3-1-2022

Physical Meaning Often Leads to Natural Derivations in Elementary Mathematics: On the Examples of Solving Quadratic and Cubic Equations

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Comments:

Technical Report: UTEP-CS-22-42

Physical Meaning Often Leads to Natural Derivations in Elementary Mathematics: On the Examples of Solving Quadratic and Cubic Equations

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Abstract Usual derivation of many formulas of elementary mathematics – such as the formulas for solving quadratic equation – often leave un unfortunate impression that mathematics is a collection of unrelated unnatural trick. In this paper, on the example of formulas for solving quadratic and cubic equations, we show that these derivations can be made much more natural if we take physical meaning into account.

1 Formulation of the problem

In elementary mathematics, many derivations feel unnatural. Derivations of many formulas or elementary mathematics are usually presented as useful tricks. These tricks result from insights of ancient geniuses.

A formula for solving quadratic equations is a good example. These derivations do not naturally follow from the formulation of the problem.

Resulting problem. The perceived un-naturalness of these derivations leads to two problems.

- Short-term problem: the perceived un-naturalness makes these derivations – and thus, the resulting formulas – difficult to remember.

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- Long-term problem: this un-naturalness adds to the unfortunate students' impression that mathematics is an un-natural collection of unrelated tricks.

From this viewpoint, it is desirable to come up with natural – or at least more natural – derivation of such formulas.

What we do in this paper. In this paper, we show that taking into account physical meaning can help.

To be more precise:

- We are not proposing radically new derivations.
- We just show that physical meaning makes some of the available derivations natural.

2 Solving quadratic equations

Simplest case. It is straightforward to solve simple quadratic equations, of the type

$$a \cdot x^2 + c = 0.$$

Indeed:

- First, similarly to linear equations, we subtract c from both sides and divide both sides by a.
- This way, we get $x^2 = -c/a$.
- Then we extract the square root, and get $x = \pm\sqrt{-c/a}$.

Need for a general case. But what if we have a generic equation $a \cdot x^2 + b \cdot x + c = 0$? This happens, e.g., when:

- we know the area $s = x \cdot y$ and the perimeter $p = 2x + 2y$ of a rectangular region, and
- we want to find its sizes x and y.

In this case, we can plug in $y = p/2 - x$ into $s = x \cdot y$ and get a quadratic equation

$$x \cdot \left(\frac{p}{2} - x\right) = s.$$

By opening the parentheses and moving all the terms to the right-hand side, we get

$$x^2 - \frac{p}{2} \cdot x + s = 0.$$

Physical meaning. In real-life applications, x is often a numerical value of some physical quantity. Numerical values depend:

- on the choice of the measuring unit and

- on the choice of a starting point.

So, a natural idea is to select a new scale in which the equation would get a simpler form.

What happens if we change a measuring unit? If we replace a measuring unit by a new one which is λ times larger, we get new value x' for which $x = \lambda \cdot x'$. Substituting this expression instead of x into the equation, we get a new equation

$$a \cdot (\lambda \cdot x)^2 + b \cdot \lambda \cdot x + c = 0.$$

However, this does not help to solve this equation.

What happens if we change a starting point? Selecting a new starting point changes the original numerical value x to the new value x' for which $x = x' + x_0$. Substituting this expression for x into the given quadratic equation, we get

$$a \cdot (x' + x_0)^2 + b \cdot (x + x_0) + c =$$

$$a \cdot (x')^2 + (2a \cdot x_0 + b) \cdot x' + (a \cdot x_0^2 + b \cdot x_0 + c) = 0.$$

For an appropriate x_0 – namely, for $x_0 = -b/(2a)$ – we can make the coefficient at x' equal to 0.

Thus, we get a simple quadratic equation – which we know how to solve.

3 Cubic equations

General description. A general cubic equation has the form

$$a \cdot x^3 + b \cdot x^2 + c \cdot x + d = 0.$$

Same idea helps, but does not yet lead to a solution. We can use the same idea as for the quadratic equation – namely, change from x to $x' = x + a_0$, and get a simplified equation

$$a \cdot (x')^3 + c' \cdot x' + d' = 0.$$

In contrast to the quadratic case, however, the resulting equation is still difficult to solve. We still need to reduce it to an easy-to-solve case with no linear terms.

Additional physics-related idea. To simplify the equation, let us take into account another natural physics-related idea: that often, a quantity is the sum of several ones.

For example, the mass of a complex system is the sum of the masses of its components.

This idea works. Let us consider the simplest case $x' = u + v$.

Substituting this expression into the above equations, we get

$$a \cdot (u+v)^3 + c' \cdot (u+v) + d' =$$

$$a \cdot (u^3 + v^3) + 3a \cdot u \cdot v \cdot (u+v) + c' \cdot (u+v) + d' =$$

$$a \cdot (u^3 + v^3) + (3 \cdot a \cdot u \cdot v + c') \cdot (u+v) + d' = 0.$$

When $3a \cdot u \cdot v = -c'$, we get the desired reduction, so

$$u^3 + v^3 = -d'/a.$$

So, we need to select u and v for which

$$u^3 + b^3 = -d'/a \text{ and } u \cdot v = -c'.$$

The second equality leads to $u^3 \cdot v^3 = (-d'/a')^3$.

We know the sum and product of u^3 and v^3, so, as we have mentioned earlier, we can find both by solving the corresponding quadratic equation. Thus, we can find u, v, and finally $x' = u + v$.

4 Acknowledgments

This work was supported in part by the National Science Foundation grants:

- 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and
- HRD-1834620 and HRD-2034030 (CAHSI Includes).

It was also supported:

- by the AT&T Fellowship in Information Technology, and
- by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.