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How to Solve the Apportionment Paradox

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Christopher Reyes and Vladik Kreinovich

Abstract In the ideal world, the number of seats that each region or each community gets in a representative body should be exactly proportional to the population of this region or community. However, since the number of seats allocated to each region or community is whole, we cannot maintain the exact proportionality. Not only this leads to a somewhat unfair situation, when residents of one region get more votes per person than residents of another one, it also leads to paradoxes – e.g., sometimes a region that gained the largest number of people loses a number of seats. To avoid this unfairness (and thus, to avoid the resulting paradoxes), we propose to assign, to each representative, a fractional number of votes, so that the overall number of votes allocated to a region will be exactly proportional to the region's population. This approach resolves the fairness problem, but it raises a new problem: in a secret vote, if we – as it is usually done – disclose the overall numbers of those who voted For and those who voted Against, we may reveal who voted how. In this paper, we propose a way to avoid this disclosure.

1 Formulation of the Problem

Apportionment: a problem. In a democratic system, where voters elect a representative body, an important issue is how many seats to allocate to each region (or each community). Ideally, the number of seats should be exactly proportional to the number of votes in a given region. However, since the number of representatives is an integer, we cannot have an exact proportion.

For example, at our University of Texas at El Paso, faculty elect the Faculty Senate. On average, every 10 faculty members should elect one member of the Faculty Senate to represent them. However, some departments have fewer than 10 faculty

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members, e.g., 6. For them, the exact proportion would mean that this department has 0.6 voting members, which is not possible – we can have either 0 or 1. Similarly, a department with 11 or 19 members can have either 1 or 2 representative.

In our university:

- every department with up to 10 members has a single representative,
- every department with 11 to 20 members has two representatives, etc.

This is not exactly fair:

- For example, if a department with 10 faculty splits into two equally sized departments with 5 faculty members in each, they would suddenly gain 2 votes instead of 1.
- Vice versa, if two departments with 11 and 5 faculty members decide to merge into a single 16-strong larger department, then instead of the original $2+1=3$ votes, they will be left with only 2 votes.

Not only this practice is perceived as not fair, it can lead to paradoxes, when, within the fixed number of overall seats, a department that gained more new members than another one can get fewer seats – and historically, this has happened in the US apportionment for the House of Representatives; see, e.g., [3].

This is not just a feature of any specific scheme, it was shown that such paradoxes can occur in any apportionment scheme where each region (or each community) gets a whole number of votes; see, e.g., [1, 2, 3].

2 What We Propose

Our suggestion. What we propose is:

- to still assign a whole number of representatives, but
- to give each representative not necessarily 1 vote (as now), but a possibly fractional number of votes, so that the overall number of votes is exactly proportional to the corresponding population.

Then, in voting, a motion passes if we sum of the votes of those who voted for exceed the desired threshold – 50% or 2/3 or whatever is needed for the motion to pass.

Examples. In the above Faculty Senate case:

- The representative of a 10-member department will get exactly 1 vote.
- To maintain the exact proportionality, the representative of a 4-member department will get 0.4 votes.
- Each of the two representatives of a 12-member department will get 0.6 votes, so that their overall number of votes 1.2 is exactly proportional to the department size.

Our proposal is not as radical as it may seem. At first glance, our proposal may sound very radical, but it is not.

Different number of votes for each representative is, in effect, how people vote at shareholders' meetings: the number of votes each representative has is exactly proportional to the number of shares owned by this voter – or by a company that this voter represents.

3 Problem of the Proposed Approach and How to Resolve It

What if we have a secret vote. Sometimes, the vote is open, but sometimes, there may be a need for a secret vote. During a secret vote, the information about how each representative voted remains confidential, but the overall number of those who voted for and those who voted against is usually made public.

The problem is that if representatives have fractional number of votes, then the public disclosure of the overall number of votes for a proposal may disclose who exactly voted for this proposal. To explain this possibility, let us consider a version of the above university example in which all departments has a whole number of votes except for one department *D* whose representative has 0.4 votes. In this case:

- if the overall number of votes for the motion is a whole number, this means that the representative of the department *D* did not vote for this proposal, while
- if the overall number of votes for the motion is not a whole number, this means that the representative of the department *D* did vote for this proposal.

How to resolve this problem. To resolve this problem, a natural solution is to round the number of votes to the nearest whole number – and if the resulting rounded numbers of votes For and Against are equal, subtract 1 from the count of the group that got more votes.

Comment. Why do we need to subtract 1?

In the above situation, we may get a situation in which, out of 41 members with the overall number of 40.4 votes:

- 21 (including the representative of the department *D*) voted for this proposal, while
- 20 others voted against.

In this case, we have 20.4 votes for and 20 votes against.

If we simply round both numbers to the nearest integer, we will get 20 For and 20 Against. Since we know that the motion passed, this would indicate that the representative of the department *D* voted for this proposal.

To avoid this disclosure of voter confidentiality, we need to modify the rounded numbers. We can do it either by adding 1 to the winning side, or by subtracting 1 from the losing side.

- If we add 1 to the winning side, we will get 21 For and 20 Against. One can easily see that this arrangement can only happen if D voted for this proposal: if D voted against, we would have 21 For and 19.4 Against, so rounding would lead to 21:19 – no need for adding.
- On the other hand, if we subtract 1 from the losing side, we will get 20:19. This can happen when the representative of the department D voted for this proposal. This can also happen if this representative voted against this proposal – as well as 19 other representatives, while 20 other representatives voted For. So, in this case, confidentiality of voting is preserved.

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References

1. M. L. Balinski and H. P. Young, *Fair Representation: Meeting the Ideal of One Man, One Vote*, Yale University Press, New Haven, 1982.
2. M. L. Balinski and H. P. Young, *Fair Representation: Meeting the Ideal of One Man, One Vote*, Brookings Institution Press, Washington, DC, 2001.
3. J. D. Stein, *How Math Explains the World: A Guide to the Power of Numbers, from Car Repair to Modern Physics*, Smithsonian Books, New York, 2008.