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From Historically First “Unary” Numbers, Through Egyptian Fractions, Roman Numerals, Leibniz's Binary Numbers and Kepler's Fractions to Modern Ideas Such as Calkin-Wilf Tree: A Unified Approach to Representing Natural Numbers and Fractions

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Abstract

In elementary mathematics classes, students are often overwhelmed by different representations of numbers and corresponding operations: usual fractions, decimal representations, binary numbers, etc. What often helps is when students learn the history of these representations, see the limitations of seemingly reasonable representations like Roman numerals, and how other representations overcame these limitations. Still, history was developed somewhat randomly, so the historical sequence is still somewhat chaotic. We believe that providing a unified approach for all these representations would help describe their sequence in a more logical way and thus, help the students even more.

In our analysis, we explore the relation between the foundations of arithmetic, especially related definitions of numbers, and the resulting notations. For example, widely used Peano axioms describe natural numbers as containing 0 and containing, for each x, the next number x + 1. The corresponding definition naturally leads to the historically first representation of natural numbers as I, II, III, IIII, etc. Allowing addition and multiplication by 2 (and powers of 2) leads to binary numbers, allowing general multiplication to decimal numbers, etc. It turns out that a similar foundational description can be found for most historical representations of natural and fractions. For example, allowing the division of 1 by a natural number leads to Egyptian fractions, allowing generic division of integers leads to common fractions, etc.

We believe that exposing students (and teachers) to at least some of these results will help them better understand the relation between different number representations, and thus, will make it easier for them to master the corresponding techniques.
Keywords: natural numbers and fractions, history of mathematics, foundations of mathematics, binary numbers, Egyptian fractions

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Introduction

In elementary mathematics classes, students are often overwhelmed by different representations of numbers and corresponding operations: usual fractions, decimal representations, binary numbers, etc. What often helps is when students learn the history of these representations, learn the limitations of seemingly reasonable representations like Roman numerals, and learn how other representations overcame these limitations.

Still, history was developed somewhat randomly, so the historical sequence is still somewhat chaotic. We believe that providing a systematic unified approach for all these representations would help describe their sequence in a more logical way and thus, help the students even more.

Purpose and objectives of the study

The main purpose of this study is to provide a systematic unified approach to historical number representations. We believe that exposing students (and teachers) to such a unified approach will help them better understand the relation between different number representations, and thus, will make it easier for them to master the corresponding techniques.

Literature review

There exist many books and paper on historical number systems and number representations; see, e.g., (Boyer & Merzbach, 2011), (Kosheleva & Villaverde, 2018) and references therein. These books and papers describe different representation of numbers, describe the practical motivation for these representations, describe how these representations historically evolved, and what were their strengths and limitations.

The problem is that most of these papers concentrated on a single historical representation – or, at best, on a few related historical representations. These papers do not provide a systematic unified view of different representations and thus, make it difficult to use several different representations when teaching students.

Methodology

In our analysis, we explore the relation between the foundations of arithmetic, especially related definitions of numbers, and the resulting representations.

Analysis of the problem

A set of objects is usually described by:

- listing basic objects and
- listing allowed operations, operations that enable us to construct new objects.

For numbers, this means that:

- we list several basic numbers, and
- we list operations that can be used to construct new numbers.

Let us describe what types of numbers and what operations we will consider in this paper.

In this paper, we will focus of two types of numbers:

- non-negative integers, also known as *natural* numbers 0, 1, 2, …, and
- non-negative fractions, also known as non-negative *rational* numbers.

For integers, natural arithmetic operations are addition, subtraction, and multiplication – and it is also natural to consider powers, since a power is nothing else but a repeated multiplication. For rational numbers, there is an additional arithmetic operation: division.

Addition, subtraction, and multiplication are exactly operations which are directly hardware supported in modern computers; see, e.g., (Knuth, 1980). To be more precise, only addition and multiplication are directly hardware supported, subtraction a – b is reduced to addition a + (--b) by using so-called 2's-complement representation of negative numbers.

All other computations are performed, in effect, as a sequence of additions and multiplications. For example, when we ask a computer to compute the value of a special function like exp(x) or sin(x), the computer actually computes the sum of the first few terms of the corresponding Taylor series, i.e., computes the value of the corresponding polynomial by performing the appropriate sequence of additions and multiplications.

How is division represented in a computer? Since computers use binary numbers, the simplest division for computers is division by 2 – it simply shifts the binary point, similarly to how division by 10 is the easiest for us humans, who use decimal system. A general division a / b is implemented in a computer as the product a * (1 / b), where the inverse 1 / b is actually computed as a sequence of additions and multiplications.

Also, in the computers, simplest special cases of addition and subtraction – namely, the operations of adding one and subtracting one – are usually hardware supported as separate operations, which in computer science are usually denoted by ++ (next integer) and -- -- (previous integer). These separate operations make sense: indeed, adding one is a very frequent operation corresponding to counting, and since adding one is much faster than the general addition, implementing adding 1 as a separate hardware supported operation enables us to decrease computation time.

Let us analyze what number systems we can get if we use some of these operations. We will start with number systems describing natural numbers, and then move to number systems describing non-negative fractions.

It is important to notice that while this way, we do describe, in a systematic way, practically all historically proposed number systems, the resulting systematic order in which we present these systems will be sometimes different from the historic order – i.e., the order in which these systems are usually represented in books on history of mathematics.

What if we only use ++ to represent natural numbers: from historically first representation of numbers to Peano arithmetic

Let us start with the case when, to construct new natural numbers, we use only one arithmetic operation, namely the simplest arithmetic operation ++ of adding 1. What basic natural numbers do we need to have in such a description? Since we want to represent 0, and 0 cannot be described as a natural number + 1, we therefore have to have 0 as one of the basic numbers. There is no need to have any other basic number: once we have 0, we can construct any natural number by adding an appropriate number if 1s: 1 is 0 + 1, 2 is 0 + 1 + 1, etc.

This is exactly how natural numbers are described in so-called Peano arithmetic, the most well-known and the most widely used formal description of natural numbers:

- 0 is a natural number, and
- if n is a natural number, then the next value – which in Peano arithmetic is usually denoted by s(n) – is also a natural number.

What representation of natural numbers do we get this way? In this approach, each natural number can be described by listing all adding-one operations that we need to apply to 0 to get this number. If we denote one application of the adding-one operation by a vertical line, then numbers 1, 2, 3, 4, etc. are represented as, correspondingly, I, II, III, IIII, etc. This is known as the *unary* number system, since we use only one symbol to represent all natural numbers -- as opposed to, e.g., the usual decimal system where we use ten different symbols 0, 1, …, 9. The unary system is exactly how our ancestors counted: e.g., if they needed to record that someone borrowed 13 sheep, they scratched 13 lines on a stick. For small numbers, this is how 1, 2, 3, and sometimes 4 were represented in Roman numerals – and this is how such numbers are marked on grandfather's clocks.

An important feature is that in this representation, there is exactly one way to represent a given natural number: namely, to represent a number n, we write down II…I, where I is repeated n times.

What if we also use addition: Biblical numbers

Let us recall that the simplest arithmetic operation is addition, and the ++ operation of adding one is the simplest particular case of this operation. In the previous section, we considered the case when we only use ++. A limitation of this representation is that it takes too much space: e.g., to represent number 1000, we need to use 1000 symbols. To make representations shorter, a natural idea is to use other arithmetic operations. The natural next operation is full addition,

So, instead of a single symbol, we use several symbols describing several different natural numbers, and we form other natural numbers by adding the given numbers. Such systems have indeed been ubiquitous in the ancient world. For example, to describe numbers in the Hebrew Bible, the first 9 letters of the Hebrew letters of the Hebrew alphabet were assigned values 1 through 9, the next 9 letters 10, 20, …, 90, then 100, etc. This provides for a more compact representation: e.g., we only need 3 symbols to represent the number 144 – the symbols for 100, 40, and 4.

The disadvantage is that, in principle, the representation stops being unique: e.g., the number 16 can be, in principle, represented both as 10 + 7, and as 9 + 6, and such a 9-using representation was actually used; see, e.g, (Kosheleva & Kreinovich, 2019) and references therein.

What if we also allow subtraction: Roman numerals

To get an even more compact representation, it is reasonable to allow other operations. The simplest operation that we did not use was subtraction. This is how the Roman numerals were built: we start with symbols describing several natural numbers, such as I for 1, V for 5, X for 10, L for 50, C for 100, D for 500, and M for 1000, and then we use addition and subtraction to represent other natural numbers.

If a symbol for a smaller number follow the symbol for a larger number, it means addition, otherwise it means subtraction. For example, 144 was represented as CXLIV, which means 100 + (50 – 1) + (5 – 1).

This system also allows several different representations for the same number: e.g., the number 4 was sometimes represented as IIII and sometimes as IV.

What if we allow the simplest possible multiplication – multiplication by 2: binary numbers

The next-in-complexity arithmetic operation is multiplication. The simplest multiplication is, of course, multiplication by 0 or 1, but such multiplication does not lead to any new natural numbers. The simplest multiplication that leads to new natural numbers is multiplication by 2. This leads to a binary number system, in which every natural number is represented as the sum of appropriate powers of 2, i.e., as the sum of numbers obtained from 1 by consequent multiplication by 2. For example, $144 = 128 + 16 = 2^7 + 2^4$.

Strictly speaking, this should be

$$2 * 2 * 2 * 2 * 2 * 2 * 2 + 2 * 2 * 2 * 2,$$

so, in effect, binary system uses also raising to the power – namely, the simplest case of raising to the power, when we only raise number 2 to different powers.

What if we allow multiplication by several numbers: decimal numbers

If instead of multiplying by 2, we allow multiplying by all natural numbers from 2 to 10 (and instead of raising 2 to different powers we raise 10 to different powers), we get our usual decimal system, where, e.g., 144 is represented as $10 * 10 + 4 * 10 + 4 = 10^2 + 4 * 10^1 + 4 * 10^0$.

This is the first example where a natural sequence differs from a historic one: decimal numbers were known for many centuries, since the 4^{th} century or even earlier, while binary numbers were only introduced by Leibniz in the 17 century.

How to represent fractions

Let us now switch to describing how we can represent fractions. To represent fractions, we need to use division to operations describing natural numbers. As we have mentioned, in computers, division a / b is implanted as a * (1 / b). So, a natural idea is to use the operation 1 / b.

What if we only allow division by 2: binary fractions

As we have mentioned, for a computer, the simplest possible division is division by b = 2 – and also repeated divisions by 2, which are equivalent to dividing by the corresponding power of 2. Since we allow division by 2, it is reasonable to also allow similar simpler operations: multiplication by 2 and addition. This leads to binary fractions like $101.011_2 = 2^2 + 1 + 1 / 2^2 + 1 / 2^3$.

What if we only allow division by 10: decimal fractions

Instead of division by 2, we can allow division by 10. This operation is easy for us humans (as opposed to being easier for computers). In combination with multiplication by 10 (and multiplication by natural numbers smaller than 10), this leads to our usual decimal fractions, e.g.:

$$21.44 = 2 * 10 + 1 + 4 / 10 + 4 / 10^2.$$

It is important to note that if we are only interested in *proper* fractions, i.e., fractions smaller than 1, then we do not need multiplication: all we need is to start with 9 digits 1, …, 9, and allow division by 10 and addition. For example, 0.44 can be represented as $0.44 = 4 / 10 + 4 / 10^2$.

What if we allow 1 / b for all natural numbers b: Egyptian fractions

In the previous two sections, we considered the cases when we only allow division by one specific natural number: b = 2 for binary fractions and b = 10 for decimal fraction. A natural next idea is to allow divisions by any natural number, i.e., to allow fractions of the type 1 / b, where b is an arbitrary integer.

It turns out that if we are only interested in representing proper fractions, then any such fractions can be represented as the sum of numbers 1 / b of such type. For example, 2 / 3 = 1 / 2 + 1 / 6, 3 / 4 = 1 / 2 + 1 / 4, etc. Such representations are known as *Egyptian fractions*, since this is how ancient Egyptians represented fractions; see, e.g., (Boyer & Merzbach, 2011), (Kosheleva & Kreinovich, 2009), (Kosheleva & Villaverde, 2018) and references therein.

What if we allow 1 / b for all natural numbers b and multiplications by all natural numbers: regular fractions

What if, in addition to the inverse, instead of adding, we allow multiplication by any natural number? In this case, we get all possible fractions of the type a / b, i.e., all regular fractions like 2 / 3 or 3 / 4.

What if we allow 1 / b for any numbers b: Kepler's fractions

What if we allow 1 / b for all possible numbers b? Of course, if we start with a finite list of numbers and only apply the operation 1 / b, then all we get is the original numbers and their inverses. So, to be able to represent arbitrary non-negative rational numbers, we need to allow at least one more operation.

As we have mentioned earlier, the simplest arithmetic operation is adding 1. It turns out that if we start with 0 and use these two operations: the inverse 1 / b and adding one, then we can represent any non-negative fraction. This idea was first proposed by Kepler in (Kepler, 1619). Thus, such fractions are known as *Kepler fractions*; this representation is also known as *Calkin-Wilf tree.*

Interestingly, in this case we have a unique representation of each positive fraction. Indeed, if we have a fraction a / b and the numerator and the denominator have a common factor, then we can divide both a and b by this factor and get a fraction in which a and b have no common divisors. In this case, we cannot have a = b, so we must have either a < b or a > b. If a < b, then a / b < 1, so we cannot get this number by adding 1 to a positive number; thus, we can get a / b by applying inverse to b / a.

If a > b, then we can represent a / b as (a – b) / b + 1. So to represent the number a / b this way, it is sufficient to represent the fraction (a – b) / b. In each such transformation, the maximum of numerator and denominator decreases, so in finitely many steps, this procedure will stop, and we will get the desired representation.

Let us illustrate this idea on the examples of fractions 3 / 4 and 5 / 13. Let us start with 3 / 4. Since 3 / 4 < 1, we represent it as 1 / (4 / 3). Here, 4 / 3 = 1 / 3 + 1, i.e., 1 / (1 + 1 + 1) + 1. Thus,

$$3 / 4 = 1 / (1 / (1 + 1 + 1) + 1).$$

For 5 / 13, we similarly have 5 / 13 = 1 / (13 / 5). Here, 13 / 5 = 8 / 5 + 1, and 8 / 5 = 3 / 5 + 1, so

$$13 / 5 = 3 / 5 + 1 + 1.$$

In this case, $3 / 5 = 1 / (5 / 3)$, where $5 / 3 = 2 / 3 + 1$. In its turn, $2 / 3 = 1 / (3 / 2)$, where $3 / 2 = 1 / 2 + 1$, i.e., $3 / 2 = 1 / (1 + 1) + 1$. So, $2 / 3 = 1 / (1 / (1 + 1) + 1)$, $5 / 3 = 1 / (1 + 1 / (1 + 1)) + 1$, thus

$$3 / 5 = 1 / (1 / (1 / (1 + 1) + 1) + 1),$$

$$13 / 5 = 1 / (1 / (1 / (1 + 1) + 1) + 1) + 1 + 1, \text{ and}$$

$$5 / 13 = 1 / (1 / (1 / (1 / (1 + 1) + 1) + 1) + 1 + 1).$$

Similar to Peano arithmetic, we can therefore describe non-negative rational numbers as follows:

- 0 is a non-negative rational number;
- if a is a non-negative rational number, then a + 1 is a non-negative rational number, and
- if a is a non-negative rational number different from 0, then 1 / a is a non-negative rational number.

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