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Shall We Use Logical Approach or More Traditional Mamdani Approach in Fuzzy Control: Pragmatic Analysis

R. Noah Padilla
The University of Texas at El Paso, rnpadilla2@miners.utep.edu

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

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Shall We Use Logical Approach or More Traditional Mamdani Approach in Fuzzy Control: Pragmatic Analysis

R. Noah Padilla[1], Olga Kosheleva[2], and Vladik Kreinovich[1]
[1]Department of Computer Science
[2]Department of Teacher Education
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
rnpadilla2@miners.utep.edu, vladik@utep.edu

Abstract

Fuzzy control methodology transforms the experts' if-then rules into a precise control strategy. From the logical viewpoint, an if-then rule means implication, so it seems reasonable to use fuzzy implication in this transformation. However, this logical approach is not what the first fuzzy controllers used. The traditional fuzzy control approach – first proposed by Mamdani – transforms the if-then rules into a statement that only contains and's and or's, and does not use fuzzy implication at all. So, a natural question arises: shall we use logical approach or the traditional approach? In this paper, we analyze this question on the example of a simple system of if-then rules. It turns out that the answer to this question depends on what we want: if we want the smoothest possible control, we should use logical approach, but if we want the most stable control, then the traditional (Mamdani) approach is better.

Keywords: fuzzy control, Mamdani approach, logical approach, stable control, smooth control

1 Formulation of the Problem

How control problems led to fuzzy logic: a brief reminder. In the early 1960s, Lotfi Zadeh, then one of the main specialists in optimal control and a co-author of the most popular textbook on optimal control, was thinking about the puzzling fact: that in many practical situations, supposedly optimal controllers were performing worse than skilled human operators. A natural answer to this puzzle was that expert human controllers possess some additional knowledge that was not implemented in the automatic controllers.

A seemingly natural idea is therefore to elicit this additional knowledge and to implement it in the automatic systems. Control specialists were willing to share this knowledge. Moreover, a large portion of this knowledge was already available in the published form, in textbooks, manuals, papers, printed instructions, etc. However, there was a big obstacle preventing the use of this additional knowledge: this additional knowledge was usually formulated in terms of if-then rules involving imprecise ("fuzzy") natural-language words like "small". For example, a typical rule would read: "if the temperature is slightly higher than desired, turn on the cooling a little bit".

Computers were originally designed to process numbers, not natural-language words. Thus, to overcome this obstacle, it was necessary to translate this natural-language knowledge into numerical terms, in terms understandable to computers. This necessity was one of the main motivation for Zadeh's invention of fuzzy techniques.

Fuzzy techniques: main idea. To better understand the motivations behind this paper, let us briefly recall – from the above viewpoint – what fuzzy techniques are about; for details, see, e.g., [1, 2, 3, 4, 5, 7].

In fuzzy techniques, each natural-language term like "small" is described by a function that assigns, to each possible value x of the corresponding quantity, a degree $\mu(x)$ (from the interval $[0,1]$) to which the expert agrees that this value satisfies the corresponding property (e.g., is small). Here:

- 1 means that the property is definitely satisfied (i.e., is true),
- 0 means that the property is definitely not satisfied (i.e., if false), and
- values between 0 and 1 correspond to intermediate degrees of expert's confidence.

The corresponding function $\mu(x)$ is known as a *membership function* or, alternatively, a *fuzzy set*.

Fuzzy logic. Expert rules usually combine imprecise statements by using logical connectives like "and", "or", "not", and "if-then" ("implies"). For example, a rule may say "If the temperature is lightly higher than desired and the pressure is slightly lower, then switch on Controller 1 or Controller 3 a little bit".

In the usual 2-valued logic, the truth value of such logical combinations like "A and B" is uniquely determined by the truth values of the component statements A and B. In fuzzy techniques, in addition to the usual values "true" (1) and "false" (0), we also use intermediate degrees. It is therefore necessary to extend the usual logical operations from the 2-element set $\{``true'', ``false''\} = \{0,1\}$ to the whole interval $[0,1]$. The interval $[0,1]$ with such extended operations is what is usually described as fuzzy logic.

Zadeh himself proposed several such extensions – many of which are still used in fuzzy systems. For example:

- one of the ways he proposed to extend "and" to the interval $[0,1]$ was to use minimum $f_\&(a,b) = \min(a,b)$,

- one of the ways he proposed to extend "or" to the interval $[0,1]$ was to use maximum $f_\vee(a,b) = \max(a,b)$,
- one of the ways he proposed to extend negation to the interval $[0,1]$ was to use $f_\neg(a) = 1 - a$, and
- one of the ways he proposed to extend implication $A \to B$ to the interval $[0,1]$ was to use $f_\to(a,b) = \max(1-a,b)$, which corresponds to the fact in the 2-valued logic, $A \to B$ is equivalent to $\neg A \vee B$.

Comment. Actually, the first fuzzy papers only had fizzy analogs of "and", "or", and "not". Implications came much later, and this is explainable: the main objective of fuzzy techniques is to formally describe human reasoning. For "and", "or", and "not", traditional logical operations are in good accordance with common sense – so it make sense to extend the corresponding traditional logical operations from the 2-element set $\{0,1\}$ to the interval $[0,1]$.

In contrast, for implication, already the 2-valued mathematical definition of this operation is far from being intuitive. Statements like "if $2+2=5$ then there are 5 witches in this room" make perfect mathematical sense, but are not intuitive at all.

Defuzzification. As a result of applying fuzzy rules, we get, for each possible value of control u, the degree $\mu(u)$ to which this value is consistent with these rules. However, we cannot directly use this knowledge in the resulting automatic controller: we need to select a single control value $\overline{u}$. This selection transforms the fuzzy set into a *single* (non-fuzzy) number and is, thus, known as *defuzzification.*

In data processing, one of the main criterion of how well a model fits is the least squares approach – where we minimize the sum of the squares of the differences. Thus, a natural – and widely used – idea is to select the value $\overline{u}$ for which the mean square difference between the selected value $\overline{u}$ and a possible value u is the smallest possible, and to use $\mu(u)$ as the weight with which we take the corresponding square. In precise terms, the idea is to minimize the integral $\int \mu(u) \cdot (\overline{u} - u)^2\,du$. Differentiating this expression with respect to the unknown $\overline{u}$ and equating the resulting derivative to 0, we conclude that

$$\overline{u} = \frac{\int u \cdot \mu(u)\,du}{\int \mu(u)\,du}.$$

This formula is known as *centroid defuzzification.*

Logical and traditional approaches to fuzzy control. Suppose that we have several fuzzy rules $A_1(x) \to B_1(u)$, $A_2(x) \to B_2(u)$, $\ldots$, $A_m(x) \to B_m(u)$, where A_i and B_i are natural language terms. Let us denote membership functions corresponding to these terms by $a_i(x)$ and $b_i(u)$.

From the purely logical viewpoint, for a given input x, a control value u is consistent with all these rules if all these implications hold, i.e., if the following formula holds:

$$(A_1(x) \to B_1(u)) \,\&\, (A_2(x) \to B_2(u)) \,\&\, \ldots \,\&\, (A_m(x) \to B_m(u)). \quad (1)$$

Thus, if we use fuzzy logic operations $f_\&$ and $f_\to$ corresponding to & and $\to$, we get the following membership function for u (here, L stands for *L*ogical):

$$\mu_L(u) = f_\&(f_\to(a_1(x), b_1(u)), f_\to(a_2(x), b_2(u)), \ldots, f_\to(a_m(x), b_m(u))). \quad (2)$$

Interestingly, this is not what the first fuzzy controllers used. In line with our point about a somewhat counter-intuitive notion of implication, researchers tried to avoid implication and reformulate the expert rules in terms that does not include implication – only "and", "or", and "not". Such a reformulation is indeed possible if we take into account that, intuitively, a control value u is reasonable if:

- either the first rule is applicable, i.e., x satisfies the property A_1 and u satisfies the property B_1,
- or the second rule is applicable, i.e., x satisfies the property A_2 and u satisfies the property B_2,
- ...
- or the m-th rule is applicable, i.e., x satisfies the property A_m and u satisfies the property B_m.

In this interpretation, the statement that u is a reasonable control value takes a different form:

$$(A_1(x) \,\&\, B_1(u)) \vee (A_2(x) \,\&\, B_2(u)) \vee \ldots \vee (A_m(x) \,\&\, B_m(u)). \quad (3)$$

Then, if we use fuzzy logic operations $f_\&$ and $f_\vee$ corresponding to & and $\vee$, we get the following membership function for u (here, T stands for *T*raditional):

$$\mu_T(u) = f_\vee(f_\&(a_1(x), b_1(u)), f_\&(a_2(x), b_2(u)), \ldots, f_\&(a_m(x), b_m(u))). \quad (4)$$

This formula was first used by Mamdani and is thus known as *Mamdani approach* to fuzzy control.

Which approach is better? Which of the above formulas leads to a better control? The answer may depend on what we want from control. For example, if we are designing a controller for a car, we may want to have a control that provides the smoothest possible ride. On the other hand, if we are controlling a chemical plant, we may want to make that any potentially dangerous deviation from the desired regime is corrected as soon as possible – i.e., that the control is the most stable one.

Similarly, in medical applications, if a patient is in danger (e.g., has high very fever), we want the most stable control: we want to make sure that the dangerous very-high-fever state ends as soon as possible, even when the appropriate measures may not be very comfortable to the patient. On the other hand, if a patient has a minor fever, we do not want to make the patient uncomfortable by prescribing strong not-very-pleasant measure, we prefer a smoother control.

What we do in this paper. In this paper, on a simple illustrative example, we analyze both logical and traditional approaches to fuzzy control from the viewpoint of smoothness and stability of the resulting control.

We want to make our arguments as convincing as possible, so we have tried to make all the computations and discussions as simple and as clear as possible.

Comment. For the traditional Mamdani control, a similar analysis was performed in [6] to see which pairs of "and" and "or"-operations lead to the smoothest control and which pairs lead to the most stable control.

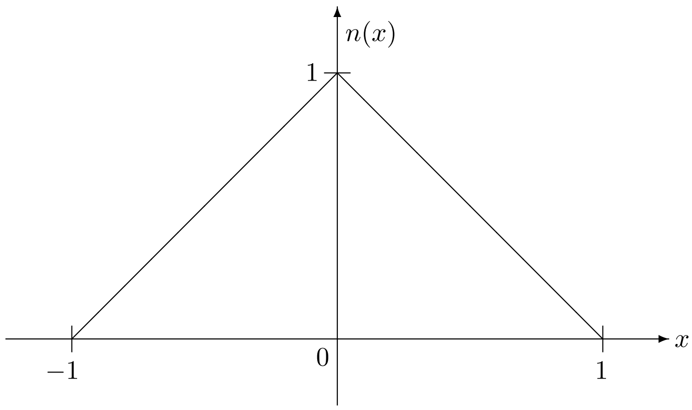
2 Simple Illustrative Example

Physical example. Let us consider – in line with one of the above examples – simple rules for thermoregulation. Suppose that there is a knob by rotating which we change the temperature: if we turn it to the right, the temperature increases, and if we turn it to the left, the temperature decreases. In this case, the input to the control is the difference $x = t - t_0$ between the current temperature t and the desired temperature t_0, and the control value u is the angle to which we turn the control knob.

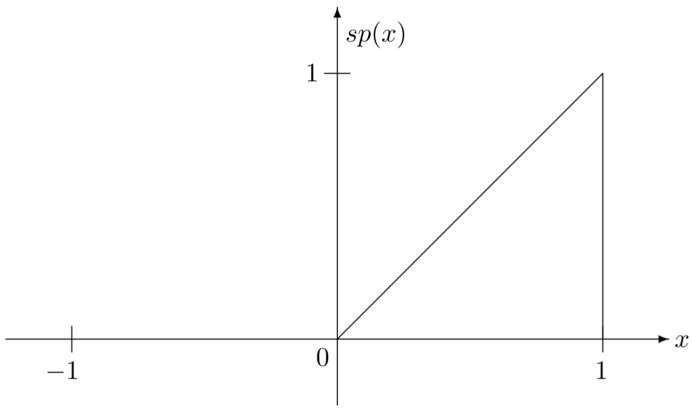
Let us consider only small deviations, when both the difference in temperature does not exceed plus minus 1 unit (e.g., 1 degree or 10 degrees), and the angle u also does not exceed 1 unit (e.g., 1 angular degree).

Appropriate natural-language terms and the resulting fuzzy sets. In the above case, from the commonsense viewpoint, both x and u can be either negligible (we will denote it by N) or small positive (we will denote it by SP) or small negative (we will denote it by SN). To numerically describe these terms, we will use the simplest – triangular – membership functions, membership functions that are obtained by linear interpolation from known values at which the property is absolutely true and absolutely false.

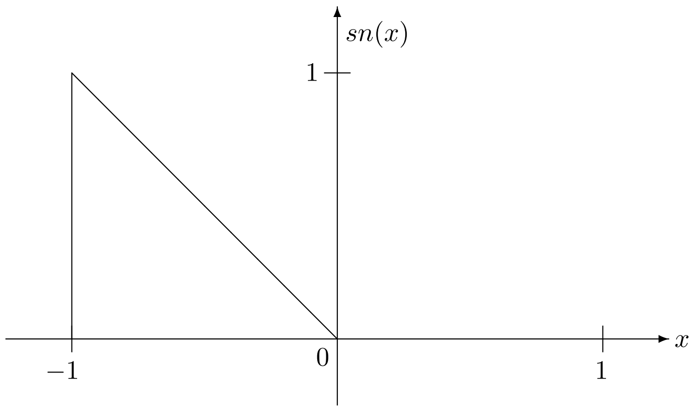
For "negligible", we know that 0 is absolutely negligible, and the borderline values ± 1 are absolutely not negligible. Thus, we get the following triangular membership function $n(x) = 1 - |x|$:



For small positive, we know that 1 is definitely small positive, and that 0 (and all negative numbers) are absolutely not small positive. In this case, linear interpolation leads to $sp(x) = x$ for $x > 0$ and $sp(x) = 0$ for $x \leq 0$:



Finally, for small negative, we know that -1 is definitely small negative, and that 0 (and all positive numbers) are absolutely not small negative. In this case, linear interpolation leads to $sn(x) = -x$ for $x < 0$ and $sn(x) = 0$ for $x \geq 0$:



Rules. From the commonsense viewpoint, we have the following three rules:

- if x is negligible, then u is negligible;
- if x is small positive, then u is small negative;
- if x is small negative, then u is small positive.

Comment. Which we consider our simplified example, these three rules are what most fuzzy rules bases use for small deviations from the desired state.

What we plan to do. Let us consider the above two approached to fuzzy control and see what these approaches lead to for small deviations x, i.e., for deviations for which $|x| \ll 1$. For both approaches, we will use the simplest fuzzy logical operations $f_{\&}(a,b) = \min(a,b)$, $f_{\vee}(a,b) = \max(a,b)$, and $f_{\to}(a,b) = \max(1-a,b)$.

Comment. Due to the symmetry of the problem, in both cases, it is sufficient to consider positive values x. Indeed, one can show that for $x < 0$ the resulting control $\overline{u}(x)$ will be minus the control corresponding to $-x > 0$: $\overline{u}(x) = -\overline{u}(-x)$.

3 Case of Traditional (Mamdani) Fuzzy Control

For the traditional fuzzy control, with the above selection of simple fuzzy logic operations, the membership function (4) takes the form

$$\mu_T(u) = \max(\min(n(x), n(u)), \min(sp(x), sn(u)), \min(sn(x), sp(u))). \quad (5)$$

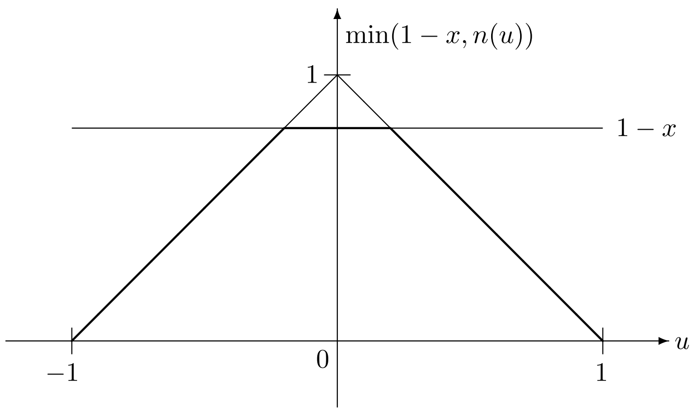
For $x > 0$, due to our selection of membership functions, we have $n(x) = 1 - x$, $sp(x) = x$, and $sn(x) = 0$, so:

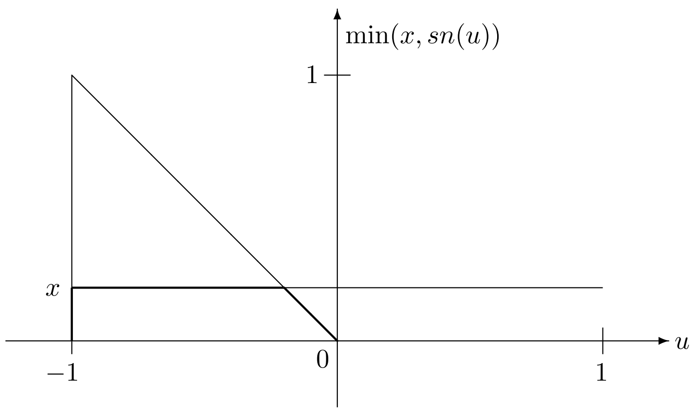
$$\mu_T(u) = \max(\min(1-x, n(u)), \min(x, sn(u)), \min(0, sp(u))). \quad (6)$$

Since $sp(u) \geq 0$, we get $\min(0, sp(u)) = 0$. Here, maximum of any number v from $[0, 1]$ and 0 is v, so we can exclude the third min-term from (6) and get

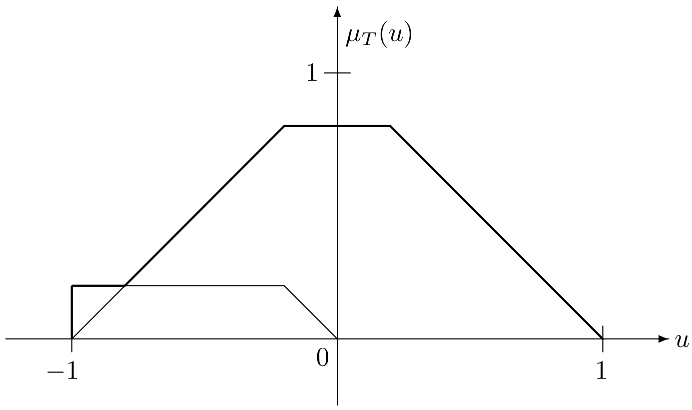
$$\mu_T(u) = \max(\min(1 - x, n(u)), \min(x, sn(u))). \tag{7}$$

The graph of the minimum of two functions can be obtained by taking, for each u, the lowest of the corresponding points from the two graphs. So, for the two remaining min-terms, we get the following graphs:





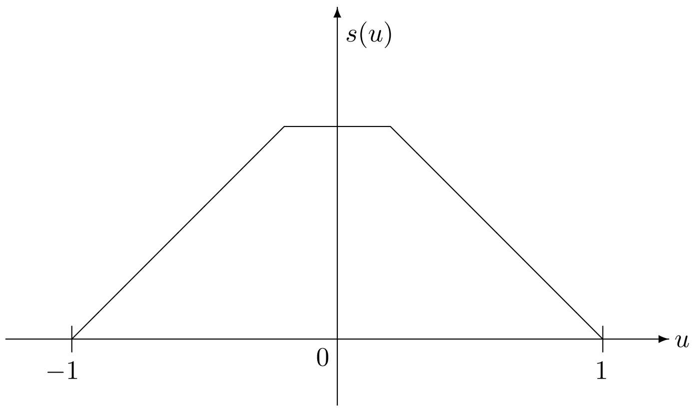
Thus, for the maximum of these two functions, we get:



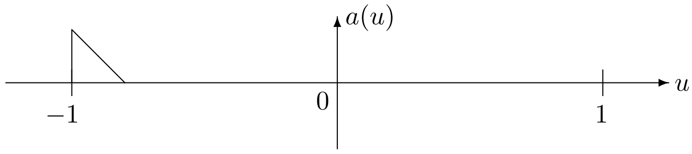
To estimate the resulting control value $\overline{u}$, we need to estimate the numerator $\int u \cdot \mu_T(u)$ and the denominator $\int \mu_T(u)\,du$. Let us start with the denominator.

For small x, the function $\mu_T(u)$ is close to the triangular function $n(u)$ for which $\int n(u)\,du$ is equal to the area of the corresponding triangle with base $1-(-1)=2$ and height 1, i.e., to 1. Thus, asymptotically, $\int \mu_T(u)\,du \sim 1$.

To estimate the numerator, let us notice that $\mu_T(u)$ can be represented as the sum of the symmetric part $s(u)$ – obtained by reflecting the right-hand side of the graph to the left:



and a small asymmetric part $a(u)$ – the difference between the left-hand side and the right-hand side:



For the symmetric function, $\int u \cdot s(u)\,du = 0$, since for every u, the terms $(-u) \cdot s(-u) = -u \cdot s(u)$ and $u \cdot s(u)$ cancel each other. Thus, the numerator is equal to $\int u \cdot a(u)\,du$. For small x, the function $a(u)$ differs from 0 only for $u \approx 1$, so asymptotically, $\int u \cdot a(u)\,du \sim \int a(u)\,du$. The integral $\int a(u)\,du$ is just an area under the triangle with base $(-1+x)-(-1) = x$ and height x, i.e., $x^2/2$. So, asymptotically,

$$\overline{u} = \frac{\int u \cdot \mu_T(u)\,du}{\int \mu_T(u)\,du} \sim \frac{x^2/2}{1} = \frac{x^2}{2}. \tag{8}$$

4 Case of Logical Control

In the case of the logical control, the formula (2) leads to

$$\mu_L(u) =$$

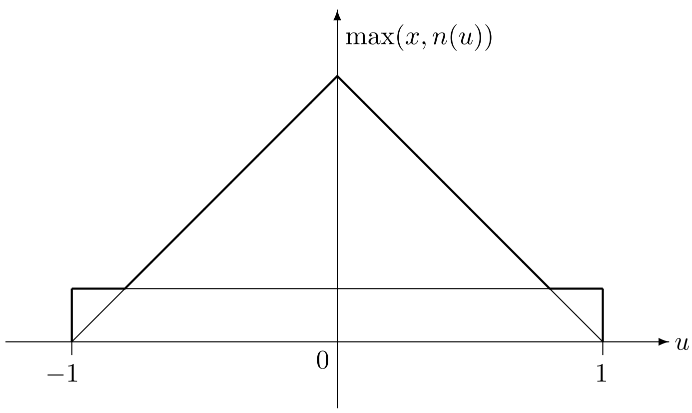
$$\min(\max(1-n(x), n(u)), \max(1-sp(x), sn(u)), \max(1-sn(x), sp(u))) =$$

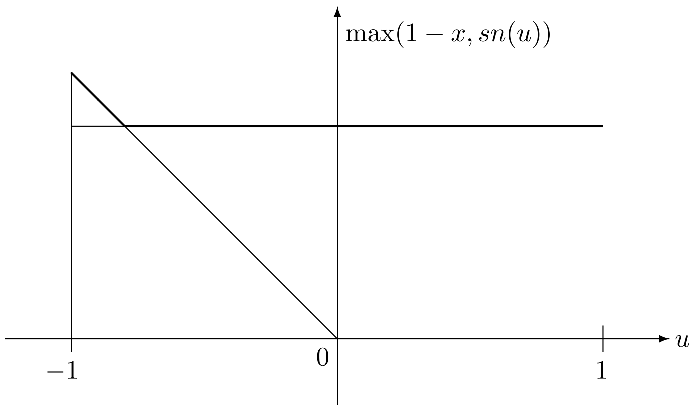
$$\min(\max(x, n(u)), \max(1-x, sn(u)), \max(1, sp(u))). \tag{9}$$

Since $sp(u) \le 1$, we get $\max(1, sp(u)) = 1$. Here, minimum of any number v from $[0,1]$ and 1 is this number v, so we can exclude the third max-term from (9) and get

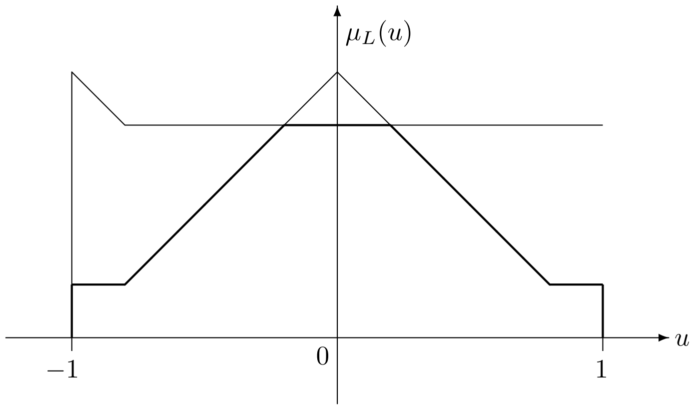
$$\mu_L(u) = \min(\max(x, n(u)), \max(1-x, sn(u))). \tag{10}$$

The two remaining max-terms have the following form:





Thus, the minimum of these two expressions has the following form:



This function is symmetric, so $\int u \cdot \mu_L(u)\,du = 0$ and thus, $\overline{u} = 0$.

5 Conclusions

General idea. According to the above discussion:

- the smoother control corresponds to smaller control values $\overline{u}$, and
- the more stable control corresponding to larger control values $\overline{u}$.

What happens here. For small x, logical control leads to $\overline{u} = 0$, while the traditional (Mamdani) control leads to $\overline{u} = x^2/2 > 0$. Thus, we arrive at the following conclusion:

- if we want the smoothest control, we should use logical fuzzy control, i.e., control based on fuzzy implication;
- on the other hand, if we want the most stable control, then we should use traditional (Mamdani) fuzzy control techniques.

Acknowledgments

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