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How to Reach a Joint Decision with the Smallest Need for Compromise

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Abstract

Usually, people's interests do not match perfectly. So when several people need to make a joint decision, they need to compromise. The more people one has to coordinate the decision with, the fewer chances that each person's preferences will be properly taken into account. Therefore, when a large group of people need to make a decision, it is desirable to make sure that this decision can be reached by dividing all the people into small-size groups so that this decision can reach a compromise between the members of each group. In this paper, we use a recent mathematical result to describe the smallest possible group size for which such a joint decision is always possible.

1 Formulation of the Problem

Need for joint decision making. In many practical situations, people need to come up with a joint decision. For example, a country may need to select financial measures to boost the economy and decrease unemployment, a city may need to improve its public transportation system, etc.

To make a joint decision, we need to select an objective function. In each such decision making process, there are some numerical characteristics $x_1, \ldots, x_n$ that describe possible decisions. For example, in a country the quality of a decision can be characterized by the resulting increase in GDP x_1, the resulting decrease in unemployment x_2, etc. For a city-wide public transportation project, x_1 can be the decrease in average commute time, x_2 can be the decrease in pollution caused by cars, etc.

In general, everyone agrees that each of these characteristics should be as large as possible. The problem is that it is usually impossible to increase all of them. For example, one way to boost the country's GDP would be to outsource low-paying jobs to other countries and save money, but this may increase unemployment. In general, it is not possible to maximize two different objective functions: usually, when we maximize one of them, this does not make other objective functions maximum. Thus, when we make a decision, we need to select a single objective function $u(x_1, \ldots, x_n)$ that we should maximize.

Possibility of linearization. In many real-life situations, we are talking about decisions that, while important, do not drastically change our lives. For example, while it is desirable to have a better public transportation system, the resulting average decrease of, e.g., 10 minute per day of commute time does not lead to a radical change in people's lives and habits. In such situations, the changes x_i are reasonably small. When the changes x_i are small, terms which are quadratic (or even higher order) in terms of x_i are much smaller than x_i and can, thus, be safely ignored in comparison with terms which are linear in x_i. Thus, we can expand the desired objective function in Taylor series and only keep linear terms in this expansion, i.e., consider objective functions of the type

$$u(x) = c_0 + c_1 \cdot x_1 + \ldots + c_n \cdot x_n. \tag{1}$$

As we have mentioned, for each of the characteristics x_i, the larger its value, the better. This means that increasing x_i should lead to a larger value of this objective function, i.e., that all coefficients c_i should be positive.

Let us simplify. If we take the formula (1) literally, this would mean that to make a joint decision, we need to select $n+1$ parameters $c_0, c_1, \ldots, c_n$. The more parameters we need to select, the more complicated the selection task. We can make this task somewhat easier if we take into account that maximizing a function $u(x_1, \ldots, x_n)$ is equivalent to maximizing a "shifted" function $u(x_1, \ldots, x_n) - c_0$. Indeed, which of the two numbers is larger will not change if we subtract the same constant from both numbers. The shifted objective function has a simplified form

$$c_1 \cdot x_1 + \ldots + c_n \cdot x_n,$$

where we only have n coefficients to determine.

We can simplify the situation even further if we take into account that which of the two numbers is larger will not change if we divide both numbers by the same positive constant. Thus, for each constant C, maximizing a function $u(x_1, \ldots, x_n)$ is equivalent to maximizing a "re-scaled" function $u(x_1, \ldots, x_n)/C$. If we take $C = c_1 + \ldots + c_n$, then we arrive at the problem of maximizing the expression

$$U(x_1, \ldots, x_n) = a_1 \cdot x_1 + \ldots + a_n \cdot x_n, \tag{2}$$

where the values

$$a_i \stackrel{\text{def}}{=} \frac{c_i}{\sum\limits_{j=1}^{n} c_j}$$

satisfy the condition

$$a_1 + \ldots + a_n = 1. \tag{3}$$

We will call the expression (2) a *normalized utility function*.

Definition 1. *Let a positive integer n be fixed; we will call it the* number of characteristics. *By a* normalized utility function*, we mean an expression (2), for which the coefficients a_i satisfy the condition (3).*

Because of the condition (2), to describe a normalized utility function, we need to select $n-1$ coefficients: e.g., once we have selected $a_1, \ldots, a_{n-1}$, then we can use the equation (3) to determine a_n as $a_n = 1 - (a_1 + \ldots + a_{n-1})$.

What we mean by a compromise. Suppose that we have a group of p participants with normalized utility functions $U_1(x), \ldots, U_p(x)$. Based on these normalized utility functions, we need to form a new normalized utility function that the group will use to make a decision. A natural requirement is that if all p participants prefer an alternative x to an alternative y, then the group should also prefer x to y. Thus, we arrive at the following definition.

Definition 2. *We say that a normalized utility function $U(x)$ is a* compromise *between p normalized utility functions $U_1(x), \ldots, U_p(x)$ when for every two alternative x and y, the following condition holds:*

- *if $U_i(x) \geq U_i(y)$ for all $i = 1, \ldots, p$,*
- *then we should have $U(x) \geq U(y)$.*

It is desirable to minimize compromising. The more people one has to coordinate a decision with, the fewer chances that each person's preferences will be properly taken into account. Therefore, when a large group of people need to make a decision, it is desirable to make sure that this decision can be reached by dividing all the people into small-size groups so that this decision can reach a compromise between the members of each group.

Resulting problem. In view of the above, it is important to find the smallest possible group size for which such a joint decision is always possible.

What we do in this paper. In this paper, we provide a solution to this problem. It turns out that this smallest size is n.

2 Our Solution

Proposition 1.

- *Each group of $N \cdot n$ normalized utility functions can be partitioned into subgroups of size n so that there exists a normalized utility function which is a compromise for each of these subgroups.*
- *For each n, there exists an integer N and a group of $N \cdot (n-1)$ normalized utility functions for which, no matter how we partition it into subgroups of size $n-1$, there does not exist a normalized utility function which is a compromise for each of these subgroups.*

3 Proof

Relation to convexity. Our proof is based on the notion of convexity; see, e.g., [4]. For every finite set of points $a^{(1)} = \left(a_1^{(1)}, \ldots, a_n^{(1)}\right)$, ..., $a^{(k)} = \left(a_1^{(k)}, \ldots, a_n^{(k)}\right)$, by their *convex combination*, we mean a point $a = (a_1, \ldots, a_n)$ for which

$$a = \alpha_1 \cdot a^{(1)} + \ldots + \alpha_k \cdot a^{(k)}$$

for some $\alpha_j \geq 0$ for which $\alpha_1 + \ldots + \alpha_k = 1$, i.e., for which, for every i from 1 to n, we have

$$a_i = \alpha_1 \cdot a_i^{(1)} + \ldots + \alpha_k \cdot a_i^{(k)}.$$

The set of all convex combinations is known as the *convex hull* of the points $a^{(1)}$, ..., $a^{(k)}$.

The relation between convexity and our problem is provided by the following lemma. To formulate this lemma, we will say that a normalized utility function $U(x) = a_1 \cdot x_1 + \ldots + a_n \cdot x_n$ is *characterized* by the point $a = (a_1, \ldots, a_n)$.

Lemma. *Let $U_0(x)$ be a normalized utility function characterized by a point $a^{(0)}$, and let $U_1(x)$, ..., $U_p(x)$ are normalized utility functions characterized points $a^{(1)}$, ..., $a^{(p)}$. Then, the following two conditions are equivalent to each other:*

- *$U_0(x)$ is a compromise between normalized utility functions $U_1(x)$, ..., $U_p(x)$, and*
- *the point $a^{(0)}$ is a convex combination of the points $a^{(1)}$, ..., $a^{(p)}$.*

Proof of the lemma. Let us first prove that if the point $a^{(0)}$ is a convex combination of the points $a^{(1)}$, ..., $a^{(p)}$, i.e., if

$$a_i^{(0)} = \alpha_1 \cdot a_i^{(1)} + \ldots + \alpha_p \cdot a_i^{(p)} \quad (4)$$

for some non-negative coefficients α_j that add up to 1, then $U_0(x)$ is a compromise between normalized utility functions $U_1(x)$, ..., $U_p(x)$.

Indeed, let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two alternatives for which $U_j(x) \geq U_j(y)$ for all i, i.e., for which

$$a_1^{(j)} \cdot x_1 + \ldots + a_n^{(j)} \cdot x_n \geq a_1^{(j)} \cdot y_1 + \ldots + a_n^{(j)} \cdot y_n.$$

If we multiply both sides of this inequality by a non-negative number $\alpha_j \geq 0$, we conclude that

$$\alpha_j \cdot a_1^{(j)} \cdot x_1 + \ldots + \alpha_j \cdot a_n^{(j)} \cdot x_n \geq \alpha_j \cdot a_1^{(j)} \cdot y_1 + \ldots + \alpha_j \cdot a_n^{(j)} \cdot y_n. \quad (5)$$

Adding the inequalities (5) corresponding to $j = 1, \ldots, p$, we get

$$\left(\sum_{j=1}^{p} \alpha_j \cdot a_1^{(j)}\right) \cdot x_1 + \ldots + \left(\sum_{j=1}^{p} \alpha_j \cdot a_n^{(j)}\right) \cdot x_n \geq$$

$$\left(\sum_{j=1}^{p} \alpha_j \cdot a_1^{(j)}\right) \cdot y_1 + \ldots + \left(\sum_{j=1}^{p} \alpha_j \cdot a_n^{(j)}\right) \cdot y_n.$$

Taking into account the formula (4), we conclude that

$$a_1^{(0)} \cdot x_1 + \ldots + a_n^{(0)} \cdot x_n \geq a_1^{(0)} \cdot y_1 + \ldots + a_n^{(0)} \cdot y_n,$$

i.e., that $U_0(x) \geq U_0(y)$. So, $U_0(x)$ is indeed a compromise between the normalized utility functions $U_1(x)$, ..., $U_p(x)$.

Vice versa, let us prove that if $U_0(x)$ is a compromise between normalized utility functions $U_1(x)$, ..., $U_p(x)$, then the point $a^{(0)}$ is a convex combination of the points $a^{(1)}$, ..., $a^{(p)}$. Let us prove this by contradiction. Let us assume that the point $a^{(0)}$ that satisfies the condition (3) is *not* a convex combination of the points $a^{(1)}$, ..., $a^{(p)}$; in other words, the point a does not belong to the convex hull of the points $a^{(1)}$, ..., $a^{(p)}$. All the points $a^{(0)}$, $a^{(1)}$, ..., $a^{(p)}$ satisfy the condition (3) and are, thus, located on a plane defined by this condition.

By the properties of convex sets, if two closed convex sets do not intersect, there exists a separating plane. A single point is, of course, a convex set, so there exists a plane that separates the point $a^{(0)}$ from the points $a^{(1)}$, ..., $a^{(p)}$. If we connect all the points from this plane to the point $(0, \ldots, 0)$, we get a plane passing through 0 that separates the point a from all the points $a^{(j)}$. In general, such a plane has the form

$$a_1 \cdot x_1 + \ldots + a_n \cdot x_n = 0$$

for some coefficients x_i. For the points on two sides of this plane, the expression $a_1 \cdot x_1 + \ldots + a_n \cdot x_n$ has different signs. Thus, the fact that the plane separates these points means that for the point $a^{(0)}$ and for the points $a^{(j)}$, this expression has different signs.

If the sign is negative for $a^{(0)}$ and positive for all the points $a^{(j)}$ with $j \geq 1$, then we have $U_j(x) > 0$ for all $j = 1, \ldots, p$ and $U_0(x) < 0$. Here, $U(\vec{0}) = 0$, where we denoted $\vec{0} \stackrel{\text{def}}{=} (0, \ldots, 0)$. Thus, we have $U_j(x) > U_j\left(\vec{0}\right)$ for all $j = 1, \ldots, p$ but $U_0(x) < U_0\left(\vec{0}\right)$. This contradicts to our assumption that $U_0(x)$ is a compromise between the normalized utility functions $U_1(x), \ldots, U_p(x)$.

Similarly, if the sign is positive for $a^{(0)}$ and negative for all the points $a^{(j)}$ with $j \geq 1$, then we have $U_j\left(\vec{0}\right) > U_j(x)$ for all $j = 1, \ldots, p$ but $U_0\left(\vec{0}\right) < U_0(x)$, which also contradicts to our assumption. Thus, the point a cannot be outside the convex hull, so it must be inside the convex full. The Lemma is proven.

Proving the first part of the proposition. Now that the lemma is proven, let us show how this lemma leads to the proof of the first part of our main result. As we have mentioned, the utility functions of $k = N \cdot (d+1)$ people can be represented by k points $(a_1, \ldots, a_n)$ from an $(n-1)$-dimensional space determined by the condition (3). It is known that any group of $N \cdot (d+1)$ points in a d-dimensional space can be partitioned into subsets of size $d+1$ so that there is a single point that belongs to the convex hull of all these subsets. This result was first proven in [2] for dimension $d = 2$, then in [3] for all dimensions d; see also [1] for the general overview of this and related results. In our case, we have a space of dimension $d = n - 1$, so the above result indeed implies the first part of our proposition.

Proving the second part of the proposition. To prove this part, let us take $N = n$ and $N \cdot (n-1) = n \cdot (n-1)$ people with the utility functions corresponding to the following points:

- we have $n-1$ people with utility function corresponding to $a^{(1)} = (1, 0, \ldots, 0)$,
- we have $n-1$ people with utility function corresponding to $a^{(2)} = (0, 1, 0, \ldots, 0)$, $\ldots$,
- for each i, we have $n-1$ people with utility function corresponding to $a^{(i)} = (0, \ldots, 0, 1, 0 \ldots, 0)$, with 1 on i-th place,$\ldots$, and
- we have $n-1$ people with utility function corresponding to $a^{(n)} = (0, \ldots, 0, 1)$.

Let us show that no matter how we partition them into groups of $n-1$, there will be no point common to the convex hulls of all these groups.

Suppose that we divide the original $n \cdot (n-1)$ points into n groups $c_1, \ldots, c_n$ each of which contains $n-1$ points. For each combination c_j of $n-1$ points, their convex hull C_j is contained in the linear space L_j generated by the corresponding vectors. So, if there is a point common to all these convex hulls, this point should belong to the intersection $L_1 \cap \ldots \cap L_n$ of the corresponding linear spaces. Each of these linear spaces is a linear combination of some collection of some points $a^{(i)}$. One can see the intersection $L_1 \cap \ldots \cap L_n$ of these linear spaces is a linear space generated by the intersection of these groups $c_1 \cap \ldots \cap c_n$.

This intersection $c_1 \cap \ldots \cap c_n$ cannot contain the vector $a^{(1)}$: indeed, this would imply that the vector $a^{(1)}$ is contained in all n groups $c_1, \ldots, c_n$ that form this partition, and this cannot be since we only have $n-1$ such vectors. Similarly, this intersection cannot contain any of the vectors $a^{(i)}$. Thus, this intersection $c_1 \cap \ldots \cap c_n$ is empty, and the linear space $L_1 \cap \ldots \cap L_n$ generated by this intersection consists of the single point $(0, \ldots, 0)$. Since the intersection $C_1 \cap \ldots \cap C_n$ of the convex hulls is contained in the intersection $L_1 \cap \ldots \cap L_n$ of linear spaces, all the elements of this intersection – i.e., all the points which are common to all convex hulls – must be contained in the 1-element set $\{(0, \ldots, 0)\}$ – so this intersection should be either empty, or consists of this point $(0, \ldots, 0)$. However, all the elements in each convex hull satisfy the condition (3), but the point $(0, \ldots, 0)$ does not satisfy this condition. Thus, the intersection of the convex hulls must be empty – which is exactly what we wanted to prove.

The second part of the Proposition is proven, and thus, the Proposition itself is proven.

4 Remaining Open Problem

In this paper, we simply prove the existence of the desired partition into small groups. It is desirable to come up with an efficient algorithm for such a subdivision.

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