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Everyone Is Above Average: Is It Possible? Is It Good?

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Abstract Starting with the 1980s, a popular US satirical radio show described a fictitious town Lake Wobegon where “all children are above average” – parodying the way parents like to talk about their children. This everyone-above-average situation was part of the fiction since, if we interpret the average in the precise mathematical sense, as average over all the town’s children, then such a situation is clearly impossible. However, usually, when parents make this claim, they do not mean town-wise average, they mean average over all the kids with whom their child directly interacts. Somewhat surprisingly, it turns out that if we interpret average this way, then the everyone-above-average situation becomes quite possible. But is it good? At first glance, this situation seems to imply fairness and equality, but, as we show, in reality, it may lead to much more inequality than in other cases.

1 Everyone Is Above Average: Formulation of the Problem

Where this expression comes from. *A Prairie Home Companion*, a satirical radio show very popular in the US from the 1980s to 2000s, regularly described events from a fictitious town of Lake Wobegon, where, among other positive things, “all children are above average".

Why this situation is usually considered to be impossible. This phrase – which is a part of the usual parent’s bragging about their children – was intended as a satire, because, of course, if all the values are above a certain level, then the average is also larger than this level – and so, this level cannot be the average level.

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But is it really impossible? It depends on what we mean by average. But is such a situation indeed impossible? It would indeed be impossible if by “average”, the speaker would mean that, e.g., the IQ level of each child is higher than the town (or state) average. However, when people brag about their kids, they usually do not mean comparison with the town or state average – especially since such an average is rarely computed and practically never known.

When we say that someone is above average, we usually mean that in terms of the desired characteristic(s), this person is above the average of his/her nearest neighbors – i.e., people with whom he/she constantly interacts and thus, has the opportunity to compare the corresponding characteristic – be it IQ, school grades, sports achievements, etc.

Resulting problem. If we interpret “above average" in this localized way, a natural question is: in this interpretation, is it possible that everyone is above average?

And if it is possible, is it good? To some extent, this situation is a version of equality – no one is left behind, but equality does not necessarily mean that the situation is good: e.g., it is not good when everyone is equally poor.

In this paper, we provide simple answers to both questions.

2 Is It Possible?

Let us formulate the problem in precise terms. To answer the above question, let us describe the problem in precise terms. First, let us consider a continuous approximation. This is a must in physics – see, e.g., [1, 3] – where matter consists of atoms, but since there are too many of them to have an efficient atom-simulating computational model, it is reasonable to consider a continuous approximation. For example, instead of the masses of individual atoms, we characterise these masses by a function $\rho(x, y)$ which is proportional to the average mass of all the particles in a small vicinity of the spatial point (x, y).

Similarly, there are too many people on Earth to have an efficient each-person-simulating computational model, so it is reasonable to consider a continuous model. In other words, we assume that the distribution of the desired quantity u among humans is described by a function $u(x, y)$ – the average value of this function in a small vicinity of the spatial point (x, y).

By the average $a(x, y)$ to which a person located at a point (x, y) is comparing his/her characteristics, it is reasonable to understand the average value of this function in some vicinity of this point. This notion becomes precise if we fix the radius $\varepsilon > 0$ of this vicinity. In this case, the average takes the form

$$a(x,y) = \frac{1}{\frac{4}{3}\cdot\pi\cdot\varepsilon^3}\cdot\int_{(X,Y):\, r\stackrel{\text{def}}{=}d((x,y),(X,Y))\le\varepsilon} u(X,Y)\,dX\,dY. \quad (1)$$

In these terms, the problem takes the following form: is it possible to have a function $u(x, y)$ for which $u(x, y) > a(x, y)$ for all points (x, y)?

Analysis of the problem. The vicinity in which people contact a lot is usually small – the overall number of people each of us seriously knows is in the hundreds, which is much, much smaller than the billions of the world's population. Since this vicinity is small, in this vicinity, we can expand the function $u(X, Y)$ in Taylor series in terms of the small differences $X - x$ and $Y - y$, and keep only the few main terms in this expansion. In other words, we take

$$u(X,Y) = a_0 + a_x \cdot (X - x) + a_y \cdot (Y - y)+$$

$$a_{xx} \cdot (X - x)^2 + a_{xy} \cdot (X - x) \cdot (Y - y) + a_{yy} \cdot (Y - y)^2, \quad (2)$$

where

$$a_0 = u(x, y), \quad a_x = \frac{\partial u}{\partial x}, \quad a_y = \frac{\partial u}{\partial y},$$

$$a_{xx} = \frac{1}{2} \cdot \frac{\partial^2 u}{\partial x^2}, \quad a_{xy} = \frac{\partial^2 u}{\partial x\, \partial y}, \quad a_{yy} = \frac{1}{2} \cdot \frac{\partial^2 u}{\partial y^2}. \quad (3)$$

Let us compute the average of this expression over the vicinity. The average of a linear combination is equal to the linear combination of the averages; thus, to find the average of the expression (2), it is sufficient to compute the averages of the terms 1, $X - x$, $Y - y$, $(X - x)^2$, $(X - x) \cdot (Y - y)$, and $(Y - y)^2$.

The average value of the constant 1 is, of course, this same constant. The average value of the term $X - x$ is 0, since the circle is invariant with respect to reflections in the line $Y = y$ and thus, for every point $(X - x, Y - y)$, we have a point $(-(X - x), Y - y)$ in the same vicinity, and the contributions of these two points to the integral describing the average cancel each other. Similar, the averages of the terms $Y - y$ and $(X - x) \cdot (Y - y)$ are zeros.

Let us find the average A of the term he average of the term $(X - x)^2$. Since the circle is invariant with respect to swapping x and y, the average of $(X - x)^2$ is equal to the average of $(Y - y)^2$, thus the average of the sum $(X - x)^2 + (Y - y)^2$ is equal to $2A$. In polar coordinates this sum is equal to r^2, thus the integral is equal to

$$\int_{(X,Y):\, r \stackrel{\text{def}}{=} d((x,y),(X,Y)) \le \varepsilon} r^2 \, dX dY$$

is equal to

$$\int_0^{\varepsilon} r^2 \cdot 2 \cdot \pi \cdot r\, dr = 2 \cdot \pi \cdot \int_0^{\varepsilon} r^3\, dr = 2 \cdot \pi \cdot \frac{\varepsilon^4}{4} = \frac{1}{2} \cdot \pi \cdot \varepsilon^4.$$

Thus, the average of this sum is equal to

$$2A = \frac{\frac{1}{2} \cdot \pi \cdot \varepsilon^4}{\frac{4}{3} \cdot \pi \cdot \varepsilon^3} = \frac{3}{8} \cdot \varepsilon.$$

Hence, the average A of each of the two terms x^2 and y^2 is equal to

$$A = \frac{3}{16} \cdot \varepsilon.$$

Thus, for the average $a(x, y)$, we get the following expression:

$$a(x, y) = u(x, y) + \frac{3}{16} \cdot \varepsilon \cdot (a_{xx} + a_{yy}).$$

Here, the sum

$$a_{xx} + a_{yy} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is known as the *Laplacian*; it is usually denoted by Δu, so

$$a(x, y) = u(x, y) + \frac{3}{16} \cdot \varepsilon \cdot \Delta u(x, y). \tag{4}$$

So when is this possible. If the Laplacian is everywhere negative, then at each point (x, y), the value $u(x, y)$ of the desired quantity is larger than the corresponding average $a(x, y)$.

This happens, e.g., when the function $u(x, y)$ is strictly concave; see, e.g., [2] – and there are many such functions.

Also, on a local level, this property holds in a vicinity of a maximum – provided that the corresponding matrix of second derivatives (known as *Jacobian*) is not degenerate.

Comment. This fits well with the fact the fictitious Lake Wobegon is located in the United States – one of the richest countries on Earth – i.e., indeed, close to the maximum.

3 It Is Possible, But Is It Good?

Not always. If everyone is above average, it may sound good, but in reality, it is not always good.

As an example, let us consider a strictly convex 1-D function, such as $u(x) = -x^2$. For a concave function, the second derivative is negative, thus, the first derivative can only decrease. Thus, as we go further in the direction in which the function decreases, its rate of decrease cannot get smaller, it only continuous to increase, and the further we go from the maximum, the smaller the value of the function $u(x)$ –

and this decrease is at least linear in terms of the distance to the function's maximum location.

In other words, in this case, we have a drastic decrease of quality – which would not see if we did not have this seemingly positive everyone-above-average situation.

Discussion. The last argument is the second time in this paper when our intuition seems to mislead us:

- First, our intuition seems to indicate that a situation in which everyone is above average is not possible. However, contrary to this intuition, it turns out that such situations are possible, and rather typical – e.g., any concave function has this property.
- Second, our intuition seems to indicate that if everyone is above average, this kind of implies fairness and equality. In reality, in this case, income – or whatever characteristic we are looking for – becomes even more unequally distributed than in situations where this everyone-above-average property is not satisfied.

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