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How to Describe Variety of a Probability Distribution: A Possible Answer to Yager’s Question

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Abstract—Entropy is a natural measure of randomness. It progresses from its smallest possible value 0 – when we have a deterministic case in which one alternative \( i \) occurs with probability 1 (\( p_i = 1 \)), to the largest possible value which is attained at a uniform distribution \( p_1 = \ldots = p_n = 1/n \). Intuitively, both in the deterministic case and in the uniform distribution case, there is not much variety in the distribution, while in the intermediate cases, when we have several different values \( p_i \), there is a strong variety. Entropy does not seem to capture this notion of variety. In this paper, we discuss how we can describe this intuitive notion.

Index Terms—Entropy, probability distribution, variety

I. VARIETY: AN INTUITIVE NOTION

For probability distributions, we have an intuitive understanding that some probability distributions are “more random” than the others. This intuitive notion of degree of randomness is captured by the formal definition of an entropy of a probability distribution; see, e.g., [1], [3]–[7]. Entropy can be defined as an average number of binary (“yes”-“no”) questions that one needs to ask to determine the exact alternative. It is known that for a distribution in which an alternative \( i \) appears with probability \( p_i \), this average number of questions can be described by Shannon’s formula

\[
S = - \sum_{i=1}^{n} p_i \cdot \log_2(p_i).
\]

For a continuous probability distribution with a probability density \( \rho(x) \), we can similarly ask how many binary questions are needed, on average, to determine \( x \) with a given accuracy \( \varepsilon \). Asymptotically, when \( \varepsilon \to 0 \), this number of questions can be described as \( S \approx - \int \rho(x) \cdot \log_2(\rho(x)) \, dx \).

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For discrete case, entropy progresses:

- from its smallest possible value 0 – when we have a deterministic case in which one alternative \( i \) occurs with probability 1 \( (p_i = 1) \),
- to the largest possible value which is attained at a uniform distribution

\[
p_1 = \ldots = p_n = \frac{1}{n}.
\]

Intuitively:

- both in the deterministic case and in the uniform distribution case, there is not much variety in the distribution, while
- in the intermediate cases, when we have several different values \( p_i \), there is a strong variety.

Entropy does not seem to capture this notion of variety. So, Ron Yager asked a question: How can we describe this intuitive notion of variety in precise terms? In this paper, we provide a possible answer to this question.

II. MAIN IDEA BEHIND OUR APPROACH

The value of entropy only depends on the values of the probability and does not depend on which alternatives have different probabilities:

- if we apply a permutation

\[
\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}
\]

to the alternatives,

- then the resulting probability distribution \( p_i \overset{\text{def}}{=} p_{\pi(i)} \) will have exactly the same entropy as the original probability distribution \( p_i \).

So, when analyzing related properties of randomness, we can assume that we only know the values \( p_1, \ldots, p_n \), but we do not know which alternative has which probability.

In general, because of the possible permutations, we can have different distributions with the same set of values \( \{p_1, \ldots, p_n\} \). In other words:

- once we fix the set of values \( \{p_1, \ldots, p_n\} \),
- we get, in general, not a single distribution but rather a variety of different distributions.

Let us see how big this variety is in different cases.

Let us first consider the deterministic case, in which all the values \( p_i \) are equal to 0 except for one value which is
equal to 1. In this case, we have \( N = n \) possible probability distributions:

- the first one in which alternative 1 occurs with probability 1,
- the second one in which alternative 2 occurs with probability 1, etc.

In the case of a uniform distribution, all the values of \( p_i \) are equal, so no matter what permutations we apply, we end up with the exact same uniform distribution. Thus, in this case, the variety consists of a single probability distribution: \( N = 1 \).

In the generic case, when all \( n \) probabilities \( p_i \) are different, we get as many probability distributions as we have permutations, i.e., \( N = n! \) different distributions.

We see that in this sense, deterministic and uniform cases indeed have low variety, while the general case has a much larger variety. It is therefore reasonable to consider the corresponding value \( N \) as the main idea behind the formalization of the intuitive notion of variety.

III. HOW TO MEASURE VARIETY: CASE OF DISCRETE DISTRIBUTIONS

In line with the above definition of entropy, it is reasonable to describe the variety as the smallest number of binary questions which are needed to uniquely determine the actual distribution.

After each binary question, we can have 2 possible answers. So:

- if we ask \( q \) binary questions,
- then, in principle, we can have \( 2^q \) possible results.

Thus:

- if we know that our (unknown) distribution is one of \( N \) distributions, and we want to uniquely pinpoint the distribution after all these questions,
- then we must have \( 2^q \geq N \).

In this case, the smallest number of questions is the smallest integer \( q \) that is \( \geq \log_2(N) \). Thus, \( \log_2(N) \) is the natural measure of variety in the discrete case.

- For the deterministic case, \( N = 1 \), so the variety is \( \log_2(1) = 0 \).
- In the uniform case, we have \( q = \log_2(n) \).
- In the general case, we have \( q = \log(n!) \).

To simplify computations, we can use the well-known Stirling formula \( n! \sim (n/e)^n \cdot \sqrt{2\pi \cdot n} \), hence

\[
q = \log_2(n!) \approx n \cdot \log_2(n).
\]

It is worth mentioning that:

- since the variety only depends on the set of probability values \( \{p_1, \ldots, p_n\} \) and not on their order,
- we can, without losing generality, assume that the values \( p_i \) are listed in increasing order

\[
p_1 \leq p_2 \leq \ldots \leq p_n.
\]

IV. HOW TO MEASURE VARIETY: CASE OF CONTINUOUS DISTRIBUTIONS

Without losing generality, we can similarly assume that the probability density \( \rho(x) \) is an increasing function of \( x \).

Similarly to entropy, a natural way to go from the discrete case to the continuous case is to take into account that in reality, we can only determine both

- the value of the variable \( x \) and
- the probability \( p \)

with a certain accuracy.

Once we fix the accuracy \( \varepsilon \) of measuring \( x \), then, within this accuracy, we have only finitely many possible values of \( x \): a value \( x_i \) covers the whole interval \([x_i - \varepsilon, x_i + \varepsilon]\), so we only need values:

- \( x_0 \) – which covers \([x_0 - \varepsilon, x_0 + \varepsilon]\),
- \( x_1 = x_0 + 2\varepsilon \) – which covers \([x_1 - \varepsilon, x_1 + \varepsilon]\) = \([x_0 + \varepsilon, x_0 + 3\varepsilon]\),
- \( x_2 = x_0 + 4\varepsilon \),
- etc.

As a result, we get a discrete problem which we already know how to handle. When the accuracy \( \varepsilon \) tends to 0, the discrete problem tends to the original continuous one.

- For entropy, it was sufficient to take into account that \( x \) cannot be measured exactly.
- For variety, since we need to distinguish between different and equal values of probability \( p_i \), we must also take into account that the probabilities can only be measured with a certain accuracy.

So, let us fix the accuracy \( \varepsilon \) with which we measure \( x \), and the accuracy \( \delta \) with which we measure probability. Once we fix \( \varepsilon \), we get values

\[
x_0, x_1 = x_0 + 2\varepsilon, x_2 = x_0 + 4\varepsilon, \ldots,
\]

\[
x_i = x_0 + i \cdot (2\varepsilon), \ldots
\]

Each of these values \( x_i \) covers an interval \([x_i - \varepsilon, x_i + \varepsilon]\), so for the probability distribution with the density \( \rho(x) \), the probability \( p_i \) of \( x_i \) is equal to

\[
p_i = \int_{x_i-\varepsilon}^{x_i+\varepsilon} \rho(x) \, dx \approx \rho(x_i) \cdot (2\varepsilon).
\]

We can only determine probabilities with accuracy \( \delta \). This means, in effect, that we divide the interval \([0, 1]\) of possible values of probability into intervals:

- \( p_0 = [0, 2\delta] \) (probabilities which are approximately equal to \( \delta \)),
- \( p_1 = [2\delta, 4\delta] \) (probabilities which are approximately equal to \( 3\delta \)),
- \ldots,
- \( p_j = [(j - 1) \cdot (2\delta), (j + 1) \cdot (2\delta)] \) (probabilities which are approximately equal to \( (j + 1/2) \cdot (2\delta) \)),
- \ldots,
• \([1 - 2\delta, 1]\) (probabilities which are approximately equal to \(1 - \delta\)),
and we consider events \(p_i\) for which the probabilities fall into the same probability interval as having (within this accuracy) the same probability.

Let \(n_j\) denote the number of events for which the corresponding probability \(p_i \approx \rho(x_i) \cdot (2\varepsilon)\) falls within the \(j\)-th probability interval \(p_j\). Then, the number of possible permutations is equal to the number of ways to subdivide the overall number of \(n = n_1 + n_2 + \ldots\) values into groups of \(n_1, n_2, \ldots\) etc.

• The total number \(C_1\) of ways to choose \(n_1\) elements out of \(n\) is well-known in combinatorics, and is equal to
\[
\binom{n}{n_1} = \frac{n!}{(n_1)! \cdot (n - n_1)!}.
\]

• After we choose these \(n_1\) elements, we have a problem in choosing \(n_2\) out of the remaining \(n - n_1\) elements; so for every selection of \(n_1\) elements we have
\[
C_2 = \binom{n - n_1}{n_2}
\]
possibilities to choose these \(n_2\) elements. Therefore, in order to get the total number of selections of \(n_1\) elements and \(n_2\) elements, we must multiply \(C_2\) by \(C_1\).

Adding selections of \(n_3, n_4, \ldots\), we get finally the following formula for \(N\):
\[
N = C_1 \cdot C_2 \cdot \ldots \cdot C_{n-1} = \frac{n!}{n_1! \cdot (n - n_1)!} \cdot \frac{n_2! \cdot (n - n_1 - n_2)!}{(n - n_1)!} \cdot \ldots = \frac{n!}{n_1! \cdot n_2! \cdot \ldots}
\]
Thus, the resulting degree of variety \(q\) is equal to
\[
q = \log_2(N) = \log(n) - \log_2(n_1!) - \log_2(n_2!) - \ldots
\]

Since \(\log_2(n!) \approx n \cdot \log(n)\), we conclude that
\[
q = n \cdot \log(n) - n_1 \cdot \log_2(n_1) - n_2 \cdot \log_2(n_2) - \ldots,
\]
where the total number of points \(n \approx L/(2\varepsilon)\) only depends on the width \(L\) of the interval on which the probability distribution is located but not on the distribution itself.

How big are the values \(n_j\)? By definition, \(n_j\) is the number of values \(x_i\) for which
\[
j \cdot (2\delta) \leq p_i = \rho(x_i) \cdot (2\varepsilon) \leq (j + 1) \cdot (2\delta),
\]
i.e., for which
\[
j \cdot \frac{\delta}{\varepsilon} \leq \rho(x_i) \leq (j + 1) \cdot \frac{\delta}{\varepsilon}.
\]

Since \(\rho(x)\) is an increasing function of \(x\), this is equivalent to \(x(j) \leq x_i \leq x(j+1)\), where
\[
x(j) \overset{\text{def}}{=} \rho^{-1}\left(\frac{j \cdot \delta}{\varepsilon}\right)
\]
and \(\rho^{-1}\) denotes the inverse function to \(\rho(x)\) — i.e., in other words,
\[
\rho(x(j)) = j \cdot \frac{\delta}{\varepsilon}.
\]
The difference \(\Delta x(j) \overset{\text{def}}{=} x(j+1) - x(j)\) between the two consequent threshold values of \(x\) can be determined from the fact that asymptotically,
\[
\rho(x(j+1)) = \rho(x(j) + \Delta x(j)) \approx \rho(x(j)) + \rho'(x(j)) \cdot \Delta x(j),
\]
where \(\rho'(x)\) denote the derivative of the density function. So from
\[
\rho(x(j)) = j \cdot \frac{\delta}{\varepsilon}
\]
and
\[
\rho(x(j+1)) = (j + 1) \cdot \frac{\delta}{\varepsilon},
\]
we conclude that
\[
\Delta x(j) \approx \frac{\delta}{\varepsilon} \cdot \frac{1}{\rho'(x(j))},
\]
(1)
On this interval, we have \(n_j \approx \Delta x(j)/(2\varepsilon)\) values \(x_i\), so
\[
n_j \approx \frac{\delta}{2\varepsilon} \cdot \frac{1}{\rho'(x(j))}.
\]
Hence,
\[
q = n \cdot \log_2(n) - \sum n_j \cdot \log_2(n_j)
\]
can be described as \(q = n \cdot \log_2(n) + \sum a(x')\), where
\[
a(x(j)) \overset{\text{def}}{=} \frac{\delta}{2\varepsilon} \cdot \frac{1}{\rho'(x(j))} \cdot \log_2 \left(\frac{\delta}{2\varepsilon} \cdot \frac{1}{\rho'(x(j))}\right).
\]
(2)
When accuracies tend to 0, this sum gets close to an integral. Since for every function \(f(x)\), the integral is approximately equal to its integral sum
\[
\int f(x) \, dx \approx \sum f(x(j)) \cdot \Delta x(j),
\]
and the smaller \(\varepsilon\) and \(\delta\), the closer the integral sum to the integral, we conclude that the sum \(\sum a(x(j))\) can be approximated as
\[
\sum b(x(j)) \cdot \Delta x(j) \approx \int b(x) \, dx,
\]
where \(b(x(j)) \overset{\text{def}}{=} a(x(j))/\Delta x(j)\). From (1) and (2), we conclude that
\[
b(x(j)) = -\frac{1}{2\varepsilon} \cdot \log_2 \left(\frac{\delta}{2\varepsilon} \cdot \frac{1}{\rho'(x(j))}\right),
\]
and hence
\[
q \approx n \cdot \log_2(n) - \int \frac{1}{2\varepsilon} \cdot \log_2 \left(\frac{\delta}{2\varepsilon} \cdot \frac{1}{\rho'(x)}\right) \, dx.
\]
Since the logarithm of the product is equal to the sum of the logarithms, we can see that
\[
q = n \cdot \log_2(n) - \left(\int \log_2 \left(\frac{\delta}{2\varepsilon} \cdot \frac{1}{\rho'(x)}\right) \, dx + \int \log_2 \left(\frac{1}{\rho'(x)}\right) \, dx\right).
\]
Thus, asymptotically, the value \( q \) can be determined once we know the value
\[ Q \stackrel{\text{def}}{=} \int \log_2(\rho'(x)) \, dx. \]

In the general case, when the function \( \rho(x) \) is not necessarily increasing, it can be decreasing as well, so we get
\[ Q \stackrel{\text{def}}{=} \int \log_2(|\rho'(x)|) \, dx. \]

V. Conclusions and Future Work

Conclusions. For a continuous probability distribution, the above measure of variety can be computed as follows:
\[ Q \stackrel{\text{def}}{=} \int \log_2(|\rho'(x)|) \, dx. \]

- For the (almost) deterministic case, when
  \[ \rho(x) \approx \frac{1}{\varepsilon} \]
on a narrow interval of width \( \varepsilon \), we have
  \[ \rho'(x) \approx \frac{\rho(x)}{\varepsilon} \approx \frac{1}{\varepsilon^2}, \]
  so \( Q \approx \varepsilon \cdot \log_2(\varepsilon^{-2}) \approx 0. \)
- For a uniform distribution \( \rho(x) = \text{const} \), we have \( \rho'(x) = 0 \), hence \( Q = -\infty. \)
- For non-uniform distributions in which \( |\rho'(x)| > 0 \), as expected, we get higher variety.

Future work. In this paper, we analyzed the case of probabilistic uncertainty. It is desirable to extend the corresponding definition and analysis to other types of uncertainty, e.g., fuzzy, interval-valued fuzzy. It is also desirable to compare this approach with alternative approaches of defining variety, such as a definition from [2] for OWA operators.

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