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Why Some Theoretically Possible Representations of Natural Numbers Were Historically Used and Some Were Not: An Algorithm-Based Explanation

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Why Some Theoretically Possible Representations of Natural Numbers Were Historically Used and Some Were Not: An Algorithm-Based Explanation

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Abstract Historically, people have used many ways to represent natural numbers: from the original "unary" arithmetic, where each number is represented as a sequence of, e.g., cuts (4 is IIII) to modern decimal and binary systems. However, with all this variety, some seemingly reasonable ways of representing natural numbers were never used. For example, it may seem reasonable to represent numbers as products – e.g., as products of prime numbers – such a representation was never used in history. So why some theoretically possible representations of natural numbers were historically used and some were not? In this paper, we propose an algorithm-based explanation for this different: namely, historically used representations have decidable theories – i.e., for each such representation, there is an algorithm that, given a formula, decides whether this formula is true or false, while for un-used representations, no such algorithm is possible.

1 Formulation of the problem

Historical representations of natural numbers: a brief overview (see, e.g., [1] for more details). There are, in principle, infinitely many natural numbers 0, 1, 2, ... To represent all such numbers, a natural idea is:

- to select one or more basic numbers, and

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- to select operations that would allow us to build new natural numbers based on the existing ones.

Which operations should we select? On the set of all natural numbers, the most natural operations are arithmetic ones. They are, in the increasing order of computational complexity: addition (of which the simplest case in adding 1), subtraction, multiplication, and division.

Historically first representation is the one corresponding to "unary" numbers, where each natural number (e.g., 5) is represented by exactly that many cuts on a stick. Roman numerals preserve some trace of this representation, since there 1 is represented as I, 2 as II, 3 as III, and sometimes even 4 as IIII. In this representation, the only basic number is 0 (or rather, since the ancient folks did not know zero, 1), and the only operation is adding 1.

Such a unary representation works well for small numbers, but for large numbers – e.g., hundreds or thousands – such a representation requires an un-realistic amount of space. A natural way to shorten the representation is to use other basic numbers in addition to 0, and use more complex operations – e.g., full addition instead of adding 1. This is exactly how, e.g., the Biblical number system worked: some letters represented numbers from 1 to 9, some letters represented numbers 10, 20, $\ldots$, 90, some letters represented 100, 200, etc., and a generic number was described by a combination of such letters – interpreted as the sum of the corresponding numbers.

A natural next step is to add the next-simplest arithmetic operation: subtraction. This is what the Romans did, and indeed, IV meaning $5-1$ is a much shorter representation of the number 4 than the addition-based IIII meaning $1+1+1+1$.

Historically the next step was the current positional system, which means adding to addition some multiplication. The simplest such system is binary arithmetic, where we only allow multiplication of 2s – i.e., in effect, computing the powers of 2 – so that any natural number is represented as the sum of the corresponding powers of 2. Decimal numbers are somewhat more complicated, since in decimal numbers, a sum presenting a natural number includes not only powers of 10, but also products of these powers with numbers 1 through 9.

Why not go further? In view of our listing of arithmetic operations in the increasing order of their complexity, a natural idea is to go further and use general multiplication – just like general addition instead of its specific case transitioned unary numbers into a more compact Biblical representation. If we add multiplication, this would means, e.g., that we can represent numbers as products, e.g., 851 could be represented as a product $37 \cdot 23$.

Such representations can be useful in many situations: e.g., if we want to divide some amount between several people. This need for making division easy is usually cited as the main reason why, e.g., Babylonians used a 60-base system: since 60 can be easily divided by 3, 4, 5, 6, 10, 12, 15, 20, and 30 equal pieces.

So why was not such a product-based representation ever proposed?

What we do in this paper. In this paper, we propose a possible answer to this question.

2 Our explanation

Main idea behind our explanation. We want to be able to reason about numbers, we want to be able to decide which properties are true and which are not. From this viewpoint, it is reasonable to select a representation of numbers for which such a decision is algorithmically possible, i.e., for which there is an algorithm that, given any formula related to such an representation, decides whether this formula is true or not.

Let us analyze the historical and potential number representations from this viewpoint.

Case of unary numbers. Let us start with the simplest case of unary numbers. In this case, the basic object is 0 and the only operation is adding 1, which we can denote by $s(n) \stackrel{\text{def}}{=} n+1$.

It is therefore reasonable to consider, as elementary formulas, equalities between terms, where a term is something that is obtained from variables and a constant 0 by using the function $s(n)$. Then, we can combine elementary formulas by using logical connectives "and", "or", "not", "implies", and quantifiers $\forall n$ and $\exists n$. The resulting "closed" formulas – i.e., formulas in which each variable is covered by some quantifier – are known as *first-order formulas in signature* $\langle 0, s\rangle$.

It is known that there is an algorithm that, given any such formula, checks whether this formula is true or not.

Case of Biblical numbers. In this case, we also consider addition, i.e., we consider signature $\langle 0, n_1, \ldots, n_k, +\rangle$, where $n_1, \ldots, n_k$ are basic numbers. From the logical viewpoint, we can describe $s(n)$ and all the numbers n_i in terms of addition:

$$m = s(n) \Leftrightarrow (\exists a\,(m = n + a \,\&\, a \neq 0) \,\&\, \neg(\exists a \exists b\,(m = n + a + b \,\&\, a \neq 0 \,\&\, b \neq 0))),$$

and n_i is simply $s(\ldots(s(0)\ldots)))$, where $s(n)$ is applied n_i times. Thus, the class of all such formulas is equivalent to the class of all the formulas in the signature $\langle 0, +\rangle$.

For this theory – known as *Presburger arithmetic* – it is also known that there is an algorithm that, that, given any such formula, checks whether this formula is true or not.

Case of Roman-type numerals. The only think that Roman numerals add is subtraction. However, from the logical viewpoint, subtraction can be described in terms of addition: $a - b = c \Leftrightarrow a = b + c$, so this case is also reducible to Presburger arithmetic for which the deciding algorithm is possible.

Case of binary and decimal numbers. In this case, the signature is

$$\langle 0, n_1\cdot, \ldots, n_k\cdot, +, b^n\rangle,$$

where b is the base of the corresponding number system (in our examples, $b = 2$ or $b = 10$), and $n_i\cdot$ means an operation of multiplying by n_i (e.g., by 1, …, 9 in the decimal case).

Each operation $n \mapsto n_i \cdot n$ is simply equivalent to the addition repeated n_i times: $n_i \cdot n = n + \ldots + n$. Thus, from the logical viewpoint, the signature can be reduced simply to $\langle 0, +, b^n \rangle$. For such a signature, the existence of the deciding algorithm was, in effect, proven in [2].

Comment. To be more precise, the paper [2] explicitly states the existence of the deciding algorithm for $b = 2$. For $b > 2$, the existence of the deciding algorithm follows from the general result from [2] about the signature $\langle 0, +, f(n) \rangle$ for exponentially growing functions $f(n)$.

What if we add multiplication? If we also allow general multiplication, i.e., if we consider the signature $\langle 0, +, \cdot \rangle$, then – as is well known – the corresponding first order theory is undecidable, in the sense that no general algoroth, is possible that, given a formula, would decide whether this formula is true or not. This was, in effect, proven by Kurt Gödel in his famous results.

Conclusion. So, our idea – that people select representations for which a deciding algorithm is possible – indeed explains why some representations were historically used while some weren't:

- for representations that were used, there exist algorithm that decided whether each formula us true or not;
- in contrast, for representations that were not historically used, such an algorithm is possible.

Clarification. We definitely do not claim that, e.g., Romans knew modern sophisticated deciding algorithms, what we claim is that they had intuition about this – intuition that was later confirmed by theorems.

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