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Fair Bankruptcy Solutions Under Interval Uncertainty

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Abstract If the overall amount of the company's assets is smaller than its total debts, then a fair solution is to give, to each creditor, the amount proportional to the corresponding debt, e.g., 10 center for each dollar or 50 cents for each dollar. But what if the debt amounts are not known exactly, and for some creditors, we only know the lower and upper bounds on the actual debt amount? What division will be fair in such a situation? In this paper, we show that the only fair solution is to make payments proportional to an appropriate convex combination of the bounds – which corresponds to Hurwicz optimism-pessimism criterion for decision making under interval uncertainty.

1 Formulation of the Problem

What is a bankruptcy problem. A company goes bankrupt if the total amount of its assets is smaller than the total amount of debts. Some of the debts have priority – e.g., according to the US labor law, salary needs to be paid in full, irrespective of debts to others. Once these priority debts are paid, we face a problem of how to divide the remaining assets A between the creditors to whom the company owes amounts $d_1, \ldots, d_n$.

How this problem is usually solved. In this case, a usual solution is to make payments proportional to debts, i.e., depending on the ratio between the assets and the debts, 10 cents per dollar, 50 cents per dollar, etc. In general, the amount g_i given to the i-th creditor is equal to

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$$g_i = d_i \cdot \frac{E}{\sum_{j=1}^{n} d_j}. \quad (1)$$

Need to take interval uncertainty into account. In some cases, the debt is purely monetary, and its amount d_i is known exactly. In many practical situations, however, the situation is more complicated, so for many creditors, we only know the bounds $\underline{d}_i \le d_i \le \overline{d}_i$ of the actual debt amount. How should we divide the assets in this situation?

Case of interval uncertainty: how is this problem solved now. Several papers describe how to solve the bankruptcy problem under interval uncertainty. For example, the paper [2] suggests selecting a single value d_i within each interval, and then using these values d_i to divide the assets. For example, to select d_i, we can use Hurwicz optimism-pessimism criterion [3, 6, 8]: namely, we agree on some value $\alpha \in [0,1]$ and take $d_i = \alpha \cdot \overline{d}_i + (1-\alpha) \cdot \underline{d}_i$.

A more complex scheme was proposed in [7] – following a solution to a similar problem in [15].

What we do in this paper. In this paper, we show that a natural formalization of fairness uniquely determines Hurwicz-based solutions – which are thus recommended as the fair ones.

2 How to Describe Fairness

Fairness: first requirement. Fairness means, first, that if the debt d_i to creditor i is smaller than or equal to the debt d_j to creditor j, then the payment g_i to creditor i should be smaller than or equal to the payment to creditor j.

Fairness: second requirement. Second, fairness means that two creditors should not gain or lose by joining together. In other words:

- if for debts $d_1, d_2, d_2, \ldots, d_n$, we had payments $g_1, g_2, g_3, \ldots, g_n$,
- them for debts $d_1 + d_2, d_2, \ldots, d_n$, we should have payments $g_1 + g_2, g_3, \ldots, g_n$.

Continuity. It also makes sense to require that if in two situations, debts are close, then payments should be close – i.e., that payments should be a continuous function of debts.

3 What If We Impose Fairness Requirements in Situations When We Know the Exact Amount of Debts

Before we consider the case of interval uncertainty, let us analyze what will happen if we impose fairness requirements in the situations when we know the exact amount of debt.

Definition 1. *Let $A < D$ be two positive numbers.*

- *We will call A the* amount of assets*, and we will call D* the amount of debt*.*
- *By a* solution *to the bankruptcy problem (or simply* solution*, for short), we mean a function S that maps every tuple $\langle d_1,\ldots,d_n\rangle$ of positive real numbers for which $d_1+\ldots+d_n = D$ into a tuple of non-negative real numbers $\langle g_1,\ldots,g_n\rangle$ for which*

$$g_1+\ldots+g_n = A.$$

Definition 2. *We say that the solution S is* fair *if it satisfies the following two requirements for each tuple $\langle d_1,d_2,d_3,\ldots,d_n\rangle$ and for $S(\langle d_1,\ldots,d_n\rangle) = \langle g_1,\ldots,g_n\rangle$:*

- *if $d_i \le d_j$, then $g_i \le g_j$;*
- *$S(\langle d_1+d_2,d_3,\ldots,d_n\rangle) = \langle g_1+g_2,g_3,\ldots,g_n\rangle$.*

Definition 3. *We say that the solution S is* continuous*, if for every n, if $d_i^{(k)} \to d_i$ for all i, $S(\langle d_1^{(k)},\ldots,d_n^{(k)}\rangle) = \langle g_1^{(k)},\ldots,g_n^{(k)}\rangle$, and $g_i^{(k)} \to g_i$ for all i, then*

$$S(\langle d_1,\ldots,d_n,A\rangle) = \langle g_1,\ldots,g_n\rangle.$$

Proposition 1. *For each solution S, the following two conditions are equivalent to each other:*

- *the solution is fair and continuous,*
- *the solution has the form*

$$g_i = d_i\cdot(A/D). \tag{2}$$

Comment. So, the usual solution is the only one which is fair (and continuous).

Proof. It is easy to check that the above solution is fair and continuous. So, to complete the proof, it is sufficient to prove that every fair continuous solution S has this form.

Indeed, let S be a fair and continuous solution. For every natural number N, we can consider the tuple $\langle d_1,\ldots,d_N\rangle = \langle D/N,\ldots,D/N\rangle$ consisting of N equal debt values. By the first fairness requirement, since the debts d_i are all equal, the payments g_i are also all equal. Since $g_1+\ldots+g_N = A$, this means that $N\cdot g_i = A$ hence $g_i = A/N$, and the payments tuple has the form $\langle g_1,\ldots,g_N\rangle = \langle A/N,\ldots,A/N\rangle$.

For any sequence of natural numbers $k_1,\ldots,k_n$ for which $k_1+\ldots+k_n = N$, the tuple $\langle k_1\cdot(D/N),\ldots,k_n\cdot(D/N)\rangle$ can be obtained from the tuple $\langle 1/N,\ldots,1/N\rangle$ by adding up the first k_1 terms, then the next k_2 terms, etc. So, due to the second fairness requirements, the resulting payment tuple $\langle g_1,\ldots,g_n\rangle$ can be obtained from the tuple $\langle A/N,\ldots,A/N\rangle$ by adding the first k_1 terms, then the next k_2 terms, etc. Thus, the payment tuple has the form $\langle k_1\cdot(A/N),\ldots,k_n\cdot(A/N)\rangle$. In other words, for each debt $d_i = k_i\cdot(D/N)$, the payment is equal to $g_i = k_i\cdot(A/N)$. From $d_i = k_i\cdot(D/N)$, we conclude that $k_i = d_i\cdot(N/D)$, hence $g_i = k_i\cdot(A/N) = d_i\cdot(N/D)\cdot(A/N) = d_i\cdot(A/D)$, i.e., that indeed $g_i = d_i\cdot(A/D)$.

We have proved the desired equality (2) for all the cases when for all the debts d_i, we have $d_i = k_i\cdot(D/N)$ for some integer k_i, i.e., when $d_i/D = k_i/N$. Any real number d_i/D can be approximated – with accuracy $1/N$ – by an appropriate fraction k_i/N. As N increases, the fraction tends to d_i/D. Thus, since the solution S is continuous, in the limit, we will have (2) for all possible real values d_i.

The proposition is proven.

Comment. At first glance, it may sound reasonable to also require that if we combine two bankruptcy problems together, then in the combined problem, each creditors should receive the sum of what he/she would receive in each solutions. In other words:

- if we have $S(\langle d_1,\ldots,d_n\rangle) = \langle g_1,\ldots,g_n\rangle$ and $S(\langle d'_1,\ldots,d'_n\rangle) = \langle g'_1,\ldots,g'_n\rangle$,
- then we should have $S(\langle d_1+d'_1,\ldots,d_n+d'_n\rangle) = \langle g_1+g'_1,\ldots,g_n+g'_n\rangle$.

This requirement is explicitly mentioned in [7]. Let us show, however, that the fair solution does not have this property. Indeed:

- let us take $d_1 = 4$, $d_2 = 1$, and $A = 2$, then $D = d_1+d_2 = 4+1 = 5$, so $A/D = 2/5 = 0.4$, $g_1 = d_1\cdot(A/D) = 4\cdot 0.4 = 1.6$, and $g_2 = d_2\cdot(A/D) = 1\cdot 0.4 = 0.4$;
- let us also take $d'_1 = d'_2 = 1$ and $A' = 1$, then $D' = d'_1+d'_2 = 1+1 = 2$, so $A'/D' = 1/2 = 0.5$, and $g'_i = d'_i\cdot(A'/D') = 1\cdot 0.5 = 0.5$.

On the other hand, for $d_1+d'_1 = 5$, $d_2+d'_2 = 2$, and $A+A' = 3$, we have $D+D' = 7$, so $(A+A')/(D+D') = 3/7$. Thus, for the first creditor, the payment is

$$(d_1+d'_1)\cdot((A+A')/(D+D')) = 5\cdot(3/7) = 15/7 = 2+1/7,$$

which is different from this creditor's summary payment $g_1+g'_1 = 1.6+0.5 = 2.1$ in two original situations.

4 Case of Interval Uncertainty

Interval sum and interval order: reminder. In the case of interval uncertainty, if we only know that the debt d_1 is in the interval $[\underline{d}_1,\overline{d}_1]$ and that the debt d_2 is in the interval $[\underline{d}_2,\overline{d}_2]$, then the only conclusion we can make about the summary debt d_1+d_2 to these two creditors is that this sum belongs to the interval

$$[\underline{d}_1+\underline{d}_2,\overline{d}_1+\overline{d}_2].$$

This interval is known as the *sum* $[\underline{d}_1,\overline{d}_1]+[\underline{d}_2,\overline{d}_2]$ of the two intervals $[\underline{d}_1,\overline{d}_1]$ and $[\underline{d}_2,\overline{d}_2]$; see, e.g., [4, 9, 11].

A natural order is component-wise: we say that the debt $[\underline{d}_i,\overline{d}_i]$ to creditor i is smaller than or equal to the debt $[\underline{d}_j,\overline{d}_j]$ to creditor j if $\underline{d}_i \le \underline{d}_j$ and $\overline{d}_i \le \overline{d}_j$.

Definition 4. *Let A be a positive real numbers and let $[\underline{D},\overline{D}]$ be an interval for which $0<\underline{D}$ and $A<\overline{D}$.*

- *We will call A* the amount of assets*, and we will call $[\underline{D},\overline{D}]$* the amount of debt.
- *By a* solution *to the bankruptcy problem (or simply* solution*, for short), we mean a function S that maps every tuple $\langle[\underline{d}_1,\overline{d}_1]\ldots,[\underline{d}_n,\overline{d}_n]\rangle$ of intervals for which $0\le\underline{d}_i$, numbers for which $\underline{d}_1+\ldots+\underline{d}_n=\underline{D}$, and $\overline{d}_1+\ldots+\overline{d}_n=\overline{D}$ into the same-size tuple of non-negative real numbers $\langle g_1,\ldots,g_n\rangle$ for which*

$$g_1+\ldots+g_n=A.$$

Definition 5. *We say that the solution S is* fair *if the following two requirements are satisfied when $S(\langle[\underline{d}_1,\overline{d}_1],\ldots,[\underline{d}_n,\overline{d}_n]\rangle)=\langle g_1,\ldots,g_n\rangle$:*

- *if $\underline{d}_i\le\underline{d}_j$ and $\overline{d}_i\le\overline{d}_j$, then $g_i\le g_j$;*
- *$S(\langle[\underline{d}_1+\underline{d}_2,\overline{d}_1+\overline{d}_2],[\underline{d}_3,\overline{d}_3],\ldots,[\underline{d}_n,\overline{d}_n]\rangle)=\langle g_1+g_2,g_3\ldots,g_n\rangle$.*

Definition 6. *We say that the solution S is* continuous*, if for every n, if $\underline{d}_i^{(k)}\to d_i$ and $\overline{d}_i^{(k)}\to d_i$ for all i, $S(\langle[\underline{d}_1^{(k)},\overline{d}_1^{(k)}],\ldots,[\underline{d}_n^{(k)},\overline{d}_n^{(k)}]\rangle)=\langle g_1^{(k)},\ldots,g_n^{(k)}\rangle$, and $g_i^{(k)}\to g_i$ for all i, then*

$$S(\langle[\underline{d}_1,\overline{d}_1],\ldots,[\underline{d}_n,\overline{d}_n]\rangle)=\langle g_1,\ldots,g_n\rangle.$$

Proposition 2. *For each solution S, the following two conditions are equivalent to each other:*

- *the solution is fair and continuous,*
- *for some $\alpha\in[0,1]$, the solution has the form $g_i=d_i\cdot(A/D)$, where*

$$d_i \stackrel{\text{def}}{=} \alpha\cdot\overline{d}_i+(1-\alpha)\cdot\underline{d}_i \text{ and } D \stackrel{\text{def}}{=} \alpha\cdot\overline{D}+(1-\alpha)\cdot\underline{D}.$$

Comment. So, the solutions based on Hurwicz combinations $d_i=\alpha\cdot\overline{d}_i+(1-\alpha)\cdot\underline{d}_i$ are the only one which are fair (and continuous).

Proof. It is east to check that the solution based on Hurwicz combination is fair and continuous. So, to complete the proof, it is sufficient to prove that every fair continuous solution S has this form.

Indeed, let S be a fair and continuous solution. For every natural number N, we can consider the tuple

$$\langle[\underline{D}/N,\underline{D}/N],\ldots,[\underline{D}/N,\underline{D}/N],[0,(\overline{D}-\underline{D})/N],\ldots,[0,(\overline{D}-\underline{D})/N]\rangle \qquad (3)$$

consisting of:

- N degenerate debt intervals $[\underline{D}/N, \underline{D}/N]$ and
- N intervals $[0, (\overline{D} - \underline{D})/N]$.

By the first fairness requirement, since the debts d_i are the same for all first N creditors, the payments g_i should also be all equal $g_1 = \ldots = g_N$. Similarly, the payments to the last N creditors should be the same: $g_{N+1} = \ldots = g_{2N}$.

For any two sequences of natural numbers $k_1, \ldots, k_n, \ell_1, \ldots, \ell_n$ for which

$$k_1 + \ldots + k_n = \ell_1 + \ldots + \ell_n = N,$$

the tuple

$$\langle [k_1 \cdot (\underline{D}/N), k_1 \cdot (\underline{D}/N) + \ell_1 \cdot (\overline{D} - \underline{D})/N], \ldots,$$
$$[k_n \cdot (\underline{D}/N), k_n \cdot (\underline{D}/N) + \ell_n \cdot (\overline{D} - \underline{D})/N] \rangle$$

can be obtained from the tuple (3) by adding up:

- the first k_1 intervals from the first half and the first ℓ_1 intervals from the second half, then
- the next k_2 intervals from the first half and the next ℓ_2 intervals from the second half, etc.

So, due to the second fairness requirements, the resulting payment tuple $\langle g_1, \ldots, g_n \rangle$ can be obtained from the tuple $\langle g_1, \ldots, g_1, g_{N+1}, \ldots, g_{N+1} \rangle$ by adding the corresponding payment terms. Thus, the payment tuple has the form

$$\langle k_1 \cdot g_1 + \ell_1 \cdot g_{N+1}, \ldots, k_n \cdot g_1 + \ell_n \cdot g_{N+1} \rangle.$$

In other words, for each debt interval

$$[\underline{d}_i, \overline{d}_i] = [k_i \cdot (\underline{D}/N), k_i \cdot (\underline{D}/N) + \ell_i \cdot (\overline{D} - \underline{D})/N],$$

the payment is equal to

$$g_i = k_i \cdot g_1 + \ell_i \cdot g_{N+1}. \tag{4}$$

Here, $\underline{d}_i = k_i \cdot (\underline{D}/N)$, so $k_i = \underline{d}_i \cdot (N/\underline{D})$. Similarly, $\overline{d}_i - \underline{d}_i = \ell_i \cdot ((\overline{D} - \underline{D})/N)$ so $\ell_i = (\overline{d}_i - \underline{d}_i) \cdot (N/(\overline{D} - \underline{D}))$. Substituting these expressions for k_i and ℓ_i into the formula (3), we conclude that $g_i = a \cdot \underline{d}_i + b \cdot (\overline{d}_i - \underline{d}_i)$, where we denoted $a \stackrel{\text{def}}{=} g_1 \cdot (N/\underline{D})$ and $b \stackrel{\text{def}}{=} g_{N+1} \cdot (N/(\overline{D} - \underline{D}))$. Thus, we have

$$g_i = b \cdot \overline{d}_i + (a - b) \cdot \underline{d}_i. \tag{5}$$

The first fairness requirement means that if $\overline{d}_i$ is larger then $\overline{d}_j$ while $\underline{d}_i = \underline{d}_j$, then g_i should be larger (or the same) than g_j. This implies that $a \geq 0$. Similarly, if $\underline{d}_i$ is larger then $\underline{d}_j$ while $\overline{d}_i = \overline{d}_j$, then g_i should be larger (or the same) than g_j. This implies that $a - b \geq 0$.

Let us denote the ratio b/a by α. Then, $b = a \cdot \alpha$ and $a - b = a \cdot (1 - \alpha)$. Thus, the formula (5) takes the form

$$g_i = a \cdot (\alpha \cdot \overline{d}_i + (1-\alpha) \cdot \underline{d}_i). \tag{6}$$

The sum of all the payments is equal to A, so

$$g_1 + \ldots + g_n = a \cdot (\alpha \cdot d_1 + (1-\alpha) \cdot \underline{d}_1 + \ldots + \alpha \cdot d_n + (1-\alpha) \cdot \underline{d}_n) =$$

$$a \cdot (\alpha \cdot (\underline{d}_1 + \ldots + \underline{d}_n) + (1-\alpha) \cdot (\overline{d}_1 + \ldots + \overline{d}_n)) = a \cdot (\alpha \cdot \overline{D} + (1-\alpha) \cdot \overline{D}) = a \cdot D,$$

hence $a = A/D$ and the formula (6) takes the desired form

$$g_i = (\alpha \cdot \overline{d}_i + (1-\alpha) \cdot \underline{d}_i) \cdot (A/D). \tag{7}$$

We have proved the desired equality (7) for all the cases when for all the creditors i, we have $\underline{d}_i = k_i \cdot (\underline{D}/N)$ for some integer k_i and $\overline{d}_i - \underline{d}_i = \ell_i \cdot ((\overline{D} - \underline{D})/N)$ for some integer ℓ_i. Similarly to the proof of Proposition 1, any two real numbers can be thus approximated, and the larger N, the more accurate the resulting approximation. Thus, due to continuity, in the limit $N \to \infty$, we have (7) for all possible values $\underline{d}_i$ and $\overline{d}_i$.

The proposition is proven.

First comment: what if we have fuzzy uncertainty? For each creditor, instead of a single interval, we can have different intervals $[\underline{d}_i(\alpha), \overline{d}_i(\alpha)]$ containing d_i with different degrees of uncertainty $\alpha \in [0,1]$. If we pick a narrower sub-interval, then we become less certain that d_i belongs to this sub-interval than that it belongs to the original interval. Thus, the interval corresponding to a higher degree of uncertainty is a subset of the interval corresponding to a lower degree of uncertainty. Such a sequence of embedded intervals is, in effect, an equivalent representation of a so-called *fuzzy number* (see, e.g., [1, 5, 10, 12, 13, 16]) for which the corresponding intervals are known as α-*cuts*.

In this case, to describe each creditor's debt, instead of two values $\underline{d}_i$ and $\overline{d}_i$, we need to describe infinitely many values $\underline{d}_i(\alpha)$ and $\overline{d}_i(\alpha)$ corresponding to different $\alpha \in [0,1]$. The overall debt corresponding to different α can be obtained by adding all n debts: $\underline{D}(\alpha) = \underline{d}_1(\alpha) + \ldots + \underline{d}_n(\alpha)$ and $\overline{D}(\alpha) = \overline{d}_1(\alpha) + \ldots + \overline{d}_n(\alpha)$.

Arguments similar to the ones we used in the proof of Proposition 2 lead to a conclusion that a fair solution is proportional to the linear combination d_i of these values, i.e., has the form $g_i = d_i/D$, where

$$d_i \stackrel{\text{def}}{=} \int (f_-(\alpha) \cdot \underline{d}_i(\alpha) + f_+(\alpha) \cdot \overline{d}_i(\alpha))\, d\alpha$$

for some functions (maybe generalized functions) $f_\pm(\alpha)$, and

$$D \stackrel{\text{def}}{=} \int (f_-(\alpha) \cdot \underline{D}(\alpha) + f_+(\alpha) \cdot \overline{D}(\alpha))\, d\alpha.$$

What if we have probabilistic uncertainty? What if for each d_i, we only know the probability distribution? In this case, it makes sense to use the following additional requirement on the bankruptcy solutions: that if we repeat the same division situation several (N) times, the payments in the resulting overall situation should be N times larger. In the overall situation, the debt amount D_i is equal to the sum of N independent equally distributed debt amounts: $D_i = d_i^{(1)} + \ldots + d_i^{(N)}$. According to the Large Numbers Theorem (see, e.g., [14]), for large N, the average

$$\frac{D_i}{N} = \frac{d_i^{(1)} + \ldots + d_i^{(N)}}{N}$$

tends to the mean $E[d_i]$ as N increases. Thus, for large N, the sum is getting (relatively) closer and closer to a single value – N times the mean. So, for large N, we have, in effect, the division problem in which instead of the original random variables, we have N times their means. The payments in the original problem should be N times smaller, i.e., they should be simply equal to the division corresponding to the means.

Thus, in the probabilistic case, we should simply compute the mean values $E[d_i]$ of the debt amount, and distribute the assets proportionally to these mean values:

$$g_i = \frac{E[d_i]}{E[d_1] + \ldots + E[d_n]} \cdot A.$$

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