \mathbf{v}_{i_j} \cdot \mathbf{v}_{i_{j+1}},$$

where  $\lambda_j$  and  $\mu_j$  are Lagrange multipliers.

Differentiating this expression with respect to  $\mathbf{v}_{i_2}$  and equating the derivative to 0, we conclude that

$$2\mathbf{v}_{i_4} + 2\lambda_2 \cdot \mathbf{v}_{i_2} + \mu_1 \cdot \mathbf{v}_{i_1} + \mu_2 \cdot \mathbf{v}_{i_3} = 0,$$

hence

$$\mathbf{v}_{i_4} = -\lambda_2 \cdot \mathbf{v}_{i_2} - \frac{1}{2} \cdot \mu_1 \cdot \mathbf{v}_{i_1} - \frac{1}{2} \cdot \mu_2 \cdot \mathbf{v}_{i_3}.$$

So, the vector  $\mathbf{v}_{i_4}$  belongs to the linear space generated by vectors  $\mathbf{v}_{i_2}$ ,  $\mathbf{v}_{i_3}$ , and  $\mathbf{v}_{i_1}$ . Let us denote the unit vectors in the directions of  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_3}$  by, correspondingly,

$$\mathbf{e}_2 = \frac{\mathbf{v}_{i_2}}{v_{i_2}} \text{ and } \mathbf{e}_3 = \frac{\mathbf{v}_{i_3}}{v_{i_3}}.$$

Since the vectors  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_3}$  are orthogonal, the unit vectors  $\mathbf{e}_2$  and  $\mathbf{e}_3$  are orthogonal too, so they can be viewed as two vectors from the orthonormal basis in the linear space generated by the vectors  $\mathbf{v}_{i_2}$ ,  $\mathbf{v}_{i_3}$ , and  $\mathbf{v}_{i_1}$ .

- If this linear space is 3-dimensional, in this 3-D space we can select the third unit vector  $\mathbf{e}$  which is orthogonal to both  $\mathbf{e}_2$  and  $\mathbf{e}_3$ .
- If the above linear space is 2-dimensional – i.e., if  $\mathbf{v}_{i_1}$  lies in the 2-D space generated by  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_3}$  – then let us take, as  $\mathbf{e}$ , any unit vector which is orthogonal to both  $\mathbf{e}_2$  and  $\mathbf{e}_3$ .

In both cases, vectors  $\mathbf{v}_{i_1}$  and  $\mathbf{v}_{i_4}$  belong to the linear space generated by the vectors  $\mathbf{e}_2$ ,  $\mathbf{e}_3$ , and  $\mathbf{e}$ .

In particular, this means that  $\mathbf{v}_{i_1} = c_{12} \cdot \mathbf{e}_2 + c_{13} \cdot \mathbf{e}_3 + c_1 \cdot \mathbf{e}$  for some numbers  $c_{12}$ ,  $c_{13}$ , and  $c_1$ . Since  $\mathbf{v}_{i_1} \perp \mathbf{v}_{i_2}$ , we have  $c_{12} = 0$ , so  $\mathbf{v}_{i_1} = c_{13} \cdot \mathbf{e}_3 + c_1 \cdot \mathbf{e}$ . From this formula, we conclude that  $v_{i_1}^2 = \|\mathbf{v}_{i_1}\|^2 = c_{13}^2 + c_1^2$ , so  $c_1^2 \leq v_{i_1}^2$ . Let us denote the ratio  $c_1/v_{i_1}$  by  $\beta_1$ , then  $c_1 = v_{i_1} \cdot \beta_1$  and, correspondingly,  $c_{13} = v_{i_1} \cdot \sqrt{1 - \beta_1^2}$ . So, the expression for  $\mathbf{v}_{i_1}$  takes the form

$$\mathbf{v}_{i_1} = v_{i_1} \cdot \sqrt{1 - \beta_1^2} \cdot \mathbf{e}_3 + v_{i_1} \cdot \beta_1 \cdot \mathbf{e}.$$

Similarly, we can conclude that

$$\mathbf{v}_{i_4} = v_{i_4} \cdot \sqrt{1 - \beta_4^2} \cdot \mathbf{e}_2 + v_{i_4} \cdot \beta_4 \cdot \mathbf{e},$$

for some value  $\beta_4$  for which  $|\beta_4| \leq 1$ . For each pair of orthogonal vectors  $\mathbf{e}_2$  and  $\mathbf{v}_{i_3}$  of lengths  $v_{i_2}$  and  $v_{i_3}$ , the above-defined vectors satisfy all the constraints. So, what remains is to find the values  $\beta_1$  and  $\beta_4$  for which the expression

$$2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_3} + 2\mathbf{v}_{i_1} \cdot \mathbf{v}_{i_4} + 2\mathbf{v}_{i_2} \cdot \mathbf{v}_{i_4}$$

attains its largest value – i.e., equivalently, for which the above-defined half-of-the-maximized-expression

$$J = \mathbf{v}_{i_1} \cdot \mathbf{v}_{i_3} + \mathbf{v}_{i_1} \cdot \mathbf{v}_{i_4} + \mathbf{v}_{i_2} \cdot \mathbf{v}_{i_4}$$

attains its largest value. Substituting the above expressions for  $\mathbf{v}_{i_1}$  and  $\mathbf{v}_{i_4}$  into this formula, and taking into account that, by our choice of  $\mathbf{e}_2$  and  $\mathbf{e}_3$ , we have  $\mathbf{v}_{i_2} = v_{i_2} \cdot \mathbf{e}_2$  and  $\mathbf{v}_{i_3} = v_{i_3} \cdot \mathbf{e}_3$ , we conclude that

$$J = v_{i_1} \cdot v_{i_3} \cdot \sqrt{1 - \beta_1^2} + v_{i_2} \cdot v_{i_4} \cdot \sqrt{1 - \beta_4^2} + \beta_1 \cdot \beta_4 \cdot v_{i_1} \cdot v_{i_4}.$$

Each of the unknown  $\beta_1$  and  $\beta_4$  has values from the interval  $[-1, 1]$ . Thus, for each of the variables  $\beta_1$  and  $\beta_4$ , the maximum of this expression is attained:



- either at one of the endpoints  $-1$  and  $1$  of this interval,
- or at the point inside this interval, in which case the derivative with respect to this variable should be equal to  $0$ .

We have two cases for each of the two variables  $\beta_1$  and  $\beta_4$ , so overall, we need to consider all  $2 \cdot 2 = 4$  cases. To find the largest possible value of the expression  $J$ , we need to consider all four possible cases, and find the largest of the corresponding values. Let us consider these cases one by one.

*Case 1.* If both values  $\beta_1$  and  $\beta_4$  are equal to  $\pm 1$ , then we get  $J = \pm v_{i_1} \cdot v_{i_4}$ . The largest of these values is when the sign is positive, then the value of the quantity  $J$  is equal to  $J_1 = v_{i_1} \cdot v_{i_4}$ .

*Case 2.* Let us now consider the case when  $\beta_1 = \pm 1$  and  $\beta_4 \in (-1, 1)$ . In this case, the expression  $J$  takes the form  $J = v_{i_2} \cdot v_{i_4} \cdot \sqrt{1 - \beta_4^2} \pm v_{i_1} \cdot v_{i_4} \cdot \beta_4$ . Differentiating this expression with respect to  $\beta_4$  and equating the derivative to  $0$ , we get

$$-\frac{2\beta_4 \cdot v_{i_2} \cdot v_{i_4}}{2\sqrt{1 - \beta_4^2}} \pm v_{i_1} \cdot v_{i_4} \cdot \beta_1 = 0.$$

If we divide both sides by  $v_{i_4}$ , divide both the numerator and the denominator of the fraction by a common factor  $2$ , and multiply both sides by the denominator, we get

$$\beta_4 \cdot v_{i_2} = \pm \sqrt{1 - \beta_4^2} \cdot v_{i_1}.$$

If we square both sides, we get

$$\beta_4^2 \cdot v_{i_2}^2 = (1 - \beta_4^2) \cdot v_{i_1}^2 = v_{i_1}^2 - \beta_4^2 \cdot v_{i_1}^2.$$

So

$$\beta_4^2 \cdot (v_{i_1}^2 + v_{i_2}^2) = v_{i_2}^2$$

and

$$\beta_4^2 = \frac{v_{i_2}^2}{v_{i_1}^2 + v_{i_2}^2}.$$

Therefore,

$$1 - \beta_4^2 = \frac{v_{i_1}^2}{v_{i_1}^2 + v_{i_2}^2},$$

so

$$\beta_4 = \pm \frac{v_{i_2}}{\sqrt{v_{i_1}^2 + v_{i_2}^2}}$$

and

$$\sqrt{1 - \beta_4^2} = \pm \frac{v_{i_1}}{\sqrt{v_{i_1}^2 + v_{i_2}^2}}.$$

Substituting these expressions into the formula for  $J$ , we conclude that

$$J = \pm \frac{v_{i_2}^2 \cdot v_{i_4}}{\sqrt{v_{i_1}^2 + v_{i_2}^2}} \pm \frac{v_{i_1}^2 \cdot v_{i_4}}{\sqrt{v_{i_1}^2 + v_{i_2}^2}}.$$

The largest value of this expression is attained when both signs are positive, so we get

$$J = \frac{v_{i_2}^2 \cdot v_{i_4}}{\sqrt{v_{i_1}^2 + v_{i_2}^2}} + \frac{v_{i_1}^2 \cdot v_{i_4}}{\sqrt{v_{i_1}^2 + v_{i_2}^2}} = \frac{v_{i_4} \cdot (v_{i_1}^2 + v_{i_2}^2)}{\sqrt{v_{i_1}^2 + v_{i_2}^2}}$$

and thus, the value  $J$  is equal to

$$J_2 = v_{i_4} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}.$$

In this case, the largest value of  $\|\mathbf{v}\|^2$  is equal to:

$$\begin{aligned} \sigma_2^2 &= v_{i_1}^2 + v_{i_2}^2 + v_{i_3}^2 + v_{i_4}^2 + 2v_{i_4} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2} = \\ &= v_{i_3}^2 + \left( (v_{i_1}^2 + v_{i_2}^2) + v_{i_4}^2 + 2v_{i_4} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2} \right) = \\ &= v_{i_3}^2 + \left( \sqrt{v_{i_1}^2 + v_{i_2}^2} + v_{i_4} \right)^2. \end{aligned}$$

*Comparing Case 1 and Case 2.* Since  $v_{i_1}^2 + v_{i_2}^2 > v_{i_1}^2$ , we have  $\sqrt{v_{i_1}^2 + v_{i_2}^2} > v_{i_1}$ , hence  $J_2 = v_{i_4} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2} > v_{i_1} \cdot v_{i_4} = J_1$ . Thus, when we are looking for the largest value of the expression  $J$ , we can safely ignore Case 1, since the values obtained in Case 2 can be larger than anything we get in Case 1.

*Case 3.* Similarly, we can consider the case when  $\beta_4 = \pm 1$  and  $\beta_1 \in (-1, 1)$ . In this case, we get the largest possible value of  $J$  equal to  $J_3 = v_{i_1} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2}$ , so the largest possible value of  $\sigma^2$  is equal to:

$$\sigma_3^2 = v_{i_2}^2 + \left( v_{i_1} + \sqrt{v_{i_3}^2 + v_{i_4}^2} \right)^2.$$

*Case 4.* Finally, let us consider the case when for the pair  $(\beta_1, \beta_4)$  at which the expression  $J$  attains its largest value, both values  $\beta_1$  and  $\beta_4$  are located inside the interval  $(-1, 1)$ . In this case, to find the maximum of the expression  $J$ , we need to differentiate it with respect to the unknowns  $\beta_1$  and  $\beta_4$  and equate the resulting derivatives to 0. If we differentiate by  $\beta_1$ , we get

$$-\frac{2\beta_1 \cdot v_{i_1} \cdot v_{i_3}}{2\sqrt{1-\beta_1^2}} + v_{i_1} \cdot v_{i_4} \cdot \beta_4 = 0.$$

Thus,

$$\beta_4 = \frac{\beta_1 \cdot v_{i_3}}{\sqrt{1-\beta_1^2} \cdot v_{i_4}},$$

and

$$\beta_4^2 = \frac{\beta_1^2 \cdot v_{i_3}^2}{(1-\beta_1^2) \cdot v_{i_4}^2}.$$

Differentiating the above expression for  $J$  with respect to  $\beta_4$  and equating the derivative to 0, we conclude that

$$-\frac{2\beta_4 \cdot v_{i_2} \cdot v_{i_4}}{2\sqrt{1-\beta_4^2}} + v_{i_1} \cdot v_{i_4} \cdot \beta_1 = 0.$$

If we divide both sides by  $v_{i_4}$ , divide both the numerator and the denominator of the fraction by a common factor 2, and multiply both sides by the denominator, we get

$$\beta_4 \cdot v_{i_2} = \sqrt{1-\beta_4^2} \cdot v_{i_1} \cdot \beta_1.$$

If we square both sides, we get

$$\beta_4^2 \cdot v_{i_2}^2 = (1-\beta_4^2) \cdot v_{i_1}^2 \cdot \beta_1^2 = v_{i_1}^2 \cdot \beta_1^2 - \beta_4^2 \cdot v_{i_1}^2 \cdot \beta_1^2.$$

Substituting the above expression for  $\beta_4^2$  into this formula, we get

$$\frac{\beta_1^2 \cdot v_{i_2}^2 \cdot v_{i_2}^2}{\sqrt{1-\beta_1^2} \cdot v_{i_4}^2} = \beta_1^2 \cdot v_{i_1}^2 - \frac{\beta_1^4 \cdot v_{i_1}^2 \cdot v_{i_3}^2}{\sqrt{1-\beta_1^2} \cdot v_{i_4}^2}.$$

*Case 4, subcase when  $\beta_1 = 0$ .* Both sides of this equality contain the common factor  $\beta_1$ . So, it is possible that  $\beta_1 = 0$ , in which case  $\beta_4 = 0$ , and the expression  $J$  attains the value

$$J_0 = v_{i_1} \cdot v_{i_3} + v_{i_2} \cdot v_{i_4}.$$

In this case, the value of  $\sigma^2$  is equal to:

$$\begin{aligned} \sigma_0^2 &= v_{i_1}^2 + v_{i_2}^2 + v_{i_3}^2 + v_{i_4}^2 + 2v_{i_1} \cdot v_{i_3} + 2v_{i_2} \cdot v_{i_4} = \\ &= (v_{i_1}^2 + v_{i_3}^2 + 2v_{i_1} \cdot v_{i_3}) + (v_{i_2}^2 + v_{i_4}^2 + 2v_{i_2} \cdot v_{i_4}) = \\ &= (v_{i_1} + v_{i_3})^2 + (v_{i_2} + v_{i_4})^2. \end{aligned}$$

Case 4, subcase when  $\beta_1 \neq 0$ . If  $\beta_1 \neq 0$ , then we can divide both sides of the above equality by  $\beta_1^2$ . Multiplying both sides by the denominator, we get

$$v_{i_2}^2 \cdot v_{i_3}^2 = v_{i_1}^2 \cdot v_{i_4}^2 \cdot (1 - \beta_1^2) - \beta_1^2 \cdot v_{i_1}^2 \cdot v_{i_3}^2,$$

so

$$v_{i_2}^2 \cdot v_{i_3}^2 = v_{i_1}^2 \cdot v_{i_4}^2 - \beta_1^2 \cdot v_{i_1}^2 \cdot v_{i_4}^2 - \beta_1^2 \cdot v_{i_1}^2 \cdot v_{i_3}^2.$$

If we move all the terms containing  $\beta_1^2$  to the left-hand side and all the other terms to the right-hand side, we get:

$$\beta_1 v_{i_1}^2 \cdot (v_{i_3}^2 + v_{i_4}^2) = v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_1}^2 \cdot v_{i_3}^2,$$

thus

$$\beta_1^2 = \frac{v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_1}^2 \cdot v_{i_3}^2}{v_{i_1}^2 \cdot (v_{i_3}^2 + v_{i_4}^2)}.$$

Here:

- when  $v_{i_1} \cdot v_{i_4} < v_{i_2} \cdot v_{i_3}$ , the right-hand side is negative, so we cannot have such a case;
- when  $v_{i_1} \cdot v_{i_4} = v_{i_2} \cdot v_{i_3}$ , then  $\beta_1 = 0$ , and we have already analyzed this case.

So, the only possibility to have  $\beta_1 \neq 0$  is when  $v_{i_1} \cdot v_{i_4} > v_{i_2} \cdot v_{i_3}$ .

In general, the situation does not change if we swap 1 and 4 and swap 2 and 3. Thus, for  $\beta_4^2$ , we get a similar expression

$$\beta_4^2 = \frac{v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_2}^2 \cdot v_{i_3}^2}{v_{i_4}^2 \cdot (v_{i_1}^2 + v_{i_2}^2)}.$$

From the expressions for  $\beta_1^2$  and  $\beta_4^2$ , we conclude that

$$\begin{aligned} 1 - \beta_1^2 &= 1 - \frac{v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_2}^2 \cdot v_{i_3}^2}{v_{i_1}^2 \cdot (v_{i_3}^2 + v_{i_4}^2)} = \\ &= \frac{v_{i_1}^2 \cdot v_{i_3}^2 + v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_1}^2 \cdot v_{i_4}^2 + v_{i_2}^2 \cdot v_{i_3}^2}{v_{i_1}^2 \cdot (v_{i_3}^2 + v_{i_4}^2)} = \\ &= \frac{v_{i_3}^2 \cdot (v_{i_1}^2 + v_{i_2}^2)}{v_{i_1}^2 \cdot (v_{i_3}^2 + v_{i_4}^2)}. \end{aligned}$$

Similarly, we have

$$1 - \beta_4^2 = \frac{v_{i_2}^2 \cdot (v_{i_3}^2 + v_{i_4}^2)}{v_{i_4}^2 \cdot (v_{i_1}^2 + v_{i_2}^2)}.$$

Thus, for the expression  $J$ , we get the value

$$J_4 = v_{i_1} \cdot v_{i_3} \cdot \frac{v_{i_3} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}}{v_{i_1} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2}} + v_{i_2} \cdot v_{i_4} \cdot \frac{v_{i_2} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2}}{v_{i_4} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}} +$$

$$v_{i_1} \cdot v_{i_4} \cdot \frac{v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_2}^2 \cdot v_{i_3}^2}{v_{i_1} \cdot v_{i_4} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}}.$$

We can somewhat simplify this expression if:

- in the first term, we delete  $v_{i_1}$  in the numerator and in the denominator,
- in the second term, we delete  $v_{i_4}$  from the numerator and from the denominator, and
- in the third term, we delete both  $v_{i_1}$  and  $v_{i_4}$  from the numerator and from the denominator.

Then, we get:

$$J_4 = v_{i_3} \cdot \frac{v_{i_3} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}}{\sqrt{v_{i_3}^2 + v_{i_4}^2}} + v_{i_2} \cdot \frac{v_{i_2} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2}}{\sqrt{v_{i_1}^2 + v_{i_2}^2}} + \frac{v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_2}^2 \cdot v_{i_3}^2}{\sqrt{v_{i_3}^2 + v_{i_4}^2} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}}.$$

If we bring all the terms to the common denominator  $\sqrt{v_{i_3}^2 + v_{i_4}^2} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}$ , then we get

$$J_4 = \frac{v_{i_3}^2 \cdot (v_{i_1}^2 + v_{i_2}^2) + v_{i_2}^2 \cdot (v_{i_3}^2 + v_{i_4}^2) + v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_2}^2 \cdot v_{i_3}^2}{\sqrt{v_{i_3}^2 + v_{i_4}^2} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}}.$$

The numerator of this expression has the form

$$v_{i_1}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_4}^2 + v_{i_1}^2 \cdot v_{i_4}^2 - v_{i_2}^2 \cdot v_{i_3}^2 =$$

$$v_{i_1}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_4}^2 + v_{i_1}^2 \cdot v_{i_4}^2 =$$

$$(v_{i_1}^2 + v_{i_2}^2) \cdot (v_{i_3}^2 + v_{i_4}^2).$$

Thus, we get

$$J_4 = \frac{(v_{i_1}^2 + v_{i_2}^2) \cdot (v_{i_3}^2 + v_{i_4}^2)}{\sqrt{v_{i_3}^2 + v_{i_4}^2} \cdot \sqrt{v_{i_1}^2 + v_{i_2}^2}},$$

i.e.,

$$J_4 = \sqrt{v_{i_1}^2 + v_{i_2}^2} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2}.$$

*Comparing  $J_4$  with  $J_2$  and  $J_3$ .* One can easily see that we always have  $J_2^2 \leq J_4^2$  and  $J_3^2 \leq J_4^2$ , thus  $J_2 \leq J_4$  and  $J_3 \leq J_4$ . Thus, if the estimate  $J_4$  is possible, there is no need to consider  $J_2$  and  $J_3$ , we only need to consider  $J_4$  and  $J_0$ .

*Comparing  $J_4$  and  $J_0$ .* Let us show that we always have  $J_0 \leq J_4$ , i.e.,

$$v_{i_1} \cdot v_{i_3} + v_{i_2} \cdot v_{i_4} \leq \sqrt{v_{i_1}^2 + v_{i_2}^2} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2}.$$

Indeed, this inequality between positive numbers is equivalent to a similar inequality between their squares:

$$v_{i_1}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_4}^2 + 2v_{i_1} \cdot v_{i_2} \cdot v_{i_3} \cdot v_{i_4} \leq (v_{i_1}^2 + v_{i_2}^2) \cdot (v_{i_3}^2 + v_{i_4}^2),$$

i.e.,

$$v_{i_1}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_4}^2 + 2v_{i_1} \cdot v_{i_2} \cdot v_{i_3} \cdot v_{i_4} \leq v_{i_1}^2 \cdot v_{i_3}^2 + v_{i_1}^2 \cdot v_{i_4}^2 + v_{i_2}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_4}^2.$$

Subtracting  $v_{i_1}^2 \cdot v_{i_3}^2 + v_{i_2}^2 \cdot v_{i_4}^2$  from both sides of this inequality, we get an equivalent inequality

$$2v_{i_1} \cdot v_{i_2} \cdot v_{i_3} \cdot v_{i_4} \leq v_{i_1}^2 \cdot v_{i_4}^2 + v_{i_2}^2 \cdot v_{i_3}^2,$$

i.e., equivalently,

$$0 \leq v_{i_1}^2 \cdot v_{i_4}^2 + v_{i_2}^2 \cdot v_{i_3}^2 - 2v_{i_1} \cdot v_{i_2} \cdot v_{i_3} \cdot v_{i_4} = (v_{i_1} \cdot v_{i_4} - v_{i_2} \cdot v_{i_3})^2,$$

which is, of course, always true. Thus, when the estimate  $J_4$  is possible, we do not need to consider the value  $J_0$  either: it is sufficient to take  $J = J_4$ .

*Value of  $\sigma$  in case  $J_4$  is possible: conclusion.* So, if the value  $J_4$  is possible, we get

$$\begin{aligned} \sigma^2 &= v_{i_1}^2 + v_{i_2}^2 + v_{i_3}^2 + v_{i_4}^2 + 2\sqrt{v_{i_1}^2 + v_{i_2}^2} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2} = \\ &= (v_{i_1}^2 + v_{i_2}^2) + (v_{i_3}^2 + v_{i_4}^2) + 2\sqrt{v_{i_1}^2 + v_{i_2}^2} \cdot \sqrt{v_{i_3}^2 + v_{i_4}^2} = \\ &= \left( \sqrt{v_{i_1}^2 + v_{i_2}^2} + \sqrt{v_{i_3}^2 + v_{i_4}^2} \right)^2, \end{aligned}$$

$$\text{so } \sigma = \sqrt{v_{i_1}^2 + v_{i_2}^2} + \sqrt{v_{i_3}^2 + v_{i_4}^2}.$$

*General comment.* The desired result for this case now follows from Proposition 14.

**Proof of Proposition 24.** Since the vertices  $i_2$  and  $i_4$  are not connected, this means that we do not have any restrictions on the relative location of the vectors  $\mathbf{v}_{i_2}$  and  $\mathbf{v}_{i_4}$ , so the largest possible value of the length of the sum  $\mathbf{v}_{i_2} + \mathbf{v}_{i_4}$  is equal to the sum of the lengths  $\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_4}\|$ . Similarly, since the vertices  $i_1$  and  $i_3$  are not connected, this means that we do not have any restrictions on the relative location of the vectors  $\mathbf{v}_{i_1}$  and  $\mathbf{v}_{i_3}$ , so the largest possible value of the length of the sum  $\mathbf{v}_{i_1} + \mathbf{v}_{i_3}$  is equal to the sum of the lengths  $\|\mathbf{v}_{i_1}\| + \|\mathbf{v}_{i_3}\|$ .

Each of the vertices  $i_1$  and  $i_3$  is connected to both  $i_2$  and  $i_4$ , which means that the measurement errors corresponding to  $i_1$  and  $i_3$  are independent of the errors corresponding to  $i_2$  and  $i_4$ . Thus, as we have described earlier, the largest possible length of the sum

$$\mathbf{v}_{i_1} + \mathbf{v}_{i_2} + \mathbf{v}_{i_3} + \mathbf{v}_{i_4} = (\mathbf{v}_{i_1} + \mathbf{v}_{i_3}) + (\mathbf{v}_{i_2} + \mathbf{v}_{i_4})$$

is equal to

$$\sqrt{(\|\mathbf{v}_{i_1}\| + \|\mathbf{v}_{i_3}\|)^2 + (\|\mathbf{v}_{i_2}\| + \|\mathbf{v}_{i_4}\|)^2}.$$

The desired result now follows from Proposition 14.

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